Nonlinear Differential Equations and Applications NoDEA

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On the viscosity solutions to Trudinger's equation

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Abstract. We study the existence of positive viscosity solutions to Trudinger's equation for cylindrical domains $\Omega \times [0, T)$, where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain, $T > 0$ and $2 \leq p < \infty$. We show existence for general domains Ω, when $n < p < \infty$. For $2 \leq p \leq n$, we prove existence for domains Ω that satisfy a uniform outer ball condition. We achieve this by constructing suitable sub-solutions and super-solutions and applying Perron's method.

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1. Introduction

In this work, we study the existence of positive viscosity solutions to Trudinger's equation. This is a follow-up of the work in [2] where we studied a doubly nonlinear parabolic differential equation involving the infinity-Laplacian. Our goal in the current work is to adapt and apply the ideas developed in [2] to show existence of solutions to an analoguous equation involving the p-laplacian.

To make our discussion more precise, we introduce some definitions and notations. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain and $T > 0$. Define $\Omega_T = \Omega \times (0, T)$ and $P_T = (\Omega \times \{0\}) \cup (\partial\Omega \times [0, T))$ its parabolic boundary.

Let $f \in C(\Omega, \mathbb{R}^+)$, $g \in C(\partial\Omega \times [0, T), \mathbb{R}^+)$ and $u : \Omega_T \times P_T \to \mathbb{R}^+$. We take $2 \leq p < \infty$. Our goal here is to show the existence of a positive viscosity solution u, continuous on $\Omega_T \cup P_T$, to the equation

$$\operatorname{div}(|Du|^{p-2}Du) - (p-1)u^{p-2}u_t = 0 \quad \text{in } \Omega_T , \tag{1.1}$$

$$u(x, 0) = f(x) \text{ and } u(x, t) = g(x, t), \ \forall (x, t) \in \partial\Omega \times [0, T).$$

This doubly nonlinear parabolic differential equation is referred to as Trudinger's equation, see [5,6,8,13] and the references therein, where results have been stated in the context of weak solutions. Our effort is to discuss existence in the context of viscosity solutions, see [3,4]. The operator

$$\Delta_p u = \mathrm{div}(|Du|^{p-2}Du), \quad 2 \le p < \infty,$$

is referred to as the p-Laplacian and is degenerate elliptic. Parabolic equations involving the p-Laplacian have been extensively studied and a detailed discussion may be found in [5]. The equation in (1.1) is doubly nonlinear and such equations are also of great interest.

In our study of (1.1) a central role is played by the following parabolic equation:

$$\Delta_p \eta + (p-1)\,|D\eta|^p - (p-1)\eta_t = 0, \quad \text{in } \Omega_T, \quad 2 \le p < \infty, \qquad (1.2)$$

with $\eta = \hat{f}$, on $\Omega \times \{0\}$, and $\eta = \hat{g}$, on $\partial\Omega \times [0,T)$, where both $\hat{f}$ and $\hat{g}$ are continuous. As a matter of fact, we will show that if u solves (1.1) and $\eta = \log u$ then (1.2) holds. A great part of this work employs this equivalence to prove the existence of u. One may find a detailed study of this equation in [5,11,12], for instance. Incidentally, the related equation $\Delta_p u - u_t = 0$ has also been studied in the viscosity setting, see for instance [7,10] and references therein. A discussion in the weak solution setting may be found in [5]. Some of our results do hold for this equation, however, our focus is primarily the study of (1.1) and (1.2).

We state our main results as two theorems. The first result addresses the case $n < p < \infty$ and we show existence for general domains. The second result states the existence result for $2 \le p \le n$ and holds for domains Ω that satisfy a uniform outer ball condition. At this time, it is not clear to us how to extend the result to general domains. Our proofs of existence make use of the Perron method, see [3,4] for more details. In order to do so we prove a comparison principle for (1.1) and (1.2). Incidentally, the comparison principle for (1.2) implies a quotient type comparison for positive solutions of (1.1). The major part of this work is devoted to the construction of suitable sub-solutions and super-solutions for (1.1). More precisely, at each point on P_T, we construct sub-solutions and super-solutions which are arbitrarily close to the given data at the point. Using the parabolic counterpart of Theorem 4.1 in [4] (see [3]), we conclude existence. For a short description, see Sect.5 in [2]. To achieve our goal the equation in (1.2) proves particularly useful when working with side conditions along $\partial\Omega \times [0,T)$.

For ease of presentation, we set

$$h(x,t) = \begin{cases} f(x), & \forall x \in \Omega, \quad t = 0, \\ g(x,t), & \forall (x,t) \in \partial\Omega \times [0,T). \end{cases} \qquad (1.3)$$

We obtain

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a bounded domain and $T > 0$. Let $h \in C(P_T, \mathbb{R}^+)$, with $\inf_{P_T} h > 0$, and $n < p < \infty$. The problem*

$$\Delta_p u = (p-1)u^{p-2}u_t, \quad and \quad u(x,t) = h(x,t), \forall (x,t) \in P_T.$$

has a unique positive viscosity solution u that is continuous on $\Omega_T \cup P_T$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain and $T > 0$. Let $h \in C(P_T, \mathbb{R}^+)$, with $\inf_{P_T} h > 0$, and $2 \leq p \leq n$. If Ω satisfies an uniform outer ball condition then the problem*

$$\Delta_p u = (p-1) u^{p-2} u_t, \quad and \quad u(x,t) = h(x,t), \; \forall (x,t) \in P_T.$$

has a unique positive viscosity solution u that is continuous on $\Omega_T \cup P_T$.

As corollaries, our work implies existence of solutions of (1.2) for any bounded continuous initial and boundary data.

We now describe the general layout of the work. In Sect. 2, we provide definitions and a change of variables formula showing the equivalence of (1.1) and (1.2). Some additional change of variables results are also stated. These will be followed by a result on a separation of variables. Section 3 contains a maximum principle, a comparison principle and their consequences. The maximum principle requires no sign conditions, however the main comparison principle holds only for positive solutions. This leads to a quotient version of the comparison for solutions to (1.1), see [2]. In Sect. 4, we construct sub-solutions and super-solutions for the initial data ($t = 0$) for $2 \leq p < \infty$. For this purpose, we work directly with the parabolic equation in (1.1). This work is valid for general domains. We discuss the side conditions in Sects. 5 and 6 and make use of the parabolic equation in (1.2). We address the case $n < p < \infty$ for general domains in Sect. 5. Section 6 takes up the case $2 \leq p \leq n$ for domains that satisfy a uniform outer ball condition.

2. Notations, definitions and some preliminaries

In what follows, $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain and $\partial\Omega$ its boundary. Let $\overline{A}$ denote the closure of a set A and A^c the complement of A in $\mathbb{R}^n$. The letters x, y, z are used for points in $\mathbb{R}^n$ and we reserve the letter o for the origin in $\mathbb{R}^n$. The letters s, t denote points in $\mathbb{R}^+ \cup \{0\}$. We use the standard notation for $x \in \mathbb{R}^n$, i.e. $x = (x_1, x_2, \ldots, x_n)$. For $T > 0$, we define the cylinder

$$\Omega_T = \Omega \times (0, T) = \{(x,t) \in \Omega \times \mathbb{R} : \; x \in \Omega, \; 0 < t < T\}. \qquad (2.1)$$

The parabolic boundary P_T of Ω_T is the set $(\Omega \times \{0\}) \cup (\partial\Omega \times [0, T))$.

Let $B_r(x) \subset \mathbb{R}^n$ be the open ball of radius r, centered at x. For $r > 0$ and $\tau > 0$, we define the following open cylinder

$$D_{r,2\tau}(x,t) = B_r(x) \times (t - \tau, t + \tau). \qquad (2.2)$$

In this work we will always take $2 \leq p < \infty$. We now define the operators Π and Γ as follows.

$$\Pi(\phi) = \Delta_p \phi - (p-1)\phi^{p-2}\phi_t \quad and \quad \Gamma(\eta) = \Delta_p \eta + (p-1)|D\eta|^p - (p-1)\eta_t. \qquad (2.3)$$

For studying viscosity solutions of (1.1) and (1.2), we define expressions which are useful in this context and will apply to the operators in (2.3). Let $\mathcal{S}(n)$ be the set of symmetric $n \times n$ matrices and $tr(X)$ denote the trace of a

matrix $X \in S(n)$. For any $(r, a, q, X) \in I\!\!R \times I\!\!R \times I\!\!R^n \times S(n)$, we define the expressions $L_p(q, X)$ and $T_p(r, a, q, X)$ as follows. Define, for $2 \leq p < \infty$,

$$L_p(q, X) = \begin{cases} |q|^{p-2}tr(X) + (p-2)|q|^{p-4}q_i q_j X_{ij}, & \text{for } q \neq 0, \\ tr(X), & p = 2, \\ 0, & \text{for } q = 0, \ p \neq 2. \end{cases} \quad (2.4)$$

The quantity $L_p(q, X)$ is the version for the differentiated p-Laplacian and $L_2(p, X) = tr(X)$. Related to the operator Π in (2.3) is the following expression:

$$T_p(r, a, q, X) = L_p(q, X) - (p-1)|r|^{p-2}a,$$
$$\forall (r, a, q, X) \in I\!\!R \times I\!\!R \times I\!\!R^n \times S(n). \quad (2.5)$$

Similarly, related to Γ, we have

$$K_p(a, q, X) = L_p(q, X) + (p-1)|q|^p - (p-1)a,$$
$$\forall (a, q, X) \in I\!\!R \times I\!\!R^n \times S(n). \quad (2.6)$$

We now recall the definition of viscosity sub-solutions and super-solutions, both via semi-jets and test functions, see [4]. From hereon, $usc(A)$ and $lsc(A)$ denote the sets of all functions that are upper semi-continuous and lower semi-continuous on a set A, respectively.

Let $u : \Omega_T \cup P_T :\to I\!\!R$, and $(y, s) \in \Omega_T$. We recall the definitions of semi-jets $\mathcal{P}^+_{\Omega_T} u(y, s)$ and $\mathcal{P}^-_{\Omega_T} u(y, s)$. An element $(a, q, X) \in I\!\!R \times I\!\!R^n \times \mathcal{S}(n)$ is in $\mathcal{P}^+_{\Omega_T} u(y, s)$ if

$$u(x, t) \leq u(y, s) + a(t - s) + \langle p, x - y \rangle$$
$$+ \frac{\langle X(x - y), x - y \rangle}{2} + o\left(|t - s| + |x - y|^2\right) \quad (2.7)$$

as $(x, t) \to (y, s)$, where $(x, t) \in \Omega_T$. We define $\mathcal{P}^-_{\Omega_T} u = -\mathcal{P}^+_{\Omega_T}(-u)$. For the sets $\overline{\mathcal{P}}^+_{\Omega_T} u(y, s)$ and $\overline{\mathcal{P}}^-_{\Omega_T} u(y, s)$, see [4].

We present definitions of a sub-solution and a super-solution for the differential equations in (1.1) and (1.2). A function $u > 0$ is a sub-solution in Ω_T of the equation in (1.1) or $\Pi(u) \geq 0$ in Ω_T, if $u \in usc(\Omega_T)$ and

$$T_p(u(x, t), a, q, X) \geq 0, \quad \forall (a, q, X) \in \mathcal{P}^+_{\Omega_T} u(x, t) \quad \text{and} \quad \forall (x, t) \in \Omega_T. \quad (2.8)$$

Similarly, $v > 0$ is a super-solution in Ω_T or $\Pi(v) \leq 0$ in Ω_T, if $T_p(v(x, t), a, q, X) \leq 0$, $\forall (a, q, X) \in \mathcal{P}^-_{\Omega_T} u(x, t)$ and $\forall (x, t) \in \Omega_T$.

