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On the spectrum of a Baouendi–Grushin type operator: an Orlicz–Sobolev space setting approach

Dedicated to Professor Gheorghe Moroşanu on the occasion of his sixty-fifth anniversary

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Abstract. We show that the spectrum of a nonhomogeneous Baouendi–Grushin type operator subject with a homogeneous Dirichlet boundary condition is exactly the interval $(0, \infty)$. This is in sharp contrast with the situation when we deal with the "classical" Baouendi–Grushin operator (i.e., an operator of type $-\Delta_x - |x|^{\xi}\Delta_y$) when the spectrum is an increasing and unbounded sequence of positive real numbers. Our proofs rely on a symmetric mountain-pass argument due to Kajikiya. In addition, we can show that for each eigenvalue there exists a sequence of eigenfunctions converging to zero.

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1. Introduction

The study of eigenvalue problems involving degenerate differential operators goes back more than 200 years ago to the well-known *Legendre's equation*

$$-\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right] = \lambda u, \quad x \in [-1,1]. \tag{1}$$

It can be shown that for each nonnegative integer n, $\lambda = n(n+1)$ is an eigenvalue of (1) with the corresponding eigenfunction

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n],$$

known as *Legendre's polynomial*.

In high dimensional case, a typical example of degenerate differential operator was introduced in 1920's by Tricomi [29], namely

$$T := \frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial y^2},$$

and generalized later as

$$\frac{\partial^2}{\partial x^2} + x\Delta_y,$$

where Δ_y denotes the Laplace operator with respect to variable y (see, e.g. [5]).

In a similar context, we consider the so-called *Baouendi–Grushin operators* (see, [4] and [18])

$$\frac{\partial^2}{\partial x^2} + x^{2\xi}\frac{\partial^2}{\partial y^2}, \quad \xi > 0, \tag{2}$$

and

$$\Delta_\xi := \Delta_x + |x|^\xi \Delta_y, \quad x \in \mathbb{R}^n,\ y \in \mathbb{R}^m\ (n+m=N), \quad \xi > 0, \tag{3}$$

where Δ_x and Δ_y stand for the standard Laplace operators on $\mathbb{R}^n$, respectively, $\mathbb{R}^m$.

Problems involving operators of type (2) and (3) have been extensively studied over the years. We just remember the papers by Tri [28], D'Ambrosio [9,10], D'Ambrosio and Lucente [11], Monti and Morbidelli [22], Thuy and Tri [27].

Related to the above discussion we note that the natural geometrical and functional setting for a wide class of linear "Grushin"-type operators was first settled in the works by Franchi and Lanconelli [12–14] and more recently in Kogoj and Lanconelli [20]. Finally, we recall the survey paper by Sawyer and Wheeden [25] related to some classes of degenerate second order elliptic operators, containing, in particular the ones of Baouendi–Grushin-type.

Particularly, operators of type Δ_ξ on domains from $\mathbb{R}^N$ interesecting the plane $x = 0$ are not elliptic operators but *hypoelliptic* operators. The lack of ellipticity is due to the presence of the degeneracy $|x|^\xi$. However, for $-\Delta_\xi$ we can still use the classical theory of compact and self-adjoint operators in order to show that, subject to a homogeneous Dirichlet boundary condition, it possesses a *discrete* spectrum consisting in an unbounded sequence of positive real numbers (see, e.g. [3, page 3])

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty, \quad \text{as}\, n \to \infty\,.$$

Operator $-\Delta_\xi$ represents the basic source of inspiration in defining a nonhomogeneous Baouendi–Grushin type operator in the next section of this paper for which we will analyze its spectrum subject to a homogeneous Dirichlet boundary condition. Our main result will show that in this new situation the spectrum is *continuous*, being exactly the open interval $(0, \infty)$. This is in sharp contrast with the situation involving $-\Delta_\xi$.

The paper is organized as follows: in Sect. 2 we define a nonhomogeneous Baouendi–Grushin type operator; in Sect. 3 we collect some preliminary results and we introduce some notations which will be used in the following; in Sect. 4 we establish some auxiliary results that will be used later in the analysis of the spectrum of the operator introduced in Sect. 2; in Sect. 5 we give the statement and the proof of the main result of this paper; the equivalence of some norms is established in the Appendix.

2. A nonhomogeneous Baouendi–Grushin type operator

Let $\Omega \subset \mathbb{R}^N$ be a bounded and smooth domain, $N = n+m$ with $n, m \geq 1$ and assume that Ω intersects the plane $x = 0$, that is the set $\{(0_n, y);\ y \in \mathbb{R}^m\}$, where 0_n is the null vector of $\mathbb{R}^n$. We denote by $\partial\Omega$ the boundary of Ω.

If $(x, y) \in \Omega$ then we denote $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$.

Consider $\xi > 0$ is a given real number and define the matrix

$$A(x) = \begin{bmatrix} I_n & O_{n,m} \\ O_{m,n} & |x|^\xi I_m \end{bmatrix} \in M_{N\times N}(\mathbb{R}),$$

where $O_{n,m}$, $O_{m,n}$ are the null matrices in $M_{n\times m}(\mathbb{R})$, respectively $M_{m\times n}(\mathbb{R})$ while I_n, I_m stand for the unit matrices in $M_{n\times n}(\mathbb{R})$, respectively $M_{m\times m}(\mathbb{R})$.

Let $a : (0, \infty) \to \mathbb{R}$ be a function such that the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = \begin{cases} a(|t|)t, & \text{if } t \neq 0, \\ 0, & \text{if } t = 0 \end{cases}$$

is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Define

$$\Phi(t) = \int_0^t \varphi(s)\, ds$$

for any $t \geq 0$.

We assume that the following relations hold true

$$1 < \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \quad \text{and} \quad \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \infty \tag{4}$$

$$(0, \infty) \ni t \longmapsto \Phi(\sqrt{t}) \quad \text{is a convex function.} \tag{5}$$

We define the degenerate operator

$$\Delta_{G_a}(\cdot) = \operatorname{div}(\nabla_{G_a} \cdot)$$

where

$$\nabla_{G_a} \cdot = A(x) \begin{bmatrix} a(|\nabla_x \cdot|)\ \nabla_x \cdot \\ a(|x|^{\xi/2}|\nabla_y \cdot|)\ \nabla_y \cdot \end{bmatrix}.$$

Thus,

$$\begin{aligned} \Delta_{G_a}(\cdot) &= \operatorname{div_x}(a(|\nabla_x \cdot|)\nabla_x \cdot) + \operatorname{div_y}(|x|^\xi a(|x|^{\xi/2}|\nabla_y \cdot|)\nabla_y \cdot) \\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(a(|\nabla_x \cdot|)\frac{\partial \cdot}{\partial x_i}\right) + \sum_{j=1}^m \frac{\partial}{\partial y_j}\left(|x|^\xi a(|x|^{\xi/2}|\nabla_y \cdot|)\frac{\partial \cdot}{\partial y_j}\right). \end{aligned}$$

This is a Baouendi–Grushin type operator since in the particular case when $a(t) = 1$ for each $t \geq 0$ we recover operator defined in (3).

The goal of this paper is to give a complete description of the spectrum of the operator $-\Delta_{G_a}$ when $\Phi^- := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} > 2$ and $\xi \in (0, 2n(\Phi^- - 2)/\Phi^-)$, subject with a homogeneous Dirichlet boundary condition. In other words, we will analyze the eigenvalue problem

$$\begin{cases} -\Delta_{G_a} u(x,y) = \lambda u(x,y), & \text{for } (x,y) \in \Omega, \\ u(x,y) = 0, & \text{for } (x,y) \in \partial\Omega. \end{cases} \tag{6}$$

3. Function space setting

In this section we provide a brief review of the basic properties of Orlicz and Orlicz–Sobolev spaces which represent the adequate function space setting where we will analyze problem (6). For more details we refer to the books by Adams [1], Adams and Hedberg [2], Musielak [23] and Rao and Ren [24], and to the papers by Clément et al. [7,8], García-Huidobro et al. [16], and Gossez [17].

