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Liouville type theorem for higher-order elliptic system with Navier boundary condition

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Abstract. This paper deals with the Liouville theorem for a higher-order elliptic system in the half-space subject to the Navier boundary value conditions. We obtain this via establishing the Liouville type theorem for the equivalent integral system by the moving plane method.

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1. Introduction

In this paper, we establish a Liouville type theorem for the $2m$-order elliptic equations coupled via the Navier boundary conditions in the half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$:

$$\begin{cases} (-\Delta)^m u(x) = v^\beta & \text{in } \mathbb{R}^n_+, \\ (-\Delta)^m v(x) = u^\alpha & \text{in } \mathbb{R}^n_+, \\ u = \Delta u = \cdots = \Delta^{m-1} u = 0 & \text{on } \partial\mathbb{R}^n_+, \\ v = \Delta v = \cdots = \Delta^{m-1} v = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{1.1}$$

where m is a positive integer satisfying $0 < 2m < n$, $\frac{n}{n-2m} < \alpha, \beta \le \frac{n+2m}{n-2m}$.

Various higher-order elliptic equations can be used to model complex spatio-temporal pattern formations [26,28,33]. They also appear in studying the so-called Paneitz–Branson operator and its generalizations, with important applications in mathematical physics [4,12,14]. There are many other applications of them for such as the hinged plate problems [25,34], the switched

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diffusion processes (in probability theory) [10], and the switching costs problems (in stochastic control) [18].

It is well known that in order to establish a priori estimates for nonlinear elliptic equations and systems of equations without variational structure, one usually uses the blow-up argument [16]. The argument, in turn, relies on nonexistence theorems for positive solutions of the limiting problems, i.e., Liouville type theorems. Thus, Liouville type theorems play a very important role in the study of nonlinear elliptic equations without variational structure.

It is known from [11,24,27,31] that the whole space problem

$$(-\Delta)^m u(x) = v^\beta, \quad (-\Delta)^m v(x) = u^\alpha \text{ in } \mathbb{R}^n \tag{1.2}$$

with $m = 1$ does not admit positive classical solutions in the following cases:

(a) $\frac{1}{\beta+1} + \frac{1}{\alpha+1} > \frac{n-2}{n}$ and either u, v are radial or $n = 3$;
(b) $\max(\frac{2(\beta+1)}{\alpha\beta-1}, \frac{2(\alpha+1)}{\alpha\beta-1}) \geq n-2$;
(c) $\alpha, \beta \leq \frac{n+2}{n-2}$ with $\frac{\alpha+\beta}{2} < \frac{n+2}{n-2}$.

This Liouville type result with $m = 1$ is still valid for the half space problem (1.1) under the same parameter regions (a)–(c) [3], and also has been extended to the biharmonic case with $m = 2$ [13].

For the polyharmonic system (1.2) ($m \neq 1$) in whole space, Guo, Liu, and Zhang [20] proved the following Liouville type results that the problem has no positive classical solutions in the following cases:

(a) $\alpha, \beta \geq 1$ with $\frac{\alpha+\beta}{2} > 1$, $\frac{1}{\beta+1} + \frac{1}{\alpha+1} > \frac{n-2}{n}$, and u, v are radial;
(b) $\alpha, \beta \geq 1$ with $\frac{\alpha+\beta}{2} > 1$, $(n-2m)\alpha < \frac{n}{\beta} + 2m$ or $(n-2m)\beta < \frac{n}{\alpha} + 2m$;
(c) $1 \leq \alpha, \beta \leq \frac{n+2m}{n-2m}$ with $1 < \frac{\alpha+\beta}{2} < \frac{n+2m}{n-2m}$.

In addition, Zhang [35] proved that if $n > 2m$, $\alpha, \beta \geq 1$ with $\frac{\alpha+\beta}{2} > 1$, and $\frac{2m(\beta+1)}{\alpha\beta-1}, \frac{2m(\alpha+1)}{\alpha\beta-1} \in [\frac{n-2}{2}, n-2m)$, then (1.2) has no positive classical solutions.

The half space problems have been thoroughly studied as well. The Liouville theorem for the Dirichlet type boundary conditions was shown in [15,29,30] that if u is a classical solution of

$$\begin{cases} (-\Delta)^m u(x) = u^p & \text{in } \mathbb{R}^n_+, \\ u \geq 0 & \text{in } \mathbb{R}^n_+, \\ u = \frac{\partial u}{\partial x_n} = \cdots = \frac{\partial^{m-1} u}{\partial x_n^{m-1}} = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

with $1 < p < \frac{n+2m}{n-2m}$, then $u \equiv 0$. The Liouville theorem with the Navier boundary problem of higher order equation in half space was studied in [32]. Please see [5,9,21] for the current and more general results on the Liouville theorem with the Navier boundary condition. Also mention the early important works [1,2] of Berestycki, Capuzzo Dolcetta and Nirenberg on second order equations.

In the present paper, instead of (1.1), we will at first establish a Liouville type theorem for the integral system

$$\begin{cases} u(x) = C_n \displaystyle\int_{\mathbb{R}^n_+} \left(\frac{1}{|x-y|^{n-2m}} - \frac{1}{|\bar{x}-y|^{n-2m}} \right) v^{\beta}(y)dy, \\ v(x) = C_n \displaystyle\int_{\mathbb{R}^n_+} \left(\frac{1}{|x-y|^{n-2m}} - \frac{1}{|\bar{x}-y|^{n-2m}} \right) u^{\alpha}(y)dy, \end{cases} \tag{1.3}$$

where $C_n > 0$, $\bar{x} = (x_1, \dots, x_{n-1}, -x_n)$ is the reflection of the point x about the $\partial\mathbb{R}^n_+$, and then prove that the two systems are equivalent each other under certain conditions. That is the following two theorems:

Theorem 1.1. *Let* $(u,v) \in L^p_{\mathrm{loc}}(\mathbb{R}^n_+) \times L^q_{\mathrm{loc}}(\mathbb{R}^n_+)$ *be a positive solution of the system (1.3) with* $p = \frac{\alpha-1}{\frac{n-2m}{n}\alpha-1}$ *and* $q = \frac{\beta-1}{\frac{n-2m}{n}\beta-1}$. *If* $\frac{n}{n-2m} < \alpha, \beta \le \frac{n+2m}{n-2m}$, *then* $(u,v) \equiv (0,0)$.