The definitions of a sub-solution u and a super-solution v of (1.2) are given in terms of K_p, see (2.6). In this case we write $\Gamma(u) \geq 0$ and $\Gamma(v) \leq 0$ respectively. More precisely, u is a sub-solution of the equation in (1.2) or $\Gamma(u) \geq 0$ in Ω_T if $u \in usc(\Omega_t)$ and

$$K_p(a, q, X) \geq 0, \quad \forall (x, t) \in \Omega_T \quad \text{and} \quad \forall (a, q, X) \in \mathcal{P}^+_{\Omega_T} u(x, t).$$

The definition of a super-solution v i.e., $\Gamma(v) \leq 0$ is similar.

We now present the definitions of a sub-solution and a super-solution in terms of test functions. In the rest of this work, we will always take a test function $\psi(x, t)$ to be C^2 in x and C^1 in t on Ω_T.

We say that $u > 0$ is a sub-solution of the equation in (1.1), written as $\Pi(u) \geq 0$ in Ω_T, if $u \in usc(\Omega_T)$ and for all test functions ψ such that $u - \psi$ has a local maximum at some $(y, s) \in \Omega_T$, we have

$$\Delta_p \psi(y, s) - (p-1)u^{p-2}(y, s)\psi_t(y, s) \geq 0. \tag{2.9}$$

The function $u > 0$ is a super-solution i.e., $\Pi(u) \leq 0$, if $u \in lsc(\Omega)$ and we have that $\Delta_p \psi(y, s) - (p-1)u^{p-2}(y, s)\psi_t(y, s) \leq 0$, for any ψ, a test function, and $(y, s) \in \Omega_T$ such that $u - \psi$ has a local minimum at (y, s).

We say η is a sub-solution of the equation in (1.2), written as $\Gamma(\eta) \geq 0$ in Ω_T, if, for any test function ψ and $(y, s) \in \Omega_T$ such that $\eta - \psi$ has a local maximum at (y, s), we have $\Gamma(\psi)(y, s) \geq 0$. Next, η is a super-solution i.e., $\Gamma(\eta) \leq 0$ in Ω_T, if for any test function ψ and $(y, s) \in \Omega_T$ such that $\eta - \psi$ has a local minimum at (y, s), we have $\Gamma(\psi)(y, s) \leq 0$.

We define u to be a sub-solution of the problem in (1.1) if $u \in usc(\overline{\Omega} \times [0, T))$ and

$$\Pi(u) \geq 0, \quad \text{in } \Omega_T, \quad u(x, t) \leq h(x, t), \quad \forall (x, t) \in P_T, \tag{2.10}$$

see (1.3) for the definition of h. Similarly, u is a super-solution of (1.1) if $u \in lsc(\Omega_T \cup P_T)$ and

$$\Pi(u) \leq 0, \quad \text{in } \Omega_T, \quad u(x, t) \geq h(x, t), \quad \forall (x, t) \in P_T \tag{2.11}$$

We say u solves (1.1) if both (2.10) and (2.11) hold. In this case, $u \in C(\Omega_T \cup P_T)$, $\Pi(u) = 0$ in Ω_T and $u = h$ on P_T. Similar definitions can be provided for the problem in (1.2).

From hereon, all differential equations and inequalities will be understood in the viscosity sense.

We now present Lemma 2.1 that addresses the change of variables needed for the equivalence of the equations in (1.1) and (1.2). This is followed by Remark 2.2 that contains some useful observations about change of variables involving the independent variables.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ and $T > 0$. Suppose that $\phi : \Omega_T \to \mathbb{R}$ and $\phi > 0$. Let the operators Π and Γ be as in (2.3). Set $\eta = \log \phi$; then the following hold in Ω_T:*

(i) $\phi \in usc(\Omega_T)$ and $\Pi(\phi) \geq 0$ in Ω_T, if and only if $\eta \in usc(\Omega_T)$
    and $\Gamma(\eta) \geq 0$. Similarly,

(ii) $\phi \in lsc(\Omega_t)$ and $\Pi(\phi) \leq 0$ in Ω_T, if and only if $\eta \in lsc(\Omega_T)$
    and $\Gamma(\eta) \leq 0$ in Ω_T.

Thus, ϕ solves $\Pi(\phi) = 0$ in Ω_T if and only if η solves $\Gamma(\eta) = 0$ in Ω_T.

Proof. We prove part (i), the proof of part (ii) is analogous. Suppose that $\phi > 0$ solves $\Pi(\phi) \geq 0$ in the sense of viscosity. Let $\psi(x, t)$ be a test function and $(y, s) \in \Omega_T$ be such that $\eta - \psi$ has a maximum at (y, s). The inequality $(\eta - \psi)(x, t) \leq (\eta - \psi)(y, s)$ is equivalent to

$$\phi(x, t) \leq \phi(y, s)e^{\psi(x,t) - \psi(y,s)} \tag{2.12}$$

Set $\xi(x,t) = \phi(y,s)\left\{e^{\psi(x,t)-\psi(y,s)} + \psi(y,s) - 1\right\}$ and observe that $\xi(y,s) = \phi(y,s)\psi(y,s)$ and $\xi_t(y,s) = \phi(y,s)\psi_t(y,s)$. Using (2.12) we see that

$$(\phi - \xi)(x,t) \leq \phi(y,s)\left[1 - \psi(y,s)\right] = (\phi - \xi)(y,s). \tag{2.13}$$

We observe

$$\left(\Delta_p e^{\psi}\right)(y,s) = e^{(p-1)\psi(y,s)}\left(\Delta_p\psi(y,s) + (p-1)|D\psi|^p(y,s)|\right).$$

Employing the above, we obtain

$$\Delta_p\xi(y,s) = \left(\frac{\phi(y,s)}{e^{\psi(y,s)}}\right)^{p-1}\left(\Delta_p e^{\psi}\right)(y,s)$$

$$= (\phi(y,s))^{p-1}\left(\Delta_p\psi(y,s) + (p-1)|D\psi|^p(y,s)|\right) \tag{2.14}$$

Since ϕ is a sub-solution, i.e., $\Delta_p\xi(y,s) - (p-1)\phi^{p-2}(y,s)\xi_t(y,s) \geq 0$ (see 2.9 and 2.13), the definition of ξ implies that

$$\Delta_p\xi(y,s) - (p-1)\phi^{p-2}(y,s)\xi_t(y,s)$$
$$= \phi(y,s)^{p-1}\left\{\Delta_p\psi(y,s) + (p-1)|D\psi|^p(y,s) - (p-1)\psi_t(y,s)\right\} \geq 0.$$

Thus, $\Gamma(\psi)(y,s) \geq 0$ and the claim holds.

We prove the converse. Let $\Gamma(\eta) \geq 0$ and suppose that $\psi(x,t)$ is a test function and and $(y,s) \in \Omega_T$ is such that $\phi - \psi$ has a maximum at (y,s). Using $\eta = \log\phi$, we have

$$\eta(x,t) \leq \log\left(\phi(y,s) + \psi(x,t) - \psi(y,s)\right).$$

Set

$$\zeta(x,t) = \log\left(\phi(y,s) + \psi(x,t) - \psi(y,s)\right) - \frac{\psi(y,s)}{\phi(y,s)}.$$

Using the bound on η, one sees that $(\eta - \zeta)(x,t) \leq (\eta - \zeta)(y,s) = \psi(y,s)/\phi(y,s)$. Thus, $\Gamma(\zeta)(y,s) \geq 0$, see (2.3) and (2.9). Differentiating,

$$D\zeta(y,s) = \frac{D\psi(y,s)}{\phi(y,s)}, \quad \zeta_t(y,s) = \frac{\psi_t(y,s)}{\phi(y,s)} \quad \text{and}$$

$$\Delta_p\zeta = \frac{\Delta_p\psi(y,s)}{\phi(y,s)^{p-1}} - \frac{(p-1)|D\psi|^p(y,s)}{\phi(y,s)^p}.$$

Using these expressions in $\Gamma(\zeta)(y,s) \geq 0$, we obtain

$$\Delta_p\psi(y,s) - (p-1)\phi^{p-2}(y,s)\psi_t(y,s) \geq 0.$$

$\square$

Remark 2.2. Assume that $o \in \Omega$. We discuss some additional change of variables formulas. Let α, β, λ, σ be positive constants and ϕ and η be defined on Ω. Set $z = \alpha x$, $\omega = \beta t$ and $\Omega_{\beta T}^{\alpha} = \{(\alpha x, \beta t) : (x,t) \in \Omega\}$.

(a) Suppose that η solves

$$\Delta_p\eta + (p-1)|D\eta|^p - (p-1)\eta_t \geq (\leq)0, \quad \text{in } \Omega_T. \tag{2.15}$$

Set $\varphi(z,\omega) = \eta(x,t)/\lambda$, differentiating $D_x\eta = \lambda\alpha D_z\varphi$, $D_{xx}\eta = \lambda\alpha^2 D_{zz}\varphi$, $D_t\eta = \lambda\beta D_\omega\varphi$ and $\Delta_p\eta = \lambda^{p-1}\alpha^p\Delta_p\varphi$.

(i) From (2.15), we see that

$$\Delta_p \varphi + \lambda(p-1)|D\varphi|^p - \frac{(p-1)\beta}{\alpha^p \lambda^{p-2}} \varphi_\omega \geq (\leq)0, \quad \text{in } \Omega_{\beta T}^\alpha.$$

(ii) Taking $\lambda = 1$ and $\beta = \alpha^p$ in part (i), we get

$$\Delta_p \varphi + (p-1)|D\varphi|^p - (p-1)\varphi_\omega \geq (\leq)0, \quad \text{in } \Omega_{\alpha^p T}^\alpha.$$

Clearly, the Trudinger equation is invariant under this change of variables. Selecting $\lambda = (p-1)^{-1}$ and $\beta = \alpha^p \lambda^{p-1}$ in part (i), (2.15) yields $\Delta_p \varphi + |D\varphi|^p - \varphi_\omega \geq (\leq)0$, in $\Omega_{\beta T}^\alpha$.

(iii) In (i) we choose $\beta = \lambda^{p-2}\alpha^p$. Then (2.15) leads to

$$\Delta_p \varphi + \lambda(p-1)|D\varphi|^p - (p-1)\varphi_\omega \geq (\leq)0.$$

(b) Finally, assume that $\phi > 0$ solves

$$\Delta_p \phi - (p-1)\phi^{p-2}\phi_t \geq (\leq)0, \quad \text{in } \Omega_T.$$

For the change of variables described above, set $\psi(z, \omega) = \phi(x, t)^{1/\theta}$, where $\theta > 0$. Then $D_x \phi = \theta\alpha\psi^{\theta-1}D_z\psi$ and $\phi_t = \theta\beta\psi^{\theta-1}\psi_\omega$. Hence,

$$\begin{aligned}
\Delta_p \phi &- (p-1)\phi^{p-2}\phi_t \\
&= \alpha^p \theta^{p-1}\text{div}(\psi^{(\theta-1)(p-1)}|D\psi|^{p-2}D\psi) - (p-1)\theta\beta\psi^{\theta(p-2)+\theta-1}\psi_\tau \\
&= \alpha^p \theta^{p-1}\psi^{(\theta-1)(p-1)}\left(\Delta_p\psi + (\theta-1)(p-1)\frac{|D\psi|^p}{\psi}\right) - (p-1)\theta\beta\psi^{\theta(p-1)-1}\psi_\omega \\
&= \alpha^p \theta^{p-1}\psi^{(\theta-1)(p-1)}\left(\Delta_p\psi + (\theta-1)(p-1)\frac{|D\psi|^p}{\psi} - \frac{(p-1)\beta}{\alpha^p\theta^{p-2}}\psi^{p-2}\psi_\omega\right),
\end{aligned}$$

since $\theta(p-1)-1 = (\theta-1)(p-1)+p-2$. Taking $\beta = \alpha^p \theta^{p-2}$ and $\theta = p/(p-1)$, we get that

$$\Delta_p\psi + \frac{|D\psi|^p}{\psi} - (p-1)\psi^{p-2}\psi_\omega \geq (\leq)0.$$

Analogues of some of our results in this work hold for the partial differential equations in (i)–(iii) in part (a) and part (b). However, our primary interest will be Trudinger's equation. $\qquad \square$

Finally, we state a separation of variables result that will be used in constructing sub-solutions and super-solutions in Sect. 4.

