Nonlinear Differ. Equ. Appl. 22 (2015), 87–103

1021-9722/15/010087-17
published online May 15, 2014
DOI 10.1007/s00030-014-0276-z

Nonlinear Differential Equations and Applications NoDEA

A Hessenberg–Jacobi isospectral flow

Alessandro Arsie and Christian Ebenbauer

Abstract. In this paper we introduce an isospectral flow (Lax flow) that deforms real Hessenberg matrices to Jacobi matrices isospectrally. The Lax flow is given by

$$\frac{dA}{dt} = [[A^T, A]_{du}, A],$$

where brackets indicate the usual matrix commutator, $[A, B] := AB - BA$, A^T is the transpose of A and the matrix $[A^T, A]_{du}$ is the matrix equal to $[A^T, A]$ along diagonal and upper triangular entries and zero below diagonal. We prove that if the initial condition A_0 is upper Hessenberg with simple spectrum and subdiagonal elements different from zero, then $\lim_{t\to+\infty} A(t)$ exists, it is a tridiagonal symmetric matrix isospectral to A_0 and it has the same sign pattern in the codiagonal elements as the initial condition A_0. Moreover we prove that the rate of convergence is exponential and that this system is the solution of an infinite horizon optimal control problem. Some simulations are provided to highlight some aspects of this nonlinear system and to provide possible extensions to its applicability.

Mathematics Subject Classification (2010). 34C40, 34D05, 34C45, 15B35, 15A29.

Keywords. Isospectral flow, Jacobi matrices, Omega-limit set, Differential equations on manifolds.

1. Introduction

Isospectral flows for (pseudo)-differential operators were introduced at the end of the 1960s by Peter Lax, providing an explanation for the existence of the infinite sequence of conservation laws appearing in certain classes of nonlinear evolutionary PDEs like the Korteweg–de Vries equation (see [1]). After their introduction, it was immediately realized that they could provide a suitable set-up for certain classes of finite dimensional integrable systems, like the Toda lattice (see [2]) and the Calogero–Moser systems (see for instance [3]). In this framework, isospectral flows (Lax equations) appear as ODEs of the following form:

$$\frac{dA}{dt} = [B, A], \tag{1.1}$$

where A is a $n \times n$ matrix, B is a matrix function whose entries are functions of the entries of A, $[A, B] := AB - BA$ is the matrix commutator. The pair (B, A) is usually called a Lax pair. If the matrix function B has entries that are smooth functions of the entries of A, the standard existence and uniqueness theorem for ODEs shows that the Cauchy problem for (1.1) has locally a solution and this solution is unique. In our case, the entries of B will be quadratic forms in the entries of A.

Besides their use in the field of integrable systems (see [2,4,5]) and their relation with representation theory via coadjoint orbits (see [6]), systems of Lax type started to be used in numerical linear algebra at the beginning of the 1980s when it was realized that suitably constructed isospectral flows provide a continuous interpolation for discrete algorithms like the QR-factorization (see [5]). This area of research has been vastly expanded in subsequent years (see [7]), including also the realization of continuous algorithms for which no discrete version is currently available. For many more examples in this set-up see the paper [8] and the book [9].

Our paper follows this line of investigation. More specifically we construct a Lax flow that deforms real Hessenberg matrices to tridiagonal matrices isospectrally. Moreover using this flow it is possible to construct tridiagonal matrices with prescribed characteristic polynomials as presented in the Sect. 5.

The flow we consider leaves invariant the vector space of upper Hessenberg matrices and we prove that if the initial condition A_0 is upper Hessenberg and lower triangular (so that the spectrum of A_0 can be readily identified from the diagonal elements of A_0), with simple spectrum and with non-zero subdiagonal elements with given sign pattern, then $\lim_{t\to+\infty} A(t)$ exists and it is a tridiagonal symmetric matrix, isospectral to A_0 and having the same sign pattern for codiagonal elements as A_0, (the codiagonal elements are the elements immediately above or below the main diagonal). The fact that it is isospectral to A_0 follows from the structural properties of any Lax equation.

Let us recall the following definition:

Definition 1.1. A $n \times n$ real Jacobi matrix is a symmetric tridiagonal matrix with strictly positive codiagonal elements. The set of all Jacobi matrices will be denoted with $\mathcal{J}$.

With this definition in mind, we can state the main result of our paper (in the following brackets indicate the usual matrix commutator, $[A, B] := AB - BA$, A^T is the transpose of A and the matrix $[A^T, A]_{du}$ is the matrix equal to $[A^T, A]$ along diagonal and upper triangular entries and zero below diagonal):

Theorem 1.2. *The flow associated to the following nonlinear system of ODEs*

$$\frac{dA}{dt} = [[A^T, A]_{du}, A], \tag{1.2}$$

provides an explicit deformation of a real upper Hessenberg matrix to a Jacobi matrix with the same spectrum.