Theorem 1.2. *Let* (u,v) *be a classical positive solution of system (1.1) with* $\alpha, \beta \ge 1$, *then the differential system (1.1) is equivalent to the integral system (1.3).*

Theorems 1.1 and 1.2 yield the main result of the paper:

Theorem 1.3. *Under the conditions of Theorems 1.1 and 1.2, the classical positive solutions of system (1.1) must be trivial.*

To prove Theorem 1.1, we will explore the moving plane method in integral forms by Chen–Li–Ou [8]. Corresponding to the half space problem (1.3), the Liouville type theorem to the whole space problem was established by Ma and Chen [22]. Refer to [7,19,23] and the references therein. Theorem 1.2 can be simply treated by using a technique introduced by Chen and Fang [9] for the scalar case of higher-order equations.

via the equivalent system coupled by $2m$ elliptic equations of 2-order and the related system of $2m$ integral equations, .

This paper is arrange as follows. We will give some preliminaries in Sect. 2, and then prove the main results of the paper in Sect. 3.

2. Preliminaries

We introduce a series lemmas for the integral system (1.3) as preliminaries, and let $C_n = 1$ there for simplicity in the sequence.

Denote

$$G(x,y) = \frac{1}{|x-y|^{n-2m}} - \frac{1}{|\bar{x}-y|^{n-2m}}, \quad x, y \in \mathbb{R}^n_+, \tag{2.1}$$

with $\bar{x}$ reflecting x about the $\partial\mathbb{R}^n_+$. Let $x^{\lambda,1} = (2\lambda - x_1, x_2, \dots, x_n)$ be the reflection of the point x about the plane $T_{\lambda,1} = \{x \in \mathbb{R}^n_+ \mid x_1 = \lambda\}$, and denote $u_{\lambda,1}(x) = u(x^{\lambda,1})$, $v_{\lambda,1}(x) = v(x^{\lambda,1})$. Define $H_{\lambda,1} = \{x \in \mathbb{R}^n_+ \mid x_1 < \lambda\}$, $H^c_{\lambda,1} = \mathbb{R}^n_+ \setminus H_{\lambda,1}$. Generally, some global integrability of solutions is required for the moving-plane method in integral form. In the present paper, the solutions are just assumed locally integrable. We employ the Kelvin transformation

$$\bar{u}(x)=\frac{1}{|x-z^0|^{n-2m}}u\left(\frac{x-z^0}{|x-z^0|^2}+z^0\right),\ \bar{v}(x)=\frac{1}{|x-z^0|^{n-2m}}v\left(\frac{x-z^0}{|x-z^0|^2}+z^0\right) \tag{2.2}$$

with $z_0 \in \partial\mathbb{R}^n_+$. We have an equivalent integral system for (2.2):

Lemma 2.1. *Let (u,v) be a positive solution of system (1.3). Then $(\bar{u},\bar{v})$ solves*

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^n_+} G(x,y)\frac{1}{|y-z^0|^{(n+2m)-\beta(n-2m)}}v^\beta(y)dy, \\ v(x) = \displaystyle\int_{\mathbb{R}^n_+} G(x,y)\frac{1}{|y-z^0|^{(n+2m)-\alpha(n-2m)}}u^\alpha(y)dy. \end{cases} \tag{2.3}$$

Proof. Notice

$$\left|\left(\frac{x-z^0}{|x-z^0|}\right)|t-z^0|-\left(\frac{t-z^0}{|t-z^0|}\right)|x-z^0|\right|^2 = |(x-z^0)-(t-z^0)|^2 = |x-t|^2,$$

$$\left|\left(\frac{\bar{x}-z^0}{|\bar{x}-z^0|}\right)|t-z^0|-\left(\frac{t-z^0}{|t-z^0|}\right)|\bar{x}-z^0|\right|^2 = |(\bar{x}-z^0)-(t-z^0)|^2 = |\bar{x}-t|^2,$$

and $|x-z^0| = |\bar{x}-z^0|$ for $z^0 \in \partial\mathbb{R}^n_+$. Let $y = \frac{t-z_0}{|t-z^0|^2} + z^0$. We have for the solution (u,v) of (2.3) by a simple computation that

$$\begin{aligned}\bar{u}(x) &= \frac{1}{|x-z^0|^{n-2m}}\int_{\mathbb{R}^n_+}\left(\frac{1}{|\frac{x-z^0}{|x-z^0|^2}+z^0-y|^{n-2m}}-\frac{1}{|\frac{\bar{x}-z^0}{|\bar{x}-z^0|^2}+z^0-y|^{n-2m}}\right)v^\beta(y)dy \\ &= \int_{\mathbb{R}^n_+}\left(\frac{1}{|x-t|^{n-2m}}-\frac{1}{|\bar{x}-t|^{n-2m}}\right)\frac{\bar{v}^\beta(t)}{|t-z^0|^{(n+2m)-\beta(n-2m)}}dt,\end{aligned}$$

and a similar result for $\bar{v}$. □

Remark 1. Due to Lemma 2.1, it suffices to prove the Liouville type conclusion for the system (2.3).