Assume $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and define

$$\Phi(t) = \int_0^t \varphi(s)\, ds \quad \text{and} \quad \Phi^\star(t) = \int_0^t \varphi^{-1}(s)\, ds$$

for any $t \geq 0$. Then Young's inequality holds true

$$st \leq \Phi(s) + \Phi^\star(t), \quad \forall s, t \geq 0. \tag{7}$$

Letting

$$\Phi^- := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \quad \text{and} \quad \Phi^+ := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}$$

we will assume that

$$1 < \Phi^- \leq \frac{t\varphi(t)}{\Phi(t)} \leq \Phi^+ < \infty, \quad \forall t > 0. \tag{8}$$

Then, relation (8) and [15, Lemma 2.5, (2.7)] imply that

$$1 < \frac{\Phi^+}{\Phi^+ - 1} \leq \frac{t\varphi^{-1}(t)}{\Phi^\star(t)} \leq \frac{\Phi^-}{\Phi^- - 1} < \infty, \quad \text{for all } t > 0, \tag{9}$$

and, actually, we have $(\Phi^\star)^- = \frac{\Phi^+}{\Phi^+ - 1}$ and $(\Phi^\star)^- = \frac{\Phi^-}{\Phi^- - 1}$.

We point out some examples of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, and for which (8) holds. For more details, the reader can consult [8, Examples 1–3, p. 243].

(1) $\varphi(t) = |t|^{p-2}t$, with $p > 1$. It can be showed that $\varphi^- = \Phi^+ = p$;
(2) $\varphi(t) = \log(1 + |t|^r)|t|^{p-2}t$, with $p, r > 1$. In this case $\Phi^- = p$, and $\Phi^+ = p + r$;
(3) $\varphi(t) = \frac{|t|^{p-2}t}{\log(1+|t|)}$ if $t \neq 0$, $\varphi(0) = 0$, with $p > 2$. In this case it turns out that $\Phi^- = p - 1$ and $\Phi^+ = p$.

Next, we recall some well-known inequalities regarding Φ and $\Phi^\star$, namely

$$\Phi^\star(\varphi(s)) \leq (\Phi^+ - 1)\, \Phi(s), \quad \forall s \geq 0 \tag{10}$$

(see, e.g. [6, Lemma 2.1]) and

$$\alpha(s)\Phi(t) \leq \Phi(st) \leq \beta(s)\Phi(t) \quad \forall s, t \geq 0, \tag{11}$$

where

$$\alpha(s) := \begin{cases} s^{\Phi^+}, & \text{if } s \in (0,1], \\ s^{\Phi^-}, & \text{if } s > 1, \end{cases} \quad \text{and} \quad \beta(s) := \begin{cases} s^{\Phi^-}, & \text{if } s \in (0,1], \\ s^{\Phi^+}, & \text{if } s > 1, \end{cases} \tag{12}$$

(see, e.g. [15, Lemma 2.1, (2.1)]).

We define the *Orlicz space*

$$L^\Phi(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that } \int_\Omega \Phi(|u(x,y)|)\, dxdy < \infty \right\}$$

which is a Banach space endowed with the *Luxembrug norm*

$$\|u\|_{L^\Phi(\Omega)} := \inf \left\{ k > 0;\ \int_\Omega \Phi\left(\frac{u(x)}{k}\right)\, dx \leq 1 \right\}.$$

We can also introduced the *Orlicz–Sobolev space* $W^{1,\Phi}(\Omega)$, the space of all functions u such that u and its distributional derivatives up to order 1 lie in the Orlicz space $L^\Phi(\Omega)$. More exactly, we define the Orlicz–Sobolev space

$$W^{1,\Phi}(\Omega) = \left\{ u \in L^\Phi(\Omega);\ |\nabla u| \in L^\Phi(\Omega) \right\}$$

which becomes a Banach space with respect to the following norm

$$\|u\|_{W^{1,\Phi}(\Omega)} := \|u\|_{L^\Phi(\Omega)} + \||\nabla u|\|_{L^\Phi(\Omega)},$$

where $\nabla u = (\nabla_x u, \nabla_y u)$. We denote by $W_0^{1,\Phi}(\Omega)$ the closure of $C_0^1(\Omega)$ in $W^{1,\Phi}(\Omega)$. On this space we can consider the equivalent norm

$$\|u\|_{W_0^{1,\Phi}(\Omega)} := \|\, |(\nabla_x u, \nabla_y u)|\, \|_{L^\Phi(\Omega)}.$$

Under assumption (8) it is well-known that the spaces $(L^\Phi(\Omega), \|\cdot\|_{L^\Phi(\Omega)})$, $(W^{1,\Phi}(\Omega), \|\cdot\|_{W^{1,\Phi}(\Omega)})$ and $(W_0^{1,\Phi}(\Omega), \|\cdot\|_{W_0^{1,\Phi}(\Omega)})$ are separable and reflexive Banach spaces. Moreover, the Hölder's inequality holds true (see [24, Inequality 4, page 79])

$$\int_\Omega u(x)v(x)\, dx \leq 2\|u\|_{L^\Phi(\Omega)}\|v\|_{L^{\Phi^\star}(\Omega)}$$

for any $u \in L^\Phi(\Omega)$ and $v \in L^{\Phi^\star}(\Omega)$.

Furthermore, if $u \in L^\Phi(\Omega)$, then

$$\|u\|_{L^\Phi(\Omega)} < 1 \Longrightarrow \|u\|^{\Phi^+}_{L^\Phi(\Omega)} \leq \int_\Omega \Phi(|u(x)|)\, dx \leq \|u\|^{\Phi^-}_{L^\Phi(\Omega)}, \tag{13}$$

$$\|u\|_{L^\Phi(\Omega)} > 1 \Longrightarrow \|u\|^{\Phi^-}_{L^\Phi(\Omega)} \leq \int_\Omega \Phi(|u(x)|)\, dx \leq \|u\|^{\Phi^+}_{L^\Phi(\Omega)}, \tag{14}$$

$$\|u\|_{L^\Phi(\Omega)} = 1 \Longleftrightarrow \int_\Omega \Phi(|u(x)|)\, dx = 1. \tag{15}$$

Let $p > 1$, and define $\varphi(t) = |t|^{p-2}t, \quad t \in \mathbb{R}$. As we already mentioned in example 1 above, it can be shown that in this case we have $\Phi^- = \Phi^+ = p$, and the corresponding Orlicz space $L^\Phi(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$, while the Orlicz–Sobolev space $W_0^{1,\Phi}(\Omega)$ becomes the Sobolev space $W_0^{1,p}(\Omega)$.

On the other hand, it is obvious that if Ω intersects the plane $x = 0$, the space $(W_0^{1,\Phi}(\Omega), \|\cdot\|_{W_0^{1,\Phi}(\Omega)})$ is no longer adequate for seeking solutions of Eq. (6). In this context, the natural functional space where we can investigate problem (6) is defined as the closure of $C_0^1(\Omega)$ under the norm

$$\|u\| := \Big\| |\nabla_x u| + |x|^{\xi/2}|\nabla_y u| \Big\|_{L^\Phi(\Omega)}. \tag{16}$$

We will denote this Orlicz–Sobolev-type space by $W_{0,\xi}^{1,\Phi}(\Omega)$. Standard arguments can be used to show that $\left(W_{0,\xi}^{1,\Phi}(\Omega), \|\cdot\|\right)$ is a reflexive Banach space. We note that the norm $\|\cdot\|$ is equivalent with the norm $\|\cdot\|_{W_0^{1,\Phi}(\Omega)}$ provided that there exists a positive constant c such that for each $(x, y) \in \Omega$ we have $|x| \geq c$ (see Appendix). Thus, the space $W_{0,\xi}^{1,\Phi}(\Omega)$ is a natural generalization of the classical Orlicz–Sobolev space $W_0^{1,\Phi}(\Omega)$.

4. Auxiliary results

In this section we establish two important auxiliary results which will be crucial in establishing the main result from the next section. The first result, given in Theorem 1, represents a generalization of Poincaré's inequality to the function space setting used in this paper while the second result, stated in Theorem 2, assures that under suitable conditions $W_{0,\xi}^{1,\Phi}(\Omega)$ is compactly embedded in $L^2(\Omega)$.