Lemma 2.3. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain,* $T > 0$ *and* $\lambda \in \mathbb{R}$*. Suppose that* $\eta(t) \in C^1$ *with* $\eta > 0$*. If* $\phi(x) \in usc(lsc)(\Omega)$*,* $\phi > 0$*, solves* $\Delta_p\phi + \lambda\phi^{p-1} \geq (\leq)0$*, in* Ω*, and* $\psi(x, t) = \phi(x)\eta(t)$ *then*

$$\Pi(\psi) := \Delta_p\psi - (p-1)\psi^{p-2}\psi_t \geq (\leq) - \psi^{p-1}(t)\left(\lambda + (p-1)\frac{\eta'}{\eta}\right), \quad \text{in } \Omega_T.$$

In particular, if $\eta(t) = e^{\ell t}$ *and* $\ell + \lambda/(p-1) \leq (\geq)0$ *then* $\Pi(\psi) \geq (\leq)0$*, in* Ω_T*.*

Proof. We prove the claim when $\Delta_p\phi + \lambda\phi^{p-1} \geq 0$ in Ω. Let ζ be a test function and $(y, s) \in \Omega_T$ be such that $\psi - \zeta$ has a maximum at (y, s), i.e.,

$$\phi(x)\eta(t) - \zeta(x, t) \leq \phi(y)\eta(s) - \zeta(y, s). \tag{2.16}$$

Taking $x = y$ in (2.18), we obtain $\phi(y)(\eta(t) - \eta(s)) \leq \zeta(y, t) - \zeta(y, s)$, $\forall\, 0 < t < T$. This yields $\zeta_t(y, s) = \phi(y)\eta'(s)$. Next, taking $t = s$, we obtain

$$\phi(x) - \frac{\zeta(x, s)}{\eta(s)} \leq \phi(y) - \frac{\zeta(y, s)}{\eta(s)}, \quad \forall x \in \Omega.$$

Thus, $\Delta_p\zeta(y, s) + \lambda\psi^{p-1}(y, s) \geq 0$. Applying these observations,

$$\begin{aligned}
\Delta_p\zeta(y, s) &- (p - 1)\psi(y, s)^{p-2}\zeta_t(y, s) \\
&\geq -\lambda\psi^{p-1}(y, s) - (p - 1)\psi(y, s)^{p-2}\phi(y)\eta'(s) \\
&= -\psi^{p-1}\left(\lambda + (p - 1)\frac{\eta'(s)}{\eta(s)}\right)
\end{aligned}$$

If $\eta(t) = e^{\ell t}$ then $\eta'/\eta = \ell$. The claim holds. $\qquad\square$

For a fixed $z \in \mathbb{R}^n$ set $r = |x - z|$. If $f(r) = f(x)$ it is well-known that

$$\Delta_p f(r) = |f'(r)|^{p-2}\left((p - 1)f''(r) + \frac{n - 1}{r}f'(r)\right), \quad 1 < p < \infty. \quad (2.17)$$

We now record a simple calculation which will be used in Sects. 5 and 6.

Remark 2.4. Let γ, $\lambda \in \mathbb{R}$ and $c > 0$. Set $\Lambda = (p - n)(p - 1)^{-1} - \gamma$. Using (2.17) we calculate, in $r > 0$,

$$\begin{aligned}
\Delta_p(\pm cr^\gamma) &= \pm c^{p-1}|\gamma r^{\gamma-1}|^{p-2}\left\{(p - 1)\gamma(\gamma - 1)r^{\gamma-2} + (n - 1)\gamma r^{\gamma-2}\right\} \\
&= \pm(p - 1)c^{p-1}\gamma|\gamma|^{p-2}r^{(p-2)(\gamma-1)+(\gamma-2)}\left((\gamma - 1) + \frac{n - 1}{p - 1}\right) \\
&= \pm(p - 1)c^{p-1}\gamma|\gamma|^{p-2}r^{p(\gamma-1)-\gamma}\left(-\Lambda\right) \\
&= \pm(p - 1)c^{p-1}|\gamma|^p r^{p(\gamma-1)}\left(\frac{-\Lambda}{\gamma r^\gamma}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta_p(\pm cr^\gamma) &+ \lambda(p - 1)|D(\pm cr^\gamma)|^p \\
&= \lambda(p - 1)c^p|\gamma|^p r^{p(\gamma-1)} \pm (p - 1)c^{p-1}|\gamma|^p r^{p(\gamma-1)}\left(\frac{-\Lambda}{\gamma r^\gamma}\right). \\
&= (p - 1)c^{p-1}|\gamma|^p r^{p(\gamma-1)}\left\{c\lambda \pm \left(\frac{-\Lambda}{\gamma r^\gamma}\right)\right\}. \quad (2.18)
\end{aligned}$$

$\qquad\square$

3. Maximum and comparison principles

In this section, we prove a maximum principle and some comparison principles for the equation in (1.1). The maximum principle is stated for a slightly modified version of the equation in (1.1) and holds without placing any sign restrictions. The comparison principle is proven using the equation in (1.2) and implies a quotient type comparison principle for positive solutions to (1.1). As a consequence, this implies uniqueness for positive viscosity solutions to (1.1).

See [2] for an analogue for a doubly nonlinear parabolic equation involving the infinity-Laplacian.

Lemma 3.1. (Weak maximum principle) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain and $T > 0$.*
(i) If $\phi \in usc(\Omega_T \cup P_T)$ solves $\Delta_p \phi - (p-1)|\phi|^{p-2}\phi_t \geq 0$, in Ω_T, then

$$\sup_{\Omega_T} \phi \leq \sup_{P_T} \phi = \sup_{\Omega_T \cup P_T} \phi.$$

(ii) If $\phi \in lsc(\Omega_T \cup P_T)$ and $\Delta_p \phi - (p-1)|\phi|^{p-2}\phi_t \leq 0$, in Ω_T, then

$$\inf_{\Omega_T} \phi \geq \inf_{P_T} \phi = \inf_{\Omega_T \cup P_T} \phi.$$

Proof. We prove part (i). We note that since Ω is bounded, we choose a $\xi \in \mathbb{R}^n \setminus \overline{\Omega}$ and an $0 < R < \infty$ such that $\Omega \subset B_R(\xi)$. Fix ξ and set $r = |x - \xi|$.

Fix τ close to T with $\tau < T$. We first show that the weak maximum principle holds in Ω_τ for any $\tau < T$. Set

$$m = \sup_{\Omega_\tau} \phi, \quad \ell = \sup_{P_\tau} \phi, \quad c = \sup_{\overline{\Omega}} \phi(x, \tau), \quad \delta = m - \ell \quad \text{and} \quad k = \max(\delta, c - \ell). \tag{3.1}$$

We argue by contradiction and assume that $\delta > 0$. Since Ω_τ is an open set there is a point $(y, s) \in \Omega_\tau$ such that $\phi(y, s) > \ell + 3\delta/4$ and $0 < s < \tau$. Define

$$g(t) = \begin{cases} 0, & 0 \leq t \leq s, \\ (t - s)^4/(\tau - s)^4, & s \leq t \leq \tau. \end{cases}$$

Select $0 < \varepsilon \leq \min(\delta/4, 1/2)$. Set

$$\psi(x, t) = \psi(r, t) = \ell + \frac{\varepsilon}{4} + kg(t) - \frac{\varepsilon r^2}{160R^2}, \quad \forall (x, t) \in \overline{\Omega}_\tau.$$

Then $\psi(x, t) \geq \ell + \varepsilon/8$, $\forall (x, t) \in \overline{\Omega}_\tau$, and $\psi(x, \tau) \geq c + \varepsilon/8$, $\forall x \in \overline{\Omega}$. Moreover,

$$\phi(y, s) - \psi(y, s) \geq \ell + \frac{3\delta}{4} - \ell - \frac{\varepsilon}{4} = \frac{3\delta}{4} - \frac{\varepsilon}{4} \geq \frac{3\delta}{4} - \frac{\delta}{16} > \frac{\delta}{4} > 0.$$

Since $\phi - \psi \leq 0$ on $\partial \overline{\Omega}_\tau$, the function $\phi - \psi$ has a positive maximum at some point $(z, \theta) \in \Omega_\tau$. Setting $\rho = |\xi - z|$ and using (2.17) and (3.1), we get

$$\Delta_p \psi(z, \theta) = |\psi'|^{p-2} \left((p-1)\psi'' + \frac{n-1}{r}\psi' \right)(z, \theta)$$

$$= -\left(\frac{\varepsilon}{80R^2} \right)^{p-1} \rho^{p-2}(n - p - 2).$$

Since $g'(t) \geq 0$, we have

$$\Delta_p \psi(z, \theta) < 0 \leq (p-1)|\phi(z, \theta)|^{p-2}\psi_t(z, \theta). \tag{3.2}$$

We obtain a contradiction and our assertion holds in Ω_τ for any $\tau < T$.

If $\sup_{\Omega_T} \phi > \sup_{P_T} \phi$ then there is a point $(y, s) \in \Omega_T$ (with $0 < s < T$) such that $\phi(y, s) > \sup_{P_T} \phi$. Select $s < \bar{s} < T$. Then, $\sup_{P_T} \phi < \phi(y, s) \leq \sup_{\Omega_{\bar{s}}} \phi \leq \sup_{P_{\bar{s}}} \phi \leq \sup_{P_T} \phi$. This is a contradiction and the lemma holds. Part (ii) may be proven similarly. □

Remark 3.2. If $\phi \in C(\Omega_T \cup P_T)$ then equality holds in the conclusions of Lemma 3.1.

Lemma 2.1 implies that a version of Lemma 3.1 holds for $\Delta_p \eta + (p - 1)|D\eta|^p - (p-1)\eta_t = 0$. It is also clear from the proof that Lemma 3.1 applies to $\Delta_p \eta - \eta_t = 0$, see (3.2). □

Next, we prove a comparison principle for (1.1) under the condition that the sub-solutions and the super-solutions are positive in Ω_T, i.e, we require the positivity of their respective infima on $\Omega_T \cup P_T$.

Theorem 3.3. (Comparison principle) *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain and $T > 0$. Let $u \in usc(\Omega_T \cup P_T)$ and $v \in lsc(\Omega_T \cup P_T)$ satisfy*

$$\Delta_p u - (p-1)u^{p-2}u_t \geq 0, \quad and \quad \Delta_p v - (p-1)v^{p-2}v_t \leq 0, \quad in \ \Omega_T.$$

Assume that $\min(\inf_{\Omega_T \cup P_T} u, \inf_{\Omega_T \cup P_T} v) > 0$. If $\sup_{P_T} v < \infty$ and $u \leq v$ on P_T, then $u \leq v$ in Ω_T.

In particular, if u and v are solutions and $u = v$ on P_T then $u = v$ in Ω_T.