Another interesting result about this flow is that it is the solution of an infinite time-horizon optimal control problem. This is given by the following result proved in Sect. 4:

Theorem 1.3. *Consider the following deterministic optimal control problem over an infinite horizon:*

$$\min_U \int_0^{+\infty} \operatorname{trace}\left(([A^T, A]_{du})^T([A^T, A]_{du})\right) + \operatorname{trace}(U^T U)\, ds,$$
$$\text{subject to } \frac{dA}{dt} = [U, A], \tag{1.3}$$

where $U(t)$ is a sufficiently smooth function taking value in the Lie algebra of upper triangular matrices. Then the optimal value function is given by $V(A) = \operatorname{trace}(A^T A)$ and the optimal feedback is given by $U = [A^T, A]_{du}$, i.e the flow (1.2) is the solution of this infinite horizon optimal control problem.

The paper is structured as follows. In Sect. 2 we introduce some notations and we prove some preliminary results about the flow under study. These are used to prove the convergence of the flow whenever the initial condition is suitably chosen. Section 3 is the technical part of our work, where we prove convergence. In particular, we show that if the flow is suitably initialized, then the corresponding ω-limit set is indeed a single matrix of the desired form.

In Sect. 4, we show that the Hessenberg–Jacobi flow is the solution of an infinite horizon optimal control problem. In the final Sect. 5 some simulations, obtained using `MatLab`$^{\text{TM}}$ are presented. In particular, we show how to construct tridiagonal matrices with prescribed characteristic polynomial using the flow. We also show how the flow can be used to construct even dimensional *real skew-symmetric tridiagonal matrices* with given simple imaginary spectrum and with given sign pattern for the codiagonal elements.

In the sequel, we will work exclusively over the real field. All the matrices involved will be real and will have real spectrum, except in the case of Sect. 5, where one simulation will deal with real matrices with purely imaginary spectrum.

2. Some preliminary results

Given any $n \times n$ matrix A, we denote with A_d the matrix equal to A along diagonal entries and zero everywhere else, with A_u, the matrix equal to A along strictly upper diagonal elements and zero everywhere else, and with A_l the matrix equal to A along strictly lower diagonal elements and zero everywhere else. Here the subfix d stands for diagonal, u for strictly upper triangular and l for strictly lower triangular. Moreover, we combine this notation in the following way: the notation A_{du} stands for a matrix equal to A along diagonal and strictly upper diagonal elements and zero everywhere else and similarly for A_{dl}.

Let us indicate with H^+ the vector space of upper real Hessenberg matrices. Recall that a matrix $A \in H^+$ if and only if $A_{ij} = 0$ for $i > j + 1$. In other terms A is an upper Hessenberg matrix if it is entirely zero below the first subdiagonal. A completely analogous definition holds for a lower Hessenberg matrix. Observe that H^+ is not a Lie algebra under matrix commutator.

We will also denote with $\mathcal{G}_0$ the connected component containing the identity of the Lie group of invertible upper triangular matrices with determinant equal to one and with $\mathfrak{g}_0$ the corresponding Lie algebra, consisting of upper triangular matrices with trace equal to zero. Notice that matrices in $\mathcal{G}_0$ have necessarily positive diagonal entries.

As usual we denote with A^T the transposition of A. Moreover, given a matrix A_0 with simple real spectrum Λ, we will denote with $\mathcal{S}_\Lambda$ the compact manifold consisting of all symmetric matrices isospectral to A_0 and with $\mathcal{T}_\Lambda$ the manifold of all symmetric tridiagonal matrices isospectral to A_0. In particular, $\mathcal{T}_\Lambda$ is a compact manifold of dimension $n-1$ (for more information about $\mathcal{T}_\Lambda$ see [10] and [11]).

In this work we introduce the following nonlinear system of ODEs in Lax form:

$$\frac{dA}{dt} = [[A^T, A]_{du}, A] \tag{2.1}$$

and we show that we can use (2.1) to obtain an explicit deformation from real upper Hessenberg matrices to Jacobi matrices. For this reason we call it the *Hessenberg–Jacobi isospectral flow*. We used it previously to diagonalize some non-symmetric matrices with special structure in [12].

Notice that (2.1) can be viewed as a polynomial vector field on the vector space of all $n \times n$ matrices with real coefficients; therefore the classical Theorem of Existence and Uniqueness implies that the corresponding Cauchy problem has always a unique (local) solution.

It is also well known that the flow associated to (2.1), being a Lax flow is isospectral (see [13]), meaning that the eigenvalues of $A(t)$ are first integrals.

