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The Dirichlet problem for prescribed principal curvature equations

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Abstract. In this paper the Dirichlet problem for the equation which prescribes the ith principal curvature of the graph of a function u is considered. A Comparison principle is obtained within the class of semiconvex subsolutions by a local perturbation procedure combined with a fine Lipschitz estimate on the elliptic operator. Existence of solutions is stated for the Dirichlet problem with boundary conditions in the viscosity sense; further assumptions guarantee that no loss of boundary data occurs. Some conditions relating the geometry of the domain and the prescribing data which are sufficient for existence and uniqueness of solutions are presented.

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1. Introduction

In this paper we study the Dirichlet problem for the prescribed i-th principal curvature equation

$$\kappa_i[u](x) = f(x) \quad \forall x \in \Omega, \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^d$, $\kappa_i[u](x)$ denotes the i-th principal curvature of the graph of $u : \Omega \to \mathbb{R}$ at $x \in \Omega$, $i = 1, \ldots, d$, and $f : \Omega \to \mathbb{R}$. The set of Eq. (1) can be rewritten as

$$F^i(Du(x), D^2u(x)) = f(x) \quad \text{in } \Omega, \tag{2}$$

where $F^i : \mathbb{R}^d \times \mathcal{S} \to \mathbb{R}$ is

$$F^i(p, M) = \lambda_i(A(p)M),$$
$$A(p) = \frac{1}{v(p)}\left(I - \frac{p \otimes p}{v^2(p)}\right), \quad v(p) = \sqrt{1 + |p|^2} \tag{3}$$

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for all $p \in \mathbb{R}^d$, $M \in \mathcal{S}$ and $i = 1, \ldots, d$. Explicitly, (1) and (2) become

$$\lambda_i \left(\frac{1}{\sqrt{1 + |Du(x)|^2}} \left(I - \frac{Du(x) \otimes Du(x)}{1 + |Du(x)|^2} \right) D^2 u(x) \right) = f(x),$$

$\lambda_1(S) \leq \ldots \leq \lambda_d(S)$ being the ordered real eigenvalues of a symmetric matrix $S \in \mathcal{S}$.

The operator F^i is degenerate elliptic, i.e. $-F^i(p, M + P) \leq -F^i(p, M)$ for all $p \in \mathbb{R}^d$, $M \in \mathcal{S}$ and $P \geq 0$. Moreover, $-F^i(p, M + rI) \leq -F^i(p, M) - rv^{-3}(p)$ for all $r > 0$, so it satisfies a non-total degeneracy condition (see [1]) within any set of functions having bounded Lipschitz constant; however, we point out that the non-total degeneracy constant $v^{-3}(p)$ goes to zero as $|p| \to \infty$. The regularity of F^i with respect to its entries is inherited from the regularity properties of $\lambda_i(\cdot)$, which is locally Lipschitz for all $i = 1, \ldots, d$.

Equation (1) might be considered as a particular case of so-called spectral equations, being of the form $\mathcal{F}(\kappa_1(x), \ldots, \kappa_d(x)) = f(x)$, where $\mathcal{F} : \mathbb{R}^d \to \mathbb{R}$. In the context of such equations, existence and uniqueness for the Dirichlet problem has been extensively studied when $\mathcal{F}$ is a symmetric function or the ratio between symmetric functions, see [6,12,15–17,20] and references therein. It is worth mentioning that this family of equations embraces the more classical prescribed mean curvature and Gaussian curvature equations, for which we refer for example to [5,11,21]. The Dirichlet problem for an equation which prescribes the first principal curvature of level sets of u is considered in [4]. To the best of our knowledge, the Dirichlet problem for (1) has never been investigated in the literature. With respect to the case of symmetric functions of principal curvatures, we observe that less information on the shape of the graph of u is a-priori prescribed in our problem.

In the setting of degenerate elliptic equations, a general approach in the non-total degenerate case is proposed in [1]. Problems showing non-degeneracy in one direction arise frequently in the subelliptic environment, see [2,19] and references therein. Many ideas used here are borrowed from [3], a seminal paper for equations without zeroth order terms.

Being our problem degenerate, we will make use of the modern viscosity theory started by Crandall, Lions and Evans (for details we refer to the User's Guide to viscosity solutions [8]). In trying to obtain a comparison principle, the key ingredient for uniqueness and existence of solutions, we have to face the lack of strict monotonicity of F^i with respect to u. In [3], the procedure to work around this problem is to transform the original equation into new equation which enjoys coercivity through a suitable change of variables. Another approach, proposed in the User's Guide, is to exploit the structure of the operator to perturb subsolutions into strict subsolutions (or supersolutions into strict supersolutions). We will implement the latter method, pointing out that a direct perturbation technique (as in [1,2,14]) does not seem to work due to poor regularity and ellipticity properties of F^i. Indeed, if u is any twice differentiable subsolution of $-F^i = -f$ and $\phi \in C^2$, setting $u_\epsilon = u + \epsilon\phi$, non-degenerate ellipticity and Lipschitz regularity implies that for some $\delta, L > 0$

$$\begin{aligned} -F^i(Du+\epsilon D\phi, D^2u+\epsilon D^2\phi) &\le -F^i(Du+\epsilon D\phi, D^2u) - \delta\epsilon\lambda_1(D^2\phi) \\ &\le -f + L|\epsilon D\phi| - \delta\epsilon\lambda_1(D^2\phi). \end{aligned}$$

In order for $L|\epsilon D\phi| - \delta\epsilon\lambda_1(D^2\phi)$ to be strictly negative, the contribution from non-total degenerate ellipticity $-\delta\lambda_1(D^2\phi)$ must be strong enough to beat the error term $L|D\phi|$. Even if a ϕ realizing the inequality $L|D\phi| - \delta\lambda_1(D^2\phi) < 0$ might exist, we are not able to control L if u is in general a subsolution in the viscosity sense. Moreover, $\delta \to 0$ as $|Du|$ grows. We then restrict our space of subsolutions to semiconvex functions u, which enjoy the properties of having locally bounded gradient and $\lambda_1(D^2u)$ controlled from below. In addition, we focus our analysis to points of maxima of $u - v$, (where v is any supersolution); in such points, $\lambda_i(D^2u)$ turns out to be controlled from above. We show in Proposition 3.1 that this information is sufficient to bound the Lipschitz constant L of F^i with respect to p, which should depend a-priori on the whole spectrum of D^2u. It is therefore possible to implement a local perturbation procedure. The main result of Sect. 3 is the comparison principle stated in the following theorem, under the hypothesis of continuity of f.

Theorem 1.1. *Suppose that*

(h0) *Ω is a bounded domain of class C^2 and $f \in C(\overline{\Omega})$.*

If u is a C-semiconvex subsolution of (2) and v is a C-supersolution of (2) in Ω (see Definition 2.7) such that $u \le v$ on $\partial\Omega$, then

$$u \le v \quad \text{in } \overline{\Omega}.$$

We avoid the method of doubling of variables and follow the ideas of [3], as the dependance of the equation on x is outside the elliptic operator F^i.

Existence for the Dirichlet problem is considered in Sect. 4. It is well-known that the geometry of the boundary plays an important role, as non-existence of solutions or loss of boundary data is very likely to occur when dealing with spectral equations (see for example [9,21]). We propose in Theorem 4.1 an existence argument which is based on the Perron's method introduced by Ishii; we adapt the version presented in [2], as the restriction to semiconvex subsolutions requires some modifications of the standard technique. An additional assumption on the geometry of $\partial\Omega$ guarantees that the solution obtained by means of Perron's method does not suffer of loss of boundary data; to show this we adapt an argument of Da Lio [9].