Lemma 2.2. *Let (u,v) be a positive solution of (2.3). Then for any $x \in H_{\lambda,1}$, we have*

$$\begin{aligned} u(x)-u_{\lambda,1}(x) = \int_{H_{\lambda,1}} &[G(x^{\lambda,1},y^{\lambda,1})-G(x,y^{\lambda,1})] \\ &\times\left[\frac{v^\beta(y)}{|y-z^0|^{(n+2m)-\beta(n-2m)}}-\frac{v^\beta_{\lambda,1}(y)}{|y^{\lambda,1}-z^0|^{(n+2m)-\beta(n-2m)}}\right]dy, \end{aligned} \tag{2.4}$$

$$\begin{aligned} v(x)-v_{\lambda,1}(x) = \int_{H_{\lambda,1}} &[G(x^{\lambda,1},y^{\lambda,1})-G(x,y^{\lambda,1})] \\ &\times\left[\frac{u^\alpha(y)}{|y-z^0|^{(n+2m)-\alpha(n-2m)}}-\frac{u^\alpha_{\lambda,1}(y)}{|y^{\lambda,1}-z^0|^{(n+2m)-\alpha(n-2m)}}\right]dy. \end{aligned} \tag{2.5}$$

Proof. By the first equation of (2.3),

$$u(x) = \int_{H_{\lambda,1}} G(x,y)\frac{v^\beta(y)}{|y-z^0|^{(n+2m)-\beta(n-2m)}}dy + \int_{H_{\lambda,1}} G(x,y^{\lambda,1})\frac{v^\beta_{\lambda,1}(y)}{|y^{\lambda,1}-z^0|^{(n+2m)-\beta(n-2m)}}dy,$$

$$u_{\lambda,1}(x) = \int_{H_{\lambda,1}} G(x^{\lambda,1},y)\frac{v^\beta(y)}{|y-z^0|^{(n+2m)-\beta(n-2m)}}dy + \int_{H_{\lambda,1}} G(x^{\lambda,1},y^{\lambda,1})\frac{v^\beta_{\lambda,1}(y)}{|y^{\lambda,1}-z^0|^{(n+2m)-\beta(n-2m)}}dy.$$

Since

$$G(x,y) = G(x^{\lambda,1},y^{\lambda,1}),\ G(x^{\lambda,1},y) = G(x,y^{\lambda,1}),\ x,y\in H_{\lambda,1},\ x\neq y,$$

we get (2.4) directly. The same is true for (2.5). □

Denote $\Sigma_{\mu,n} = \{x\in\mathbb{R}^n_+ \mid 0<x_n<\mu\}$, $T_{\mu,n} = \{x\in\mathbb{R}^n_+ \mid x_n=\mu\}$, $\tilde\Sigma_{\mu,n} = \{x^{\mu,n} \mid x\in\Sigma_{\mu,n}\}$, $\Sigma^c_{\mu,n} = \mathbb{R}^n_+\setminus\Sigma_{\mu,n}$. The following lemma on the Green function $G(x,y)$ in $\Sigma_{\mu,n}$ was known.

Lemma 2.3. (*Lemma 2.1 in [6]*)
(i) *For any $x,y\in\Sigma_{\mu,n}$, $x\neq y$, we have*

$$G(x^{\mu,n},y^{\mu,n}) > \max G(x^{\mu,n},y), G(x,y^{\mu,n}),$$
$$G(x^{\mu,n},y^{\mu,n}) - G(x,y) > |G(x^{\mu,n},y) - G(x,y^{\mu,n})|.$$

(ii) *For any $x\in\Sigma_{\mu,n}$, $y\in\Sigma^c_{\mu,n}$, it holds that*

$$G(x^{\mu,n},y) > G(x,y).$$

Lemma 2.4. *Let (u,v) be a positive solution of (1.3). For any $x\in\Sigma_{\mu,n}$, we have*

$$u(x) - u_{\mu,n}(x) \le \int_{\Sigma_{\mu,n}} G(x^{\mu,n},y^{\mu,n})(v^\beta - v^\beta_{\mu,n})(y)dy,$$
$$v(x) - v_{\mu,n}(x) \le \int_{\Sigma_{\mu,n}} G(x^{\mu,n},y^{\mu,n})(u^\alpha - u^\alpha_{\mu,n})(y)dy.$$

Proof. We only deal with the first inequality. Since

$$u(x) = \int_{\Sigma_{\mu,n}} G(x,y)v^\beta(y)dy + \int_{\Sigma_{\mu,n}} G(x,y^{\mu,n})v^\beta_{\mu,n}(y)dy + \int_{\Sigma^c_{\mu,n}\setminus\tilde\Sigma_{\mu,n}} G(x,y)v^\beta(y)dy,$$

$$u_{\mu,n}(x) = \int_{\Sigma_{\mu,n}} G(x^{\mu,n},y)v^\beta(y)dy + \int_{\Sigma_{\mu,n}} G(x^{\mu,n},y^{\mu,n})v^\beta_{\mu,n}(y)dy + \int_{\Sigma^c_{\mu,n}\setminus\tilde\Sigma_{\mu,n}} G(x^{\mu,n},y)v^\beta(y)dy,$$

we have by Lemma 2.3 that

$$
\begin{aligned}
u(x)-u_{\mu,n}(x) \leq & \int_{\Sigma_{\mu,n}} [G(x^{\mu,n},y^{\mu,n})-G(x,y^{\mu,n})]v^\beta(y)dy \\
& -\int_{\Sigma_{\mu,n}} [G(x^{\mu,n},y^{\mu,n}) \\
& -G(x,y^{\mu,n})]v^\beta_{\mu,n}(y)dy+\int_{\Sigma^c_{\mu,n}\setminus\tilde{\Sigma}_{\mu,n}} [G(x,y)-G(x^{\mu,n},y)]v^\beta(y)dy \\
\leq & \int_{\Sigma_{\mu,n}} [G(x^{\mu,n},y^{\mu,n})-G(x,y^{\mu,n})][v^\beta-v^\beta_{\mu,n}](y)dy.
\end{aligned}
$$