The next lemma highlights some of the relevant properties of (2.1):

Lemma 2.1. *For the vector field* (2.1) *the following properties hold:*

1. *H^+ is an invariant vector space, namely if the initial condition $A(0)$ is an upper Hessenberg matrix, $A(t)$ will remain upper Hessenberg for all times for which the solution is defined.*
2. *The flow is forward complete, so $A(t)$ exists for all $t \geq 0$. More precisely we have $A(t) \in \mathcal{B}(0, R]$, for all $t \geq 0$ where $\mathcal{B}(0, R]$ is the closed ball in $\mathbb{R}^{n \times n}$ centered at zero, with radius $R := \sqrt{\operatorname{trace}(A(0)^T A(0))}$.*
3. *Equilibria are symmetric matrices, hence tridiagonal within the vector space of upper Hessenberg matrices.*
4. *If $A(0) \in H^+$ and with simple real spectrum Λ, then the ω-limit set $\Omega(A(0))$ is contained in the vector space of symmetric tridiagonal matrices, in particular $\Omega(A(0)) \subset \mathcal{T}_\Lambda$.*

Proof. To prove the first point, it is sufficient to observe that the right-hand side of (2.1) is upper Hessenberg because the upper Hessenberg matrices form a vector space and because the product of an upper triangular matrix and an upper Hessenberg matrix is necessarily upper Hessenberg.

To prove the second claim, we show that the Frobenius norm of A is actually monotonically decreasing along (2.1), as long as $[A(t)^T, A(t)] \neq 0$. Consider the positive definite function

$$V(A) := \|A\|_F^2 := \operatorname{trace}(A^T A). \tag{2.2}$$

Differentiating (2.2) with respect to (2.1) and using the facts $\operatorname{trace}(AB) = \operatorname{trace}(BA)$, $\operatorname{trace}(A^T) = \operatorname{trace}(A)$ one obtains:

$$\dot{V}(A) = -2\operatorname{trace}([A^T, A][A^T, A]_{du}). \tag{2.3}$$

Since $[A^T, A]$ is a symmetric matrix, we can use the following fact to prove that $\dot{V}(A)$ is negative as long as $[A(t)^T, A(t)] \neq 0$. For a symmetric matrix X the following decomposition holds:

$$XX_{du} = \left((X_{du})^T + X_u\right) X_{du} \quad \text{and} \quad \operatorname{trace}(X_u X_{du}) = 0.$$

This implies that

$$\begin{aligned}\dot{V}(A) &= -2\operatorname{trace}([A^T, A][A^T, A]_{du}) = -2\operatorname{trace}\left(([A^T, A]_{du})^T [A^T, A]_{du}\right)\\ &= -2\|[A^T, A]_{du}\|_F^2 \leq 0. \end{aligned} \tag{2.4}$$

Therefore the Lie derivative of $V(A)$ along the vector field (2.1) is negative definite, as long as $[A^T, A]_{du}$ is nonzero, or equivalently as long as $[A^T, A]$ is nonzero. Indeed, since $[A^T, A]$ is symmetric, $[A^T, A]$ and $[A^T, A]_{du}$ contain the same information. In particular the Frobenius norm of A remains bounded and so the flow (2.1) is forward complete, since $V(A)$ is a proper function. The statement about $A(t) \in \mathcal{B}(0, R]$ follows immediately.

As for the third point, we observe that at an equilibrium $\dot{V}(A) = 0$, so from (2.4) we get $[A^T, A]_{du} = 0$, which in turn implies $[A^T, A] = 0$. Therefore A is normal and together with the simplicity and reality of the spectrum we conclude that A is symmetric (see [14]). Vice versa, if $\dot{V}(A) = 0$, then A is an equilibrium as it is immediate to see.

Finally, for the fourth item we observe the following: since $A(t)$ stays in a closed ball and $\|A(t)\|_F$ is strictly decreasing as long as $A(t)$ is not an equilibrium, we can extract a sequence of times $\{t_n\}$ converging to infinity, such that $A(t_n)$ converges to a matrix $A_\infty \in \Omega(A(0))$, with $\|A_\infty\|_F = \inf \|A(t_n)\|_F$. If A_∞ were not an equilibrium, restarting the flow with initial data A_∞ and then taking $\epsilon > 0$ and sufficiently small, we would get a matrix $A_\infty(\epsilon)$ with $\|A_\infty(\epsilon)\|_F < \|A_\infty\|_F$. On the other hand, $A_\infty(\epsilon)$ is the limit of the matrices $A(t_n + \epsilon)$, as it is immediate to see using the properties of the flow map, but $\|A(t_n + \epsilon)\|_F > \|A_\infty\|_F$, which contradicts $\|A_\infty(\epsilon)\|_F < \|A_\infty\|_F$. Therefore, any accumulation point $A_\infty \in \Omega(A(0))$ is an equilibrium, hence symmetric by the previous item. Moreover, since the flow preserves upper Hessenberg matrices and $A(0)$ is upper Hessenberg, then any accumulation point is actually given by a *tridiagonal symmetric matrix*. In particular, the ω-limit set $\Omega(A(0))$ is contained in $\mathcal{S}_\Lambda$, the compact manifold consisting of all symmetric matrices isospectral to $A(0)$. $\square$

While the forward completeness of flow (2.1) is easily proved in the previous lemma, it is not possible to conclude directly that $T(t)$ and $T(t)^{-1}$ remain bounded from the fact that $A(t) = T(t)A_0T(t)^{-1}$, where $T(t) \in \mathcal{G}_0$.

