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# On a class of nonhomogeneous fractional quasilinear equations in $\mathbb{R}^n$ with exponential growth

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**Abstract.** This paper examines a class of nonlocal equations involving the fractional $p$-Laplacian, where the nonlinear term is assumed to have exponential growth. More specifically, by using a suitable Trudinger–Moser inequality for fractional Sobolev spaces, we establish the existence of weak solutions for these equations.

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**Keywords.** Fractional $p$-Laplacian, Critical growth, Trudinger–Moser inequality, Fixed point result, Discontinuous nonlinearity.

## 1. Introduction

Our aim is to show the existence of weak solutions for the following class of equations:

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u) + \lambda h \quad \text{in} \quad \mathbb{R}^n, \tag{1.1}$$

where $p \geq 2$, $0 < s < 1$, $n \geq 1$, $\lambda$ is a positive parameter and $(-\Delta)_p^s$ is the fractional $p$-Laplacian defined by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} \, \mathrm{d}y, \quad x \in \mathbb{R}^n. \tag{1.2}$$

When $p = 2$ the Eq. (1.1) arises when one seeks for standing waves solutions of the following nonlinear fractional Schrödinger equation

$$i\frac{\partial \varphi}{\partial t} = (-\Delta)^s \varphi + W(x)\varphi - g(x,|\varphi|)\varphi, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n. \tag{1.3}$$

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The special case when $(-\Delta)^s\varphi+W(x)\varphi-g(x,|\varphi|)\varphi$ does not depend explicitly on the time is of great importance for physical applications (see Laskin [17]). In this case there exists a special solution of the fractional Schrödinger equation of the form

$$\varphi(t,x)=u(x)e^{-iEt} \quad \text{(so-called standing wave solution)},$$

where $u(x)$ satisfies

$$(-\Delta)^s u+V(x)u=\widetilde{f}(x,u), \quad x\in\mathbb{R}^n,$$

with $V(x)=W(x)-E$ and $\widetilde{f}(x,u)=g(x,|u|)u$ for a suitable $E>0$ (see N. Laskin [16,17]).

By examining the literature, we notice that many authors, by considering different ways, have established the existence of solutions for the equation above in the last decades, see for instance [3,5,6,12,13,18]. It worth to mention that for $s=1$, the fractional Schrödinger equation becomes the standard Schrödinger equation:

$$i\,\frac{\partial\varphi}{\partial t}=-\Delta\varphi+W(x)\varphi-g(x,|\varphi|)\varphi, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^n. \tag{1.4}$$

The existence of standing wave solutions for (1.4) is discussed in papers such as [1,11,22,24] and references therein.

In order to obtain the existence of weak solutions for (1.1), let us recall some results related to the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$. First of all, define the Gagliardo seminorm by

$$[u]_{s,p}=\left(\int_{\mathbb{R}^{2n}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\right)^{1/p},$$

where $u:\mathbb{R}^n\to\mathbb{R}$ is a measurable function. Now, consider that the fractional Sobolev space given by

$$W^{s,p}(\mathbb{R}^n)=\{u\in L^p(\mathbb{R}^n)\,:\,u \text{ measurable and } [u]_{s,p}<\infty\}$$

is assumed to be endowed with norm

$$\|u\|_{s,p}=([u]_{s,p}^p+\|u\|_p^p)^{1/p},$$

where the fractional critical exponent is defined by

$$p_s^*=\begin{cases}\dfrac{np}{n-sp}, & \text{if} \quad sp<n;\\ \infty, & \text{if} \quad sp\geq n.\end{cases}$$

The classical Sobolev embedding states that $W^{s,p}(\mathbb{R}^n)\hookrightarrow L^{p_s^*}(\mathbb{R}^n)$ for $sp<n$. Furthermore, $W^{n/p,\,p}(\mathbb{R}^n)\hookrightarrow L^q(\mathbb{R}^n)$ for $p\leq q<\infty$; however, $W^{n/p,\,p}(\mathbb{R}^n)\not\hookrightarrow L^\infty(\mathbb{R}^n)$. For a detailed account on the properties of $W^{s,p}(\mathbb{R}^n)$ we refer to [10].