□

In addition, we need the Weighted Hardy-Littlewood-Sobolev inequality:

Lemma 2.5. ([17]) *Let* $1<l,m<\infty$, $0<\nu<n$, $\tau+\kappa\geq 0$, $\frac{1}{l}+\frac{1}{m}+\frac{\nu+\kappa+\tau}{n}=2$, *and* $1-\frac{1}{m}-\frac{\nu}{n}<\frac{\tau}{n}<1-\frac{1}{m}$. *Then*

$$
\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^\tau|x-y|^\nu|y|^\kappa}dxdy\Big|\leq C\|f\|_{L^m}\|g\|_{L^l}
$$

with $C=C(\tau,\kappa,l,\nu,n)>0$, *or equivalently,*

$$
\|Tg(x)\|_{L^\gamma}=\sup_{\|f\|_{L^m}=1}\langle Tg(x),f(x)\rangle\leq C\|g\|_{L^l}
$$

with $Tg(x)=\int_{\mathbb{R}^n}\frac{g(y)}{|x|^\tau|x-y|^\nu|y|^\kappa}dy$, $\frac{1}{l}+\frac{\nu+\kappa+\tau}{n}=1+\frac{1}{\gamma}$, $\frac{1}{m}+\frac{1}{\gamma}=1$.

3. Proof of Theorem of the main results

The techniques for the proof of Theorem 1.1 are motivated by those in [15] for a scalar problem with the Dirichlet boundary condition. We begin with two lemmas.

Lemma 3.1. *Let* $(\bar{u},\bar{v})\in L^p(\mathbb{R}^n_+\backslash B_\epsilon(z^0))\times L^q(\mathbb{R}^n_+\backslash B_\epsilon(z^0))$ *be a positive solution of (2.3) with* $\epsilon>0$, $p=\frac{\alpha-1}{\frac{n-2m}{n}\alpha-1}$, *and* $q=\frac{\beta-1}{\frac{n-2m}{n}\beta-1}$. *If* $\frac{n}{n-2m}<\alpha,\beta\leq\frac{n+2m}{n-2m}$ *with* $\frac{\alpha+\beta}{2}<\frac{n+2m}{n-2m}$, *or* $(\bar{u},\bar{v})$ *is singular at* z^0 *with* $\frac{\alpha+\beta}{2}=\frac{n+2m}{n-2m}$, *then* $(\bar{u},\bar{v})$ *is rotationally symmetric about any line parallel to the* x_n*-axis and passing through* z^0.

Proof. We apply the moving-plane method in two steps:

1. *Prepare to move the plane from near* $x_1=-\infty$

Compare the values of $(\bar{u}_{\lambda,1}(x),\bar{v}_{\lambda,1}(x))$ and $(\bar{u}(x),\bar{v}(x))$. Denote $w_{\lambda,1}(x)=\bar{u}(x)-\bar{u}_{\lambda,1}(x)$, $g_{\lambda,1}(x)=\bar{v}(x)-\bar{v}_{\lambda,1}(x)$. For λ sufficiently negative, we are going to prove that

$$
w_{\lambda,1}(x),g_{\lambda,1}(x)\leq 0 \quad \text{for a.e. } x\in\tilde{H}_{\lambda,1}=H_{\lambda,1}\setminus B_\varepsilon((z^0)^\lambda). \tag{3.1}
$$

It suffices to show that both

$$H^u_{\lambda,1} = \{x \in \tilde{H}_{\lambda,1} \mid w_{\lambda,1}(x) > 0\} \quad \text{and} \quad H^v_{\lambda,1} = \{x \in \tilde{H}_{\lambda,1} \mid g_{\lambda,1}(x) > 0\}$$

has measure zero.

Noticing $|y-z^0| > |y^{\lambda,1}-z^0|$, and $(n+2m)-\beta(n-2m), (n+2m)-\alpha(n-2m) > 0$, by Lemma 2.2 with the mean value theorem, we have for sufficiently negative values of λ and $x \in H^u_{\lambda,1}$ that

$$\begin{aligned}
0 &< w_{\lambda,1}(x) \\
&= \int_{H^v_{\lambda,1}} + \int_{H_{\lambda,1}\setminus H^v_{\lambda,1}} [G(x^{\lambda,1}, y^{\lambda,1}) - G(x, y^{\lambda,1})] \left[\frac{\bar{v}^{\beta}(y)}{|y-z^0|^{(n+2m)-\beta(n-2m)}}\right. \\
&\quad \left. - \frac{\bar{v}^{\beta}_{\lambda,1}(y)}{|y^{\lambda,1}-z^0|^{(n+2m)-\beta(n-2m)}}\right] dy \\
&\le \int_{H^v_{\lambda,1}} [G(x^{\lambda,1}, y^{\lambda,1}) - G(x, y^{\lambda,1})] \left[\frac{\bar{v}^{\beta}(y)}{|y-z^0|^{(n+2m)-\beta(n-2m)}}\right. \\
&\quad \left. - \frac{\bar{v}^{\beta}_{\lambda,1}(y)}{|y^{\lambda,1}-z^0|^{(n+2m)-\beta(n-2m)}}\right] dy \\
&\le \beta \int_{H^v_{\lambda,1}} \frac{[\bar{v}^{\beta-1}(\bar{v}-\bar{v}_{\lambda,1})](y)}{|x-y|^{n-2m}|y-z^0|^{(n+2m)-\beta(n-2m)}} dy.
\end{aligned}$$