We remark also that by the fourth point of Lemma 2.1, if A_0 is upper Hessenberg with simple real spectrum, then $\lim_{t\to+\infty} A(t)$ converges to the manifold $\mathcal{T}_\Lambda$, that properly contains the set $\mathcal{J}$ of Jacobi matrices.

The following lemma provides some further information about the evolution of subdiagonal elements of an upper Hessenberg matrix A_0 subject to the evolution of (2.1):

Lemma 2.2. *Assume $A_0 := A(0)$ is an upper Hessenberg matrix and suppose it evolves according to (2.1). Then each subdiagonal element $A_{i+1,i}$ $i = 1,\dots,n-1$ evolves in the following way: if $(A_0)_{i+1,i} = 0$ then $(A(t))_{i+1,i} = 0$ for all future times, and if $(A_0)_{i+1,i} \neq 0$, then $(A(t))_{i+1,i}$ can not change sign.*

Proof. The proof is based on the following claim: the equation for the time evolution of a subdiagonal element $A_{i+1,i}$ has the form

$$\frac{dA_{i+1,i}}{dt} = A_{i+1,i}\ f(A_{k,l}),$$

where f is a suitable function of the entries of A. From this claim the lemma follows immediately: if $(A_0)_{i+1,i} = 0$, then $A_{i+1,i}$ stays zero, and if $(A_0)_{i+1,i} \neq 0$, then $A_{i+1,i}$ can not change sign. Indeed, a simple computation left to the reader shows that in our case we have:

$$\frac{dA_{i+1,i}}{dt} = A_{i+1,i}\left[\left([A^T,A]_{du}\right)_{i+1,i+1} - \left([A^T,A]_{du}\right)_{i,i}\right], \quad i = 1,\dots,n-1,$$

and the lemma is proved. □

3. Convergence

We study the convergence properties of the flow (2.1) starting with an initial datum A_0 which is lower triangular, upper Hessenberg with simple spectrum Λ and subdiagonal elements with given sign pattern (in particular they are all different from zero). We call such an initial condition A_0 admissible. We could equally well use upper Hessenberg matrices with simple real spectrum Λ and subdiagonal elements with given sign pattern, the only difference is that for the initial data we consider the spectrum can be immediately read from the matrix.

By Lemma 2.1 we know that if the flow (2.1) is initialized with an admissible initial condition A_0 with spectrum Λ, then the ω-limit set $\Omega(A_0)$ is contained in the compact manifold of symmetric tridiagonal matrices isospectral to A_0, i.e. $\Omega(A_0) \subset \mathcal{T}_\Lambda$. However, we still do not know if $\Omega(A_0)$ is actually a singleton and if it is a matrix of the desired form, since in principle, it could also be a matrix in which some of the codiagonal elements are zero. Indeed, it is not immediate to prove that the solution of (2.1) with an admissible initial condition A_0 will converge to a tridiagonal symmetric matrix with the *given* *sign pattern for the codiagonal elements* and to achieve this goal we will use a series of intermediate results.

First of all, we can give a more precise description of the ω-limit set after the following lemma:

Lemma 3.1. *Let A_0 be an admissible initial condition for (2.1). Then there is no T upper triangular and with determinant one and belonging to $\mathcal{G}_0$ such that $(TA_0T^{-1})_{i+1,i} = 0$, for $i = 1, \ldots, n-1$. Moreover, for all T upper triangular, with determinant one and belonging to $\mathcal{G}_0$, we have that* $\operatorname{sign}((A_0)_{i+1,i}) = \operatorname{sign}((TA_0T^{-1})_{i+1,i})$, *$i = 1, \ldots, n-1$.*

Proof. A simple computation left to the reader shows that

$$(TA_0T^{-1})_{i+1,i} = T_{i+1,i+1}(A_0)_{i+1,i}(T^{-1})_{ii}, \tag{3.1}$$

for $i = 1, \ldots, n-1$. Since $(A_0)_{i+1,i} \neq 0$ and T has determinant one, all its diagonal entries are different from zero and the same is true for T^{-1}. In this way the first claim is proved. Since T belongs to the connected component containing the identity of the Lie group of upper triangular matrices with determinant one, it follows that all of its eigenvalues are positive, and the same is true for its inverse T^{-1}. Therefore from Eq. (3.1) the second claim about preservation of signs follows. The second claim follows also directly from Lemma 2.2. □

In the following we outline and prove that $T(t)$ and $T(t)^{-1}$, with $A(t) := T(t)A_0T(t)^{-1}$ remain bounded for $t \geq 0$ based on the fact that $\|[A^T(t), A(t)]_{du}\|_F$ converges to zero exponentially fast.