Theorem 1. *Assume conditions* (4) *and* (5) *are fulfilled. Then, there exists a positive constant C such that*

$$\begin{aligned}\int_\Omega [\Phi(|u(x,y)|) + \Phi(|x|^{\xi/2}|u(x,y)|)]\, dxdy \\ \leq C \int_\Omega \Big[\Phi(|\nabla_x u(x,y)|) + \Phi(|x|^{\xi/2}|\nabla_y u(x,y)|)\Big]\, dxdy\end{aligned} \tag{17}$$

for all $u \in C_0^1(\Omega)$.

Proof. Let $u \in C_0^1(\Omega)$ be given. Elementary computations show that

$$\mathrm{div}[\Phi(|u(x,y)|)(x, 0_m)] = n\, \Phi(|u(x,y)|) + \varphi(|u(x,y)|)\, x \cdot \nabla_x |u(x,y)|$$

for almost every $(x, y) \in \Omega$, where 0_m denotes the null vector in $\mathbb{R}^m$. Similarly, we have

$$\operatorname{div}[\Phi(|x|^{\xi/2}|u(x,y)|)(0_n,y)] = m\ \Phi(|x|^{\xi/2}|u(x,y)|) \\ +|x|^{\xi/2}\varphi(|x|^{\xi/2}|u(x,y)|)\ y\cdot\nabla_y|u(x,y)|$$

for almost every $(x,y)\in\Omega$, where 0_n is the null vector in $\mathbb{R}^n$.

After integrating these two identities over ω, an open set and sufficiently smooth subset of Ω such that supp $u\subset\omega$, we obtain

$$\int_\omega \operatorname{div}[\Phi(|u(x,y)|)(x,0_m)]\,dxdy = n\int_\omega \Phi(|u(x,y)|)dxdy + \int_\omega \varphi(|u(x,y)|)\ x \\ \cdot\nabla_x|u(x,y)|\ dxdy$$

and

$$\int_\omega \operatorname{div}[\Phi(|x|^{\xi/2}\ |u(x,y)|)(0_n,y)]\,dxdy \\ = m\int_\omega \Phi(|x|^{\xi/2}|u(x,y)|)\,dxdy \\ + \int_\omega |x|^{\xi/2}\ \varphi(x|^{\xi/2}|u(x,y|)\ y\cdot\nabla_y|u(x,y)|\,dxdy.$$

Then the flux-divergence theorem implies that

$$\int_\omega \operatorname{div}[\Phi(|u(x,y)|)(x,0_m)]\,dxdy = \int_{\partial\omega}\Phi(|u(x,y)|)(x,0_m)\cdot\overrightarrow{n}(x,y)\ d\sigma(x,y) \\ = 0$$

and

$$\int_\omega \operatorname{div}[\Phi(|x|^{\xi/2}|u(x,y)|)(0_n,y)]\,dxdy \\ = \int_{\partial\omega}\Phi(|x|^{\xi/2}|u(x,y)|)(0_n,y)\cdot\overrightarrow{n}(x,y)\ d\sigma(x,y) = 0,$$

where $\overrightarrow{n}(x,y)$ stands for the unit outward normal at $\partial\omega$.

Combining all the above pieces of information and since $\varphi(t)\geq 0$ for all $t\geq 0$, we infer that

$$\int_\omega [n\Phi(|u(x,y)|) + m\Phi(|x|^{\xi/2}|u(x,y)|)]\,dxdy \\ \leq \int_\omega \varphi(|u(x,y)|)\ |x|\ \ |\nabla_x|u(x,y)||\ dxdy \\ + \int_\omega |x|^{\xi/2}\ \varphi(|x|^{\xi/2}|u(x,y|)\ |y|\ \ |\nabla_y|u(x,y)||\ dxdy.$$

Thus, using the fact that supp $u\subset\omega\subset\Omega$ and taking into account the properties of Φ, we obtain

$$\int_\Omega [n\Phi(|u(x,y)|) + m\Phi(|x|^{\xi/2}|u(x,y)|)]\,dxdy \\ \leq \int_\Omega \varphi(|u(x,y)|)\ |x|\ |\nabla_x u(x,y)|\ dxdy \\ + \int_\Omega |x|^{\xi/2}\varphi(|x|^{\xi/2}|u(x,y|)\ |y|\ |\nabla_y u(x,y)|\ dxdy.$$

We point out that the Young-type inequality (7) implies that for all $\epsilon \in (0,1)$, we have

$$\int_\Omega \varphi(|u(x,y)|)\,|x|\,|\nabla_x u(x,y)|\,dxdy \le \int_\Omega \Phi(\epsilon^{-1}|x|\,|\nabla_x u(x,y)|)\,dxdy + \int_\Omega \Phi^\star(\epsilon\varphi(|u(x,y)|))\,dxdy$$

and

$$\int_\Omega |x|^{\xi/2}\varphi(|x|^{\xi/2}|u(x,y)|)\,|y|\,|\nabla_y u(x,y)|\,dxdy \le \int_\Omega \Phi(\epsilon^{-1}|x|^{\xi/2}|y|\,|\nabla_y u(x,y)|)\,dxdy + \int_\Omega \Phi^\star(\epsilon\varphi(|x|^{\xi/2}|u(x,y)|))\,dxdy.$$

Taking into account the above pieces of information and then using inequalities (10), (11) and (13) and the fact that $|x| \le |(x,y)|$ and $|y| \le |(x,y)|$, we find

$$[n - \epsilon^{(\Phi^\star)^-}(\Phi^+ - 1)] \int_\Omega \Phi(|u(x,y)|)\,dxdy + [m - \epsilon^{(\Phi^\star)^-}(\Phi^+ - 1)] \int_\Omega \Phi(|x|^{\xi/2}|u(x,y)|)\,dxdy$$
$$\le (\epsilon^{-1})^{\Phi^+}\int_\Omega \Phi(|x||\nabla_x u(x,y)|)\,dxdy + (\epsilon^{-1})^{\Phi^+}\int_\Omega \Phi(|y||x|^{\xi/2}|\nabla_y u(x,y)|)\,dxdy$$
$$\le (\epsilon^{-1})^{\Phi^+}(1+\mathrm{diam}(\Omega))^{\Phi^+}\int_\Omega [\Phi(|\nabla_x u(x,y)|) + \Phi(|x|^{\xi/2}|\nabla_y u(x,y)|)]\,dxdy,$$

where $\mathrm{diam}(\Omega)$ is the diameter of Ω. The proof of Theorem 1 is complete. □

Theorem 2. *Assume that the hypotheses of Theorem 1 are fulfilled and the domain Ω intersects that plane $x = 0$. Furthermore, assume that $\Phi^- > 2$ and $0 < \xi < 2n(\Phi^- - 2)/\Phi^-$. Then $W^{1,\Phi}_{0,\xi}(\Omega)$ is compactly embedded in the Lebesgue space $L^2(\Omega)$.*

Proof. Let $\{u_p\}$ be a bounded sequence in the Orlicz–Sobolev space $W^{1,\Phi}_{0,\xi}(\Omega)$. Since Ω intersects the plane $x = 0$, for each $\epsilon \in (0, \min\{1, \frac{1}{2}\ \mathrm{diam}(\Omega)\})$ we have

$$D_\epsilon := \{(x,y) \in \Omega;\ |x| < \epsilon\} \subset \Omega,$$

where $\mathrm{diam}(\Omega)$ is the diameter of the domain Ω.