Proof. Clearly, $u > 0$ in Ω_T, and since $u \leq v$ on P_T, by Lemma 3.1, u is bounded. Define $\eta(x,t) = \log u(x,t)$ and $\zeta(x,t) = \log v(x,t)$. Then η and ζ are both bounded, in particular, from below. By Lemma 2.1,

$$\Delta_p \eta + (p-1)|D\eta|^p - (p-1)\eta_t \geq 0 \quad and$$
$$\Delta_p \zeta + (p-1)|D\zeta|^p - (p-1)\zeta_t \leq 0, \quad in \ \Omega_T,$$

with $\eta \leq \zeta$ on P_T. The conclusion follows by an adaptation of Theorem 8.2 in [4]. □

Thus, Theorem 3.3 implies uniqueness of positive solutions of (1.1) and solutions of (1.2). We derive now some further consequences. For ease of presentation, we recall the notation $\Gamma(u) = \Delta_p u + (p-1)|Du|^p - (p-1)u_t$, see (2.3).

Corollary 3.4. *Let $\eta \in usc(\Omega_T \cup P_T)$ and $\zeta \in lsc(\Omega_T \cup P_T)$. Suppose that*

$$\Gamma(\eta) \geq 0 \quad and \quad \Gamma(\zeta) \leq 0, \quad in \ \Omega_T.$$

If η and ζ are bounded in $\Omega_T \cup P_T$ then $\sup_{\Omega_T}(\eta - \zeta) \leq \sup_{P_T}(\eta - \zeta)$.

Moreover, if η and ζ are solutions then $\eta, \zeta \in C(\Omega_T \cup P_T)$ and $\sup_{P_T} |\eta - \zeta| = \sup_{\Omega_T} |\eta - \zeta|$. In particular, if $\eta = \zeta$ on P_T then $\eta = \zeta$ in Ω_T.

Proof. Let $k = \sup_{P_T}(\eta - \zeta)$ and $\zeta_k = \zeta + k$. Since $\Gamma(\zeta_k) \leq 0$ and $\zeta_k \geq \eta$ on P_T, Theorem 3.3, implies that $\sup_{\Omega_T}(\eta - \zeta_k) \leq \sup_{P_t}(\eta - \zeta_k) = 0$. □

Theorem 3.3 and Corollary 3.4 imply a quotient comparison principle.

Corollary 3.5. *Let u and v be as in Theorem 3.3, then*

$$\frac{u}{v} \leq \sup_{P_T} \frac{u}{v}, \quad or \quad \frac{u-v}{v} \leq \sup_{P_T} \frac{u-v}{v}, \quad in \ \Omega_T.$$

Proof. This follows from Corollary 3.4 by writing $\eta = \log u$ and $\zeta = \log v$. For solutions, we obtain

$$\sup_{P_T} \frac{u}{v} = \sup_{\Omega_T} \frac{u}{v} \quad \text{and} \quad \sup_{P_T} \frac{v}{u} = \sup_{\Omega_T} \frac{v}{u}.$$

$\square$

4. Proofs of Theorems 1.1 and 1.2: Initial data $2 \leq p < \infty$

Our proof of the existence of solutions to (1.1) involves constructing sub-solutions and super-solutions for the problem (see 2.10, 2.11) that are arbitrarily close, in a local sense, to the data specified on the parabolic boundary P_T. For this purpose, we divided our work into three sections. In this section we take up the construction for the initial data at $t = 0$. Our work is valid for $2 \leq p < \infty$ and any bounded Ω. We take up the side conditions i.e, data specified along $\partial\Omega \times [0, T)$, in Sects. 5 and 6. The ideas used to construct the sub-solutions and super-solutions are quite similar to those in [2].

We recall the definition of h from (1.3),

$$h(x, t) = \begin{cases} f(x), & \forall x \in \Omega, \quad t = 0, \\ g(x, t), & \forall (x, t) \in \partial\Omega \times [0, T). \end{cases}$$

Note that $h(x, t)$ is continuous on P_T. Set

$$m = \inf_{P_T} h \quad \text{and} \quad M = \sup_{P_T} h. \tag{4.1}$$

Assume that $0 < m < M < \infty$. If $m = M$ then $u(x, t) = M$ is the unique solution of (1.1). In this section we work directly with the operator

$$\Pi(\phi) = \Delta_p \phi - (p - 1)\phi^{p-2}\phi_t, \quad \text{in } \Omega_T. \tag{4.2}$$

Also, recall from Lemma 2.3 that if $\phi(x)$, $\phi > 0$, solves $\Delta_p \phi + \lambda\phi^{p-1} \geq (\leq)0$ and $\psi(x, t) = \phi(x)e^{-\ell t}$ then

$$\Pi(\psi) = \Delta_p \psi - (p - 1)\psi^{p-2}\psi_t \geq (\leq)\psi^{p-1}\{\ell(p - 1) - \lambda\}. \tag{4.3}$$

Using (2.17) we also note that if $r = |x - y|$, for some $y \in \mathbb{R}^n$, then

$$\Delta_p r^2 = \sigma_p 2^{p-1} r^{p-2}, \quad \text{in } \mathbb{R}^n, \tag{4.4}$$

where $\sigma_p = (p + n - 2)$. In what follows, $\varepsilon > 0$ is small so that $m - 2\varepsilon > 0$.

Part I. Sub-solutions: $t = 0$

Let $y \in \overline{\Omega}$. Assume that $h(y, 0) > m$, otherwise take the sub-solution to be m in all of Ω_T. We discuss the cases (a) $y \in \Omega$ and (b) $y \in \partial\Omega$ separately.

Case (a): Let $y \in \Omega$. By continuity, there is a $0 < \delta \leq \text{dist}(y, \partial\Omega)$ such that

$$h(y, 0) - \varepsilon \leq h(x, 0) \leq h(y, 0) + \varepsilon, \quad \forall x \in B_\delta(y).$$

Set $r = |x - y|$ and take

$$\phi(x) = h(y, 0) - 2\varepsilon - (h(y, 0) - m)\frac{r^2}{\delta^2}, \quad \forall x \in \overline{B}_\delta(y), \tag{4.5}$$

Clearly, $\phi(y) = h(y,0) - 2\varepsilon$ and $\phi|_{\partial B_\delta(y)} = m - 2\varepsilon$. Moreover, by (4.4) and taking $r = \delta$, we have

$$\Delta_p \phi = -\frac{\sigma_p 2^{p-1} r^{p-2}}{\delta^{2(p-1)}}(h(y,0)-m)^{p-1} \geq -\frac{\sigma_p 2^{p-1}}{\delta^p}(h(y,0)-m)^{p-1}, \quad \text{in } B_\delta(y).$$
(4.6)

Now take

$$\lambda = \frac{\sigma_p 2^{p-1}}{\delta^p}\left(\frac{h(y,0)-m}{m-2\varepsilon}\right)^{p-1}.$$

Since $m - 2\varepsilon \leq \phi \leq h(y,0) - 2\varepsilon$, using (4.6) we get that

$$\Delta_p \phi + \lambda \phi^{p-1} \geq \lambda(m-2\varepsilon)^{p-1} - \frac{\sigma_p 2^{p-1}}{\delta^p}(h(y,0)-m)^{p-1} \geq 0, \quad \text{in } B_\delta(y).$$
(4.7)

Taking $\psi(x,t) = \phi(x)e^{-\lambda t/(p-1)}$ in $B_\delta(y) \times (0,T)$, it is clear by (4.3) that $\Pi(\psi) \geq 0$. Also, $\forall (x,t) \in \overline{B}_\delta(y) \times [0,T]$,

(i) $\psi(y,0) = h(y,0) - 2\varepsilon$, (ii) $m - 2\varepsilon \leq \psi(x,0) \leq h(x,0) - \varepsilon$, and   (4.8)
(iii) $(m-2\varepsilon)e^{-\ell t} \leq \psi(x,t) \leq \psi(y,t) = (h(y,0)-2\varepsilon)e^{-\ell t}$.

Call $R = \overline{B}_\delta(y) \times [0,T]$; extend ψ as follows:

$$\psi_y(x,t) = \begin{cases} \psi(x,t), & \forall (x,t) \in R, \\ (m-2\varepsilon)e^{-\lambda t/(p-1)}, & \forall (x,t) \in \Omega_T \setminus R. \end{cases}$$
(4.9)

A simple calculation shows that ψ_y is a sub-solution in $\Omega_T \setminus R$. We only need to check that ψ is a sub-solution in $\partial B_\delta(y) \times [0,T]$.

Let $(z,s) \in \partial B_\delta(y) \times (0,T)$ and $(a,q,X) \in \mathcal{P}_{\Omega_T}^+ \psi_y(z,s)$. Thus, for $(x,t) \to (z,s)$, where $(x,t) \in \Omega_T$, we have

$$\psi_y(x,t) - \psi_y(z,s) \leq a(t-s) + \langle q, x-z \rangle$$
$$+ \frac{\langle X(x-z), x-z \rangle}{2} + o\left(|t-s| + |x-z|^2\right), \quad (4.10)$$

Taking $x = z$ in (4.10) we have $\psi_y(z,t) - \psi_y(z,s) \leq a(t-s)$, as $t \to s$. Recalling (4.8)(iii) and (4.9), we get

$$a = -\frac{(m-2\varepsilon)\lambda e^{-\lambda s/(p-1)}}{p-1} < 0.$$
(4.11)

Next, take $t = s$ in (4.10) to obtain $\psi_y(x,s) - \psi_y(z,s) \leq \langle q, x-z \rangle + o(|x-z|)$, as $x \to z$. Recalling (4.8) and (4.9), $\psi_y(x,t) \geq (m-2\varepsilon)e^{-\lambda t/(p-1)}$, in Ω_T, and $\psi_y(x,s) - \psi_y(z,s) \geq 0$. Thus, $\langle q, x-z \rangle + o(|x-z|) \geq 0$, as $x \to z$. Clearly, $q = 0$ and $X \geq 0$. Using (2.8) and (4.11)

$$|q|^{p-2}\text{tr}(X) + (p-2)|q|^{p-4}\sum q_i q_j X_{ij} - (p-1)a\psi_y^{p-2}(z,s) > 0.$$

To summarize, for every $y \in \Omega$ and $\varepsilon > 0$, small, the function $\psi_y \in C(\overline{\Omega}_T)$ is such that

(i) $0 < \psi_y \le h$ on P_T, (ii) $\psi_y(y,0) = h(y,0) - 2\varepsilon$, and (iii) $\Pi(\psi) \ge 0$, in Ω_T. See (4.8) and (4.9). For later reference, set for every $y \in \Omega$ and $\varepsilon > 0$, small,

$$\alpha_{y,\varepsilon}(x,t) = \log \psi_y(x,t), \quad \forall (x,t) \in \overline{\Omega}_T. \tag{4.12}$$

$\square$

Case (b): Let $y \in \partial\Omega$. By continuity, there are $\delta > 0$ and $\tau > 0$ such that

$$h(y,0) - \varepsilon \le h(x,t) \le h(y,0) + \varepsilon, \quad \forall (x,t) \in P_T \cap (B_\delta(y) \times [0,\tau]).$$

We choose $\phi(x)$ and λ as in Case (a). Recalling (4.3), we select $\ell \ge \lambda/(p-1)$, such that

$$(h(y,0) - 2\varepsilon)e^{-\ell\tau} \le m - 2\varepsilon.$$

Defining $\psi(x,t) = \phi(x)e^{-\ell t}$, in $\overline{B}_\delta(y) \times [0,T]$, we see that ψ is a sub-solution in $B_\delta(y) \times (0,T)$. We observe that $(m - 2\varepsilon)e^{-\ell t} \le \psi(x,t) \le \psi(y,t) = (h(y,0) - 2\varepsilon)e^{-\ell t}$, $\quad \forall (x,t) \in \overline{B}_\delta(y) \times [0,T]$. Set

$$\hat{\psi}_y(x,t) = \begin{cases} \psi(x,t), & \forall (x,t) \in \Omega_T \cap (B_\delta(y) \times [0,T)) \\ (m-2\varepsilon)e^{-\ell t}, & \forall (x,t) \in \Omega_T \setminus (B_\delta(y) \times [0,T)). \end{cases}$$

Verifying that $\hat{\psi}_y$ is a sub-solution in Ω_T is similar to Case (a). For any $y \in \partial\Omega$ and $\varepsilon > 0$, small, define

$$\hat{\alpha}_{y,\varepsilon}(x,t) = \log \hat{\psi}_y(x,t), \quad \forall (x,t) \in \Omega_T. \tag{4.13}$$

$\square$

Part II. Super-solutions: $t = 0$

As done in Part I, we discuss the two cases: (a) $y \in \Omega$, and (b) $y \in \partial\Omega$. The treatment here is quite similar to that in Part I. We assume that $h(y,0) < M$, otherwise the function $\psi = M$, in Ω_T, is a super-solution.