First observe that since A evolves following (2.1), it is immediate to see that T evolves according to

$$\frac{dT}{dt} = [A^T, A]_{du}T = \left[(TA_0T^{-1})^T, TA_0T^{-1}\right]_{du} T, \tag{3.2}$$

where (3.2) holds on $[0, t_{\max})$, the maximal interval of existence. The fact that $A(t)$ is bounded for all future times, does not imply that $T(t)$ is also bounded for all future times.

Considering $\|T\|_F^2 := \operatorname{trace}(TT^T)$ we have

$$\begin{aligned}\frac{d\|T\|_F^2}{dt} &= 2\operatorname{trace}([A^T, A]_{du}TT^T) \leq 2\,\|TT^T\|_F\ \|[A^T, A]_{du}\|_F \\ &\leq 2\,\|T\|_F^2\ \|[A^T, A]_{du}\|_F,\end{aligned}$$

using the Cauchy–Schwarz inequality for the scalar product $\langle A, B\rangle := \operatorname{trace}(AB^T)$ and the sub-multiplicative property of the Frobenius norm: $\|AB\|_F \leq \|A\|_F\|B\|_F$. After a straightforward simplification, we get immediately

$$\frac{1}{\|T(t)\|_F}\frac{d\|T(t)\|_F}{dt} \leq \|[A^T(t), A(t)]_{du}\|_F \tag{3.3}$$

and integrating both sides of (3.3) along the solution of (2.1) starting at an admissible initial condition A_0 we obtain the following estimate:

$$\ln(\|T(t)\|_F) - \ln(\|T(0)\|_F) \leq \int_0^t \|[A^T(s), A(s)]_{du}\|_F \, ds. \tag{3.4}$$

Therefore, to prove that $\|T(t)\|$ remains bounded, it is sufficient to show that the integral on the right hand side of (3.4) is convergent as t goes to $+\infty$. This is the case if $\|[A^T(t), A(t)]\|_F$ converges to zero sufficiently fast. Moreover, to prove that $\|T^{-1}(t)\|$ remains bounded, we just observe that from $TT^{-1} = \mathrm{Id}$ we get immediately $\frac{dT^{-1}}{dt} = -T^{-1}[A^T, A]_{du}$. Using the same estimates we used above we obtain

$$\frac{d\|T^{-1}\|_F^2}{dt} \leq 2\,\|T^{-1}\|_F^2 \; \|[A^T, A]_{du}\|_F,$$

from which we get the counterpart of inequality (3.4) for T^{-1}:

$$\ln(\|T(t)^{-1}\|_F) - \ln(\|T(0)^{-1}\|_F) \leq \int_0^t \|[A^T(s), A(s)]_{du}\|_F \, ds. \tag{3.5}$$

Next, we are going to prove that $\|[A^T(t), A(t)]_{du}\|_F$ converges to zero exponentially fast along a solution of (2.1) starting at an admissible initial condition A_0. To do this, we actually prove a more general statement, namely that $\|[A^T(t), A(t)]_{du}\|_F$ converges to zero exponentially fast starting from *any* A_0 with simple real spectrum Λ.

The main idea is to prove that the compact manifold $\mathcal{S}_\Lambda$ of all symmetric matrices isospectral to A_0 is exponentially attracting for the flow (2.1), where A_0 is any matrix with simple real spectrum Λ. This manifold contains all the ω-limit sets for initial data A_0 with simple real spectrum Λ due to Lemma 2.1.

Now we recall some results and definitions from [15].

Definition 3.2. [15] An invariant manifold $\mathcal{Q} \subset \mathbb{R}^n$ for a vector field $\frac{dx}{dt} = f(x)$ is called exponentially attracting for this vector field if there exists a neighborhood W of $\mathcal{Q}$ and positive constants K and γ such that for any point $x_0 = x(0) \in W$ and for $t \geq 0$ the following inequality holds:

$$d(x(t), \mathcal{Q}) \leq K e^{-\gamma t} d(x_0, \mathcal{Q}),$$

where d is a distance function.

Following [15], consider along with the vector field $\frac{dx}{dt} = f(x)$ the system

$$\frac{dx}{dt} = f(x) \quad \frac{d\xi}{dt} = Df(x)\xi, \tag{3.6}$$

where Df is the Jacobian of f. System (3.6) can be interpreted as the family of all linearizations of $\frac{dx}{dt} = f(x)$ along its solutions; in fact if $x(t)$ is a solution of $\frac{dx}{dt} = f(x)$, then $\frac{d\xi}{dt} = Df(x(t))\xi$ is a non-autonomous linear system. In the following, $x(t)$ will be assumed to be a solution lying on $\mathcal{Q}$. For each $x \in \mathcal{Q}$, let us denote with $T_x\mathcal{Q}$ the tangent space at x to $\mathcal{Q}$ and $N_x\mathcal{Q}$ its orthogonal complement (for a choice of a Riemannian metric in the environment). Let us also denote with $P_x : T_x\mathbb{R}^n \to N_x\mathcal{Q}$ the projection operator sending vector $\xi \in T_x\mathbb{R}^n$ onto $N_x\mathcal{Q}$.