In this paper, we are interested in the borderline case of the Sobolev embedding, i.e., $s=n/p$, which is usually known as the Trudinger–Moser case (cf. [19,21,26]). More precisely, we examine the existence of weak solutions for the following class of equations involving the fractional $p$-Laplacian:

$$(-\Delta)_p^{n/p}\,u+V(x)|u|^{p-2}u=f(x,u)+\lambda h \quad \text{in} \quad \mathbb{R}^n, \tag{1.5}$$

where $p \geq 2$, $0 < n/p < 1$, $\lambda$ is a positive parameter, $f$, $V$ are functions that satisfy mild conditions and $h$ belongs to the dual of an appropriated functional space (see more details below).

It is important to point out that the existence of solutions for the equation (1.1) has been discussed under various assumptions involving the potential $V$, provided that $p = 2$, $s = 1$ and $n \geq 3$ (see [1,22,24] and references therein); or $p = 2$, $0 < s < 1$ and $n \geq 2$ (cf. [5,23] and references therein). Besides, the limiting case of the Sobolev embedding with $p = n$, namely, $s = 1$, was studied in some papers as, for example [7–9,15,27]. It is also worthwhile to remark that, in these works, different hypotheses on $V$ are assumed in order to overcome the problem of "lack of compactness", which often occurs in elliptic problems associated with unbounded domains. More precisely, numerous papers assume that the potential is uniformly positive, that is, $V(x) \geq V_0 > 0$ for any $x \in \mathbb{R}^n$, and satisfies one of the following assumptions:

1. $V(x) \to +\infty$ as $|x| \to +\infty$;
2. $1/V \in L^1(\mathbb{R}^n)$;
3. for every $M > 0$ the Lebesgue measure $\mu(\{x \in \mathbb{R}^n : V(x) \leq M\})$ is finite.

Each of these conditions ensure that the space

$$E := \left\{ u \in W^{s,p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V(x)|u|^p \mathrm{d}x < \infty \right\}$$

is compactly embedded in the Lebesgue space $L^q(\mathbb{R}^n)$ for all $q \geq p$.

To our knowledge, there are a few results available in the literature which discuss sign change for the potential $V$, see, for instance [11,25,28]. However, we stress that in all the above mentioned papers, the authors require the potential $V$ to be continuous and bounded below.

We focus our efforts on treating the case in which $V$ can change sign without requiring any additional condition in order to get compactness. In particular, we do not suppose the existence of an uniform bound on the potential, which may develop singularities near zero. Physically, this corresponds to collision of particle with the center of force, see for example [20] for more details. Throughout the paper, we assume the following hypotheses on $V$: $(V_1)$ $V : \mathbb{R}^n \to \mathbb{R}$ is measurable and

$$\lambda_1 := \inf \left\{ \int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^p}{|x - y|^{2n}} \, \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^n} V^+(x)|u|^p \, \mathrm{d}x : u \in X \text{ and } \|u\|_p = 1 \right\}$$

is positive, where $X$ is defined in (1.8). $(V_2)$ $V^- \in L^\beta(\mathbb{R}^n)$ for some $1 < \beta \leq \infty$; where $V^\pm = \max\{\pm V, 0\}$.

In this paper, we consider that the nonlinearity $f(x, s)$ may be discontinuous and exhibit exponential growth. In order to improve our description of the assumptions on $f(x, s)$, we recall some well known facts related to the limiting Sobolev embedding.

After the works [14,19,21,26], authors have paid considerable attention in the limiting case of the Sobolev embedding. Roughly speaking, Kozono et al. [14] proved that if $p$ and $p'$ satisfy $1/p + 1/p' = 1$, then for all $\alpha > 0$ and $u \in W^{n/p,\,p}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \Phi_\alpha(u)\,\mathrm{d}x < \infty, \tag{1.6}$$

where

$$\Phi_\alpha(s) := e^{\alpha|s|^{p'}} - \sum_{\substack{0\le j<p-1\\ j\in\mathbb{N}}} \frac{\alpha^j |s|^{jp'}}{j!}.$$

Moreover, there exist positive constants $\alpha_{n,p}$ and $C_{n,p}$, depending only on $p$ and $n$, such that

$$\int_{\mathbb{R}^n} \Phi_\alpha(u)\,\mathrm{d}x \le C_{n,p}\,, \quad \forall\alpha \in (0, \alpha_{n,p}), \tag{1.7}$$

for all $u \in W^{n/p,\,p}(\mathbb{R}^n)$ with $\|u\|_{n/p,\,p} \le 1$.