In addition to the hypothesis on $\partial\Omega$, existence of at least a subsolution and a supersolution of the equation is needed. In Sect. 5 we state some conditions that are sufficient for such requirements to be fulfilled. They involve the size of Ω (Proposition 5.1), the principal curvatures of $\partial\Omega$ and the prescribing function f (Proposition 5.2); on the other hand, the boundary datum g can be any arbitrary continous function. A full existence and uniqueness result for (1) can be summarized in the following

Theorem 1.2. *Suppose that Ω is a C^2 strictly convex bounded domain and Ω, $f \in C(\overline{\Omega})$ satisfy*

$$\Omega \subset\subset \left\{ x \in \mathbb{R}^d : \sum_{j=1}^{\max\{i,d-i+1\}} x_j^2 < \left(\max_{\overline{\Omega}} |f| \right)^{-2} \right\}$$

and

$$-\kappa_{\Omega,d-i}(x) < f(x) < \kappa_{\Omega,i-1}(x) \quad \forall x \in \partial\Omega,$$

where $\kappa_{\Omega,1}(x) \le \cdots \le \kappa_{\Omega,d-1}(x)$ denote the principal curvatures of $\partial\Omega$ at $x \in \partial\Omega$. Then, for all $g \in C(\partial\Omega)$ there exists a unique viscosity solution (in the sense of Definition 2.12) $u \in C(\overline{\Omega})$ of (1) such that $u|_{\partial\Omega} = g$.

Under particular circumstances, we show that the conditions we obtain for existence are almost optimal, by relating (1) to prescribed Gaussian curvature equations and exploiting known results of ill-posedness of the associated Dirichlet problem.

We finally point out that the restriction to semiconvex subsolutions slightly modifies the original problem, as we obtain comparison and existence in the standard viscosity sense for the equation

$$\max\{-\lambda_1(D^2u) - C, -F^i(Du, D^2u) + f\} = 0.$$

We present an example of non-semiconvex subsolution of (1) (Example 2.6) and a non-semiconvex solution which differs from the semiconvex one obtained by our existence theorem (Example 5.5).

2. Preliminaries

Throughout this paper, Ω will be a bounded C^2 domain of $\mathbb{R}^d$. We will denote by $\mathcal{S}$ the set of d-dimensional symmetric matrices, by $\mathrm{diag}[D_1, \cdots, D_d]$ the diagonal matrix with entries $D_1, \ldots, D_d$, by $\kappa_{\Omega,1}(x) \le \ldots \le \kappa_{\Omega,d-1}(x)$ the $d-1$ principal curvatures of Ω at $x \in \partial\Omega$ and by $d(x)$ the distance function from $\partial\Omega$. We recall that, given $u \in C^2(\Omega)$, the vector $\kappa[u](x)$ of (ordered) principal curvatures of the graph of u at a point $x \in \Omega$ is the vector of eigenvalues of the matrix $A(Du(x))D^2u(x)$, where A is defined in (3). Note that $A(Du(x))D^2u(x)$ has indeed d real eigenvalues: $A(p)$ is positive definite and its square root is

$$P(p) = \frac{1}{v^{1/2}(p)} \left(I - \frac{p \otimes p}{v(p)(1 + v(p))} \right) \in \mathcal{S},$$

and $A(Du(x))D^2u(x)$ has the same eigenvalues of $P(Du(x))D^2u(x)P(Du(x))$, which is of course symmetric.

We collect now some properties of $A(p)$ and the operators F^i.

Lemma 2.1. *For all $p \in \mathbb{R}^d$ we have $A(p) \in \mathcal{S}$ and the vector of eigenvalues of $A(p)$ is*

$$\lambda(A(p)) = (v^{-3}(p), v^{-1}(p), \ldots, v^{-1}(p)).$$

Moreover, let $\mathcal{K}$ be a compact subset of $\mathbb{R}^d$. Then,

i) $(1/\alpha)I \le A(p) \le I$ *for all* $p \in \mathcal{K}$,
ii) $\|A(p) - A(r)\| \le L|p - r|$ *for all* $p, r \in \mathcal{K}$,

for some $\alpha, L > 0$.

Proof. Let $p \in \mathbb{R}^d$, for some orthogonal matrix $\mathcal{O}$ we have

$$p \otimes p = \mathcal{O}^{-1}\mathrm{diag}[0, \dots, 0, |p|^2]\mathcal{O},$$

so $A(p)$ is equivalent to

$$\begin{aligned} \frac{1}{v(p)}\mathrm{diag}\left[1, \dots, 1, 1 - \frac{|p|^2}{1 + |p|^2}\right] &= \frac{1}{v(p)}\mathrm{diag}\left[1, \dots, 1, \frac{1}{1 + |p|^2}\right] \\ &= \mathrm{diag}\left[\frac{1}{v(p)}, \dots, \frac{1}{v(p)}, \frac{1}{v^3(p)}\right]. \end{aligned} \tag{4}$$

Then i) and ii) follow from the representation (4). □

Lemma 2.2. *The operator* $-F^i$ *defined in* (2) *is (degenerate) elliptic and it satisfies*

$$-F^i(p, M + P) \le -F^i(p, M) - \frac{\lambda_1(P)}{v^3(p)}$$

for all $p \in \mathbb{R}^d$, $M \in \mathcal{S}$, $P \ge 0$.

Proof. We have, for all $p \in \mathbb{R}^d$, $M \in \mathcal{S}$, $P \ge 0$

$$\begin{aligned} -F^i(p, M + P) &= -\lambda_i(A(p)M + A(p)P) \\ &\le -\lambda_i(A(p)M) - \lambda_1(A(p)P) \le -\lambda_i(A(p)M) - \frac{\lambda_1(P)}{v^3(p)}, \end{aligned} \tag{5}$$

using Weyl's inequality (Theorem 4.3.1, [13]) for the first inequality and Ostrowski's Theorem (Theorem 4.5.9, [13]) for the second one, and recalling from Lemma 2.1 that $\lambda_1(A(p)) = v^{-3}(p)$. □

Remark 2.3. We observe that, substituting $P = rI$ in (5),

$$-F^i(p, M + rI) \le -F^i(p, M) - \frac{r}{v^3(p)}$$

for all $r \ge 0$. Inequalities in (5) cannot be improved in general and $v^{-3}(p) \to 0$ as $|p| \to +\infty$. However, $v^{-3}(p)$ is bounded away from zero if $|p|$ is bounded, so in this case $-F^i$ becomes non-totally degenerate elliptic, see (7) in [1].

We define subsolutions of (2) in the standard viscosity sense:

Definition 2.4. $u \in \mathrm{USC}(\overline{\Omega})$ is a (viscosity) subsolution of $-F^i(Du, D^2u) + f = 0$ in Ω if for all $\phi \in C^2(\Omega)$ such that $u - \phi$ has a maximum point at $x_0 \in \Omega$ we have $-F^i(D\phi(x_0), D^2\phi(x_0)) + f(x_0) \le 0$.

In order to prove a comparison principle for Eq. (2), we need the concept of C-semiconvexity; a semiconvex function is indeed locally Lipschitz, among other properties that will be used (see, for example, [7]).

Definition 2.5. A function $u \in \mathrm{USC}(\overline{\Omega})$ is C-semiconvex in Ω if

$$-\lambda_1(D^2u) - C \leq 0 \quad \text{in } \Omega$$

in the viscosity sense, namely if for all $\phi \in C^2(\Omega)$ such that $u - \phi$ has a maximum point at $x_0 \in \Omega$ we have $-\lambda_1(D^2\phi(x_0)) - C \leq 0$.

Even if we will restrict ourselves to consider C-semiconvex subsolutions, which enjoy nice regularity properties, there might be in general viscosity subsolutions of (2) that are not semiconvex:

Example 2.6. Consider on $\Omega = (-1,1)^2 \subset \mathbb{R}^2$ the function $u(x) = -|x_1|^{1/2}$. For all $\epsilon > 0$, let $u_\epsilon \in C^2(\Omega)$ depending only on x_1 be such that

$$u_\epsilon = u \quad \text{on } \Omega \setminus \{|x_1| \geq \epsilon\} \quad \text{and} \quad u_\epsilon = u_\epsilon(x_1) \leq u \quad \text{on } \Omega.$$

We have that

$$F^2(Du_\epsilon, D^2u_\epsilon) = \max\left\{\frac{(u_\epsilon)_{x_1x_1}}{(1+(u_\epsilon)^2_{x_1})^{3/2}}, 0\right\} \geq 0 \quad \text{on } \Omega,$$

so u_ϵ is a subsolution of $-F^2(Du, D^2u) = 0$. Note that

$$u^*(x) = \sup_{\epsilon>0} u_\epsilon(x) \quad \forall x \in \Omega.$$

By standard arguments u^* is a viscosity subsolution of $-F^2(Du, D^2u) = 0$. Moreover, u is continuous so $u = u^*$ and therefore u is a subsolution of the same equation, but it is not C-semiconvex for any $C \geq 0$ nor Lipschitz continuous.