Furthermore, by Lemma 2.5 with Hölder's inequality and $p^* = \frac{p}{p-1}$,

$$\begin{aligned}
\|w_\lambda\|_{p,\, H^u_{\lambda,1}} &\le \beta \sup_{\|f\|_{L^{p^*}}=1} \int_{H^v_{\lambda,1}} \frac{[\bar{v}^{\beta-1}(\bar{v}-\bar{v}_{\lambda,1})f](y)}{|x-y|^{n-2m}|y-z^0|^{(n+2m)-\beta(n-2m)}} dy \\
&\le C\|\bar{v}^{\beta-1} g_{\lambda,1}\chi_{H^v_{\lambda,1}}\|_{Q,\,\mathbb{R}^n} \\
&= C\|\bar{v}^{\beta-1} g_{\lambda,1}\|_{Q,H^v_{\lambda,1}} \\
&\le C\|\bar{v}\|^{\beta-1}_{q,H^v_{\lambda,1}} \|g_{\lambda,1}\|_{p,H^v_{\lambda,1}},
\end{aligned} \tag{3.2}$$

where $\frac{1}{Q} = 1 + \frac{1}{p} - \frac{n-2m+[(n+2m)-(n-2m)\beta]}{n}$ and throughout the paper, C is used to represent positive constants independent of x, which may change from line to line. Similarly, we have

$$\|g_{\lambda,1}\|_{p,H^v_{\lambda,1}} \le C\|\bar{u}\|^{\alpha-1}_{p,H^u_{\lambda,1}} \|w_{\lambda,1}\|_{p,H^u_{\lambda,1}}. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\|w_{\lambda,1}\|_{p,H^u_{\lambda,1}} \le C\|\bar{u}\|^{\alpha-1}_{p,H^u_{\lambda,1}} \|\bar{v}\|^{\beta-1}_{q,H^v_{\lambda,1}} \|w_{\lambda,1}\|_{p,H^u_{\lambda,1}}. \tag{3.4}$$

Since $(\bar{u}, \bar{v}) \in L^p(\mathbb{R}^n_+) \times L^q(\mathbb{R}^n_+)$, we can choose N sufficiently large such that

$$C\|\bar{u}\|^{\alpha-1}_{p,H^u_{\lambda,1}} \|\bar{v}\|^{\beta-1}_{q,H^v_{\lambda,1}} < \frac{1}{2}$$

whenever $\lambda < -N$, and thus $\|w_{\lambda,1}\|_{p,\, H^u_{\lambda,1}} = 0$ by (3.4). In the same way, $\|g_{\lambda,1}\|_{q,\, H^v_{\lambda,1}} = 0$. This proves (3.1).

2. *Move the plane to the limiting position to derive the symmetry*

Inequality (3.1) provides a starting point to move the plane $T_{\lambda,1}$. We start from the neighborhood of $-\infty$, and move the plane to the right as long as (3.1) holds. Define

$$\lambda_0 = \sup\{\lambda \le z_1^0 \mid w_{\rho,1}, g_{\rho,1} \le 0,\ \rho \le \lambda \text{ for a.e. } x \in \tilde{H}_{\rho,1}\}. \tag{3.5}$$

We at first prove that $\lambda_0 = z_1^0$. Assume for contradiction that $\lambda_0 < z_1^0$. We claim

$$w_{\lambda_0,1}(x) = g_{\lambda_0,1}(x) = 0 \text{ a.e. in } \tilde{H}_{\lambda_0,1}. \tag{3.6}$$

Otherwise, for such a λ_0, e.g., the set $E_0 = \{x|\ g_{\lambda_0,1}(x) < 0, x \in \tilde{H}_{\lambda_0,1}\}$ possesses a positive measure. By (2.3),

$$\begin{aligned}
&\bar{u}(x) - \bar{u}_{\lambda,1}(x)\\
&\le \int_{H_{\lambda,1}} (G(x^{\lambda,1}, y^{\lambda,1}) - G(x, y^{\lambda,1}))\frac{1}{|y-z^0|^{(n+2m)-\beta(n-2m)}}(\bar{v}^\beta - \bar{v}^\beta_{\lambda,1})(y)dy\\
&= \int_{E_0} (G(x^{\lambda,1}, y^{\lambda,1}) - G(x, y^{\lambda,1}))\frac{1}{|y-z^0|^{(n+2m)-\beta(n-2m)}}(\bar{v}^\beta - \bar{v}^\beta_{\lambda,1})(y)dy,
\end{aligned}$$

Consequently,

$$w_{\lambda_0,1}(x) < 0 \text{ a.e. in } \tilde{H}_{\lambda_0,1}. \tag{3.7}$$

Denote $\lambda_\epsilon = \lambda + \epsilon$ with $\epsilon > 0$ to be determined. For any small $\eta > 0$, choose R sufficiently large such that

$$\int_{\mathbb{R}^n_+ \setminus B_\varepsilon((z^0)^{\lambda_\epsilon}) \setminus B_R(0)} |\bar{u}|^p(y)dy \le \eta.$$