Case (a): Let $y \in \Omega$. Select $0 < \delta \le \mathrm{dist}(y, \partial\Omega)$ such that

$$h(y,0) - \varepsilon \le h(x,0) \le h(y,0) + \varepsilon, \quad \forall x \in B_\delta(y).$$

Set $r = |x - y|$ and consider

$$\phi(x) = \phi(r) = h(y,0) + 2\varepsilon + (M - h(y,0))\frac{r^2}{\delta^2}, \quad \forall x \in \overline{B}_\delta(y). \tag{4.14}$$

Thus, $\phi(y) = h(y,0) + 2\varepsilon$, $\phi|_{\partial B_\delta(y)} = M + 2\varepsilon$ and $0 < h(y,0) + 2\varepsilon \le \phi(x) \le M + 2\varepsilon$. Choose

$$\lambda = \frac{\sigma_p 2^{p-1}}{\delta^p}\left(\frac{M - h(y,0)}{h(y,0) + 2\varepsilon}\right)^{p-1}.$$

Using the observation made above and (4.4), we get in $0 \le r \le \delta$,

$$\Delta_p\phi - \lambda\phi^{p-1} = \frac{\sigma_p 2^{p-1}r^{p-2}}{\delta^{2(p-1)}}(M - h(y,0))^{p-1} - \lambda\phi^{p-1}$$

$$\le \frac{\sigma_p 2^{p-1}}{\delta^p}(M - h(y,0))^{p-1} - \lambda(h(y,0) + 2\varepsilon)^{p-1} = 0.$$

Set $\psi(x,t) = \phi(x)e^{\lambda t/(p-1)}$, in $\overline{B}_\delta(y) \times [0,T)$. By (4.3), ψ is a super-solution in $B_\delta(y) \times [0,T)$. Also, $\forall(x,t) \in \overline{B}_\delta(y) \times [0,T]$,

(i) $\psi(y,0) = h(y,0) + 2\varepsilon,$ (ii) $\psi(x,0) \geq h(x,0) + \varepsilon,$ $\forall x \in \overline{B}_\delta(y),$ and

(iii) $(h(y,0) + 2\varepsilon)e^{\lambda t/(p-1)} = \psi_y(y,t) \leq \psi(x,t) \leq (M + 2\varepsilon)e^{\lambda t/(p-1)}.$

$$(4.15)$$

Set $R = \overline{B}_\delta(y) \times [0,T)$. Extend ψ by setting

$$\psi_y(x,t) = \begin{cases} \psi(x,t), & \forall(x,t) \in \Omega_T \cap R, \\ (M + 2\varepsilon)e^{\lambda t/(p-1)}, & \forall(x,t) \in \Omega_T \setminus R. \end{cases} \qquad (4.16)$$

By (4.3), ψ_y is super-solution in $\Omega_T \setminus R$. We only need to check that ψ is a super-solution in $\partial B_\delta(y) \times [0,T)$.

Let $(z,s) \in \partial B_\delta(y) \times (0,T)$ and $(a,q,X) \in \mathcal{P}_{\Omega_T}^{-} \psi_y(z,s)$, i.e., as $(x,t) \to (z,s)$,

$$\psi_y(x,t) - \psi_y(z,s) \geq a(t-s) + \langle q, x-z \rangle$$
$$+ \frac{\langle X(x-z), x-z \rangle}{2} + o(|t-s| + |x-z|^2).$$

We use (4.15) and (4.16) and argueing as in Case (a) of Part I. Taking $x = z$ we get

$$a = \frac{(M + 2\varepsilon)\lambda e^{\lambda s/(p-1)}}{p-1} > 0.$$

Next, note that (4.15) and (4.16) imply $\psi_y(x,s) - \psi(z,s) \leq 0,$ $\forall x \in \Omega.$ Taking $t = s$, we get $q = 0$ and $X \leq 0$. Using (2.8) $|q|^{p-2}\mathrm{tr}(X) + (p - 2)|q|^{p-4}\sum q_i q_j X_{ij} - (p-1)a\psi_y^{p-2}(z,s) < 0$.

Summarizing: (i) $\psi_y \geq h$, on P_T, (ii) $\psi_y(y,0) = h(y,0) + 2\varepsilon$, and (iii) $\Pi(\psi_y) \leq 0$, in Ω_T, see (4.15) and (4.16). Set for every $y \in \Omega$ and $\varepsilon > 0$, small,

$$\beta_{y,\varepsilon}(x,t) = \log \psi_y(x,t), \quad \forall(x,t) \in \overline{\Omega}_T. \qquad (4.17)$$

$\square$

Case (b): Let $y \in \partial\Omega$. By continuity, there are $\delta > 0$ and $\tau > 0$ such that

$$h(y,0) - \varepsilon \leq h(x,t) \leq h(y,0) + \varepsilon, \quad \forall(x,t) \in P_T \cap (B_\delta(y) \times [0,\tau]).$$

We choose $\phi(x)$ and λ as in Case (a). Recalling (4.3) we select $\ell \geq \lambda/(p-1)$, such that

$$(h(y,0) + 2\varepsilon)e^{\ell\tau} \geq M + 2\varepsilon.$$

Defining

$$\psi(x,t) = \phi(x)e^{\ell t}, \quad \text{in } B_\delta(y) \times (0,T),$$

we see that ψ is a super-solution in $B_\delta(y) \times (0,T)$. Also, for every $(x,t) \in \overline{B}_\delta(y) \times [0,T]$,

$$(h(y,0) + 2\varepsilon)e^{\ell t} = \psi_y(y,t) \leq \psi(x,t) \leq (M + 2\varepsilon)e^{\ell t}.$$

Set

$$\hat{\psi}_y(x,t) = \begin{cases} \psi(x,t), & \forall(x,t) \in \overline{\Omega}_T \cap (B_\delta(y) \times [0,T]) \\ (M+2\varepsilon)e^{\ell t}, & \forall(x,t) \in \overline{\Omega}_T \setminus (B_\delta(y) \times [0,T]). \end{cases}$$

Verifying that $\hat{\psi}_y$ is a sub-solution in Ω_T is similar to Case (a). Now set

$$\hat{\beta}(x,t) = \log \hat{\psi}_y(x,t), \quad \forall(x,t) \in \overline{\Omega}_T. \tag{4.18}$$

$\square$

5. Side boundary: the case $n < p < \infty$ and the proof of Theorem 1.1

We now take up the task of constructing sub-solutions and super-solutions for the side boundary $\partial\Omega \times (0,T)$. Unlike the case of the initial conditions (see Sect. 4) we work with the equation in (1.2) which we recall here.

$$\Delta_p\eta + (p-1)|\eta|^p - (p-1)\eta_t = 0, \quad \text{in } \Omega_T,$$
$$\text{and} \quad \eta(x,t) = \log h(x,t), \quad \forall(x,t) \in P_T. \tag{5.1}$$

See Lemma 2.1, where the change of variable $\eta = \log u$ is discussed. We take $n < p < \infty$ and Ω is any bounded domain.

Also recall the notation $\Gamma(v) := \Delta_p v + (p-1)|Dv|^p - (p-1)v_t$, see (2.3) and also the definitions and discussion following (2.6) and (2.8). In particular, we mention that in order for $\Gamma(\eta) \geq (\leq)0$ we require that $\forall(a,p,X) \in \mathcal{P}^+_{\Omega_T}\eta(x,t)(\mathcal{P}^-_{\Omega_T}\eta(x,t))$ and $\forall(x,t) \in \Omega_T$,

$$|q|^{p-2}\text{tr}(X) + (p-2)|q|^{p-4}q_iq_jX_{ij} + (p-1)|q|^p - (p-1)a \geq (\leq)0.$$

Let m and M be as in (4.1). Fix $\varepsilon > 0$, small, so that $m - 2\varepsilon > 0$. Note that if $m = M$ then $u(x,t) = m$ is the unique solution to (1.1). We continue to assume that $0 < m < M < \infty$. Recall the notation $D_{\rho,2\theta}(x,t) = B_\rho(x) \times (t-\theta, t+\theta)$, see (2.2).

Let $(y,s) \in P_T$ where $s > 0$. There is a $\delta_0 > 0$ and $\tau_0 > 0$, depending on y and s, such that

$$h(y,s) - \varepsilon \leq h(x,t) \leq h(y,s) + \varepsilon, \quad \forall(x,t) \in \overline{D}_{\delta_0,2\tau_0}(y,s) \cap P_T. \tag{5.2}$$

For any $\gamma \in \mathbb{R}$, we set

$$\Lambda = \frac{p-n}{p-1} - \gamma. \tag{5.3}$$

Let $r = |x - y|$; recalling Remark 2.4, (2.18) and (5.3) (take $c > 0$) we have in $r > 0$,

$$\Delta_p(\pm cr^\gamma) + (p-1)|D(\pm cr^\gamma)|^p = (p-1)c^{p-1}|\gamma|^p r^{p(\gamma-1)}\left\{c \pm \left(\frac{-\Lambda}{\gamma r^\gamma}\right)\right\}, \tag{5.4}$$

Before we move on to the construction of the various functions, we state a lemma that would be used in this section and in Sect. 6.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain, $T > 0$ and $O \subset \Omega_T$ be a sub-domain. Suppose that $\alpha \in \mathbb{R}$ and $w : \Omega_T \to \mathbb{R}$ is such that $w = \alpha$ in $\Omega_T \setminus O$.*

(i) If $w \in usc(\Omega_T)$, $w \geq \alpha$,   in O,   and $\Gamma(w) \geq 0$,   in O,   then $\Gamma(w) \geq 0$,   in Ω_T.

(ii) If $w \in lsc(\Omega_T)$, $w \leq \alpha$,   in O,   and $\Gamma(w) \leq 0$,   in O,   then $\Gamma(w) \leq 0$,   in Ω_T.

Proof. It is clear that we need check the claim only at points on $\partial O \cap \Omega_T$. We prove part (i), the proof part of (ii) is similar.