Definition 3.3. [15] An invariant manifold $\mathcal{Q}$ of $\frac{dx}{dt} = f(x)$ is called exponentially stable in linear approximation if for any trajectory $x(t)$ lying on $\mathcal{Q}$ and any $\xi(0)$ the corresponding solution of system (3.6) satisfies for $t \geq 0$ the following inequality

$$\|\nu(t)\| \leq Ke^{-\beta t}\|\nu(0)\|, \quad \nu(t) = P_{x(t)}\xi(t), \quad \beta > 0 \tag{3.7}$$

where the constants K and β can be chosen to be independent on the choice of $x(0)$ and $\xi(0)$.

The following result highlights the connection between the two definitions above:

Theorem 3.4. [15] *An invariant compact manifold $\mathcal{Q}$ is exponentially attracting if and only if $\mathcal{Q}$ is exponentially stable in linear approximation.*

We apply the previous result taking $\mathcal{S}_\Lambda$ for the manifold $\mathcal{Q}$. In this case $\mathcal{Q}$ consists of a manifold of equilibria, so to check that property (3.7) holds it is sufficient to linearize the flow (2.1) at an arbitrary equilibrium point $S_0 \in \mathcal{S}_\Lambda$ and check that the normal directions to $T_{S_0}\mathcal{S}_\Lambda$ are decreasing exponentially fast.

First we need to define the projection to the normal space $N_{S_0}\mathcal{S}_\Lambda$, which is achieved in the next lemma:

Lemma 3.5. *Call S_0 any point in $\mathcal{S}_\Lambda$ and denote with $T_{S_0}\mathcal{S}_\Lambda$ the tangent space to $\mathcal{S}_\Lambda$ at S_0. For any P tangent vector to the space $M_{n\times n}$ of $n \times n$ matrices at S_0, $P = [U, S_0]$, its projection to the orthogonal complement $N_{S_0}\mathcal{S}_\Lambda$ of $T_{S_0}\mathcal{S}_\Lambda$ in the space of all matrices is given by*

$$\pi_N(P) := \frac{P - P^T}{2},$$

where the orthogonal complement $N_{S_0}\mathcal{S}_\Lambda$ of $T_{S_0}\mathcal{S}_\Lambda$ is taken with respect to the Riemannian metric $\langle A, B\rangle := \mathrm{trace}(AB^T)$.

Proof. The space $T_{S_0}\mathcal{S}_\Lambda$ is given described by vectors of the form $[V, S_0]$ where V varies in the Lie algebra of skew symmetric matrices. Therefore, the projection $\pi_T : T_{S_0}M_{n\times n} \to T_{S_0}\mathcal{S}_\Lambda$ is given by

$$\pi_T(P) = \pi_T([U, S_0]) = \left[\frac{U - U^T}{2}, S_0\right] \quad \text{with } P = [U, S_0] \in T_{S_0}M_{n\times n}.$$

Therefore, $\pi_N = \mathrm{Id} - \pi_T$ is given by

$$\pi_N(P) = P - \pi_T(P) = \left[\frac{U + U^T}{2}, S_0\right] = \frac{P - P^T}{2}.$$

It is immediate to check that $\pi_N(P)$ and $\pi_T(P)$ are orthogonal to each other with respect to $\langle A, B\rangle := \mathrm{trace}(AB^T)$ because $\pi_N(P)$ is skew-symmetric while $\pi_T(P)$ is symmetric. □

To prove that $\mathcal{S}_\Lambda$ is indeed linearly exponentially stable, we need to linearize the flow (2.1) at a point $S_0 \in \mathcal{S}_\Lambda$. Recall that such an S_0 is a symmetric matrix with simple spectrum. In order to obtain the linearization, we write a first order deformation of S_0 as $dS_0 = S_0 + P(t)$ and we obtain an equation for $P(t)$:

Lemma 3.6. *The linearization of* (2.1) *at* S_0 *is given by*

$$\dot{P}(t) = \left[([P^T, S_0] + [S_0, P])_{du}, S_0\right] = [[S_0, P - P^T]_{du}, S_0]. \tag{3.8}$$

Proof. Equation (3.8) can be obtained substituting $S_0 + P(t)$ instead of A in (2.1), and then using the fact that S_0 is an equilibrium and collecting the terms linear in P and disregarding those that have a quadratic or cubic dependence on P. □

We are interested to show that if $P(t)$ evolves according to the linearization (3.8) than its normal component

$$\nu(t) := \pi_N(P(t)) = \frac{P - P^T}{2} \in N_{S_0}\mathcal{S}_\Lambda \tag{3.9}$$

is converging to zero exponentially fast. This will allow us to conclude that $\mathcal{S}_\Lambda$ is linearly exponentially stable.