Motivated by (1.6)–(1.7), we assume that the nonlinearity $f(x,s)$ of the Eq. (1.5) satisfies the following properties:

- $(F_1)$ for each measurable function $u : \mathbb{R}^n \to \mathbb{R}$, the Nemytskii function $N_f : \mathbb{R}^n \to \mathbb{R}$, given by $N_f(x) = f(x, u(x))$, is measurable;
- $(F_2)$ for each $x \in \mathbb{R}^n$, $f(x,s)$ is nondecreasing in $s$ and satisfies

$$|f(x,s)| \le c_1 k(x)|s|^\rho + c_2\Phi_{\alpha_0}(s)|s|^\mu, \quad \forall\,(x,s) \in \mathbb{R}^n \times \mathbb{R},$$

where $k \in L^\sigma(\mathbb{R}^n)$ for some $1 < \sigma \le \infty$, $c_1, c_2 > 0$, $\alpha_0 > 0$, $\rho > p-1$ and $\mu > p-1$.

In order to obtain the existence of weak solutions for (1.5), we consider the subspace of $W^{n/p,\,p}(\mathbb{R}^n)$

$$X = \left\{u \in W^{n/p,\,p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V^+(x)|u|^p\,\mathrm{d}x < \infty\right\} \tag{1.8}$$

endowed with the norm

$$\|u\| = ([u]^p_{n/p,\,p} + \|(V^+(x))^{1/p}u\|_p^p)^{1/p}.$$

Using the condition $(V_1)$, we show that the embedding $X \hookrightarrow W^{n/p,\,p}(\mathbb{R}^n)$ is continuous (see Lemma 2.1 below). Since $W^{n/p,\,p}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ are complete spaces, one concludes that $(X, \|\cdot\|)$ is a Banach space. Moreover, by using the Clarkson's first inequality (see [2, pg. 95]), it follows that $X$ is uniformly convex. As a result, $X$ is a reflexive space.

Next, for all $p \le q < \infty$, we set

$$S_q := \inf_{\substack{u\in X\\ u\ne 0}} \frac{\|u\|^p}{\left(\int_{\mathbb{R}^n} |u|^q\,\mathrm{d}x\right)^{p/q}}. \tag{1.9}$$

Consequently, from the continuous embedding $W^{n/p,\,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, one has that $S_q > 0$. In this context, we assume that $h \in X'$ (dual space of $X$) and say that $u \in X$ is a weak solution for the Eq. (1.5) if the following equality holds:

$$\langle A(u), v\rangle + \int_{\mathbb{R}^n} V(x)|u|^{p-2}uv\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x,u)v\,\mathrm{d}x + \lambda\langle h, v\rangle, \quad \forall\, v \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $X$ and $X'$, and $A : X \to X'$ is the nonlinear operator defined by

$$\langle A(u), v \rangle = \int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{2n}} \, \mathrm{d}x\mathrm{d}y,$$

for all $u, v \in X$. It is standard to check that, if $u \in C^{0,\nu}$ (Hölder continuous) for suitable $\nu > 0$, this definition coincides with (1.2).

At this point, in addition to the hypotheses on $V$, we assume the condition

- $(V_3)$ $\|V^-\|_\beta < S_{q_0}$, where $q_0 := \beta p/(\beta - 1) > p$.

The main result of this paper is presented next:

**Theorem 1.1.** *Suppose that* $(V_1) - (V_3)$ *and* $(F_1) - (F_2)$ *hold. Consider that* $h \neq 0$. *Then, there exists* $\lambda_0 > 0$ *such that, for all* $0 < \lambda \leq \lambda_0$, *the equation* (1.5) *has a nontrivial weak solution.*

The main properties of this class of problems are: the domains is unbounded and the operator involved is nonlocal and nonlinear. Furthermore, the nonlinearity $f(x, s)$ may develop jump discontinuities in $s$ and behave like $\Phi_\alpha(s)$ at infinity. To our knowledge, the result is new even for the semilinear case $p = 2$.

**Remark 1.1.** We point out that ours results are closely related with results by Iannizzotto and Squassina [13]. It is worth to mention that the eigenvalue problem for this class of problems has been studied in [18].

**Remark 1.2.** The perturbation $h$ could belong to $(W^{n/p,p})'$ (dual space of $W^{n/p,p}(\mathbb{R}^n)$) or $L^{p/(p-1)}(\mathbb{R}^n)$. However, we consider a more general perturbation $h \in X'$.