By Theorem 1 it follows that the sequence $\{u_p\}$ is bounded in the Orlicz space $L^\Phi(\Omega)$. Consequently, $\{u_p\} \subset W^{1,\Phi^-}(\Omega\backslash\overline{D_\epsilon})$ is a bounded sequence. Since $W^{1,\Phi}(\Omega\backslash\overline{D_\epsilon}) \subset W^{1,\Phi^-}(\Omega\backslash\overline{D_\epsilon})$, we deduce that the sequence $\{u_p\}$ is bounded in $W^{1,\Phi^-}(\Omega\backslash\overline{D_\epsilon})$. The classical compact embedding theorem implies that $\{u_p\}$

possesses a convergent subsequence, still denoted by $\{u_p\}$, in Lebesgue space $L^2(\Omega\backslash\overline{D_\epsilon})$. Therefore, for any p and r large enough, we get

$$\int_{\Omega\backslash\overline{D_\epsilon}} |u_p - u_r|^2 \, dxdy < \epsilon.$$

On the other hand, using the Hölder's inequality, we have

$$\int_{D_\epsilon} |u_p - u_r|^2 \, dxdy = \int_{D_\epsilon} \frac{1}{|x|^{\xi/2}} \, |x|^{\xi/2}|u_p - u_r|^2 \, dxdy$$
$$\leq \| |x|^{-\xi/2}\chi_{D_\epsilon}\|_{L^{\frac{\Phi^-}{\Phi^- -2}}(\Omega)} \| |x|^{\xi/2}|u_p - u_r|^2\|_{L^{\Phi^-/2}(\Omega)}.$$

Next, using the fact that $|x| < \epsilon < 1$ for each $(x, y) \in D_\epsilon$ and taking into account inequalities (13)–(14), then the conclusion of Theorem 1 yields

$$\begin{aligned}
&\| |x|^{\xi/2}|u_p - u_r|^2\|_{L^{\Phi^-/2}(\Omega)} \\
&= \left(\int_\Omega |x|^{\xi\Phi^-/4}|u_p - u_r|^{\Phi^-} dxdy\right)^{2/\Phi^-} \\
&\leq \|u_p - u_r\|^2_{L^{\Phi^-}(\Omega)} \\
&\leq C_1\|u_p - u_r\|^2_{L^\Phi(\Omega)} \\
&\leq C_2\left[\left(\int_\Omega \Phi(|u_p - u_r|)\, dxdy\right)^{2/\Phi^-} + \left(\int_\Omega \Phi(|u_p - u_r|)\, dxdy\right)^{2/\Phi^+}\right] \\
&\leq C_3\left[\int_\Omega \Big(\Phi(|\nabla_x(u_p - u_r)|) + \Phi(|x|^{\xi/2}|\nabla_y(u_p - u_r)|)\Big)\, dxdy\right]^{2/\Phi^-} \\
&\quad + C_3\left[\int_\Omega \Big(\Phi(|\nabla_x(u_p - u_r)|) + \Phi(|x|^{\xi/2}|\nabla_y(u_p - u_r)|)\Big)\, dxdy\right]^{2/\Phi^+} \\
&\leq C_4 < \infty,
\end{aligned}$$

since $\{u_p\}$ is bounded in $W^{1,\Phi}_{0,\xi}(\Omega)$, where C_1, C_2, C_3 and C_4 are positive constants.

In what follows, we compute

$$\| |x|^{-\xi/2}\chi_{D_\epsilon}\|_{L^{\frac{\Phi^-}{\Phi^- -2}}(\Omega)} = \left(\int_{D_\epsilon} |x|^{\frac{-\xi\Phi^-}{2(\Phi^- -2)}} \, dxdy\right)^{\frac{\Phi^- -2}{\Phi^-}}.$$

We note that $D_\epsilon \subset \Omega_1 \times \Omega_2$ where $\Omega_1 := \{x \in \mathbb{R}^n;\ |x| < \epsilon\}$ and $\Omega_2 := \{y \in \mathbb{R}^m;\ |y| < M\}$ with $M := \sup_{(x,y)\in\Omega} |y|$.

We introduce the function $H : \Omega_1 \times \Omega_2 \to \mathbb{R}$ defined by $H(x, y) = |x|^{-\frac{\xi\Phi^-}{2(\Phi^- -2)}}$. For each $x \in \Omega_1\backslash\{0\}$ we have

$$\int_{\Omega_2} |H(x, y)|\, dy = |x|^{-\frac{\xi\Phi^-}{\Phi^- -2}}|\Omega_2| < \infty,$$

where $|\Omega_2|$ is the m-dimensional Lebesgue measure of Ω_2. On the other hand, using the co-area formula and taking into account that Ω_1 is the ball centered in origin of radius ϵ in the n-dimensional euclidian space $\mathbb{R}^n$, denoted by $B_\epsilon(0)$, we obtain

$$\int_{\Omega_1} |x|^{-\frac{\xi\Phi^-}{2(\Phi^- -2)}}\, dx = \int_0^\epsilon \omega_n\, t^{n-1-\frac{\xi\Phi^-}{2(\Phi^- -2)}}\, dt = \mathcal{C}\, \epsilon^{n-\frac{\xi\Phi^-}{2(\Phi^- -2)}},$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$ and $\mathcal{C}$ is a positive constant. As a consequence, we deduce that

$$\int_{\Omega_1} dx \int_{\Omega_2} |H(x,y)|\, dy = \mathcal{C}\, \epsilon^{n-\frac{\xi\Phi^-}{2(\Phi^- -2)}} |\Omega_2| < \infty.$$

Thus, we can apply Tonelli's theorem and we infer that $H \in L^1(\Omega_1 \times \Omega_2)$. Going further, we can apply Fubini's theorem and we get

$$\int_{\Omega_1\times\Omega_2} |x|^{-\frac{\xi\Phi^-}{2(\Phi^- -2)}}\, dxdy \le \mathcal{C}\, \epsilon^{n-\frac{\xi\Phi^-}{2(\Phi^- -2)}},$$

where $\mathcal{C}$ is a positive constant.

Taking into account the above pieces of information, we infer that

$$\||x|^{-\xi/2}\chi_{D_\epsilon}\|_{L^{\frac{\Phi^-}{\Phi^- -2}}(\Omega)} \le \left(\int_{\Omega_1\times\Omega_2} |x|^{-\frac{\xi\Phi^-}{2(\Phi^- -2)}}\, dxdy\right)^{\frac{\Phi^- -2}{\Phi^-}} \le \text{const.}\ \epsilon^{\frac{n(\Phi^- -2)}{\Phi^-}-\frac{\xi}{2}}.$$

Therefore,

$$\int_\Omega |u_p - u_r|^2\, dxdy \le \text{Const.}\left(\epsilon + \epsilon^{\frac{n(\Phi^- -2)}{\Phi^-}-\frac{\xi}{2}}\right).$$

We conclude that $\{u_p\}$ is a Cauchy sequence in $L^2(\Omega)$ and the proof of Theorem 2 is complete. □

5. Spectrum of the operator $-\Delta_{G_a}$

We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of the problem (6) if there exists $u_\lambda \in W_{0,\xi}^{1,\Phi}(\Omega)\setminus\{0\}$ such that

$$\int_\Omega \Big(a(|\nabla_x u_\lambda|)\nabla_x u_\lambda \cdot \nabla_x v + |x|^\xi a(|x|^{\xi/2}|\nabla_y u_\lambda|)\nabla_y u_\lambda \cdot \nabla_y v\Big) dxdy$$
$$= \lambda \int_\Omega u_\lambda v\, dxdy \tag{18}$$

for any $v \in W_{0,\xi}^{1,\Phi}(\Omega)$. Moreover, u_λ from the above definition is called an *eigenfunction* associated to eigenvalue λ.

Remark. Note that each $\lambda \in (-\infty, 0]$ can not be an eigenvalue of problem (6). That fact follows simply from relation (18) by testing with $v = u_\lambda$.

The following theorem describes the entire spectrum of the operator $-\Delta_{G_a}$.

Theorem 3. *The set of the eigenvalues of the problem* (6) *is exactly the interval* $(0,\infty)$. *Moreover, for each* $\lambda > 0$ *there exists a sequence of eigenfunctions* $\{u_k\} \subset W_{0,\xi}^{1,\Phi}(\Omega)$ *such that* $\lim_{k\to\infty} u_k = 0$ *in* $W_{0,\xi}^{1,\Phi}(\Omega)$.