It follows from Lusin's theorem and (3.7), for any $\theta > 0$, there exists a closed set F_δ such that $w_{\lambda_0,1}|_{F_\delta}$ is continuous, with $F_\delta \subset E := H_{\lambda_0,1} \bigcap B_R(0)$ and $m(E \setminus F_\delta) < \theta$. Since $w_{\lambda_0,1}(x) < 0$ in $\tilde{H}_{\lambda_0,1}$, we know $w_{\lambda_0,1}(x) < 0$ in F_δ. Choosing $\epsilon > 0$ sufficiently small, we have

$$w_{\lambda_\epsilon,1}(x) < 0 \quad \text{for any } x \in F_\delta$$

by continuity. Denote $D_{\lambda_\epsilon} = (\tilde{H}_{\lambda_\epsilon,1} \setminus H_{\lambda_0,1}) \cap B_R(0)$. Then

$$H^u_{\lambda_\epsilon,1} \subset M := (\mathbb{R}^n_+ \setminus B_\varepsilon((z^0)^{\lambda_\epsilon}) \setminus B_R(0)) \cup (E \setminus F_\delta) \cup D_{\lambda_\epsilon}.$$

Let R be large, θ and ϵ small such that $\int_{H^u_{\lambda_\epsilon,1}} |\bar{u}|^p(y)dy \le \int_M |\bar{u}|^p(y)dy \le \frac{1}{2}$. Similarly,

$$\int_{H^v_{\lambda_\epsilon,1}} |\bar{v}|^q(y)dy \le \frac{1}{2}.$$

By (3.4) with $\lambda = \lambda_\epsilon$, we can get

$$\|w_{\lambda_\epsilon,1}\|_{p,H^u_{\lambda_\epsilon,1}} \le C\|\bar{u}\|^{\alpha-1}_{p,H^u_{\lambda_\epsilon,1}}\|\bar{v}\|^{\beta-1}_{q,H^v_{\lambda_\epsilon,1}}\|w_{\lambda_\epsilon,1}\|_{p,H^u_{\lambda_\epsilon,1}} \le \frac{1}{4}\|w_{\lambda_\epsilon,1}\|_{p,H^u_{\lambda_\epsilon,1}},$$

which implies $\|w_{\lambda_\epsilon,1}\|_{p,H^u_{\lambda_\epsilon,1}} \equiv 0$. Thus

$$w_{\lambda_\epsilon,1}(x) \leq 0 \text{ a.e. in } \tilde{H}_{\lambda_\epsilon,1},$$

and similarly,

$$g_{\lambda_\epsilon,1}(x) \leq 0 \text{ a.e. in } \tilde{H}_{\lambda_\epsilon,1}.$$

This contradicts (3.5). So, the claim (3.6) is true.

For the subcritical case $\frac{n}{n-2m} < \alpha, \beta \leq \frac{n+2m}{n-2m}$ with $\frac{\alpha+\beta}{2} < \frac{n+2m}{n-2m}$, without loss of generality, assume $\frac{n}{n-2m} < \beta < \frac{n+2m}{n-2m}$. By Lemma 2.2 and $G(x^{\lambda_0,1}, y^{\lambda_0,1}) > G(x, y^{\lambda_0,1})$, $|y - z^0| > |y^{\lambda_0,1} - z^0|$ in $H_{\lambda_0,1}$, we have

$$\begin{aligned}
0 &\equiv \bar{u}(x) - \bar{u}_{\lambda_0,1}(x) \\
&= \int_{H_{\lambda_0,1}} [G(x^{\lambda_0,1}, y^{\lambda_0,1}) - G(x, y^{\lambda_0,1})] \left[\frac{1}{|y - z^0|^{(n+2m)-\beta(n-2m)}} \bar{v}^\beta \right. \\
&\qquad \left. - \frac{1}{|y^{\lambda_0,1} - z^0|^{(n+2m)-\beta(n-2m)}} \bar{v}^\beta_{\lambda_0,1} \right] (y) dy \\
&< \int_{H_{\lambda_0,1}} [G(x^{\lambda_0,1}, y^{\lambda_0,1}) - G(x, y^{\lambda_0,1})] \frac{1}{|y - z^0|^{(n+2m)-\beta(n-2m)}} [\bar{v}^\beta - \bar{v}^\beta_{\lambda_0,1}](y) dy \\
&\equiv 0.
\end{aligned}$$

This contradiction implies that $\lambda_0 < z^0_1$ is impossible for this case.

For the critical case $\frac{\alpha+\beta}{2} = \frac{n+2m}{n-2m}$, with $(\bar{u}, \bar{v})$ singular at z^0, the claim (3.6) contradicts the singularity of $(\bar{u}, \bar{v})$. So, $\lambda_0 = z^0_1$ also has to be true.

In summary, for both cases,

$$w_{z^0_1,1}(x), g_{z^0_1,1}(x) \leq 0 \quad \text{for a.e } x \in H_{z^0_1,1} \setminus B_\varepsilon(z^0),$$

Similarly, we can move the plane from near $x_1 = +\infty$ to the left and derive that

$$w_{z^0_1,1}(x), g_{z^0_1,1}(x) \geq 0 \quad \text{for a.e } x \in H_{z^0_1,1} \setminus B_\varepsilon(z^0).$$

This concludes that (3.6) holds with $\lambda_0 = z^0_1$.