As for supersolutions, we weaken the standard definition of viscosity supersolution, restricting the space of test functions.

Definition 2.7. A function $u \in \mathrm{LSC}(\overline{\Omega})$ is a (viscosity) (C-)supersolution of $-F^i(Du, D^2u) + f = 0$ in Ω if for all $\phi \in C^2(\Omega)$ such that $u - \phi$ has a minimum point at $x_0 \in \Omega$ and

$$-C < \lambda_1(D^2\phi(x_0))$$

we have $-F^i(D\phi(x_0), D^2\phi(x_0)) + f(x_0) \geq 0$.

Definition 2.8. u is a (viscosity) (C-)solution of (2) in Ω if it is C-semiconvex, u is a subsolution of $-F^i + f = 0$ and u is a C-supersolution of $-F^i + f = 0$ in Ω.

Remark 2.9. Note that u is a C-solution if and only if it is both a subsolution and a supersolution in the standard viscosity sense of

$$\max\{-\lambda_1(D^2u) - C, -F^i(Du, D^2u) + f\} = 0. \tag{6}$$

This is reminescent of the definition of sub/supersolution for Monge-Ampere type equations in [2] and [14].

If u is a strict C-semiconvex viscosity solution of (6), then it is twice differentiable almost everywhere due to Alexandrov Theorem and $-\lambda_1(D^2u) - C < 0$, thus $-F^i(Du, D^2u) + f = 0$ holds a.e. (everywhere if $u \in C^2(\Omega)$). However, a C-solution u is not in general a standard viscosity solution of (2), and vice-versa.

We will solve the Dirichlet problem for (2) firstly the in viscosity sense (see Sect. 7, [8]): given

$$g : \partial\Omega \to \mathbb{R}$$

let us define $G^i : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S} \to \mathbb{R}$ as

$$G^i(x, r, p, X) = \begin{cases} -F^i(p, X) + f(x) & \text{if } x \in \Omega, \\ r - g(x) & \text{if } x \in \partial\Omega \end{cases} \tag{7}$$

Definition 2.10. $u \in \mathrm{USC}(\overline{\Omega})$ is a (viscosity) subsolution of $(G^i)_* = 0$ in $\overline{\Omega}$ if it is a subsolution of $-F^i(Du, D^2u) + f = 0$ in Ω and for all $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a maximum point at $x_0 \in \partial\Omega$ we have

$$\min\{-F^i(D\phi(x_0), D^2\phi(x_0)) + f(x_0), u(x_0) - g(x_0)\} \leq 0. \tag{8}$$

Definition 2.11. $u \in \mathrm{LSC}(\overline{\Omega})$ is a (viscosity) (C-)supersolution of $(G^i)^* = 0$ in $\overline{\Omega}$ if it is a supersolution of $-F^i(Du, D^2u) + f = 0$ in Ω and for all $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a minimum point at $x_0 \in \partial\Omega$ and

$$-C < \lambda_1(D^2\phi(x_0))$$

we have

$$\max\{-F^i(D\phi(x_0), D^2\phi(x_0)) + f(x_0), u(x_0) - g(x_0)\} \geq 0. \tag{9}$$

Definition 2.12. A function $u : \overline{\Omega} \to \mathbb{R}$ is a discontinuous (viscosity) solution of the Dirichlet problem $G^i = 0$ in $\overline{\Omega}$ if for some $C \geq 0$ it is C-semiconvex in $\overline{\Omega}$, u^* is a subsolution of $(G^i)_* = 0$ and u_* is a C-supersolution of $(G^i)^* = 0$ in $\overline{\Omega}$.

A (viscosity) solution of $G^i = 0$ is a discontinuous solution of the Dirichlet problem $G^i = 0$ in $\overline{\Omega}$ such that $u \in C(\overline{\Omega})$ and

$$u = g \quad \text{on } \partial\Omega.$$

3. The comparison principle

In order to prove the comparison principle stated in Theorem 1.1, we need the following a-priori estimate on the Lipschitz constant of F^i with respect to p, which depends on a bound on p itself and the first i eigenvalues $\lambda_1, \ldots, \lambda_i$ of M.

Proposition 3.1. *Let $1 \leq i \leq d$, $p \in \mathbb{R}^d$ and $M \in \mathcal{S}$ be such that*

$$-\Lambda \leq \lambda_1(M) \leq \ldots \leq \lambda_i(M) \leq \Lambda, \quad |p| \leq K$$

for some $K, \Lambda > 0$. Then, there exists $L = L(d, i, K, \Lambda) > 0$ such that

$$F^i(r, M) \geq F^i(p, M) - L|p - r|$$

for all $|r| \leq K$, $|p - r| < 1$.

Proof. Step 1. We may suppose that M is diagonal; indeed, there exists some orthogonal $\mathcal{O}$ such that

$$M = \mathcal{O}^{-1}\Delta\mathcal{O}, \quad \Delta = \mathrm{diag}[\mu_1, \dots, \mu_d], \quad \mu_j = \lambda_j(M),\ j = 1, \dots, d,$$

where $-\Lambda \le \mu_1 \le \dots \le \mu_i \le \Lambda$, then

$$F^i(p, M) = \lambda_i(A(p)M) = \lambda_i(\mathcal{O}A(p)\mathcal{O}^{-1}\Delta) = \lambda_i(B(p)\Delta B(p)), \tag{10}$$

with $B^2(p) = \mathcal{O}A(p)\mathcal{O}^{-1} > 0$; note that B has the same properties of A stated in Lemma 2.1, i.e. there exist $\tilde{\alpha}, \tilde{L} > 0$ depending on K such that

- $(1/\tilde{\alpha})I \le B(p) \le I$, for all $|p| \le K$,
- $\|B(p) - B(r)\| \le \tilde{L}|p - r|$ for all $|p|, |r| \le K$.

Step 2. Thanks to the variational characterization of eigenvalues (Theorem 4.2.11, [13]) and (10) we may write

$$\begin{aligned}\lambda_i(A(p)M) &= \max_{S \in \mathcal{S}_{d-i+1}} \min_{|\xi|=1, \xi \in S} \xi^T B(p)\Delta B(p)\xi \\ &= \max_{S \in \mathcal{S}_{d-i+1}} \min_{v \in V_p(S)} v^T \Delta v = \max_{S \in \mathcal{S}_{d-i+1}} \min_{v \in V_p(S)} \sum_{j=1}^{d} \mu_j v_j^2,\end{aligned}$$

where $\mathcal{S}_{d-i+1}$ denotes the family of $(d-i+1)$-dimensional subspaces of $\mathbb{R}^d$ and $V_p(S) = \{B(p)\xi : |\xi| = 1, \xi \in S\}$ for all $S \subseteq \mathbb{R}^d$, $|p| \le K$.

Step 3. Let $S \in \mathcal{S}_{d-i+1}$ be fixed, we prove that

$$\min_{v \in V_p(S)} \sum_{j=1}^{d} \mu_j v_j^2 = \min_{v \in W_p(S)} \sum_{j=1}^{d} \mu_j v_j^2, \tag{11}$$

where

$$W_p(S) = V_p(S) \cap \{v : \mu_j v_j^2 \le i\Lambda + (d-1)\Lambda \text{ for all } d-i+1 \le j \le d\}.$$

Indeed, let $\hat{\xi} \in S$, $|\hat{\xi}| = 1$ be such that

$$B(p)\hat{\xi} = (\dots, \underbrace{0, 0, \dots, 0}_{d-i}). \tag{12}$$

In order to prove the existence of such $\hat{\xi}$, we consider an orthonormal basis $\{\xi^j\}$ of S and $\hat{\xi} = \sum_{j=1}^{d-i+1} \beta_j \xi^j$, where $\beta = (\beta_1, \dots, \beta_{d-i+1}) \in \mathbb{R}^{d-i+1}$. Then

$$\begin{aligned}(12) \Leftrightarrow \sum_{j=1}^{d-i+1} \beta_j \underbrace{B(p)\xi^j}_{\eta^j \in \mathbb{R}^d} &= (\dots, 0, 0, \dots, 0) \\ &\Leftrightarrow \begin{cases} \beta_1 \eta^1_{i+1} + \beta_2 \eta^2_{i+1} + \dots + \beta_{d-i+1}\eta^{d-i+1}_{i+1} = 0 \\ \vdots \\ \beta_1 \eta^1_d + \beta_2 \eta^2_d + \dots + \beta_{d-i+1}\eta^{d-i+1}_d = 0. \end{cases}\end{aligned}$$