5.1. Proof of Theorem 3

Let $\lambda \in (0, \infty)$ be arbitrary but fixed. We consider the energy functional corresponding to the problem (6) defined as $I_\lambda : W_{0,\xi}^{1,\Phi}(\Omega) \to \mathbb{R}$,

$$I_\lambda(u) = \int_\Omega \Big(\Phi(|\nabla_x u|) + \Phi(|x|^{\xi/2}|\nabla_y u|) \Big) dxdy \\ -\frac{\lambda}{2} \int_\Omega |u(x,y)|^2 \, dxdy \tag{19}$$

for all $u \in W_{0,\xi}^{1,\Phi}(\Omega)$.

Standard arguments show that the functional $I_\lambda \in C^1\big(W_{0,\xi}^{1,\Phi}(\Omega), \mathbb{R}\big)$ and its Fréchet derivative is given by

$$\langle I'_\lambda(u), v\rangle = \int_\Omega \Big(a(|\nabla_x u|)\nabla_x u \cdot \nabla_x v + |x|^\xi a(|x|^{\xi/2}|\nabla_y u|)\nabla_y u \cdot \nabla_y v \Big) dxdy \\ -\lambda \int_\Omega uv \, dxdy \tag{20}$$

for all $u, v \in W_{0,\xi}^{1,\Phi}(\Omega)$.

In view of the above relation, we note that the weak solutions of the problem (6) are exactly the critical points of the functional I_λ. Thus, λ is an eigenvalue of the problem (6) if and only if the functional I_λ has a nontrivial critical point.

The idea to prove Theorem 3 is to use the above remarks and to apply a symmetric version of the mountain pass lemma, developed by Kajikiya. In order to give the statement of Kajikiya's result we recall first some definitions and we introduce some notations.

Definition 1. Let X be a real Banach space. We say that a subset A of X is *symmetric* if $u \in A$ implies $-u \in A$. For a closed symmetric set A which does not contain the origin, we define the *genus* of A, denoted by $\gamma(A)$, as the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If does not exist such an integer k, we define $\gamma(A) = +\infty$. Moreover, we set $\gamma(\emptyset) = 0$. Finally, we denote by Γ_k the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq k$.

Theorem 4. [19, Theorem 1] *Let X be an infinite dimensional Banach space and $\Lambda \in C^1(X, \mathbb{R})$ satisfies conditions* (A1) *and* (A2) *below.*

(A1) *$\Lambda(u)$ is even, bounded from below, $\Lambda(0) = 0$ and $\Lambda(u)$ satisfies the Palais-Smale condition, that is any sequence $\{u_p\}$ in X such that $\{\Lambda(u_p)\}$ is bounded and $\Lambda'(u_p) \to 0$ in $X^\star$ as $p \to \infty$ has a convergent subsequence.*

(A2) *For each $k \in \mathbb{N}$, there exists a subset $A_k \in \Gamma_k$ such that* $\sup_{u \in A_k} \Lambda(u) < 0$.

Under the above assumptions (A1) *and* (A2), *either* (i) *or* (ii) *below holds true:*

(i) *There exists a sequence $\{u_k\}$ such that $\Lambda'(u_k) = 0$, $\Lambda(u_k) < 0$ and $\{u_k\}$ converges to zero.*

(ii) *There exist two sequences* $\{u_k\}$ *and* $\{v_k\}$ *such that* $\Lambda'(u_k) = 0$, $\Lambda(u_k) = 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$, $\Lambda'(v_k) = 0$, $\Lambda(v_k) < 0$, $\lim_{k\to\infty} \Lambda(v_k) = 0$ *and* $\{v_k\}$ *converges to a non-zero limit.*

In order to apply Theorem 4 to the functional I_λ we have to show that conditions (A1) and (A2) from this theorem are fulfilled.

Lemma 1. *The functional* I_λ *satisfies condition* (*A*1) *from Theorem* 4*.*

Proof. Clearly, $I_\lambda(-u) = I_\lambda(u)$ for any $u \in W^{1,\Phi}_{0,\xi}(\Omega)$, that is the functional I_λ is even, and $I_\lambda(0) = 0$.

On the other hand, using the second inequality (11) and the fact that Φ is a convex function, we obtain

$$\begin{aligned}\Phi\Big(|\nabla_x u| + |x|^{\xi/2}|\nabla_y u|\Big) &\le 2^{\Phi^+}\ \ \Phi\left(\frac{|\nabla_x u| + |x|^{\xi/2}|\nabla_y u|}{2}\right)\\ &\le 2^{\Phi^+-1}\big(\Phi(|\nabla_x u|) + \Phi(|x|^{\xi/2}|\nabla_y u|)\big),\\ &\qquad \forall u \in W^{1,\Phi}_{0,\xi}(\Omega).\end{aligned}$$

Next, since $W^{1,\Phi}_{0,\xi}(\Omega)$ is compactly embedded in the Lebesgue space $L^2(\Omega)$ (by Theorem 2), it follows that there exists a positive constant C such that

$$\|u\|_{L^2(\Omega)} \le C\|u\| \quad \text{for any} \ \ u \in W^{1,\Phi}_{0,\xi}(\Omega).$$

The last two inequalities combined with relations (13)–(14) show that

$$I_\lambda(u) \ge \frac{1}{2^{\Phi^+-1}}\alpha(\|u\|) - \frac{\lambda}{2}\, C^2\, \|u\|^2, \quad \forall u \in W^{1,\Phi}_{0,\xi}(\Omega), \tag{21}$$

where function α is defined in relation (12). Since $2 < \Phi^- \le \Phi^+$ we infer that I_λ is bounded from below on $W^{1,\Phi}_{0,\xi}(\Omega)$.

Finally, we prove that I_λ satisfies the Palais–Smale condition. Let $\{u_p\} \subset W^{1,\Phi}_{0,\xi}(\Omega)$ be a sequence such that $\{I_\lambda(u_p)\}$ is bounded and $\langle I'_\lambda(u_p), v\rangle \to 0$ as $p \to \infty$ for any $v \in W^{1,\Phi}_{0,\xi}(\Omega)$.

We show that the sequence $\{u_p\}$ is bounded in $W^{1,\Phi}_{0,\xi}(\Omega)$. Assume by contradiction the contrary. Then, passing eventually to a subsequence, still denoted by $\{u_p\}$, we may assume that $\|u_p\| \to \infty$ as $p \to \infty$. Combining this fact with relation (21) we find

$$\lim_{\|u_p\|\to\infty} I_\lambda(u_p) = \infty,$$

a contradiction with the assumption that $\{I_\lambda(u_p)\}$ is a bounded sequence. It follows that $\{u_p\}$ is a bounded sequence in $W^{1,\Phi}_{0,\xi}(\Omega)$. Therefore, since $W^{1,\Phi}_{0,\xi}(\Omega)$ is a reflexive Banach space, there exists $u \in W^{1,\Phi}_{0,\xi}(\Omega)$ such that a subsequence of $\{u_p\}$, still denoted by $\{u_p\}$, converges weakly to u in $W^{1,\Phi}_{0,\xi}(\Omega)$. Moreover, by Theorem 2 we deduce that $\{u_p\}$ converges strongly to u in $L^2(\Omega)$.