We have the following key result:

Proposition 3.7. *The normal component* $\nu(t) = \frac{P(t)-P(t)^T}{2}$, *where* $P(t)$ *evolves according to* (3.8) *converges to zero exponentially fast.*

Proof. To prove this we consider the Lie derivative of the Frobenius norm $\|\nu(t)\|_F^2$ along the vector field (3.8). Notice that this is proportional to $\|P - P^T\|_F^2$ up to a constant. We obtain

$$\begin{aligned}\frac{d\|\nu\|_F^2}{dt} &= 2\,\mathrm{trace}\left(\nu^T\left(-\left[[\nu, S_0]_{du} + ([\nu, S_0]_{du})^T, S_0\right]\right)\right)\\ &= 2\,\mathrm{trace}\left([\nu^T, S_0]\left([\nu, S_0]_{du} + ([\nu, S_0]_{du})^T\right)\right),\end{aligned}$$

which is equal, since $\nu^T = -\nu$ to

$$-2\,\mathrm{trace}\left([\nu, S_0]\left([\nu, S_0]_{du} + ([\nu, S_0]_{du})^T\right)\right).$$

Now $[\nu, S_0]$ is clearly symmetric, therefore, $[\nu, S_0]_{du} + ([\nu, S_0]_{du})^T = [\nu, S_0] + [\nu, S_0]_d$. Thus

$$\begin{aligned}\frac{d\|\nu\|_F^2}{dt} &= -2\,\mathrm{trace}\left([\nu, S_0]\left([\nu, S_0] + [\nu, S_0]_d\right)\right)\\ &= -2\,\mathrm{trace}\left([\nu, S_0][\nu, S_0]\right) - 2\,\mathrm{trace}\left([\nu, S_0][\nu, S_0]_d\right),\end{aligned}$$

and both terms on the right are negative semidefinite. To prove that the quadratic form obtained from $\frac{d\|\nu(t)\|_F^2}{dt}$ is indeed negative definite on $N_{S_0}\mathcal{S}_\Lambda$ it is sufficient to look at $\mathrm{trace}([\nu, S_0][\nu, S_0])$. This term is zero iff $[\nu, S_0] = 0$, but S_0 is symmetric with simple spectrum, being isospectral to A_0, while ν is skewsymmetric and therefore (see [14]) $[\nu, S_0] = 0$ iff $\nu = 0$. This means that the quadratic form $\frac{d\|\nu(t)\|_F^2}{dt}$ is indeed negative definite on $N_{S_0}\mathcal{S}_\Lambda$ and this shows that ν converges to zero exponentially fast due to Lyapunov linearization theorem. □

We can now prove the following.

Theorem 3.8. *The manifold $\mathcal{S}_\Lambda$ is exponentially attracting for the flow* (2.1)

Proof. From Proposition 3.7 we have that $\nu(t)$ is converging to zero exponentially fast because the quadratic form $\frac{d\|\nu(t)\|_F^2}{dt}$ is negative definite on $N_{S_0}\mathcal{S}_\Lambda$, for each $S_0 \in \mathcal{S}_\Lambda$. In particular the eigenvalues of the quadratic form $\frac{d\|\nu(t)\|_F^2}{dt}$ are negative and they are bounded away globally from 0 as S_0 varies on $\mathcal{S}_\Lambda$ because $\mathcal{S}_\Lambda$ is compact and the quadratic form is continuous. Therefore there exist positive constants K and β, independent on S_0 such that the bound in the Definition 3.3 holds (see for instance Observation 9.3 in [15]). Therefore $\mathcal{S}_\Lambda$ is linearly exponentially stable and by Theorem 3.4 it follows that $\mathcal{S}_\Lambda$ is exponentially attracting. □

The time evolution of $\|\nu(t)\|_F$ is related to the time evolution of $\|[A(t)^T, A(t)]_{du}\|_F$, at least in the linearization. Indeed we have the following lemma.

Lemma 3.9. *Let $A(t)$ be a solution of the flow* (2.1)*. The quantity $[A^T(t), A(t)]_{du}$ converges to zero exponentially fast if $\nu(t)$, evolving according to* (3.8) *converges to zero exponentially fast.*

Proof. First it is immediate to check that $[A^T, A] = 2[A_{\mathrm{sym}}, A_{\mathrm{sk}}]$, where A_{sym} is the symmetric part of A and A_{sk} is the skew-symmetric part of A. Then we have $\|[A^T, A]\|_F \leq 4\|A_{\mathrm{sym}}\|_F\|A_{\mathrm{sk}}\|_F$. Since $\|A_{\mathrm{sym}}\|_F$ is bounded because of $\|A_{\mathrm{sym}}\|_F^2 \leq \|A_{\mathrm{sym}}\|_F^2 + \|A_{\mathrm{sk}}\|_F^2 = \|A\|_F^2$ which is bounded by Lemma 2.1, we can write

$$\|[A^T, A]\|_F \leq C\|A_{\mathrm{sk}}\|_F, \tag{3.10}$$

for some constant $C > 0$. On the other hand, the time evolution of the linearization of the skew-symmetric part A_{sk} at S_0 is given by