By the arbitrary of the x_1-direction chosen, we have actually shown that the solution (u, v) is rotationally symmetric about any axis parallel to x_n-axis and passing through z^0. □

Lemma 3.2. *Let $(u, v) \in L^p_{\rm loc}(\mathbb{R}^n_+) \times L^q_{\rm loc}(\mathbb{R}^n_+)$ be a positive solution of (1.3), $p = \frac{\alpha-1}{\frac{n-2m}{n}\alpha-1}$, and $q = \frac{\beta-1}{\frac{n-2m}{n}\beta-1}$. Assume $\frac{n}{n-2m} < \alpha, \beta \leq \frac{n+2m}{n-2m}$, if one of $\bar{u}$ and $\bar{v}$ is not singular at z^0, then $(u, v) \equiv (0, 0)$.*

Proof. Without loss of generality, assume that there is $z^0 \in \partial\mathbb{R}^n_+$ such that $\bar{u}$ is not singular at z^0 ($\bar{v}$ may be singular at z^0 or not). We have

$$u(x) = \frac{1}{|x - z^0|^{n-2m}} \bar{u}\left(\frac{x - z^0}{|x - z^0|^2} + z^0 \right).$$

Consequently, $u(x) = O(|x|^{-(n-2m)})$, $|x| \to \infty$. Together with $u \in L^p_{\rm loc}(\mathbb{R}^n_+)$, we know $u \in L^p(\mathbb{R}^n_+)$, and so $\bar{u} \in L^p(\mathbb{R}^n_+)$. Moreover, $\bar{v} \in L^q(\mathbb{R}^n_+ \setminus B_\epsilon(z^0))$.

The rest of the proof consists of two steps.

1. We start from the position near $x_n = 0$. We will show that if μ sufficiently small, then

$$\omega_{\mu,n}(x) = \bar{u}(x) - \bar{u}_{\mu,n}(x) \le 0 \text{ a.e. in } \Sigma_{\mu,n}. \tag{3.8}$$

Denote

$$B^u_{\mu,n} = \{x \in \Sigma_{\mu,n} \mid \omega_{\mu,n}(x) > 0\}.$$

We claim that $B^u_{\mu,n}$ must be measure zero, provided μ sufficiently small. In fact, for any $x \in B^u_{\mu,n}$, similar to (3.4), we know by Lemma 2.4 that

$$\|\omega_{\mu,n}\|_{p,B^u_{\mu,n}} \le C\|\bar{u}\|^{\alpha-1}_{p,B^u_{\mu,n}} \|\bar{v}\|^{\beta-1}_{q,B^v_{\mu,n}} \|\omega_{\mu,n}\|_{p,B^u_{\mu,n}}. \tag{3.9}$$

Since $(\bar{u},\bar{v}) \in L^p(\mathbb{R}^n_+) \times L^q(\mathbb{R}^n_+ \setminus B_\epsilon(z^0))$, we can choose $\mu > 0$ small enough such that

$$C\|\bar{u}\|^{\alpha-1}_{p,B^u_{\mu,n}} \|\bar{v}\|^{\beta-1}_{q,B^v_{\mu,n}} \le \frac{1}{2},$$

and hence $\|\omega_{\mu,n}\|_{p,B^u_{\mu,n}} = 0$ by (3.9).

2. The inequality (3.8) provides a starting point to move the plane $T_{\mu,n} = \{x \in \mathbb{R}^n_+ \,|\, x_n = \mu\}$. Now we start from the neighborhood of $x_n = 0$, and move the plane up as long as (3.8) holds. Define $\mu_0 = \sup\{\mu \,|\, \omega_{\rho,n} \le 0,\ \rho \le \mu \text{ a.e. in } \Sigma_{\rho,n}\}$. If $\mu_0 < +\infty$, similarly to (3.6), with Lemma 2.4, we can obtain

$$\omega_{\mu_0,n}(x) \equiv 0 \text{ a.e. in } \Sigma_{\mu_0,n}.$$

This yields the contradiction that $\bar{u}(x) \equiv 0$ on the plane $\{x_n = 2\mu_0\}$. So, $\mu_0 = +\infty$, and hence $\bar{u}$ is strictly monotonically increasing with respect to x_n. This implies that

$$\int_{\mathbb{R}^{n-1}} \int_a^\infty |\bar{u}(x', a)|^p dx_n dx' = \infty$$

for any $a > 0$, which contradicts $\bar{u} \in L^p(\mathbb{R}^n_+)$. We conclude $\bar{u} \equiv 0$, and so $u \equiv 0$. Moreover, $v \equiv 0$ is obtained by (1.3). □

Proof of Theorem 1.1. Suppose $\frac{n}{n-2m} < \alpha, \beta \le \frac{n+2m}{n-2m}$. By Lemma 3.2, if $\bar{u}$ (or $\bar{v}$) is not singular at a point $z^0 \in \partial\mathbb{R}^n_+$, then $u, v \equiv 0$.

Now consider $\frac{\alpha+\beta}{2} < \frac{n+2m}{n-2m}$, or $\bar{u}$ and $\bar{v}$ are singular at every $z_0 \in \partial\mathbb{R}^n_+$ with $\frac{\alpha+\beta}{2} = \frac{n+2m}{n-2m}$. Let Ω be a domain with a positive distance away from z_0. Then

$$\int_\Omega \bar{u}^p dy, \int_\Omega \bar{v}^q dy < \infty.$$

By Lemma 3.1, $(\bar{u},\bar{v})$ is rotationally symmetric about any line parallel to x_n-axis and passing through z^0. With $X^i = (x^i, x_n) \in \mathbb{R}^{n-1} \times [0,\infty)$, $i = 1, 2$, let z^0 be the projection of $\bar{X} = \frac{X^1+X^2}{2}$ on $\partial\mathbb{R}^n_+$. Set $Y^i = \frac{X^i - z_0}{|X^i - z_0|^2} + z^0$, $i = 1, 2$. From the above arguments, we have $\bar{u}(Y^1) = \bar{u}(Y^2)$, $\bar{v}(Y^1) = \bar{v}(Y^2)$, since $|Y^1 - z^0| = |Y^2 - z^0|$. Hence $u(X^1) = u(X^2)$, $v(X^1) = v(X^2)$. This implies that (u,v) only depends on the x_n-variable. Denote $u(x) = u(x_n)$, $v(x) = v(x_n)$ for simplicity. For fixed $x \in \mathbb{R}^n_+$, choose R large enough such that $|x_n| < \frac{R}{2}$. We have

$$u(x_n) \geq \int_{\mathbb{R}^n_+\setminus B_R(0)} \left(\frac{1}{|x-y|^{n-2m}} - \frac{1}{|\bar{x}-y|^{n-2m}} \right) v^\beta(y_n)dy.$$