Let $(y, s) \in \partial O$ and $(a, q, X) \in \mathcal{P}^+_{\Omega_T} w(y, s)$. Since $w \geq \alpha$ and $w(y, s) = \alpha$, we have

$$0 \leq w(x, t) - w(y, s) \leq a(t - s) + \langle q, x - y \rangle \tag{5.5}$$
$$+ \frac{\langle X(x - y), x - y \rangle}{2} + o(|t - s| + |x - y|^2),$$

as $(x, t) \to (y, s)$, where $(x, t) \in \Omega_T$. Taking $x = y$ in (5.5) we have that $a(t - s) + o(|t - s|) \geq 0$, as $t \to s$, implying that $a = 0$. Next, taking $t = s$ in (5.5), we see that $\langle q, x - y \rangle + o(|x - y|) \geq 0$, as $x \to z$. We obtain $q = 0$, $X \geq 0$ and as a result

$$|q|^{p-2} \mathrm{tr}(X) + (p - 2)|q|^{p-4} q_i q_j X_{ij} + (p - 1)|q|^p - (p - 1)a = 0,$$

proving that $\Gamma(w) \geq 0$ in Ω_T. The lemma is proved. $\square$

Part I: Sub-solutions

Recall (5.2) and fix y, s, δ_0 and τ_0. We construct a sub-solution in a region R that lies in $\overline{D}_{\delta_0, 2\tau_0}(y, s)$ and then extend it to the rest of Ω_T as a sub-solution. In what follows, the quantities $0 < \tau \leq \tau_0$ and $0 < \delta \leq \delta_0$, positive constants k, γ and c are such that

$$k = \frac{1}{\tau} \log \left(\frac{h(y, s) - 2\varepsilon}{m - 2\varepsilon} \right), \quad \gamma = \frac{p - n}{p - 1}, \quad \text{and} \quad \delta = \left(\frac{k\tau}{c} \right)^{1/\gamma}. \tag{5.6}$$

Here we take

$$c \geq \frac{(k\tau)^\mu}{\gamma^\gamma \tau^{\gamma/p}}, \quad \text{where} \quad \mu = \frac{\gamma + p(1 - \gamma)}{p}. \tag{5.7}$$

By (5.6) and (5.7), it is clear that $\delta \to 0$ if $\tau \to 0$. We now fix a value of τ such that $0 < \tau \leq \tau_0$ and then a value of c such that $0 < \delta \leq \delta_0$. This also fixes the value of k. Our choice of γ shows that $\Lambda = 0$, see (5.3). We comment that choosing $0 < \gamma < (p - n)/(p - 1)$ will also work.

We now describe the region R. Firstly, $R \subset \overline{D}_{\delta, 2\tau}(y, s)$ and is the union of two cusp-like regions R^+ and R^-, where R^+ and R^- are as shown in Fig. 1.
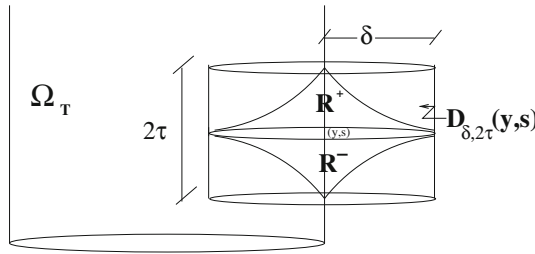
FIGURE 1. Cusps

We now describe these more precisely. Set $r = |x - y|$; define, in $0 \leq r \leq \delta$ and $s - \tau \leq t \leq s + \tau$,

$$R^+ \text{ is the cusp: } k\tau + k(s - t) - cr^\gamma \geq 0, \quad s \leq t \leq s + \tau, \quad \text{and}$$
$$R^- \text{ is the cusp: } k\tau + k(t - s) - cr^\gamma \geq 0, \quad s - \tau \leq t \leq s. \tag{5.8}$$

The choice of the various constants in (5.6) and (5.7) implies that $\overline{R}$ lies in the cylinder $\overline{B}_\delta(y) \times [s - \tau, s + \tau]$. The region R^- is the reflection of R^+ about $t = s$. The base $R^+ \cap R^-$, common to both the cusp regions, is at $t = s$ and is the spatial ball given by $0 \leq r \leq \delta$.

In $\overline{\Omega}_T$, define the *bump function*

$$\eta(x, t) = \eta(r, t) = \begin{cases} \log(m - 2\varepsilon) + k\tau + k(s - t) - cr^\gamma, & \forall (x, t) \in R^+, \\ \log(m - 2\varepsilon) + k\tau + k(t - s) - cr^\gamma, & \forall (x, t) \in R^-, \\ \log(m - 2\varepsilon), & \forall (x, t) \in \overline{\Omega}_T \setminus R. \end{cases} \tag{5.9}$$

From (5.2), (5.6) and (5.9), we see that

(i) $\eta(y, s) = \log(h(y, s) - 2\varepsilon)$, $\quad (ii)$ $\eta(x, t) \geq \log(m - 2\varepsilon)$, $\quad \forall (x, t) \in \overline{\Omega}_T$,

(iii) $\log(m - 2\varepsilon) \leq \eta(x, t) \leq \log(h(y, s) - 2\varepsilon) \leq \log h(x, t)$,

$\quad \forall (x, t) \in R \cap P_T$, and $\quad (iv)$ $\eta(x, t) \leq \log h(x, t)$, $\quad \forall (x, t) \in P_T$.

If we show that η is a sub-solution in Ω_T, the observations (i)–(iv), listed above, would then imply that η is a sub-solution of (1.2). We first show that η is a sub-solution in $R \cap \Omega_T$. Lemma 5.1 will then show that η is a sub-solution in Ω_T. We consider: (a) $t \neq s$, and (b) $t = s$. In the following we take $0 < r < \delta$, the case $r = \delta$ is contained in Lemma 5.1.

(a) $t \neq s$: The function η is C^∞ in the interior of $R \cap \Omega_T$ $(t \neq s)$, since $r > 0$ in Ω_T. Noting from (5.3) that $\Lambda = 0$ and $\gamma < 1$, using (5.4) and (5.9), we get in $0 < r < \delta$,

$$\Delta_p \eta + (p - 1)|D\eta|^p - (p - 1)\eta_t = (p - 1)\left(c^p \gamma^p r^{p(\gamma - 1)} \pm k\right)$$
$$\geq (p - 1)\left(\frac{c^p \gamma^p}{r^{p(1 - \gamma)}} - k\right) \geq (p - 1)\left(\frac{c^p \gamma^p}{\delta^{p(1 - \gamma)}} - k\right) \tag{5.10}$$

We now calculate using (5.6), (5.7) and the definition of μ,

$$
\frac{c^p \gamma^p}{\delta^{p(1-\gamma)}} = c^p \gamma^p \left(\frac{c}{k\tau} \right)^{p(1-\gamma)/\gamma} = c^{p/\gamma} \frac{\gamma^p}{(k\tau)^{p(1-\gamma)/\gamma}}
$$

$$
\geq \left(\frac{(k\tau)^\mu}{\gamma^\gamma \tau^{\gamma/p}} \right)^{p/\gamma} \frac{\gamma^p}{(k\tau)^{p(1-\gamma)/\gamma}} = \frac{(k\tau)^{\{\gamma + p(1-\gamma)\}/\gamma}}{\tau(k\tau)^{p(1-\gamma)/\gamma}} = k.
$$

Hence, (5.10) implies that η is a sub-solution.

(b) $t = s$: Let $(z, s) \in R$ and $(a, q, X) \in \mathcal{P}_R^+ \eta(z, s)$. Then $(x, t) \in R$ and for $(x, t) \to (z, s)$,

$$
\eta(x, t) \leq \eta(z, s) + a(t - s) + \langle q, x - z \rangle
$$
$$
+ \frac{\langle X(x - z), x - z \rangle}{2} + o(|t - s| + |x - z|^2). \tag{5.11}
$$

We now compute a, q and X. Since η is C^∞ in $r > 0$, using $t = s$ in (5.11) we get $q = D\eta(z, s)$ and $X \geq D^2 \eta(z, s)$. Using (5.9) and taking $x = z$ in (5.11)

$$
a(t - s) + o(|t - s|) \geq \begin{cases} k(t - s), & t \leq s, \\ k(s - t), & t \geq s, \end{cases} \quad \text{as} \quad t \to s,
$$

Hence, $-k \leq a \leq k$. Thus, from (5.9) and the discussion following (5.10), we get

$$
|q|^{p-2} \mathrm{tr}(X) + (p-2)|q|^{p-4} q_i q_j X_{ij} + (p-1)|q|^p - a(p-1)
$$
$$
\geq \Delta_p \eta(z, s) + (p-1)|D\eta(z, s)|^p - k(p-1) \geq 0.
$$

Thus η is a sub-solution in $R \cap \Omega_T$. Define for any $(y, s) \in \partial\Omega \times (0, T)$ and $\varepsilon > 0$, small,

$$
\nu_{(y,s),\varepsilon}(x, t) = \eta(x, t), \quad \forall (x, t) \in \overline{\Omega}_T. \tag{5.12}
$$

$\square$

Part II: Super-solutions

Let τ_0 and δ_0 be as in (5.2). For ease of presentation, set $\alpha = (p-n)/(p-1)$, then $\Lambda = \alpha - \gamma$. Let $0 < \tau \leq \tau_0$, $0 < \delta \leq \delta_0$ and set

$$
k = \frac{1}{\tau} \log \left(\frac{M + 2\varepsilon}{h(y, s) + 2\varepsilon} \right), \quad \gamma = \frac{\alpha}{1 + 2k\tau}, \quad \text{and} \quad \delta = \left(\frac{k\tau}{c} \right)^{1/\gamma}. \tag{5.13}
$$

We take

$$
c \geq \left(\frac{2(k\tau)^\mu}{\tau \Lambda \gamma^{p-1}} \right)^{\gamma/p}, \quad \text{where} \quad \mu = \frac{p(1-\gamma)}{\gamma} + 2. \tag{5.14}
$$

It is clear from (5.13) and (5.14) that $\delta \to 0$ as $\tau \to 0$. Fix a value $0 < \tau \leq \tau_0$ and then a value of c such that $\delta \leq \delta_0$. Also, our choice of γ implies that

$$
\Lambda = \alpha - \gamma = \frac{2k\tau\alpha}{1 + 2k\tau} \quad \text{and} \quad c\gamma\delta^\gamma = \gamma k\tau = \frac{k\tau\alpha}{1 + 2k\tau} = \frac{\Lambda}{2}. \tag{5.15}
$$

We construct the super-solution in a region R the union of two cusps R^+ and R^-. These are defined as follows.

R^+ is the cusp-region: $k\tau + k(s-t) - cr^\gamma \geq 0$, $\quad s \leq t \leq s+\tau$, $\quad$ and

R^- is the cusp-region: $k\tau + k(t-s) - cr^\gamma \geq 0$, $\quad s-\tau \leq t \leq s$.

Clearly, $\overline{R} \subset \overline{B}_\delta(y) \times [s-\tau, s+\tau]$. Define the *indent function* in Ω_T as follows:

$$\eta(x,t) = \begin{cases} \log(M+2\varepsilon) + cr^\gamma - k\tau - k(s-t), & \forall (x,t) \in R^+, \\ \log(M+2\varepsilon) + cr^\gamma - k\tau - k(t-s), & \forall (x,t) \in R^-, \\ \log(M+2\varepsilon), & \forall (x,t) \in \overline{\Omega}_T \setminus R. \end{cases} \quad (5.16)$$

From (5.13), (5.14) and (5.16), we see that

(i) $\eta(y,s) = \log(h(y,s) + 2\varepsilon)$, $\quad$ (ii) $\eta(x,t) \leq \log(M+2\varepsilon)$, $\quad \forall (x,t) \in \overline{\Omega}_T$,

(iii) $\log h(x,t) \leq \log(h(y,s) + 2\varepsilon) \leq \eta(x,t) \leq \log(M+2\varepsilon)$,

$\quad \forall (x,t) \in R \cap P_T$, and $\quad$ (iv) $\eta(x,t) \geq \log h(x,t)$, $\quad \forall (x,t) \in P_T$.

If we show that η is a super-solution in Ω_T, the observations (i)–(v), listed above, would then imply that η is a super-solution of (1.2). We first show that η is a super-solution in $R \cap \Omega_T$. Lemma 5.1 then shows that η is a super-solution in Ω_T. We consider the cases: (a) $t \neq s$, and (b) $t = s$.