**Remark 1.3.** It is worth to mention that the Theorem 1.1 guarantees the existence of a nontrivial weak solution only for nonhomogeneous case.

**Remark 1.4.** Motivated by (1.7), we say that a function $f(x, s)$ has *subcritical growth* if

$$\lim_{|s| \to \infty} f(x, s) e^{-\alpha |s|^{p'}} = 0, \quad \text{uniformly in } x \in \mathbb{R}^n,$$

for all $\alpha > 0$, and has *critical growth* if there exists $\alpha_0 > 0$ such that

$$\lim_{|s| \to \infty} f(x, s) e^{-\alpha |s|^{p'}} = \begin{cases} 0, & \text{for all } \quad \alpha > \alpha_0; \\ +\infty, & \text{for all } \quad \alpha < \alpha_0, \end{cases}$$

uniformly in $x \in \mathbb{R}^n$. We point out that the hypothesis $(F_2)$ allows the nonlinearity $f(x, s)$ to have subcritical or critical growth.

*The outline of the paper is as follows:* Section 2 contains some preliminary results, which are required in the proof of our main result; and in Section 3 we show Theorem 1.1.

Hereafter, $C$, $C_0$, $C_1$, $C_2$, ... denote positive (possibly different) constants and we use the notation $\| \cdot \|_p$ for the standard $L^p(\mathbb{R}^n)$-norm.

## 2. Preliminary results

In this section, we obtain some technical results and we establish the appropriate setting to prove Theorem 1.1.

**Lemma 2.1.** *By condition* $(V_1)$ *the space* $X$ *is continuously embedded in* $W^{n/p,\,p}(\mathbb{R}^n)$*. In particular, the embedding* $X \hookrightarrow L^q(\mathbb{R}^n)$ *is continuous for each* $q \in [p, \infty)$*.*

*Proof.* By condition $(V_1)$ we have

$$\|u\|_p^p \le \frac{1}{\lambda_1}\left([u]^p_{n/p,\,p} + \|(V^+(x))^{1/p}u\|_p^p\right), \quad \forall\, u \in X.$$

Thus,

$$\|u\|^p_{n/p,p} = [u]^p_{n/p,\,p} + \|u\|_p^p \le [u]^p_{n/p,\,p} + \frac{1}{\lambda_1}\left([u]^p_{n/p,\,p} + \|(V^+(x))^{1/p}u\|_p^p\right),$$

which implies

$$\|u\|_{n/p,p} \le \sqrt[p]{\left(1+\frac{1}{\lambda_1}\right)}\|u\|.$$

Therefore, lemma is proved. □

Next as a consequence of (1.7), we establish in our function space framework $X$ an estimate that will be essential in our arguments. Next, we set

$$\kappa := \inf_{\substack{u\in X\\ u\neq 0}} \frac{\|u\|}{\|u\|_{n/p,\,p}}. \tag{2.1}$$

**Lemma 2.2.** *Suppose that* $(V_1)$ *holds. Let* $u, v \in X$ *and* $\alpha, \mu > 0$*. If* $\|u\| \le M$ *and* $\alpha(M/\kappa)^{p'} < \alpha_{n,p}$*, then there exists a constant* $C = C(\alpha, M, \mu) > 0$ *such that*

$$\int_{\mathbb{R}^n} \Phi_\alpha(u)|u|^\mu|v|\,\mathrm{d}x \le C\|u\|^\mu\|v\|.$$

*Proof.* First, we choose $q > 1$ sufficiently close to 1 satisfying $q\alpha(M/\kappa)^{p'} < \alpha_{n,p}$ and

$$\overline{\sigma} := \frac{2q}{q-1} > \max\{p, p/\mu\}.$$

Thus, $1/q + 1/\overline{\sigma} + 1/\overline{\sigma} = 1$ and applying the generalized Hölder inequality and Lemma 2.1 one has

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi_\alpha(u)|u|^\mu|v|\,\mathrm{d}x &\le \left(\int_{\mathbb{R}^n} [\Phi_\alpha(u)]^q\,\mathrm{d}x\right)^{1/q}\|u\|^\mu_{\mu\overline{\sigma}}\|v\|_{\overline{\sigma}}\\ &\le C_1\left(\int_{\mathbb{R}^n} \Phi_{q\alpha}(u)\,\mathrm{d}x\right)^{1/q}\|u\|^\mu\|v\|\\ &\le C_2\left[\int_{\mathbb{R}^n} \Phi_{q\alpha\|u\|^{p'}_{n/p,\,p}}\left(\frac{u}{\|u\|_{n/p,\,p}}\right)\mathrm{d}x\right]^{1/q}\|u\|^\mu\|v\|.\end{aligned} \tag{2.2}$$