The (homogeneous) system has $d-i$ equations and $d-i+1$ unknowns β_j, so there exists a linear non-trivial subspace of solutions; we pick a solution β in order to have $|\hat{\xi}| = 1$, and so $\hat{v} := B(p)\hat{\xi} \in V_p(S)$. Due to the particular choice of $\hat{v}$ we deduce

$$\min_{v \in V_p(S)} \sum_{j=1}^{d} \mu_j v_j^2 \leq \sum_{j=1}^{d} \mu_j \hat{v}_j^2 = \sum_{j=1}^{i} \mu_j \hat{v}_j^2 \leq i\Lambda,$$

since $|\hat{v}_j| \leq |\hat{v}| = |B(p)\hat{\xi}| \leq \|B(p)\| \leq 1$. Let now $v \in V_p(S)$; if $\mu_{\bar{j}} v_{\bar{j}}^2 > i\Lambda + (d-1)\Lambda$ for some $\bar{j}$ such that $d - i + 1 \leq \bar{j} \leq d$, then

$$\sum_{j=1}^{d} \mu_j v_j^2 = \sum_{j \neq \bar{j}} \mu_j v_j^2 + \mu_{\bar{j}} v_{\bar{j}}^2 > -(d-1)\Lambda + i\Lambda + (d-1)\Lambda = i\Lambda,$$

and therefore (11) holds.

Step 4. We want to compare now $\lambda_i(A(p)M)$ and $\lambda_i(A(p+q)M)$, $q \in \mathbb{R}^d$. We fix $S' \in \mathcal{S}_{d-i+1}$; let $S = B^{-1}(p+q)B(p)S' \in \mathcal{S}_{d-i+1}$ and $v \in W_{p+q}(S)$ (that is $v = B(p+q)\xi$ for some $\xi \in S$, $|\xi| = 1$). Set

$$\xi' := \frac{B^{-1}(p)B(p+q)\xi}{|B^{-1}(p)B(p+q)\xi|}, \quad w := B(p)\xi';$$

we have $|\xi'| = 1$ and $\xi' \in S'$, so $w \in V_p(S')$. Hence

$$w = \frac{1}{|B^{-1}(p)B(p+q)\xi|} B(p+q)\xi = (1+\delta)v$$

for $\delta = \delta(p, q, \xi) = |B^{-1}(p)B(p+q)\xi|^{-1} - 1$. It then follows that

$$\begin{aligned} \sum_{j=1}^{d} \mu_j v_j^2 - \sum_{j=1}^{d} \mu_j w_j^2 &= \sum_{j=1}^{d} \mu_j (v_j - w_j)(v_j + w_j) = \sum_{j=1}^{d} -\mu_j \delta v_j (2+\delta) v_j \\ &= -\delta(2+\delta) \sum_{j=1}^{d} \mu_j v_j^2 \geq -C(d, i, \Lambda)\delta \end{aligned} \tag{13}$$

if $|\delta| < 1$, since $v \in W_{p+q}(S)$, so

$$\sum_{j=1}^{d} \mu_j v_j^2 = \sum_{j=1}^{i} \mu_j v_j^2 + \sum_{j=i+1}^{d} \mu_j v_j^2 \leq i\Lambda + (d-i)(i\Lambda + (d-1)\Lambda).$$

We have then, by (13),

$$\sum_{j=1}^{d} \mu_j v_j^2 \geq \sum_{j=1}^{d} \mu_j w_j^2 - C\delta \geq \min_{w \in V_p(S')} \sum_{j=1}^{d} \mu_j w_j^2 - C\delta,$$

and since $v \in W_{p+q}(S)$ was arbitrarily chosen,

$$\min_{v \in W_{p+q}(S)} \sum_{j=1}^{d} \mu_j v_j^2 \geq \min_{w \in V_p(S')} \sum_{j=1}^{d} \mu_j w_j^2 - C\delta,$$

therefore

$$\max_{S \in \mathcal{S}_{d-i+1}} \min_{v \in W_{p+q}(S)} \sum_{j=1}^{d} \mu_j v_j^2 \geq \min_{w \in V_p(S')} \sum_{j=1}^{d} \mu_j w_h^2 - C\delta.$$

Being also $S' \in \mathcal{S}_{d-i+1}$ arbitrarily chosen we conclude

$$\max_{S \in \mathcal{S}_{d-i+1}} \min_{v \in W_{p+q}(S)} \sum_{j=1}^{d} \mu_j v_j^2 \geq \max_{S' \in \mathcal{S}_{d-i+1}} \min_{w \in V_p(S')} \sum_{j=1}^{d} \mu_j w_j^2 - C\delta,$$

and the inequality $\lambda_i(A(p+q)M) \geq \lambda_i(A(p)M) - C\delta$ follows using steps *2* and *3*.

Step 5. We are left with the estimation of the constant δ. By the hypothesis of Lipschitz continuity of $B(p)$,

$$B(p+q) = B(p) + R, \quad \|R\| \leq \tilde{L}|q|$$

if $|p+q| \leq K$. Hence, for all ξ, $|\xi| = 1$,

$$\begin{aligned} |B^{-1}(p)B(p+q)\xi| &= |B^{-1}(p)(B(p)+R)\xi| = |\xi + B^{-1}(p)R\xi| \\ &\geq 1 - |B^{-1}(p)R\xi| \geq 1 - \|B^{-1}(p)R\| \geq 1 - \tilde{L}\tilde{\alpha}|q|, \end{aligned}$$

the last inequality because $\|B^{-1}\| \leq \tilde{\alpha}$ (see Step 1). Hence,

$$1 + \delta = |B^{-1}(p)B(p+q)\xi|^{-1} \leq (1 - L\tilde{\alpha}|q|)^{-1} \leq 1 + 2\tilde{L}\tilde{\alpha}|q|,$$

so $\delta \leq 2\tilde{L}\tilde{\alpha}|q|$, which implies

$$\lambda_i(A(p+q)M) \geq \lambda_i(A(p)M) - 2C\tilde{L}\tilde{\alpha}|q|.$$

$\square$

We are now ready to prove the comparison principle.

Proof of Theorem 1.1 Step 1. Suppose by contradiction that $\bar{x} \in \Omega$ is a maximum point of $u - v$. We choose $\Omega' \subset\subset \Omega$ so that $\bar{x} \in \Omega'$; u is bounded and C-semiconvex, so there exists a constant $D > 0$ such that

$$|Du(x)| \leq D \quad \forall x \in \Omega', \tag{14}$$

almost everywhere (see, for example, Remark 2.1.8 [7]), and D depends on $C, \sup_{\Omega''} |u|$, $\mathrm{dist}(\Omega', \partial\Omega)$, where $\Omega' \subset\subset \Omega'' \subset \Omega$. Moreover, by Lemma 2.1, i)

$$(1/\alpha)I \leq A(p) \leq I \quad \forall |p| \leq D + 1 \tag{15}$$

for some $\alpha = \alpha(D) > 0$. Being $A(p)$ Lipschitz uniformly on compact sets (Lemma 2.1, ii)), it is true by Proposition 3.1 that for all $p, q \in \mathbb{R}^d$, $M \in \mathcal{S}$ satisfying

$$|p|, |q| \leq D + 1 \quad -\Lambda \leq \lambda_1(M) \leq \cdots \leq \lambda_i(M) \leq \Lambda,$$

where $\Lambda = \max\{C, \alpha \max_{\overline{\Omega}} |f| + 1\}$, there exists L' depending on D, C and $\alpha \max_{\overline{\Omega}} |f|$ only (so depending on u and f) such that

$$\lambda_i(A(r)M) \geq \lambda_i(A(p)M) - L'|p - r|$$

for all $|p|, |r| \leq D + 1$, $|p - r| < 1$.