(a) $t \neq s$: Note that $\eta_t = \pm k$. Using (5.4), (5.13), (5.14) and (5.15), we calculate in $0 < r \leq \delta$,

$$\Delta_p \eta + (p-1)|D\eta|^p - (p-1)\eta_t = (p-1)\left\{ c^{p-1}\gamma^{p-1} r^{p(\gamma-1)-\gamma}(c\gamma r^\gamma - \Lambda) \pm k \right\}$$

$$\leq (p-1)\left\{ c^{p-1}\gamma^{p-1} r^{p(\gamma-1)-\gamma}(c\gamma\delta^\gamma - \Lambda) + k \right\}$$

$$\leq (p-1)\left\{ k - \frac{c^{p-1}\gamma^{p-1}\Lambda}{2\delta^{p(1-\gamma)+\gamma}} \right\}. \quad (5.17)$$

Recalling (5.14), (5.15) and noting that $p-1+1+p(1-\gamma)/\gamma = p/\gamma$, we have

$$\frac{c^{p-1}\gamma^{p-1}\Lambda}{2\delta^{p(1-\gamma)+\gamma}} = c^{p-1}c^{1+p(1-\gamma)/\gamma}\frac{\gamma^{p-1}\Lambda}{2(k\tau)^{1+p(1-\gamma)/\gamma}} = c^{p/\gamma}\frac{\gamma^{p-1}\Lambda}{2(k\tau)^{1+p(1-\gamma)/\gamma}}$$

$$\geq \left(\frac{2(k\tau)^\mu}{\tau\Lambda\gamma^{p-1}}\right)\left(\frac{\gamma^{p-1}\Lambda}{2(k\tau)^{1+p(1-\gamma)/\gamma}}\right) = k.$$

This together with (5.17) yields that $\Gamma(\eta) \leq 0$.

(b) $t = s$: We now show that η is a super-solution in $R \cap \Omega_T$ when $t = s$ and $0 < r < \delta$. Let $(z,s) \in R \cap \Omega_T$ and $(a,q,X) \in \mathcal{P}_{\Omega_T}^- \overline{\eta}(z,\theta)$, i.e., as $(x,t) \to (z,s)$, $(x,t) \in \Omega_T$,

$$\eta(x,t) - \eta(z,s) \geq a(t-s) + \langle q, x-z \rangle$$

$$+ \frac{\langle X(x-z), x-z \rangle}{2} + o(|t-s| + |x-y|^2). \quad (5.18)$$

We take $x = z$ in (5.18) and use (5.16) to see that

$$a(t-s) + o(|t-s|) \leq \begin{cases} k(t-s), & s \leq t \leq s+\tau, \\ k(s-t), & s-\tau \leq t \leq s. \end{cases}$$

Thus, $-k \leq a \leq k$. Since η is C^2 in $r > 0$, $q = D\eta(z,s)$ and $X \leq D^2\eta(z,s)$. Using the calculations done in (a), see (5.17), we have

$$|q|^{p-2}\mathrm{tr}X + (p-2)|q|^{p-4}\sum q_i q_j X_{ij} + (p-1)|q|^p - (p-1)a$$
$$\leq \Delta_p\eta + (p-1)|D\eta|^p + (p-1)k \leq 0.$$

This shows that η is a super-solution in the interior of $R \cap \Omega_T$. For every $(y,s) \in \partial\Omega \times (0,T)$ and $\varepsilon > 0$, small, define

$$\hat{\nu}_{(y,s),\varepsilon} = \eta, \quad \text{in } \overline{\Omega}_T. \tag{5.19}$$

$\square$

Recall that the functions $\alpha_{y,\varepsilon}$, $\hat{\alpha}_{y,\varepsilon}$ and $\nu_{(y,s),\varepsilon}$ are the required sub-solutions for (1.2), see (4.12), (4.13) and (5.12). Next, $\beta_{y,\varepsilon}$, $\hat{\beta}_{y,\varepsilon}$ and $\hat{\nu}_{(y,s),\varepsilon}$ are the required super-solutions for (1.2), see (4.17) and (4.18). These six functions are in $C(\overline{\Omega}_T)$ and the Perron method implies Theorem 1.1.

6. Side conditions: the case $2 \leq p \leq n$ and proof of Theorem 1.2

In this section, we assume that Ω satisfies a uniform outer ball condition. To be more precise: there is a $\rho_0 > 0$ such that for each $y \in \partial\Omega$, if $0 < \rho \leq \rho_0$ then there is a $z \in \mathbb{R}^n \setminus \Omega$ such that ball $B_\rho(z) \subset \mathbb{R}^n \setminus \Omega$ and $y \in \partial B_\rho(z) \cap \partial\Omega$. The center of the region R is the center of the outer ball and lies outside $\overline{\Omega}_T$. Moreover, R lies in a cylindrical shell. Fix $\varepsilon > 0$, small, such that $m - 2\varepsilon > 0$.

Let $(y,s) \in P_T$ where $s > 0$. Recall the notation $D_{\rho,2\theta}(x,t) = B_\rho(x) \times (t-\theta, t+\theta)$. There is a $\delta_0 > 0$ and $\tau_0 > 0$, small, depending on y and s, such that

$$h(y,s) - \varepsilon \leq h(x,t) \leq h(y,s) + \varepsilon, \quad \forall (x,t) \in \overline{D}_{\delta_0,2\tau_0}(y,s) \cap P_T. \tag{6.1}$$

For any $\gamma > 0$, we define (see (5.3))

$$\Lambda = \gamma - \frac{n-p}{p-1}. \tag{6.2}$$

Recall that $m = \inf_{P_T} h$, $M = \sup_{P_T} h$, and assume that $0 < m \leq M < \infty$.

Part I: Sub-solutions

By our hypothesis, there are $z \in \mathbb{R}^n \setminus \Omega$ and $0 < \rho \leq \rho_0$ such that $B_\rho(z) \subset \Omega^c$ and $y \in \partial B_\rho(z) \cap \partial\Omega$. Note that z depends on ρ. Set $r = |x - z|$; the region R will be in the cylindrical shell $(\overline{B}_{\rho+\delta}(z) \setminus B_\rho(z)) \times [s - \tau, s + \tau]$, where ρ, δ and τ will be determined below. To begin with we require that this shell be in $D_{\delta_0,2\tau_0}(y,s)$ and this is achieved if $\rho + \delta \leq \delta_0/2$. We fix a value of $0 < \tau \leq \tau_0$ and impose that

$$\gamma > \frac{n-p}{p-1}, \quad k = \frac{1}{\tau}\log\left(\frac{h(y,s) - 2\varepsilon}{m - 2\varepsilon}\right) \quad \text{and} \quad \delta = \rho\left\{\left(\frac{1}{1 - \rho^\gamma k\tau}\right)^{1/\gamma} - 1\right\}. \tag{6.3}$$
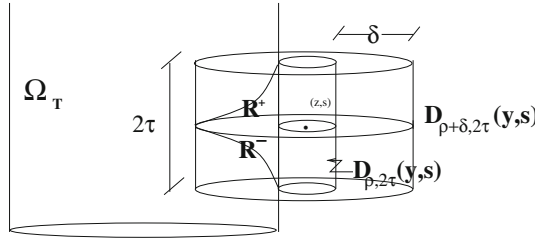
FIGURE 2. Regions R^+ and R^-

Choose

$$0 < \rho \leq \min \left\{ \rho_0,\ (k\tau)^{-1/\gamma},\ \left(\frac{A}{1+k\tau A} \right)^{1/\gamma} \right\}, \quad \text{and} \quad A = \left(\frac{\gamma^p}{k} \right)^{\gamma/(p(1+\gamma))} \tag{6.4}$$

where ρ is small enough to ensure that $\rho + \delta \leq \delta_0/2$. This is possible since the function $\delta = \delta(\rho)$ is increasing and its range is $[0, \infty)$.

　　The region R is the union of two regions R^+ and R^-. We now describe these more precisely. For $0 < \rho \leq r \leq \rho + \delta$ and $s - \tau \leq t \leq s + \tau$, we define the regions

$$R^+ \ \text{is the cusp:} \ \ k\tau + k(s-t) + r^{-\gamma} - \rho^{-\gamma} \geq 0, \quad s \leq t \leq s+\tau, \quad \text{and}$$
$$R^- \ \text{is the cusp:} \ \ k\tau + k(t-s) + r^{-\gamma} - \rho^{-\gamma} \geq 0, \quad s-\tau \leq t \leq s. \tag{6.5}$$

The region R^- is the reflection of R^+ about $t = s$. The base $R^+ \cap R^-$, common to both the regions, is at $t = s$ and is the spatial annulus given by $\rho \leq r \leq \rho + \delta$. See Fig. 2. From (6.3), $\overline{R} \subset (\overline{B}_{\rho+\delta}(z) \setminus B_\rho(z)) \times [s - \tau, s + \tau]$.

　　Define the following *bump function* in Ω_T:

$$\eta(x,t) = \eta(r,t)$$
$$= \begin{cases} \log(m - 2\varepsilon) + k\tau + k(s-t) + r^{-\gamma} - \rho^{-\gamma}, & \forall (x,t) \in R^+, \\ \log(m - 2\varepsilon) + k\tau + k(t-s) + r^{-\gamma} - \rho^{-\gamma}, & \forall (x,t) \in R^-, \\ \log(m - 2\varepsilon) & \forall (x,t) \in \overline{\Omega}_T \setminus R \end{cases} \tag{6.6}$$

From (6.3), (6.5) and (6.6), we see that

(i) $\eta(y,s) = \log(h(y,s) - 2\varepsilon)$, 　(ii) $\eta(x,t) \geq \log(m - 2\varepsilon)$, 　$\forall (x,t) \in \overline{\Omega}_T$,
(iii) $\log(m - 2\varepsilon) \leq \eta(x,t) \leq \log(h(y,s) - 2\varepsilon) \leq \log h(x,t)$,
　　$\forall (x,t) \in R \cap P_T$, 　and 　(iv) $\eta(x,t) \leq \log h(x,t)$, 　$\forall (x,t) \in P_T$.

If we show that η is a sub-solution in Ω_T, the observations (i)–(iv), listed above, would then imply that η is a sub-solution of (1.2). We first show that η is a sub-solution in $R \cap \Omega_T$. We consider: (a) $t \neq s$, and (b) $t = s$. We show (a), the proof of (b) is similar to that in Part I of Sect. 5. Lemma 5.1 then shows η is a sub-solution in Ω_T.

(a) $t \neq s$**:** The function η is C^∞ in the interior of $R \cap \Omega_T$, for $t \neq s$. Using (5.4) and (6.6), we get in $\rho \leq r \leq \rho + \delta$, $s - \tau \leq t \leq s + \tau$,

$$\Delta_p \eta + (p-1)|D\eta|^p - (p-1)\eta_t = (p-1) \left[\frac{\gamma^p}{r^{p(\gamma+1)}} \left\{ 1 + \frac{r^\gamma \Lambda}{\gamma} \right\} \pm k \right]$$

$$\geq (p-1) \left(\frac{\gamma^p}{r^{p(\gamma+1)}} - k \right)$$

$$\geq (p-1) \left(\frac{\gamma^p}{(\rho+\delta)^{p(\gamma+1)}} - k \right). \qquad (6.7)$$

Using (6.3) and (6.4),

$$\rho + \delta = \left(\frac{\rho^\gamma}{1 - \rho^\gamma k\tau} \right)^{1/\gamma} \leq A^{1/\gamma} = \left(\frac{\gamma^p}{k} \right)^{1/(p(1+\gamma))}.$$

Thus, (6.7) implies $\Delta_p \eta + (p-1)|D\eta|^p - (p-1)\eta_t \geq 0$, in $R \cap \Omega_T$, $t \neq s$. Set for each $(y,s) \in P_T$,

$$\hat{\nu}_{(y,s),\varepsilon} = \eta \quad \text{on } \overline{\Omega}_T. \qquad (6.8)$$

Part II: Super-solutions

Our treatment differs slightly from the one in Sect. 5 in that we employ scaling and work with an altered equation.