Since $q\alpha\|u\|_{n/p,\,p}^{p'} \le q\alpha(M/\kappa)^{p'} < \alpha_{n,p}$, in view of the Trudinger–Moser inequality (1.7) we conclude that

$$\int_{\mathbb{R}^n} \Phi_\alpha(u)|u|^\mu |v|\,\mathrm{d}x \le C\|u\|^\mu \|v\|$$

and the proof is finished. □

For the convenience of the reader, in the sequel, we recall some basic concepts and notations. Let $X$ be a real Banach space. A nonempty subset $X_+ \neq \{0\}$ of $X$ is called an order cone if the following hold:

i) $X_+$ is closed and convex;
ii) if $u \in X_+$ and $\alpha \ge 0$, then $\alpha u \in X_+$;
iii) if $u \in X_+$ and $-u \in X_+$, then $u = 0$.

We observe that an order cone $X_+$ induces in a natural way a partial order in $X$ as follows: $x \preceq y$ if and only if $y - x \in X_+$, and $(X, \preceq)$ is called an ordered Banach space. If in addition, $\inf\{x, y\}$ and $\sup\{x, y\}$ exist for all $x, y \in X$ with respect to $\preceq$ then we say that $(X, \|\cdot\|)$ is a *lattice*. Furthermore, if $\|x^\pm\| \le \|x\|$ for all $x \in X$, with $x^+ := \sup\{0, x\}$ and $x^- := -\inf\{0, x\}$ then $(X, \|\cdot\|)$ is called a *Banach semilattice*.

Special examples of Banach semilattices are the Lebesgue spaces $L^p(\mathbb{R}^n)$ and Sobolev spaces $W^{n/p,p}(\mathbb{R}^n)$ when we consider the natural partial order ($u \preceq v$ if and only if $u \le v$ a.e. in $\mathbb{R}^n$).

Let $(X, \preceq)$ and $(\widetilde{X}, \lhd)$ be ordered Banach spaces. We say that an operator $G : X \to \widetilde{X}$ is increasing iff for all $x, y \in X$, $x \preceq y$ implies $Gx \lhd Gy$. A subset $B$ of $X$ is said to have the *fixed point property* if every increasing operator $S : B \to B$ has a fixed point.

Now, we present a version of the fixed point result due to S. Carl and S. Heikkilä (see Corollary 2.2 in [4]) which we use to prove Theorem 1.1.

**Lemma 2.3.** *Let $X$ be a Banach semilattice which is reflexive. Then any closed ball of $X$ has the fixed point property.*

For more details with respect to definitions and results about ordered Banach spaces, we refer to [4] and references therein.

## 3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. To do this, we need to introduce some appropriate operators in order to apply Lemma 2.3. First, we consider the operator $L : X \to X'$ defined by

$$\langle Lu, v\rangle = \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{2n}}\,\mathrm{d}x\mathrm{d}y \\ + \int_{\mathbb{R}^n} V^+(x)|u|^{p-2}uv\,\mathrm{d}x,$$

for all $u, v \in X$. Note that for each $u \in X$, $Lu$ is a linear map. Moreover, we deduce from Hölder inequality that

$$|\langle Lu, v\rangle| \leq C\|u\|^{p-1}\|v\|, \quad \text{for some} \quad C > 0,$$

which shows that $Lu \in X'$ and therefore $L$ is well defined. Furthermore, we have

**Lemma 3.1.** *Under the hypothesis* $(V_1)$*, the operator* $L : X \to X'$ *is continuous and invertible.*

*Proof.* Let $(u_k)$ in $X$ such that $u_k \to u$ in $X$. Using Hölder inequality, for $v \in X$ with $\|v\| \leq 1$ we obtain

$$\begin{aligned}
&|\langle Lu_k - Lu, v\rangle| \\
&\leq \left(\int_{\mathbb{R}^{2n}} \left|\frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))-|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{2n}}\right|^{\frac{p}{p-1}} \mathrm{d}x\mathrm{d}y\right)^{\frac{p-1}{p}} \\
&\quad + \left(\int_{\mathbb{R}^n} V^+(x)\left||u_k|^{p-2}u_k - |u|^{p-2}u\right|^{\frac{p}{p-1}} \mathrm{d}x\right)^{\frac{p-1}{p}} \\
&:= I_1 + I_2.
\end{aligned}$$