Moreover, [18], Lemma 4.1 states that for all $\epsilon > 0$ there exists $\psi \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

i) $u - v - \psi$ has a maximum in $\tilde{x} \in \Omega$, where $|\tilde{x} - \bar{x}| < \epsilon$.
ii) $D^2\psi(\tilde{x})$ is negative definite and its eigenvalues satisfy $-\epsilon < \lambda_j(D^2\psi(\tilde{x})) < 0$ for all j.
iii) $|D\psi(\tilde{x})| < \epsilon \min_j |\lambda_j(D^2\psi(\tilde{x}))|$.

Pick $\epsilon > 0$ sufficiently small to have $\tilde{x} \in \Omega'$ and $\epsilon < 1/(16\alpha L')$. Setting $\phi(x) = \psi(x) + |x - \tilde{x}|^4$, the function $u - v - \phi$ has $\tilde{x}$ as the unique strict maximum in $\overline{\Omega'}$. Set $\hat{\sigma} := \min_j |\lambda_j(D^2\psi(\tilde{x}))|/2$, so

ii') $D^2\psi(\tilde{x}) < -\hat{\sigma} I$.

Let now $\underline{v}_\mu$ be the inf-convolution of v:

$$\underline{v}_\mu(x) = \inf_{\xi \in \mathbb{R}^d} \left\{ v(\xi) + (\mu/2)|x - \xi|^2 \right\}.$$

By standard arguments $\underline{v}_\mu$ is μ-semiconcave; moreover, there exists a sequence $\mu_j \to +\infty$ such that the maximum points x_{μ_j} of $u - \underline{v}_{\mu_j} - \phi$ satisfy $x_{\mu_j} \to \tilde{x}$. By continuity of $D\phi$ and $D^2\phi$ we may choose j large enough such that properties ii') and iii) are true also for $D\phi(x_{\mu_j})$ and $D^2\phi(x_{\mu_j})$; we may increase j so that,

$$\omega(1/\mu_j) < \frac{\hat{\sigma}}{4\alpha}, \tag{16}$$

where ω is a continuity modulus of f (on $\overline{\Omega'}$). For simplicity we set $\underline{v} = \underline{v}_{\mu_j}$ and $\hat{x} = x_{\mu_j}$.

Step 2. There exists a sequence $t_m \to 0$ such that the function

$$x \mapsto u(x) - \underline{v}(x) - \phi(x) - \langle t_m, x \rangle$$

has a strict maximum at x_m and $x_m \to \hat{x}$. By Jensen's Lemma (see Lemma A.3, [8]), we know that if $r > 0$ is sufficiently small there exists $\bar{\rho} > 0$ such that the set of maxima of

$$x \mapsto u(x) - \underline{v}(x) - \phi(x) - \langle t_m, x \rangle - \langle q, x \rangle \tag{17}$$

in $B_r(x_m)$, $q \in B_\rho(0)$ with $\rho < \bar{\rho}$ contains a set of positive measure. Due to Aleksandrov's Theorem (see, for example, Theorem A.2, [8]) u and $\underline{v}$ are twice differentiable almost everywhere, therefore, for all $\rho, r > 0$ (small) we may choose $z = z_m \in B_r(x_m)$ and $q \in B_\rho(0)$ such that z is a maximum point of (17) and u, $\underline{v}$ are twice differentiable. Note that $Du(z) - D\underline{v}(z) - D\phi(z) - t_m - q = 0$ holds, so (for ϵ small enough)

$$|D\underline{v}(z)| \le D + \max_{B_r(x_m)} |D\phi| + \bar{\rho} + |t_0| \le D + 1.$$

Moreover

$$D^2u(z) \le D^2\underline{v}(z) + D^2\phi(z),$$

so $-C\, I \le D^2u(z) < -D^2\phi(z) + D^2u(z) \le D^2\underline{v}(z)$ by ii') and C-semiconvexity of u. Moreover, v is a C-supersolution of $-F^i + f = 0$, and it is standard that $\underline{v}$ is a C-supersolution of

$$-F^i(D\underline{v}, D^2\underline{v}) + f(x) = \omega(1/\mu_j),$$

therefore by the estimate on $|D\underline{v}(z)|$, (15) and Ostrowski's Theorem (Theorem 4.5.9, [13]),

$$\begin{aligned} \max_{\overline{\Omega}} |f| &\ge f(z) - \omega(1/\mu_j) \\ &\ge F^i(D\underline{v}(z), D^2\underline{v}(z)) = \lambda_i(A(D\underline{v}(z))\, D^2\underline{v}(z)) \ge (1/\alpha)\lambda_i(D^2\underline{v}(z)). \end{aligned}$$

We obtain, again using $D^2u(z) \le D^2\underline{v}(z) + D^2\phi(z)$ and Weyl's inequality (Theorem 4.3.1, [13]), that

$$\begin{aligned} -C &\le \lambda_1(D^2u(z)) \le \lambda_k(D^2u(z)) \le \lambda_k(D^2\underline{v}(z) + D^2\phi(z)) \\ &\le \lambda_i(D^2\underline{v}(z) + D^2\phi(z)) \le \alpha \max_{\overline{\Omega}} |f| + \lambda_d(D^2\phi(z)) \quad \forall k = 1, \ldots, i, \end{aligned}$$

which implies $-\Lambda \le \lambda_k(D^2u(z)) \le \Lambda$ for all $\forall k = 1, \ldots, i$.

Step 3. Finally, we let $\rho, r \to 0$ and $m \to +\infty$; being $|Du(z_m)|$ and $|D\underline{v}(z_m)|$ bounded, we extract a subsequence such that $z_m \to \hat{x}$ and $p_m = Du(z_m) \to \bar{p}$, $q_m = D\underline{v}(z_m) \to \bar{q}$. Setting $X_m = D^2u(z_m)$ and $Y_m = D^2\underline{v}(z_m)$, we have

$$\begin{aligned} &(p_m, X_m) \in \overline{J}^{2,+}_{\Omega} u(z_m), (q_m, Y_m) \in \overline{J}^{2,-}_{\Omega} \underline{v}(z_m) \\ &p_m = q_m + D\phi(z_m) + o(1) \\ &X_m \le Y_m + D^2\phi(z_m) \\ &-\Lambda \le \lambda_1(X_m) \le \ldots \le \lambda_i(X_m) \le \Lambda, \end{aligned}$$

hence

$$\begin{aligned} F^i(p_m, X_m) &\ge f(z_m) \\ F^i(q_m, Y_m) &\le f(z_m) + \omega(1/\mu_j). \end{aligned}$$

Therefore

$$\begin{aligned} f(z_m) + \omega(1/\mu_j) &\ge F^i(q_m, Y_m) = \lambda_i(A(q_m)Y_m) \ge \lambda_i(A(q_m)(X_m - D^2\phi(z_m)) \\ &\ge \lambda_i(A(q_m)X_m) + \lambda_1(-A(q_m)D^2\phi(z_m)) \\ &\ge \lambda_i(A(p_m)X_m) + \beta_m = F^i(p_m, X_m) + \beta_m \ge f(z_m) + \beta_m. \end{aligned}$$

where $\beta_m = -L'|D\phi(z_m) + o(1)| + \lambda_1(-A(q_m)D^2\phi(z_m))$. By the choice of ϵ

$$\begin{aligned} &|D\phi(z_m) + o(1)| \to |D\phi(\hat{x})| < \epsilon(2\hat{\sigma}) < \frac{\hat{\sigma}}{8\alpha L'}, \\ &D^2\phi(z_m) \to D^2\phi(\hat{x}) < -\hat{\sigma} I \Rightarrow -A(q_m)D^2\phi(z_m) \to -A(\bar{q})D^2\phi(\hat{x}) > \frac{\hat{\sigma}}{\alpha} I, \end{aligned}$$

simplifying we have

$$\omega(1/\mu_j) \ge \beta_m \ge -\frac{\hat{\sigma}}{4\alpha} + \frac{\hat{\sigma}}{2\alpha} = \frac{\hat{\sigma}}{4\alpha},$$

for m large enough, which contradicts (16). $\square$

4. Existence for the Dirichlet problem

We now state a general existence result for the Dirichlet problem for (2). Our abstract assumptions are

(h1) There exist a C-semiconvex subsolution $\underline{u}$ of $(G^i)_* = 0$ in $\overline{\Omega}$ and a bounded C-supersolution $\bar{u}$ of $(G^i)^* = 0$ in $\overline{\Omega}$ such that $\underline{u} \le \bar{u}$ on $\overline{\Omega}$.