We utilize the change of variables described in part (a) (iii) of Remark 2.2. Let $\lambda > 0$; set $\omega = \lambda^{p-2}t$ and $\Omega_{\lambda^{p-2}T} = \{(x, \lambda^{p-2}t) : (x,t) \in \Omega_T\}$. Suppose that $w(x,\omega) = v(x,t)/\lambda$ then the following holds: $\Delta_p v + (p-1)|Dv|^p - (p-1)v_t \geq (\leq)0$, in Ω_T, if and only if

$$\Delta_p w + \lambda(p-1)|Dw|^p - (p-1)w_\omega \geq (\leq)0, \text{in } \Omega_{\lambda^{p-2}T}. \qquad (6.9)$$

Let $(y,s) \in P_T$, $s > 0$, and call

$$\alpha = \log \left(\frac{M + 2\varepsilon}{h(y,s) + 2\varepsilon} \right). \qquad (6.10)$$

We fix a value of $\lambda > 0$, small, such that

$$\alpha\lambda < 1. \qquad (6.11)$$

Set

$$\omega = \lambda^{p-2}t, \quad \hat{s} = \lambda^{p-2}s, \quad \hat{T} = \lambda^{p-2}T \quad \text{and} \quad \hat{h}(x,\omega) = h(x,t)^{1/\lambda}.$$

Our goal is to construct a super-solution $\varphi(x,\omega)$ of (6.9), i.e.,

$$\Delta_p \varphi + \lambda(p-1)|D\varphi|^{p-2} - (p-1)\varphi_\omega \leq 0, \quad \text{in } \Omega_{\hat{T}}, \quad \varphi \geq \log \hat{h} \text{ on } P_{\hat{T}} (6.12)$$

such that $\varphi(y,\hat{s})$ is close to $\hat{h}(y,\hat{s})$. By (6.9) the function $\eta(x,t) = \lambda\varphi(x,\omega)$ is then a super-solution of (1.2) with $\eta(y,s)$ close to $h(y,s)$.

Recalling (6.10) and (6.11), choose $0 < \hat{\epsilon} \leq \varepsilon$, small, such that

$$\hat{h}(y,\hat{s}) + 2\hat{\epsilon} \leq (h(y,s) + 2\varepsilon)^{1/\lambda} \quad \text{and} \quad \lambda \log \left(\frac{M^{1/\lambda} + 2\hat{\epsilon}}{\hat{h}(y,\hat{s}) + 2\hat{\epsilon}} \right) < 1. \quad (6.13)$$

Next, there are $\tau_0 > 0$ and $\delta_0 > 0$, small, such that

$$\hat{h}(y,\hat{s}) - \hat{\epsilon} \leq \hat{h}(x,\omega) \leq \hat{h}(y,\hat{s}) + \hat{\epsilon}, \quad \forall (x,\omega) \in \overline{D}_{\delta_0,\tau_0}(y,\hat{s}) \cap P_{\hat{T}}. \quad (6.14)$$

We fix $0 < \tau \leq \tau_0$ and set $\hat{M} = M^{1/\lambda}$. Define

$$k = \frac{1}{\tau} \log \left(\frac{\hat{M} + 2\hat{\epsilon}}{\hat{h}(y,\hat{s}) + 2\hat{\epsilon}} \right). \quad (6.15)$$

Recalling (6.13) fix values of $0 < \theta < 1$ and $\gamma > (n-p)/(p-1)$ such that

$$\lambda k \tau = \frac{\theta^2 \Lambda}{\gamma}, \quad (6.16)$$

also see (6.2). For $c > 0$, to be chosen later, set (use (6.16))

$$\rho^\gamma = \frac{c\theta}{k\tau} = \frac{c\lambda\gamma}{\theta\Lambda} \quad \text{and}$$

$$\delta = \left(\frac{c\theta}{k\tau} \right)^{1/\gamma} \left\{ \left(\frac{1}{1-\theta} \right)^{1/\gamma} - 1 \right\} = \rho \left\{ \left(\frac{1}{1-\theta} \right)^{1/\gamma} - 1 \right\}. \quad (6.17)$$

The expression for δ follows by setting $k\tau = c(\rho^{-\gamma} - (\rho+\delta)^{-\gamma})$. Note we will select c, small, so that $\rho + \delta \leq \delta_0/2$.

We now describe the region R. By the outer ball condition, for each $0 < \rho \leq \rho_0$, there is a $z \in \mathbb{R}^n \setminus \overline{\Omega}$ (depending perhaps on ρ) such that $B_\rho(z) \subset \mathbb{R}^n \setminus \overline{\Omega}$ and $y \in \partial B_\rho(z)$. Define $r = |x - z|$: set in $\rho \leq r \leq \rho + \delta$ and $\hat{s} - \tau \leq \omega \leq \hat{s} + \tau$,

R^+ is the region: $\quad k(\hat{s} - \omega) + k\tau + cr^{-\gamma} - c\rho^{-\gamma} \geq 0, \quad \hat{s} \leq \omega \leq \hat{s} + \tau, \quad$ and

R^- is the region: $\quad k(\omega - \hat{s}) + k\tau + cr^{-\gamma} - c\rho^{-\gamma} \geq 0, \quad \hat{s} - \tau \leq \omega \leq \hat{s}.$

$$(6.18)$$

By (6.17) the region R lies in the cylindrical shell $(\overline{B}_{\rho+\delta}(z) \setminus B_\rho(z)) \times [\hat{s} - \tau, \hat{s} + \tau]$.

In Ω_T, define the *indent function*

$$\varphi(x,\omega) = \begin{cases} \log(\hat{M} + 2\hat{\epsilon}) - k(\hat{s} - \omega) - k\tau + c\rho^{-\gamma} - cr^{-\gamma}, & \forall (x,\omega) \in R^+, \\ \log(\hat{M} + 2\hat{\epsilon}) - k(\omega - \hat{s}) - k\tau + c\rho^{-\gamma} - cr^{-\gamma}, & \forall (x,\omega) \in R^-, \\ \log(\hat{M} + 2\hat{\epsilon}), & \forall (x,\omega) \in \overline{\Omega}_{\hat{T}} \setminus R. \end{cases}$$

$$(6.19)$$

From (6.15), (6.17), (6.18) and (6.19) we see that

(i) $\varphi(y,\hat{s}) = \log(\hat{h}(y,\hat{s}) + 2\hat{\epsilon}), \quad$ (ii) $\varphi(x,t) \leq \log(\hat{M} + 2\hat{\epsilon}), \quad \forall (x,\omega) \in \overline{\Omega}_{\hat{T}},$

(iii) $\log \hat{h}(x,t) \leq \log(\hat{h}(y,\hat{s}) + 2\hat{\epsilon}) \leq \varphi(x,\omega) \leq \log(\hat{M} + 2\hat{\epsilon}),$

$\forall (x,t) \in R \cap P_{\hat{T}}, \quad$ and $\quad$ (iv) $\varphi(x,\omega) \geq \log \hat{h}(x,\omega), \quad \forall (x,\omega) \in P_{\hat{T}}.$

If we show that φ is a super-solution in $\Omega_{\hat{T}}$, the observations (i)–(iv), listed above, would then imply that $\eta(x,t) = \lambda\varphi(x,\omega)$ is a super-solution of (1.2). To do this we first show that φ is a super-solution in $R \cap \Omega_{\hat{T}}$. We consider: (a)

$t \neq s$, and (b) $t = s$. Lemma 5.1 will then show that φ is a super-solution in $\Omega_{\hat{T}}$. Proof of (b) is similar to the proof in Part II of Sect. 5.

For ease of presentation, call

$$\Gamma_\lambda(\phi) = \Delta_p\phi + \lambda(p-1)|D\phi|^p - (p-1)\phi_\omega.$$

(a) $t \neq s$**:** Applying (2.18), (5.4) and using (6.19) in $\rho \leq r \leq \rho + \delta$ and $\hat{s} - \tau \leq t \leq \hat{s} + \tau$,

$$\Gamma_\lambda(\varphi) = \frac{(p-1)c^{p-1}\gamma^p}{r^{p(1+\gamma)}}\left(c\lambda - \frac{r^\gamma\Lambda}{\gamma}\right) \pm (p-1)\hat{k}$$

$$\leq (p-1)\left\{\frac{c^{p-1}\gamma^{p-1}}{r^{p(1+\gamma)-\gamma}}\left(\frac{c\lambda\gamma}{r^\gamma} - \Lambda\right) + k\right\}$$

$$\leq (p-1)\left\{\frac{(c\gamma)^{p-1}}{r^{p(1+\gamma)-\gamma}}\left(\frac{c\lambda\gamma}{\rho^\gamma} - \Lambda\right) + k\right\}. \tag{6.20}$$

Using (6.17), (6.20) leads to, in $\rho \leq r \leq \rho + \delta$ and $\hat{s} - \tau \leq t \leq \hat{s} + \tau$,

$$\Gamma_\lambda(\varphi) \leq (p-1)\left\{k - \frac{(1-\theta)\Lambda(c\gamma)^{p-1}}{r^{p(1+\gamma)-\gamma}}\right\} \leq (p-1)\left\{k - \frac{(1-\theta)\Lambda(c\gamma)^{p-1}}{(\rho+\delta)^{p(1+\gamma)-\gamma}}\right\}. \tag{6.21}$$

Call $\vartheta = p(1+\gamma) - \gamma$ and observe that

$$1 + \frac{\vartheta}{\gamma} = \frac{p(1+\gamma)}{\gamma} \quad \text{and} \quad p - 1 - \frac{\vartheta}{\gamma} = -\frac{p}{\gamma}. \tag{6.22}$$

Recall from (6.17) that $\rho + \delta = \rho(1-\theta)^{-1/\gamma}$. Calculating (see (6.21)) using (6.17) and (6.22) we obtain

$$\frac{(1-\theta)(c\gamma)^{p-1}\Lambda}{(\rho+\delta)^\vartheta} = \frac{(1-\theta)(1-\theta)^{\vartheta/\gamma}(c\gamma)^{p-1}\Lambda}{\rho^\vartheta} = \Lambda\gamma^{p-1}(1-\theta)^{p(1+\gamma)/\gamma}\left(\frac{c^{p-1}}{\rho^\vartheta}\right)$$

$$= \Lambda\gamma^{p-1}(1-\theta)^{p(1+\gamma)/\gamma}\left\{c^{p-1}\left(\frac{k\tau}{c\theta}\right)^{\vartheta/\gamma}\right\}$$

$$= \Lambda\gamma^{p-1}(k\tau)^{\vartheta/\gamma}\left(\frac{(1-\theta)^{p(1+\gamma)}}{\theta^\vartheta}\right)^{1/\gamma}\left(\frac{c^{p-1}}{c^{\vartheta/\gamma}}\right) = \frac{L}{c^{p/\gamma}},$$

where $L = L(\Lambda, \gamma, p, \theta, k\tau)$. Thus, we obtain from (6.21)

$$\Gamma_\lambda(\varphi) \leq (p-1)\left(k - \frac{L}{c^{p/\gamma}}\right) \leq 0,$$

if we choose c small enough. Our choice now determines ρ, δ and satisfies $\rho + \delta \leq \delta_0/2$. Set for each $\varepsilon > 0$ and $(y,s) \in P_T$, $\hat{\mu}_{(y,s),\varepsilon} = \eta$ on Ω_T.

Recall that the functions $\alpha_{y,\varepsilon}$, $\hat{\alpha}_{y,\varepsilon}$ and $\hat{\nu}_{(y,s),\varepsilon}$ are the required sub-solutions for (1.2), see (4.12), (4.13) and (6.8). Next, $\beta_{y,\varepsilon}$, $\hat{\beta}_{y,\varepsilon}$ and $\hat{\nu}_{(y,s),\varepsilon}$

are the required super-solutions for (1.2), see (4.17) and (4.18). These are in $C(\overline{\Omega}_T)$ and the Perron method implies Theorem 1.2.

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