Now by using the following inequality

$$||a|^{p-2}a - |b|^{p-2}b| \leq 2^{p-2}(p-1)|a-b|(|a|+|b|)^{p-2}, \quad \forall a, b \in \mathbb{R},$$

we obtain the following estimate

$$\begin{aligned}
&I_1^{\frac{p}{p-1}} \leq C_1 \\
&\times \int_{\mathbb{R}^{2n}} \frac{|(u_k(x)-u_k(y))-(u(x)-u(y))|^{\frac{p}{p-1}}(|u_k(x)-u_k(y)| + |u(x)-u(y)|)^{\frac{p(p-2)}{p-1}}}{|x-y|^{2n}} \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Using Hölder inequality we get

$$I_1^{\frac{p}{p-1}} \leq C_2[u_k - u]^{\frac{p}{p-1}}([u_k] + [u])^{\frac{p(p-1)}{p-2}} \leq C_2\|u_k - u\|^{\frac{p}{p-1}}(\|u_k\| + \|u\|)^{\frac{p(p-1)}{p-2}}.$$

Similarly,

$$\begin{aligned}
I_2^{\frac{p}{p-1}} &\leq C_3\|(V^+(x))^{\frac{1}{p}}(u_k - u)\|_p^{\frac{p}{p-1}}(\|(V^+(x))^{\frac{1}{p}}u_k\|_p + \|(V^+(x))^{\frac{1}{p}}u\|_p)^{\frac{p(p-1)}{p-2}} \\
&\leq C_3\|u_k - u\|^{\frac{p}{p-1}}(\|u_k\| + \|u\|)^{\frac{p(p-1)}{p-2}}.
\end{aligned}$$

Since $u_k \to u$ in $X$, by using the last estimates, we conclude that

$$\|Lu_k - Lu\|_{X'} = \sup_{v \in X,\, \|v\| \leq 1} |\langle Lu_k - Lu, v\rangle| \leq I_1 + I_2 \to 0$$

and the continuity of $L$ is proved. Furthermore, from $\langle Lu, u\rangle = \|u\|^p$ for all $u \in X$ and $p \geq 2$ we get

$$\lim_{\|u\| \to \infty} \frac{\langle Lu, u\rangle}{\|u\|} = \infty.$$

On the other hand, using the well known inequality,

$$(|a|^{p-2}a - |b|^{p-2}b)(a-b) \geq C_p|a-b|^{p-2}, \quad a, b \in \mathbb{R}, \tag{3.1}$$

one has

$$\langle Lv_1 - Lv_2, v_1 - v_2\rangle > 0$$

for all $v_1, v_2 \in X$, $v_1 \neq v_2$. Thus, by the Minty–Browder theorem (see for instance Theorem 5.16 in [2]), the operator $L$ is invertible and this completes the proof. □

At this point, we consider another operator $T : X \to X'$ given by

$$\langle Tu, v\rangle = \int_{\mathbb{R}^n} [V^-(x)|u|^{p-2}u + f(x, u)]v\,\mathrm{d}x + \lambda\langle h, v\rangle,\ u, v \in X.$$

It is clear that for each fixed $u \in X$, $Tu$ is a linear map. As a consequence of Lemma 2.2 we have the following estimate:

**Lemma 3.2.** *Assume* $(V_1)$–$(V_2)$ *and* $(F_1)$–$(F_2)$*. Let* $M > 0$ *be such that* $\alpha_0(M/\kappa)^{p'} < \alpha_{n,p}$*. There exist positive constants* $C_1$ *and* $C_2$ *such that if* $\|u\| \leq M$*, then*

$$|\langle Tu, v\rangle| \leq \left(S_{q_0}^{-1}\|V^-\|_\beta\|u\|^{p-1} + C_1\|u\|^\rho + C_2\|u\|^\mu + \lambda\|h\|_{X'}\right)\|v\|,$$

*for all* $v \in X$*.*

*Proof.* Here, let us consider $1 < \beta < \infty$ and $1 < \sigma < \infty$. The case $\beta = \infty$ or $\sigma = \infty$ are more simple and they are treated similarly. Note that

$$\frac{1}{\beta} + \frac{p-1}{q_0} + \frac{1}{q_0} = 1 \Leftrightarrow q_0 = \frac{p\beta}{\beta - 1}.$$