Next, we show that $\{u_p\}$ converges strongly to u in $W^{1,\Phi}_{0,\xi}(\Omega)$. Taking into account that the mapping $J_\lambda : W^{1,\Phi}_{0,\xi}(\Omega) \to \mathbb{R}$ defined by $J_\lambda(v) = \langle I'_\lambda(u), v\rangle$

for any $v \in W_{0,\xi}^{1,\Phi}(\Omega)$, is linear and continuous, the above facts imply that $J_\lambda(u_p - u) \to 0$ as $p \to \infty$, namely

$$\lim_{p\to\infty} \langle I'_\lambda(u), u_p - u\rangle = 0. \tag{22}$$

On the other hand, since $\langle I'_\lambda(u_p), v\rangle \to 0$ as $p \to \infty$ for any $v \in W_{0,\xi}^{1,\Phi}(\Omega)$, we have

$$\lim_{p\to\infty} \langle I'_\lambda(u_p), u_p - u\rangle = 0. \tag{23}$$

Using Hölder's inequality and the facts that $\{u_p\}$ converges strongly to u in $L^2(\Omega)$ and $\{u_p\}$ is a bounded sequence in $W_{0,\xi}^{1,\Phi}(\Omega)$ and also in $L^2(\Omega)$, we infer that

$$\lim_{p\to\infty} \int_\Omega u(u_p - u)\, dxdy = 0 \tag{24}$$

and

$$\lim_{p\to\infty} \int_\Omega u_p(u_p - u)\, dxdy = 0. \tag{25}$$

Adding relations (23) and (22) and taking into consideration (24)–(25), we obtain

$$\begin{aligned}\lim_{p\to\infty} \Bigg[&\int_\Omega \big(a(|\nabla_x u_p|)\nabla_x u_p - a(|\nabla_x u|)\nabla_x u\big) \cdot \nabla_x(u_p - u)\, dxdy \\ &+ \int_\Omega |x|^\xi \big(a(|x|^{\xi/2}|\nabla_y u_p|)\nabla_y u_p - a(|x|^{\xi/2}|\nabla_y u|)\nabla_y u\big) \cdot \nabla_y(u_p - u)\, dxdy \Bigg] \\ = 0.&\end{aligned} \tag{26}$$

Next, using the fact that Φ is a convex function, we deduce

$$\Phi(|\nabla_x u_p|) \le \Phi\left(\left| \frac{\nabla_x u_p + \nabla_x u}{2} \right| \right) + a(|\nabla_x u_p|)\nabla_x u_p \cdot \frac{\nabla_x u_p - \nabla_x u}{2}, \quad \forall p \in \mathbb{N}$$

and

$$\Phi(|\nabla_x u|) \le \Phi\left(\left| \frac{\nabla_x u + \nabla_x u_p}{2} \right| \right) + a(|\nabla_x u|)\nabla_x u \cdot \frac{\nabla_x u - \nabla_x u_p}{2}, \quad \forall p \in \mathbb{N}.$$

Upon adding the last two inequalities term by term and integrating over Ω, we get

$$\begin{aligned} 2\int_\Omega \Phi(|\nabla_x u|)\, dxdy &+ 2\int_\Omega \Phi(|\nabla_x u_p|)\, dxdy - 4\int_\Omega \Phi\left(\left| \frac{\nabla_x u_p + \nabla_x u}{2} \right| \right) dxdy \\ &\le \int_\Omega \Big[a(|\nabla_x u_p|)\nabla_x u_p - a(|\nabla_x u|)\nabla_x u \Big] \cdot (\nabla_x u_p - \nabla_x u)\, dxdy, \quad \forall p \in \mathbb{N}. \end{aligned} \tag{27}$$

Since condition (5) is assumed we deduce by [21, Theorem 2.1] that

$$\frac{1}{2}\int_\Omega \Big(\Phi(|\nabla_x u_p|)+\Phi(|\nabla_x u|)\Big)\,dxdy \geq \int_\Omega \Phi\left(\left|\frac{\nabla_x u_p+\nabla_x u}{2}\right|\right)dxdy+\int_\Omega \Phi\left(\left|\frac{\nabla_x u_p-\nabla_x u}{2}\right|\right)dxdy, \quad \forall p\in\mathbb{N}. \tag{28}$$

Relations (27), (28) and (11) imply

$$\begin{aligned}\int_\Omega \Big[a(|\nabla_x u_p|)\nabla_x u_p - a(|\nabla_x u|)\nabla_x u\Big]\cdot(\nabla_x u_p-\nabla_x u)\,dxdy \\ \geq 4\int_\Omega \Phi\left(\left|\frac{\nabla_x u_p-\nabla_x u}{2}\right|\right)dxdy \\ \geq 2^{2-\Phi^+}\int_\Omega \Phi(|\nabla_x(u_p-u)|)\,dxdy, \quad \forall p\in\mathbb{N}.\end{aligned} \tag{29}$$

Similar way, we can show that

$$\int_\Omega |x|^\xi \Big[a(|x|^{\xi/2}|\nabla_y u_p|)\nabla_y u_p - a(|x|^{\xi/2}|\nabla_y u|)\nabla_y u\Big]\cdot(\nabla_y u_p-\nabla_y u)\,dxdy \geq 2^{2-\Phi^+}\int_\Omega \Phi(|x|^{\xi/2}|\nabla_y(u_p-u)|)\,dxdy, \quad \forall p\in\mathbb{N}. \tag{30}$$

Combining (29), (30), (26) and relations (13)–(14) we conclude that $\{u_p\}$ converges strongly to u in $(W_{0,\xi}^{1,\Phi}(\Omega),\|\cdot\|)$. Therefore, I_λ satisfies the Palais–Smale condition.

The proof of Lemma 1 is complete. □

Lemma 2. *The functional I_λ satisfies condition* $(A2)$ *from Theorem* 4.

Proof. We construct a sequence of subsets $A_k\in\Gamma_k$ such that $\sup_{u\in A_k} I_\lambda(u)<0$ for each $k\in\mathbb{N}$.

We fix $k\in\mathbb{N}$ arbitrary. Let $(x_1,y_1)\in\Omega$ and r_1 be a positive real number such that $\overline{B_{r_1}(x_1,y_1)}\subset\Omega$ and $|\overline{B_{r_1}(x_1,y_1)}|<\frac{1}{2}|\Omega|$, where $B_{r_1}(x_1,y_1)$ is the ball in $\mathbb{R}^N$ centered in (x_1,y_1) of radius r_1. Consider $v_1\in C_0^1(\Omega)$ be a function with $\mathrm{supp}(v_1)=\overline{B_{r_1}(x_1,y_1)}$. Define $\Omega_1:=\Omega\backslash\overline{B_{r_1}(x_1,y_1)}$. Let $(x_2,y_2)\in\Omega$ and r_2 be a positive real number such that $\overline{B_{r_2}(x_2,y_2)}\subset\Omega_1$ and $|\overline{B_{r_2}(x_2,y_2)}|<\frac{1}{2}|\Omega_1|$. Consider $v_2\in C_0^1(\Omega)$ such that $\mathrm{supp}(v_2)=\overline{B_{r_2}(x_2,y_2)}$. Next, define $\Omega_2:=\Omega_1\setminus\overline{B_{r_2}(x_2,y_2)}$ and let $(x_3,y_3)\in\Omega$ and $r_3>0$ be a real number such that $\overline{B_{r_3}(x_3,y_3)}\subset\Omega_2$ and $|\overline{B_{r_3}(x_3,y_3)}|<\frac{1}{2}|\Omega_2|$. Consider $v_3\in C_0^1(\Omega)$ such that $\mathrm{supp}(v_3)=\overline{B_{r_3}(x_3,y_3)}$.

Continuing the process described above we can construct by recurrence a sequence of functions $v_1,v_2,\ldots,v_k\in C_0^1(\Omega)$ such that $\mathrm{supp}(v_i)\neq\mathrm{supp}(v_j)$ for $i\neq j$ and $|\mathrm{supp}(v_i)|>0$ for any $i,j\in\{1,\ldots,k\}$.

We define the finite dimensional subspace of $W_{0,\xi}^{1,\Phi}(\Omega)$,

$$V:=\mathrm{span}\{v_1,v_2,v_3,\ldots,v_k\}.$$

Clearly, dim $V = k$ and $\int_\Omega |v|^2\,dxdy > 0$ for every $v \in V\setminus\{0\}$. We denote by $\mathbb{S}^1$ the unit sphere in $W^{1,\Phi}_{0,\xi}(\Omega)$, namely

$$\mathbb{S}^1 := \{v \in W^{1,\Phi}_{0,\xi}(\Omega);\ \|v\| = 1\}.$$

For any real number $t \in (0,1)$, we define the set

$$A_k(t) := t\,(\mathbb{S}^1 \cap V) = \{tv;\ v \in \mathbb{S}^1 \cap V\}.$$

By [26, Proposition 5.2] we have that for any bounded symmetric neighborhood B of the origin in $\mathbb{R}^k$ there holds $\gamma(\partial B) = k$. Thus, we deduce that $\gamma(A_k(t)) = k$ for any $t \in (0,1)$.