(h2) The domain Ω is strictly convex and for all $x \in \partial\Omega$

$$i) \begin{cases} \liminf_{y\to x,\alpha\searrow 0} \left[-F^i\left(\frac{Dd(y)+o_\alpha(1)}{\alpha}, -\frac{1}{\alpha^2}Dd(y)\otimes Dd(y) + \frac{o_\alpha(1)}{\alpha^2}\right) + f(y)\right] > 0 \\ or \\ \liminf_{y\to x,\alpha\searrow 0} \left[-F^i\left(\frac{Dd(y)+o_\alpha(1)}{\alpha}, \frac{1}{\alpha}D^2d(y) + \frac{o_\alpha(1)}{\alpha}\right) + f(y)\right] > 0, \end{cases}$$

and

$$ii) \begin{cases} \limsup_{y\to x,\alpha\searrow 0} \left[-F^i\left(\frac{-Dd(y)+o_\alpha(1)}{\alpha}, \frac{1}{\alpha^2}Dd(y)\otimes Dd(y) + \frac{o_\alpha(1)}{\alpha^2}\right) + f(y)\right] < 0 \\ or \\ \limsup_{y\to x,\alpha\searrow 0} \left[-F^i\left(\frac{-Dd(y)+o_\alpha(1)}{\alpha}, -\frac{1}{\alpha}D^2d(y) + \frac{o_\alpha(1)}{\alpha}\right) + f(y)\right] < 0, \end{cases}$$

where $o_\alpha(1) \to 0$ as $\alpha \searrow 0$ and $d(x)$ denotes the distance function from $\partial\Omega$ (which is C^2 in a neighbourhood of $\partial\Omega$).

We need (h1) to have a well-defined Perron family and (h2) to guarantee that there is no loss of boundary data for the solution of the Dirichlet problem, that we will find in a generalized (viscosity) sense (Definition 2.12). We point out that the boundary function g does not enter into (h2), which is just a condition on the geometry of the boundary Ω and the datum f, coupled via the elliptic operator $-F^i$.

Theorem 4.1. *Suppose that (h0) and (h1) hold. Then, for every $g \in C(\partial\Omega)$ there exists a discontinuous solution u of $G^i = 0$ in $\overline{\Omega}$. If also (h2) holds, then u is the unique solution of the Dirichlet problem $G^i = 0$ in $\overline{\Omega}$.*

Before proving the theorem, we state a result that assures that some inequality is satisfied if loss of boundary data for the Dirichlet problem $(G^i)_* \leq 0$ happens. We borrow this result from [9], adapting the proof for our particular definition of supersolution.

Proposition 4.2. *Let Ω be a strictly convex C^2 domain, $g \in C(\overline{\Omega})$ and $u \in$ USC$(\overline{\Omega})$ a viscosity subsolution of $(G^i)_* = 0$ in $\overline{\Omega}$ (resp.* LSC$(\overline{\Omega})$ *supersolution of $(G^i)^* = 0$) and suppose that $u(x_0) > g(x_0)$ at $x_0 \in \partial\Omega$ (resp. $u(x_0) < g(x_0)$). Then the following two conditions hold:*

$$\begin{cases} \liminf_{y\to x,\alpha\searrow 0} \left[-F^i\left(\frac{Dd(y)+o_\alpha(1)}{\alpha}, -\frac{1}{\alpha^2}Dd(y)\otimes Dd(y) + \frac{o_\alpha(1)}{\alpha^2}\right) + f(y)\right] \leq 0, \\ \liminf_{y\to x,\alpha\searrow 0} \left[-F^i\left(\frac{Dd(y)+o_\alpha(1)}{\alpha}, \frac{1}{\alpha}D^2d(y) + \frac{o_\alpha(1)}{\alpha}\right) + f(y)\right] \leq 0 \end{cases}$$

(resp.

$$\begin{cases} \limsup_{y\to x,\alpha\searrow 0} \left[-F^i\left(\frac{-Dd(y)+o_\alpha(1)}{\alpha}, \frac{1}{\alpha^2}Dd(y)\otimes Dd(y) + \frac{o_\alpha(1)}{\alpha^2}\right) + f(y)\right] \geq 0, \\ \limsup_{y\to x,\alpha\searrow 0} \left[-F^i\left(\frac{-Dd(y)+o_\alpha(1)}{\alpha}, -\frac{1}{\alpha}D^2d(y) + \frac{o_\alpha(1)}{\alpha}\right) + f(y)\right] \geq 0 \end{cases}$$

), where $o_\alpha(1) \to 0$ as $\alpha \searrow 0$.

Proof. This is Proposition 3.1 of [9] if $u \in \mathrm{USC}(\overline{\Omega})$ is a subsolution of $(G^i)_* = 0$.

If u is a supersolution the proof goes along the same lines, but since the space of test functions is restricted we have to check that the test function $\Psi(x) := -\frac{|x-x_0|^4}{\epsilon^4} + \psi\left(\frac{d(x)}{\alpha}\right)$ is admissible (i.e. semiconvex, at least for $\alpha << \epsilon$ and in a neighbourhood of x_0); here $\alpha, \epsilon > 0$ and ψ is a negative smooth function such that $\psi(t) \to -\infty$ as $t \to \infty$, $\psi(0) = 0$, $\psi'(0) = -1$, $\psi''(0) = 1$. We take a minimum point $x = x_{\epsilon,\alpha}$ of $u - \Psi$ on $\overline{\Omega}$. By direct computations

$$D^2\Psi(x) = \psi'\left(\frac{d(x)}{\alpha}\right)\frac{D^2 d(x)}{\alpha} + \psi''\left(\frac{d(x)}{\alpha}\right)\frac{Dd(x)}{\alpha} \otimes \frac{Dd(x)}{\alpha} + \frac{o_\epsilon(1)}{\epsilon^2} \quad \text{as } \epsilon \to 0.$$

α may be chosen small enough compared to ϵ so that $(o_\epsilon(1)/\epsilon^2)$ is also $(o_\alpha(1)/\alpha)$ as $\alpha \searrow 0$ (take for example $\alpha = \epsilon^2$). Moreover $d(x)/\alpha \to 0$ and in particular $\psi'(d(x)/\alpha) = -1 + o_\alpha(1)$, $\psi''(d(x)/\alpha) = 1 + o_\alpha(1)$, hence

$$D^2\Psi(x) = \frac{1}{\alpha^2}\left[(1 + o_\alpha(1))Dd(x) \otimes Dd(x) + \alpha(-1 + o_\alpha(1))D^2 d(x) + o_\alpha(\alpha)\right].$$

We recall that by a change of coordinates (see [10], Lemma 14.17), Dd, $D^2 d$ have the form

$$Dd(x) = (0, \ldots, 0, 1) \quad D^2 d(x) = \operatorname{diag}\left[\frac{-\kappa_1}{1 - \kappa_1 d(x)}, \ldots, \frac{-\kappa_{d-1}}{1 - k_{d-1} d(x)}, 0\right],$$

where $\kappa_1, \ldots, \kappa_{d-1}$ are the principal curvatures of $\partial\Omega$ at a point $y_0 \in \partial\Omega$ such that $d(x) = |x - y_0|$; therefore

$$D^2\Psi(x) = \frac{1}{\alpha^2}\operatorname{diag}[\alpha(\kappa_1 + o_\alpha(1)), \ldots, \alpha(\kappa_{d-1} + o_\alpha(1)), 1 + o_\alpha(1)] \geq 0$$

if α is small enough, as the curvatures $\kappa_1, \ldots, \kappa_{d-1}$ are positive.