By the mean value formula, we know for $y \in \mathbb{R}^n_+ \setminus B_R(0)$ that

$$\begin{aligned}\frac{1}{|x-y|^{n-2m}} - \frac{1}{|\bar{x}-y|^{n-2m}} &= (2m-n)\frac{(\xi_1 - y_1, \ldots, \xi_n - y_n)\cdot(0,\cdots,2x_n)}{|\xi-y|^{n-2m+2}}\\ &\geq \frac{c}{|y|^{n-2m+2}},\end{aligned}$$

with $c = c(x_n, R) > 0$. Set $|y'| = r$, $|y_n| = a$. We have

$$\begin{aligned}u(x_n) &\geq \int_{\mathbb{R}^n_+\setminus B_R(0)} c\frac{1}{|y|^{n-2m+2}} v^\beta(y)dy\\ &= c\int_R^\infty \frac{v^\beta(y_n)|y_n|^{2m-2}}{|y_n|}dy_n \int_R^\infty \frac{(\frac{r}{a})^{n-2}}{[|\frac{r}{a}|^2+1]^{\frac{n}{2}}} d\left(\frac{r}{a}\right)\\ &\geq c\int_R^\infty \frac{v^\beta(y_n)}{y_n^{2-2m}}dy_n. \end{aligned} \tag{3.10}$$

Due to the integer $m \in (0, n/2)$, there exists a sequence $y_n^i \to \infty$ as $i \to \infty$, such that $\lim_{i\to\infty} v(y_n^i) = 0$.

For simplicity, denote $u(x_n) = u(t)$, $v(x_n) = v(t)$.

Notice $m \in \mathbb{N}$ satisfying $0 < 2m < n$. If $m = 2k$ with $k \in \mathbb{N}$, we have

$$\begin{aligned}u^{(2m)}(t) = (-\Delta)^m u(x) &= \int_{\mathbb{R}^n_+} (-\Delta)^m \left(\frac{1}{|x-y|^{n-2m}} - \frac{1}{|\bar{x}-y|^{n-2m}} \right) v^\beta(y)dy\\ &= v^\beta(x) > 0,\end{aligned}$$

which implies $u^{(2m-1)}(t)$ is increasing.

We claim that $u^{(2m-1)}(t) \leq 0$. Otherwise, there is $t_0 > 0$ such that $u^{(2m-1)}(t_0) > 0$, and thus

$$u^{(2m-1)}(t) \geq u^{(2m-1)}(t_0) > 0 \quad \text{for } t \geq t_0 > 0.$$

Integrating several times, and then letting $t \to \infty$, we have $u(t) \to \infty$. This contradicts the fact that $u(y_n^i) \to 0$. The claim implies $u^{(2m-2)}(t)$ is nonincreasing.

By deduction, we derive that $u(t)$ is nonincreasing. Together with the positivity of u and $u(0) = 0$, we conclude $u \equiv 0$. Therefore, $v \equiv 0$ as well due to the second equation of (1.3).

If $m = 2k - 1$ with $k \in \mathbb{N}$, we can show by a similar procedure that $v(t)$ is nondecreasing. By (3.10), we have

$$+\infty > u(x) = u(x_n) \geq c\int_R^\infty \frac{v^\beta(y_n)}{y_n^{2-2m}}dy_n dy_n \geq cv^\beta(R)\int_R^\infty \frac{1}{y_n^{2-2m}}dy_n dy_n = +\infty,$$

and hence $v \equiv 0$. In addition, $u \equiv 0$ by (1.3). $\square$

Finally, it suffices to show the equivalence between the differential system (1.1) and the integral system (1.3) (Theorem 1.2). The required proof is identical mutatis mutandis to the same result for the single equation case.

It is easy to see that the higher-order PDEs problem (1.1) can be rewritten as the following second-order system

$$\begin{cases} -\Delta v_i = v_{i+1},\ v_i|_{\partial\mathbb{R}^n_+} = 0,\ i = 0, 1, \ldots, m-1,\ \text{with } v_0 = u,\ v_m = v^\beta, \\ -\Delta u_i = u_{i+1},\ u_i|_{\partial\mathbb{R}^n_+} = 0,\ i = 0, 1, \ldots, m-1,\ \text{with } u_0 = v,\ u_m = u^\alpha. \end{cases} \tag{3.11}$$

On the other hand, rewrite the integral system (1.3) as

$$\begin{cases} v_i = \displaystyle\int_{\mathbb{R}^n_+} G(x, y, 2) v_{i+1}(y) dy,\ i = 0, 1, \ldots, m-1,\ \text{with } v_0 = u,\ v_m = v^\beta, \\ u_i = \displaystyle\int_{\mathbb{R}^n_+} G(x, y, 2) u_{i+1}(y) dy,\ i = 0, 1, \ldots, m-1,\ \text{with } u_0 = v,\ u_m = u^\alpha, \end{cases} \tag{3.12}$$

where $G(x, y, 2)$ is defined by (2.1) with $m = 1$.

Consequently, by using the technique introduced by Chen and Fang [9] for the scalar case of higher-order equations, we can establish the equivalence between the $2m$ second-order elliptic equations (3.11) and the $2m$ integral system (3.12). We omit the details.

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