It remains to show that for each integer $k \in \mathbb{N}$ there exists $t_k \in (0,1)$ such that

$$\sup_{u\in A_k(t_k)} I_\lambda(u) < 0.$$

Note that for any $t \in (0,1)$, we have

$$\begin{aligned}
&\sup_{u\in A_k(t)} I_\lambda(u)\\
&\quad= \sup_{v\in\mathbb{S}^1\cap V} I_\lambda(t\,v)\\
&\quad= \sup_{v\in\mathbb{S}^1\cap V} \left\{ \int_\Omega [\Phi(t|\nabla_x v|) + \Phi(t|x|^{\xi/2}|\nabla_y v|)]dxdy - \frac{\lambda}{2}\int_\Omega |tv|^2\,dxdy \right\}\\
&\quad\le \sup_{v\in\mathbb{S}^1\cap V} \left\{ t^{\Phi^-} \int_\Omega [\Phi(|\nabla_x v|) + \Phi(|x|^{\xi/2}|\nabla_y v|)]dxdy - \frac{\lambda\,t^2}{2}\int_\Omega |v|^2\,dxdy \right\}\\
&\quad\le \sup_{v\in\mathbb{S}^1\cap V} \left\{ 2\,t^{\Phi^-} \int_\Omega \Phi(|\nabla_x v| + |x|^{\xi/2}|\nabla_y v|)dxdy - \frac{\lambda\,t^2}{2}\int_\Omega |v|^2\,dxdy \right\}\\
&\quad\le \sup_{v\in\mathbb{S}^1\cap V} \left\{ t^{\Phi^-}\left(2 - \frac{\lambda}{2t^{\Phi^- -2}}\int_\Omega |v|^2\,dxdy\right)\right\}.
\end{aligned}$$

Since $\mathbb{S}^1 \cap V$ is a compact set, we get

$$m := \min_{v\in\mathbb{S}^1\cap V} \int_\Omega |v|^2\,dxdy > 0.$$

Taking into account that $2 < \Phi^-$ we deduce that we can choose $t_k \in (0,1)$ small enough such that

$$2 - \frac{\lambda}{2}\,t_k^{2-\Phi^-} m < 0.$$

The above relation yields

$$\sup_{u\in A_k(t_k)} I_\lambda(u) < 0.$$

Thus, the proof of Lemma 2 is complete. □

Proof of Theorem 3. Using Lemmas 1 and 2 and applying Theorem 4 for the functional I_λ we deduce that, in any case (i) and (ii), there exists a sequence $\{u_k\} \subset W_{0,\xi}^{1,\Phi}(\Omega)$ such that $\langle I_\lambda'(u_k), v\rangle = 0$ for any $v \in W_{0,\xi}^{1,\Phi}(\Omega)$, $I_\lambda(u_k) \leq 0$, $u_k \neq 0$ for each k and $\{u_k\}$ converges to zero in $W_{0,\xi}^{1,\Phi}(\Omega)$. Thus, each $\lambda > 0$ is an eigenvalue of the problem (6) with the corresponding eigenfunctions $\{u_k\}$ converging to 0. The proof of Theorem 3 is complete. □

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Appendix

In this section we show that the norms

$$\|u\| := \Big\| |\nabla_x u| + |x|^{\xi/2} |\nabla_y u| \Big\|_{L^\Phi(\Omega)},$$
$$\|u\|_1 := \big\|(\nabla_x u, \nabla_y u)\big\|_{L^\Phi(\Omega)}$$

and

$$\|u\|_2 := \big\|(\nabla_x u, |x|^{\xi/2} \nabla_y u)\big\|_{L^\Phi(\Omega)}$$

are equivalent on the Orlicz–Sobolev space $W_{0,\xi}^{1,\Phi}(\Omega)$, provided that there exists a positive constant c such that for every $(x, y) \in \Omega$ we have $|x| \geq c$. Particularly, that fact shows that the functional space $W_{0,\xi}^{1,\Phi}(\Omega)$ is a natural generalization of the classical Orlicz–Sobolev space $W_0^{1,\Phi}(\Omega)$. More precisely, we will prove:

Proposition 1. *Let $\Omega \subset \mathbb{R}^N$ $(N = n + m)$ be a bounded and smooth domain for which there exists a positive constant c such that for each $(x, y) \in \Omega$ we have $|x| \geq c$. Then the norms*

$$\|u\| := \Big\| |\nabla_x u| + |x|^{\xi/2} |\nabla_y u| \Big\|_{L^\Phi(\Omega)},$$
$$\|u\|_1 := \big\|(\nabla_x u, \nabla_y u)\big\|_{L^\Phi(\Omega)}$$

and

$$\|u\|_2 := \left\|(\nabla_x u, |x|^{\xi/2}\nabla_y u)\right\|_{L^\Phi(\Omega)}$$

are equivalent.

Proof. First, we prove that the norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent. Simple computations show that we have

$$\frac{1}{\sqrt{2}}\left(|\nabla_x u| + |x|^{\xi/2}|\nabla_y u|\right) \leq \sqrt{|\nabla_x u|^2 + |x|^\xi|\nabla_y u|^2} \tag{31}$$

for any $u \in W^{1,\Phi}_{0,\xi}(\Omega)$ and $(x, y) \in \Omega$. The definition of Luxemburg norm leads to

$$\int_\Omega \Phi\left(\frac{\sqrt{|\nabla_x u|^2 + |x|^\xi|\nabla_y u|^2}}{\|u\|_2}\right)\, dxdy \leq 1$$

and taking into account that Φ is an increasing function and inequality (31) holds true, we find that

$$\int_\Omega \Phi\left(\frac{|\nabla_x u| + |x|^{\xi/2}|\nabla_y u|}{\sqrt{2}\|u\|_2}\right)\, dxdy \leq 1$$

which implies that $\|u\| \leq \sqrt{2}\ \|u\|_2$ for any $u \in W^{1,\Phi}_{0,\xi}(\Omega)$. Next, using again the definition of the Luxemburg norm, we get

$$\int_\Omega \Phi\left(\frac{|\nabla_x u| + |x|^{\xi/2}|\nabla_y u|}{\|u\|}\right)\, dxdy \leq 1.$$

It is clear that the following inequality

$$\sqrt{|\nabla_x u|^2 + |x|^\xi|\nabla_y u|^2} \leq |\nabla_x u| + |x|^{\xi/2}|\nabla_y u|$$

holds true for any $u \in W^{1,\Phi}_{0,\xi}(\Omega)$ and $(x, y) \in \Omega$. Combining the fact that Φ is an increasing function with the last two inequalities we deduce

$$\int_\Omega \Phi\left(\frac{\sqrt{|\nabla_x u|^2 + |x|^\xi|\nabla_y u|^2}}{\|u\|}\right)\, dxdy \leq 1$$

or, $\|u\|_2 \leq \|u\|$ for any $u \in W^{1,\Phi}_{0,\xi}(\Omega)$.

Therefore, we have

$$\frac{1}{\sqrt{2}}\ \|u\| \leq \|u\|_2 \leq \|u\| \quad \text{for all } u \in W^{1,\Phi}_{0,\xi}(\Omega),$$

which shows the equivalence of norms $\|\cdot\|$ and $\|\cdot\|_2$.

Second, we show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent provided that there exists a positive constant c such that for any $(x, y) \in \Omega$ we have $|x| \geq c$. We start with the simple remark that

$$a(|\nabla_x u|^2 + |\nabla_y u|^2) \leq |\nabla_x u|^2 + |x|^\xi|\nabla_y u|^2 \leq b(|\nabla_x u|^2 + |\nabla_y u|^2) \tag{32}$$

for any $u \in W_{0,\xi}^{1,\Phi}(\Omega)$ and $(x, y) \in \Omega$, where $a := \min_{(x,y)\in\Omega}\left\{1, |x|^{\xi}\right\}$ and $b := \max_{(x,y)\in\Omega}\left\{1, |x|^{\xi}\right\}$.