By hypothesis we have $u(x_0) > g(x_0)$, g is continuous and $u(x) \to u(x_0)$, so $u(x) > g(x)$ if $x \in \partial\Omega$ and α is small; Ψ is therefore an admissible test function, being convex in a neighbourhood of x, hence

$$-F^i(D\Psi(x), D^2\Psi(x)) + f(x) \geq 0.$$

Substituting $D\Psi, D^2\Psi$ we take the limit as $\alpha \searrow 0$ and use the ellipticity of $-F^i$ to obtain the assertion (as in [9]). □

Proof of Theorem 4.1 As in [3], we implement the Perron's method for viscosity solutions with the variant of Da Lio [9], considering the boundary conditions in the generalized viscosity sense. We define the Perron family

$$\mathcal{W} := \{w : \underline{u} \leq w \leq \bar{u}, w \text{ is a } C-\text{semiconvex subsolution of } (G^i)_* = 0\},$$

which is non-empty by (h1), and define a candidate solution as

$$u(z) := \sup_{w \in \mathcal{W}} w(z), \quad z \in \overline{\Omega}.$$

The fact that u^* is C-semiconvex is standard, as it the supremum of a family of C-semiconvex functions. Then $u^* \in C(\Omega)$, so $u = u^*$ in Ω, but no continuity is assured up to the boundary. Moreover, u^* is a subsolution of $(G^i)_* = 0$ (see [9], Lemma 6.1).

In order to prove that u_* is a C-supersolution we use the usual method of "bump" functions, arguing by contradiction: if u_* fails to be a C-supersolution, then it is possible to construct a function $V_\epsilon \in \mathcal{W}$ such that $V_\epsilon > u$ at some point in $\overline{\Omega}$. The proof is similar to the one of Theorem 6.1 [9], but some care has to be taken since we have restricted the class of test functions for supersolutions. Let then $\phi \in C^2(\overline{\Omega})$ be such that $u_* - \phi$ has a global minimum at $\bar{z}$, $-C < \lambda_1(D^2\phi(\bar{z}))$, and

$$(G^i)^*(\bar{z}, u_*(\bar{z}), D\phi(\bar{z}), D^2\phi(\bar{z})) < 0.$$

Assume without loss of generality that $u_*(\bar{z}) = \phi(\bar{z})$ and $u_*(z) \geq \phi(z)$ for all $z \in \overline{\Omega}$. We consider for all $\epsilon > 0$

$$V_\epsilon(z) = \max\{u(z), \phi_\epsilon(z)\}, \quad \phi_\epsilon(z) := \phi(z) + \epsilon - |z - \bar{z}|^4.$$

In order to conclude, we just have to show that $V_\epsilon(z)$ is C-semiconvex. We have that $V_\epsilon = u$ except perhaps in $B(\bar{z}, \epsilon^{1/4})$ and u is C-semiconvex. Moreover, $D^2(|z - \bar{z}|^4) \to 0$ as $z \to \bar{z}$, and $-C < \lambda_1(D^2\phi(z))$ for all $z \in B(\bar{z}, 2\epsilon^{1/4})$ if ϵ is small enough by continuity of the second derivatives of ϕ, so ϕ_ϵ is C-semiconvex in $B(\bar{z}, 2\epsilon^{1/4})$, possibly reducing ϵ. It is then standard that the maximum (considered in $B(\bar{z}, 2\epsilon^{1/4})$) between C-semiconvex functions is C-semiconvex. Therefore u_* is a C-supersolution of $(G^i)^* = 0$, and the first assertion of the theorem is proved.

If also (h2) holds, then (8) and (9) reduce to $u^* \leq g$ and $u_* \geq g$ on the boundary $\partial\Omega$ by Proposition 4.2, so there is no loss of boundary data. By Theorem 1.1 we can conclude that $u^* \leq u_*$ on $\overline{\Omega}$, and so $u = u^* = u_*$ by the definition of envelopes. Then $u \in C(\overline{\Omega})$ and $u = g$ on $\partial\Omega$.

Through a standard argument, uniqueness for the Dirichlet problem follows using the comparison principle stated in Theorem 1.1. □

5. Some sufficient conditions for existence and uniqueness

We shall look now for sufficient conditions for (h1), (h2). Let us define the sets

$$\Gamma_k^R := \left\{ x \in \mathbb{R}^d : \sum_{j=1}^{k} x_j^2 < R^2 \right\},$$

$R > 0$ and $k = 1, \ldots, d$. Note that

$$\Gamma_d^R \subset \Gamma_{d-1}^R \subset \cdots \subset \Gamma_1^R,$$

and that only Γ_d^R is bounded.

The next proposition shows that if Ω is contained in Γ_k^R, for some suitable k, R depending on f, i, then it is possible to write explicitly a subsolution and a supersolution of $(G^i)_* = 0$.

Proposition 5.1. *Suppose that $|f(x)| \leq M$ on $\overline{\Omega}$ for some $M > 0$, g is bounded on $\partial\Omega$ and Ω is a domain such that*

$$\Omega \subset\subset \Gamma_k^R, \tag{18}$$

where $R = M^{-1}$ and $k = \max\{i, d - i + 1\}$. Then, (h1) holds.

Proof. As a subsolution, we take

$$\underline{u} := -\sqrt{R^2 - \sum_{j=1}^{d-i+1} x_j^2} + \underline{C},$$

which is well-defined since (18) holds; we set $\underline{C} = \inf_{\partial\Omega} g$. Then $\underline{u} \in C^2(\Omega)$, and a computation yields

$$-F^i(D\underline{u}(x), D^2\underline{u}(x)) + f(x) = -\frac{1}{R} + f(x) \le 0$$

for all $x \in \Omega$, so $\underline{u}$ is a (classical) subsolution of $-F^i(Du, D^2u) + f = 0$ in Ω. Moreover it is convex, so it is C-semiconvex for all $C \ge 0$. Finally,

$$\underline{u}(x) \le \underline{C} = \inf_{\partial\Omega} g \le g(x)$$

for all $x \in \partial\Omega$, so $\underline{u}$ is a C-semiconvex subsolution of $(G^i)_* = 0$.

As a supersolution,

$$\bar{u} := \sqrt{R^2 - \sum_{j=1}^{i} x_j^2} + \overline{C},$$

$\overline{C} = \sup_{\partial\Omega} g$. Similarly,

$$-F^i(D\bar{u}(x), D^2\bar{u}(x)) + f(x) = \frac{1}{R} + f(x) \ge 0.$$

for all $x \in \Omega$, so $\bar{u}$ is a classical supersolution of $-F^i(Du, D^2u) + f = 0$ in Ω, and thus it is a bounded C-supersolution of $(G^i)^* \ge 0$ in $\overline{\Omega}$, satisfying also $\underline{u} \le \bar{u}$. □

We now state a sufficient condition on f and the principal curvatures of $\partial\Omega$ for (h2).

Proposition 5.2. *Suppose that Ω is a strictly convex C^2 domain and $f \in C(\overline{\Omega})$. Let $\kappa_{\Omega,1}(x), \ldots, \kappa_{\Omega,d-1}(x)$ be the principal curvatures of $\partial\Omega$ at $x \in \partial\Omega$ and let $\kappa_{\Omega,0} = 0$. If*

$$-\kappa_{\Omega,d-i}(x) < f(x) < \kappa_{\Omega,i-1}(x) \quad \forall x \in \partial\Omega, \tag{19}$$

then (h2) holds.

Proof. We recall that principal curvatures are rotationally invariant, and by a change of coordinates ([10], Lemma 14.17), Dd, D^2d have the form

$$\begin{aligned} Dd(y) &= (0, \ldots, 0, 1), \\ D^2d(y) &= \operatorname{diag}\left[\frac{-\kappa_{\Omega,1}(\bar{y})}{1 - \kappa_{\Omega,1}(\bar{y})d(y)}, \ldots, \frac{-\kappa_{\Omega,d-1}(\bar{y})}{1 - \kappa_{\Omega,d-1}(\bar{y})d(y)}, 0\right], \end{aligned}$$

where $\bar{y} \in \partial\Omega$ is such that $|\bar{y} - y| = d(y)$. By computation,

$$\begin{aligned}
&F^i\left(\frac{Dd(y) + o_\alpha(1)}{\alpha}, \frac{1}{\alpha}D^2 d(y) + \frac{o_\alpha(1)}{\alpha}\right)\\
&\quad = \lambda_i\left(\frac{I - Dd(y) \otimes Dd(y) + o_\alpha(1)}{1 + o_\alpha(1)}(D^2 d(y) + o_\alpha(1))\right)\\
&\quad \to \lambda_i(\mathrm{diag}[-\kappa_{\Omega,1}(x), \ldots, -\kappa_{\Omega,d-1}(x), 0]) = -\kappa_{\Omega,d-i}(x)
\end{aligned}$$

as $\alpha \to 0$ and $y \to x$ (so $\bar{y} \to x$). Hence

$$\begin{aligned}
&\lim_{y\to x, \alpha\searrow 0}\left[-F^i\left(\frac{Dd(y) + o_\alpha(1)}{\alpha}, \frac{1}{\alpha}D^2 d(y) + \frac{o_\alpha(1)}{\alpha}\right) + f(y)\right]\\
&\quad = \kappa_{\Omega,d-i}(x) + f(x) > 0,
\end{aligned}$$

and (h2), i) is proved.