Using the generalized Hölder inequality together with the inequality $\|u\|_{q_0}^p \leq S_{q_0}^{-1}\|u\|^p$ one has

$$\left|\int_{\mathbb{R}^n} V^-(x)|u|^{p-2}uv\,\mathrm{d}x\right| \leq S_{q_0}^{-1}\|V^-\|_\beta\|u\|^{p-1}\|v\|. \tag{3.2}$$

Analogously, setting $\tau := \sigma(\rho+1)/(\sigma-1) > p$ we see that

$$\frac{1}{\sigma} + \frac{\rho}{\tau} + \frac{1}{\tau} = 1,$$

from which follows that

$$\left|\int_{\mathbb{R}^n} k(x)|u|^\rho v\,\mathrm{d}x\right| \leq \|k\|_\sigma\|u\|_\tau^\rho\|v\|_\tau \leq C\|k\|_\sigma\|u\|^\rho\|v\|. \tag{3.3}$$

On the other hand, Lemma 2.2 with $\alpha = \alpha_0$ yields

$$\int_{\mathbb{R}^n} \Phi_{\alpha_0}(u)|u|^\mu|v|\,\mathrm{d}x \leq C\|u\|^\mu\|v\|. \tag{3.4}$$

Since $|\langle h, v\rangle| \leq \|h\|_{X'}\|v\|$, using estimates (3.2), (3.3), (3.4) and condition $(F_2)$, we obtain the desired result. □

Analyzing the previous proof, we observe that (3.2) and (3.3) holds for each $u \in X$ as well as estimate (2.2). This fact together with hypothesis $(F_2)$ shows that the operator $T$ is well defined.

Now, we define the operator $S : X \to X$ by $S = L^{-1} \circ T$.

In order to address the existence of a fixed point for $S$, we need the following lemma:

**Lemma 3.3.** *Under the hypotheses of Theorem 1.1, for any $\lambda < \lambda_0$, there exists $R > 0$ such that*

$$S(B_X[0,R]) \subset B_X[0,R],$$

*where $B_X[0,R] = \{u \in X : \|u\| \leq R\}$.*

*Proof.* Let $u \in X$ and set $v = (L^{-1} \circ T)u = Su$. Since $\langle Lv, v\rangle = \|v\|^p$ we get

$$\|Su\|^p = \langle Tu, Su\rangle \leq \|Tu\|_{X'}\|Su\|.$$

Choosing $R \leq M$, if $\|u\| \leq R$ then by Lemma 3.2 one has

$$\begin{aligned}\|Su\|^{p-1} &\leq \|Tu\|_{X'}\\ &\leq S_{q_0}^{-1}\|V^-\|_\beta\|u\|^{p-1} + C_1\|u\|^\rho + C_2\|u\|^\mu + \lambda\|h\|_{X'}\\ &\leq S_{q_0}^{-1}\|V^-\|_\beta R^{p-1} + C_1R^\rho + C_2R^\mu + \lambda\|h\|_{X'},\end{aligned}$$

which implies

$$\frac{\|Su\|^{p-1}}{R^{p-1}} \leq S_{q_0}^{-1}\|V^-\|_\beta + C_1R^{\rho-p+1} + C_2R^{\mu-p+1} + \lambda\frac{\|h\|_{X'}}{R^{p-1}}. \tag{3.5}$$

Now, choose $R > 0$ sufficiently small so that

$$C_1R^{\rho-p+1} + C_2R^{\mu-p+1} \leq \frac{1 - S_{q_0}^{-1}\|V^-\|_\beta}{2}.$$

Setting

$$\lambda_0 := \frac{R^{p-1}(1 - S_{q_0}^{-1}\|V^-\|_\beta)}{2\|h\|_{X'}},$$

for all $0 < \lambda \leq \lambda_0$, we deduce from (3.5) that

$$\frac{\|Su\|^{p-1}}{R^{p-1}} \leq 1.$$

Therefore, $S(B_X[0,R]) \subset B_X[0,R]$ and the proof is complete. □

At this point, we are ready to prove our main result.