By the definition of the Luxemburg norm, we deduce

$$\int_{\Omega} \Phi\left(\frac{\sqrt{|\nabla_x u|^2 + |\nabla_y u|^2}}{\|u\|_1}\right) dxdy \leq 1 \tag{33}$$

and

$$\int_{\Omega} \Phi\left(\frac{\sqrt{|\nabla_x u|^2 + |x|^{\xi}|\nabla_y u|^2}}{\|u\|_2}\right) dxdy \leq 1. \tag{34}$$

• If $a \geq 1$, by (32) we get $|\nabla_x u|^2 + |\nabla_y u|^2 \leq |\nabla_x u|^2 + |x|^{\xi}|\nabla_y u|^2$ for any $u \in W_{0,\xi}^{1,\Phi}(\Omega)$ and $(x, y) \in \Omega$. Combining this inequality with the fact that Φ is an increasing function and inequality (34) holds true, we obtain

$$\int_{\Omega} \Phi\left(\frac{\sqrt{|\nabla_x u|^2 + |\nabla_y u|^2}}{\|u\|_2}\right) dxdy \leq 1$$

or $\|u\|_1 \leq \|u\|_2$.

• If $a < 1$, by the first inequality in (32) and (34) we deduce that

$$\int_{\Omega} \Phi\left(\frac{\sqrt{|\nabla_x u|^2 + |\nabla_y u|^2}}{\sqrt{a}\|u\|_2}\right) dxdy \leq 1$$

which implies that $\|u\|_1 \leq \frac{1}{\sqrt{a}}\|u\|_2$.

• If $b \leq 1$, inequality (32) leads to

$$|\nabla_x u|^2 + |x|^{\xi}|\nabla_y u|^2 \leq |\nabla_x u|^2 + |\nabla_y u|^2, \quad \forall u \in W_{0,\xi}^{1,\Phi}(\Omega)$$

which helps us to prove that $\|u\|_2 \leq \|u\|_1$ as a consequence of inequality (33).

• If $b > 1$, by the second inequality in (32), inequality (33) and the fact that Φ is an increasing function, we get

$$\int_{\Omega} \Phi\left(\frac{\sqrt{|\nabla_x u|^2 + |x|^{\xi}|\nabla_y u|^2}}{\sqrt{b}\|u\|_1}\right) dxdy \leq 1$$

or $\|u\|_2 \leq \sqrt{b}\|u\|_1$.

In brief, for any $u \in W_{0,\xi}^{1,\Phi}(\Omega)$, we proved that

• if $a < 1 < b$, then $\sqrt{a}\|u\|_1 \leq \|u\|_2 \leq \sqrt{b}\|u\|_1$;
• if $1 \leq a < b$, then $\|u\|_1 \leq \|u\|_2 \leq \sqrt{b}\|u\|_1$;
• if $a < b \leq 1$, then $\sqrt{a}\|u\|_1 \leq \|u\|_2 \leq \|u\|_1$.

Consequently, the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent provided that there exists a positive constant c such that for each $(x, y) \in \Omega$ we have $|x| \geq c$.

The proof of Proposition 1 is complete. □

References

[1] Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)

[2] Adams, D.R., Hedberg, L.I.: Function Spaces and Potential Theory, Grundlehren Math. Wiss., vol. 314. Springer, Berlin (1996)

[3] Anh, C.T., Hung, P.Q., Ke, T.D., Phong, T.T.: Global attractor for a semilinear parabolic equation involving Grushin operatot. Electron. J. Differ. Equ. **32**, 1–11 (2008)

[4] Baouendi, M.S.: Sur une Classe d'Operateurs Elliptiques Degeneres. Bull. Soc. Math. France **95**, 45–87 (1967)

[5] Barros-Neto, J., Cardoso, F.: Bessel integrals and fundamental solutions for a generalized Tricomi operator. J. Funct. Anal. **183**, 472–497 (2001)

[6] Bocea, M., Mihăilescu, M.: A Caffarelli–Kohn–Nirenberg inequality in Orlicz–Sobolev spaces and applications. Appl. Anal. **91**, 1649–1659 (2012)

[7] Clément, P.H, Garía-Huidobro, M., Manáservich, R., Schmitt, K.: Mountain pass type solutons for quasilinear elliptic equations. Calc. Var. PDEs **11**, 33–62 (2000)

[8] Clément, P.H, de Pagter, B., Sweers, G., de Thélin, F.: Existence of solutions to a semilinear elliptic system through Orlicz–Sobolev spaces. Mediterr. J. Math. **1**, 241–267 (2004)

[9] D'Ambrosio, L.: Hardy inequalities related to Grushin type operators. Proc. Am. Math. Soc. **132**, 725–734 (2004)

[10] D'Ambrosio, L.: Hardy-type inequalities related to degenerate elliptic differential operators. Ann. Sc. Norm. Super. Pisa Cl. Sci. **4**, 451–486 (2005)

[11] D'Ambrosio, L., Lucente, S.: Nonlinear Liouville theorems for Grushin and Tricomi operators. J. Differ. Equ. **193**, 511–541 (2003)

[12] Franchi, B., Lanconelli, E.: Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **10**(4), 523–541 (1983)

[13] Franchi, B., Lanconelli, E.: An embedding theorem for Sobolev spaces related to nonsmooth vector fields and Harnack inequality. Commun. Partial Differ. Equ. **9**(13), 1237–1264 (1984)

[14] Franchi, B., Lanconelli, E.: Une métrique associée à une classe d'opérateurs elliptiques dégénérés. (French) [A metric associated with a class of degenerate elliptic operators]. In: Conference on Linear Partial and Pseudodifferential Operators (Torino, 1982). Rend. Sem. Mat. Univ. Politec. Torino 1983, pp. 105–114 (1984) (special issue)

[15] Fukagai, N., Ito, M., Narukawa, K.: Positive solutions of quasilinear elliptic equations with critical Orlicz–Sobolev nonlinearity on $\mathbb{R}^N$. Funkcial. Ekvac. **49**, 235–267 (2006)

[16] Garcia-Huidobro, M., Le, V.K., Manásevich, R., Schmitt, K.: On principal eigennvalues for quasilinear elliptic differential operators: an Orlicz–Sobolev setting. NoDEA Nonlinear Differ. Equ. Appl. **6**, 207–225 (1999)

[17] Gossez, J.P.: Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients. Trans. Am. Math. Soc. **190**, 163–205 (1974)

[18] Grushin, V.V.: On a class of hypoelliptic operators. Math. USSR-Sb **12**, 458–476 (1970)

[19] Kajikiya, R.: A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations. J. Funct. Anal. **225**, 352–370 (2005)

[20] Kogoj, A.E., Lanconelli, E.: On semilinear Δ_λ-Laplace equations. Nonlinear Anal. Theory Methods Appl. **75**, 4637–4649 (2012)

[21] Lamperti, J.W.: On the isometries of certain function-spaces. Pac. J. Math. **8**, 459–466 (1958)

[22] Monti, R., Morbidelli, D.: Kelvin transform for Grushin operator and critical semilinear equations. Duke Math. J. **131**, 167–202 (2006)

[23] Musielak, J.: Orlicz Spaces and Modular Speces. Academic Press, New York (1975)

[24] Rao, M.M., Ren, Z.D.: Theory of Orlicz Spaces. Marcel Dekker Inc., New York (1991)

[25] Sawyer, E.T., Wheeden, R.L.: Hölder continuity of weak solutions to subelliptic equations with rough coefficients. Mem. Am. Math. Soc. **180**(847), x+157 (2006)

[26] Struwe, M.: Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Springer, Heidelberg (1996)

[27] Thuy, P.T., Tri, N.M.: Nontrivial solutions to boundary value problems for semilinear strongly degenerate elliptic differential equations. Nonlinear Differ. Equ. Appl. **19**, 279–298 (2012)

[28] Tri, N.M.: On Grushin's equation. Math. Notes **63**, 84–93 (1998)

[29] Tricomi, F.G.: Sulle Equazioni Lineari alle Derivate Parziali di 2^o Ordine di Tipo Misto. Mem Lincei **14**, 133–247 (1923)

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