Similarly,

$$\begin{aligned}
&F^i\left(-\frac{Dd(y) + o_\alpha(1)}{\alpha}, -\frac{1}{\alpha}D^2 d(y) + \frac{o_\alpha(1)}{\alpha}\right)\\
&\quad = \lambda_i\left(\frac{I - Dd(y) \otimes Dd(y) + o_\alpha(1)}{1 + o_\alpha(1)}(-D^2 d(y) + o_\alpha(1))\right)\\
&\quad \to \lambda_i(\mathrm{diag}[0, \kappa_{\Omega,1}(x), \ldots, \kappa_{\Omega,d-1}(x)]) = \kappa_{\Omega,i-1}(x),
\end{aligned}$$

so

$$\begin{aligned}
&\lim_{y\to x, \alpha\searrow 0}\left[-F^i\left(\frac{-Dd(y) + o_\alpha(1)}{\alpha}, -\frac{1}{\alpha}D^2 d(y) + \frac{o_\alpha(1)}{\alpha}\right) + f(y)\right]\\
&\quad = -\kappa_{\Omega,i-1}(x) + f(x) < 0,
\end{aligned}$$

that leads to (h2), ii). □

Proof of Theorem 1.2. By the two Propositions 5.1, 5.2 we have proved and Theorem 4.1, we are able to conclude that the existence and uniqueness statements of Theorem 1.2 hold. □

We now show that, for the case $i = 1$ condition (19), which becomes $-\kappa_{\Omega,d-1} < f < 0$, turns out to be almost optimal for the solvability of the Dirichlet problem, at least considering classical solutions. Indeed, it is solvable if f is negative at the boundary and does not admit any solution if f is positive or going to zero slowly enough.

Proposition 5.3. *Let Ω be a C^2 uniformly convex domain in $\mathbb{R}^d$ and f be a positive function on Ω satisfying*

$$f(x) \geq \epsilon d^b(x) \quad \forall x \in \mathcal{N}_y,$$

where $\mathcal{N}_y$ is a neighbourhood of some point $y \in \partial\Omega, \epsilon > 0$ and $b < 1/d$. Then, there exists a function $g \in C^\infty(\partial\Omega)$ for which the Dirichlet problem for $F^1(Du, D^2u) = f$ is not solvable for convex $u \in C(\overline{\Omega}) \cap C^2(\Omega)$.

Proof. Suppose that $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ solves $\kappa_1[u] = F^1(Du, D^2u) = f$ (recall that $\kappa_j[u](x)$ denotes the jth curvature of the graph of u at a point x) in $\mathcal{N}_y$

and $u = g$ for some g that will be constructed later. Then, u is convex. Indeed, $f \geq 0$, and therefore $D^2 u$ has to be non-negative on Ω. Moreover,

$$\begin{aligned} \frac{\det D^2 u(x)}{(1+|Du(x)|^2)^{(d+2)/2}} &= \kappa_1[u](x)\, \kappa_2[u](x) \cdots \kappa_d[u](x) \\ &\geq (\kappa_1[u](x))^d = (f(x))^d \geq \left(\frac{\epsilon}{2}\right)^d d^{bd}(x), \end{aligned} \tag{20}$$

for all $x \in \mathcal{N}(y)$, so

$$\det D^2 u(x) =: \overline{F}(x, Du) \geq \left(\frac{\epsilon}{2}\right)^d d^{bd}(x)\, (1+|Du(x)|^2)^{(n+2)/2}. \tag{21}$$

We now exploit a non-existence result for the Dirichlet problem for the prescribed Gaussian curvature equation. Theorem 1.3 of [21] states that if $bd < 1$ (that is true by hypothesis), then there exists $g \in C^\infty(\partial\Omega)$ for which the Dirichlet problem for (21) is not solvable for convex $u \in C(\overline{\Omega}) \cap C^2(\Omega)$. For that boundary data g is then impossible to solve the Dirichlet problem for $\kappa_1[u] = f$. $\square$

Remark 5.4. In the last proposition, we applied the observation that if u is a subsolution of the prescribed first curvature equation, i.e. $-\kappa_1[u] + f = -F^1(Du, D^2) + f \leq 0$ and $f \geq 0$, then u is also a subsolution of a prescribed Gaussian curvature equation, namely

$$\det D^2 u \geq f^d\, (1+|Du|^2)^{(d+2)/2}.$$

If $i = 1$ it is easy to derive a necessary condition for existence of a solution of (2), at least if $f > 0$, using standard knowledge on the Gaussian curvature equation. Indeed, suppose that u is a viscosity solution of $\kappa_1[u] \geq f$ on Ω (i.e. a subsolution of $\kappa_1[u] = f$), then u has to be (strictly) convex on Ω since $D^2 u > 0$ (in viscosity sense). Therefore

$$\frac{\det D^2 u(x)}{(1+|Du(x)|^2)^{(d+2)/2}} = \kappa_1[u](x) \cdots \kappa_n[u](x) \geq (\kappa_1[u](x))^d \geq (f(x))^d$$

a.e. on Ω. By integrating and through the change of variables $z = Du(x)$ (Du is one-to-one) we get

$$\int_\Omega (f(x))^d dx \leq \int_{\mathbb{R}^d} \frac{1}{(1+|z|^2)^{(d+2)/2}} dz. \tag{22}$$

This shows that, in space dimension one, the geometric condition (18) on Ω becomes (nearly) optimal: let $d = 1$ and $\Omega = (-a, a)$ for some $a > 0$ and $f \equiv M > 0$. Then (18) is

$$(-a, a) \subset\subset \Gamma_1^{\frac{1}{M}} = \left(-\frac{1}{M}, \frac{1}{M}\right),$$

i.e. $a < 1/M$, and the necessary condition (22) reads

$$2Ma = \int_{-a}^{a} M \leq \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{3/2}} dx = 2,$$

that is $a \leq 1/M$.

We end the section with an example which shows that there exist standard viscosity solutions and C-solutions (see Definition 2.12) of the Dirichlet problem for (2) that do not coincide. General existence (and uniqueness?) of solutions in the standard viscosity sense and for functions u which are not necessarily semiconvex is at this stage an open problem.

Example 5.5. Suppose that $d = 4$, $i = 2$, $\Omega = B_1(0)$, $f \equiv 0$ and

$$g(x) = (\operatorname{sgn}(x_1)\sqrt{1 - (|x_1| - 1)^2})|_{\partial B_1(0)}.$$

Theorem 4.1 guarantees the existence of a convex solution of (2), indeed $\bar{u} = 1$ and $\underline{u} = -1$ satisfty (h1). Moreover, (h2) holds because of Proposition 5.2 ($\kappa_{\Omega,1} = \kappa_{\Omega,2} = \kappa_{\Omega,3} = 1$ and $-\kappa_{\Omega,2} < 0 < \kappa_{\Omega,1}$); (h0) is easily satisfied.

Consider now $u(x) = \operatorname{sgn}(x_1)\sqrt{1 - (|x_1| - 1)^2}$. It satisfies the Dirichlet boundary conditions and it is a standard viscosity solution of (2), but it is not convex (nor C-semiconvex for all $C \geq 0$); indeed, if $x_1 \neq 0$ then u is twice differentiable at x and $F^2(Du(x), D^2u(x)) = \lambda_2(\operatorname{diag}[c, 0, 0, 0]) = 0$, where $c = -1$ if $x_1 > 0$ and $c = 1$ if $x_1 < 0$. If $x_1 = 0$ there are no test functions ϕ such that $u - \phi$ has a maximum or a minimum at x, so the definition of viscosity sub/supersolution is satisfied trivially.

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