*Proof of Theorem 1.1.* In order to apply Lemma 2.3, we consider the following partial order in X:

$$u_1, u_2 \in X,\ u_1 \preceq u_2 \Leftrightarrow u_1 \leq u_2 \text{ a.e. in } \mathbb{R}^n. \tag{3.6}$$

It is clear that $(X, \preceq)$ is an ordered Banach space and for all $u, v \in X$, there exist $\sup\{u, v\}$ and $\inf\{u, v\}$ with respect to the order $\preceq$. Moreover, recalling that $u^+ = \sup\{u, 0\}$ and $u^- = -\inf\{u, 0\}$, by the order (3.6), $u^+$ and $u^-$ are the positive and negative parts of $u$. Since $|u^\pm(x)| \leq |u(x)|$ and $|u^\pm(x) - u^\pm(y)| \leq |u(x) - u(y)|$ almost everywhere in $\mathbb{R}^n$ we see that $\|u^\pm\| \leq \|u\|$. Hence $(X, \preceq)$ is a Banach semilattice which is reflexive. We also observe that the dual space $X'$, endowed with the order:

$$\varphi_1, \varphi_2 \in X',\ \varphi_1 \lhd \varphi_2 \Leftrightarrow \langle\varphi_1, v\rangle \leq \langle\varphi_2, v\rangle,\ \forall\, v \in X_+$$

where $X_+ := \{v \in X : v \geq 0 \text{ a.e. in } \mathbb{R}^n\}$, is an ordered Banach space.

We claim that $L^{-1} : (X', \lhd) \to (X, \preceq)$ is an increasing operator. Indeed, let $\varphi_1, \varphi_2 \in X'$ such that $\varphi_1 \lhd \varphi_2$. Setting $u_1 = L^{-1}\varphi_1$ and $u_2 = L^{-1}\varphi_2$, for $v \in X_+$ one has

$$\begin{aligned}
&0 \leq \langle \varphi_2 - \varphi_1, v\rangle = \langle Lu_2 - Lu_1, v\rangle \\
&= \int_{\mathbb{R}^{2n}} \frac{\left(|u_2(x)-u_2(y)|^{p-2}(u_2(x)-u_2(y)) - |u_1(x)-u_1(y)|^{p-2}(u_1(x)-u_1(y))\right)(v(x)-v(y))}{|x-y|^{2n}}\, \mathrm{d}x\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n} V^+(x)\left(|u_2|^{p-2}u_2 - |u_1|^{p-2}u_1\right) v\, \mathrm{d}x.
\end{aligned}$$

From this, taking $v = (u_2 - u_1)^- \in X_+$ and using (3.1), there exists $C > 0$ such that

$$0 \leq \langle \varphi_2 - \varphi_1, (u_2 - u_1)^-\rangle \leq -C\|(u_2 - u_1)^-\|^p \leq 0$$

whence we conclude that $(u_2 - u_1)^- = 0$ and so $u_1 \leq u_2$ almost everywhere in $\mathbb{R}^n$, that is, $L^{-1}\varphi_1 \preceq L^{-1}\varphi_2$. Now, we prove that $T : (X, \preceq) \to (X', \lhd)$ is increasing. To this, let $u_1, u_2 \in X$ such that $u_1 \leq u_2$ almost everywhere in $\mathbb{R}^n$. Since the function $l(s) := |s|^{p-2}s$ is increasing it follows from assumption $(F_2)$ that

$$\begin{aligned}
\langle Tu_1, v\rangle &= \int_{\mathbb{R}^n} [V^-(x)|u_1|^{p-2}u_1 + f(x,u_1)]v\, \mathrm{d}x + \lambda\langle h, v\rangle \\
&\leq \int_{\mathbb{R}^n} [V^-(x)|u_2|^{p-2}u_2 + f(x,u_2)]v\, \mathrm{d}x + \lambda\langle h, v\rangle = \langle Tu_2, v\rangle,\ \forall\, v \in X_+,
\end{aligned}$$

that is, $Tu_1 \lhd Tu_2$. Consequently, by definition, the operator $S : (X, \preceq) \to (X, \preceq)$ is also increasing. By Lemma 2.3, $B_X[0, R]$ has the *fixed point property* and in view of Lemma 3.3 there exists $u_0 \in B_X[0, R]$ such that $Su_0 = u_0$. Since $S = L^{-1} \circ T$ we have

$$\langle Lu_0, v\rangle = \langle Tu_0, v\rangle, \quad \forall\, v \in X.$$

Thus, by the definitions of $L$ and $T$, since $h \neq 0$, $u_0$ is a nontrivial weak solution of (1.5) and the theorem is proved. □

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