$$\frac{1}{2}\left[\frac{dU}{dt}, S_0\right] - \frac{1}{2}\left[\frac{dU}{dt}, S_0\right]^T = \frac{\dot{P} - \dot{P}^T}{2} = \frac{d\nu}{dt},$$

since $A_{\mathrm{sk}} = \frac{A-A^T}{2}$ and due to Eqs. (3.8) and (3.9). By Proposition 3.7 we have that $\|\nu(t)\|_F$ is converging to zero exponentially fast, and likewise it is $\|[A^T, A]\|_F$ by inequality (3.10). Therefore $\|[A^T, A]_{du}\|_F$ is also converging to zero exponentially fast, since $\|[A^T, A]_{du}\|_F \leq \|[A^T, A]\|_F$. □

Let us remark that Lemma 3.9 proves that the integral $\int_0^t \|[A^T(s), A(s)]_{du}\|_F \, ds$ is convergent for $t \to +\infty$ along a solution of (2.1), for any initial data A_0 with simple spectrum, not just for the class of A_0 which are upper Hessenberg and lower diagonal with simple real spectrum. However, we apply this convergence result to that case.

Combining the results so far obtained, we can prove the following.

Theorem 3.10. *Let $A(t)$ be the solution of* (2.1) *starting from an admissible initial condition A_0. Then $\lim_{t\to+\infty} A(t)$ converges exponentially fast to the set of tridiagonal symmetric matrices isospectral to A_0 with the given sign pattern for codiagonal elements.*

Proof. By Lemma 3.9 we know that $\|[A(t)^T, A(t)]_{du}\|_F$ is converging exponentially fast to zero. In particular, this implies that $\|T(t)\|_F$ and $\|T^{-1}(t)\|_F$ remain bounded because of inequality (3.4) and inequality (3.5), respectively. Therefore the eigenvalues of $T(t)$ and $T^{-1}(t)$ remain bounded and bounded away from zero. This allows us to apply Lemma 3.1 in the limit for $t \to +\infty$ and to conclude that $A(t)$ can not converge to the set of diagonal matrices, in particular $\lim_{t\to+\infty} A(t)_{i+1,i} \neq 0$ for $i = 1, \dots, n-1$. Moreover, since by Lemma 2.1 $A(t)$ converges to the set of symmetric tridiagonal matrices isospectral to A_0 and the subdiagonal elements of $A(t)$ can not change sign by Lemma 2.2, we have that $A(t)$ has to converge to the set of tridiagonal symmetric matrices isospectral to A_0 with the same sign pattern for codiagonal elements as the sign pattern of subdiagonal elements of A_0. □

Our final goal is to show that the ω-limit for an admissible initial condition A_0 is indeed a single point. Since $\mathcal{S}_\Lambda$ is exponentially attracting, it is in particular normally hyperbolic, so one could invoke the theory of normal hyperbolic manifolds to claim that the ω-limit set is in this case a singleton. However, in our case, we prefer to give a self-consistent elementary proof of this fact.

We conclude with the following:

Theorem 3.11. *Let A_0 be an admissible initial condition for (2.1). Then $\Omega(A_0)$ is a singleton and it is a symmetric tridiagonal matrix, isospectral to A_0 with codiagonal elements having the same sign pattern as the subdiagonal elements of A_0. Therefore the flow (2.1) performs an explicit deformation from a upper Hessenberg matrix to a symmetric tridiagonal matrix preserving the spectrum.*

Proof. The only claim we need to prove is that $\Omega(A_0)$ is a singleton. Since the Frobenius norm is sub-multiplicative we obtain

$$\|[[A^T, A]_{du}, A]_F\| \leq 2\|[A^T, A]_{du}\|_F \, \|A\|_F.$$

Therefore we have:

$$\begin{aligned}
&\lim_{t\to+\infty} \int_0^t \|[[A^T(s), A(s)]_{du}, A(s)]\|_F \, ds \\
&\quad \leq 2 \lim_{t\to+\infty} \int_0^t \|[A^T(s), A(s)]_{du}\|_F \|A(s)\|_F \, ds \\
&\quad \leq \lim_{t\to+\infty} \int_0^t \|[A^T(s), A(s)]_{du}\|_F K \, ds < +\infty,
\end{aligned}$$

since $\|A\|$ can be bounded by a constant K due to Lemma 2.1, and the integral of $\|[A^T, A]_{du}\|$ along a solution is convergent by Lemma 3.9. This is enough to conclude that $\Omega(A_0)$ is indeed a singleton, since the convergence of the improper integral above says that the length of the solution curve is finite. □

4. Optimality of the flow

Let us observe that Eq. (1.1) can be also viewed as a realization of the choice of a feedback for a controlled Lax system of the form

$$\frac{dA}{dt} = [U, A], \tag{4.1}$$

where matrix function $U = U(t)$ is the control input. Then one can view Eq. (4.1) as a general control system; to obtain a specific behavior, one has not only to select a admissible initial condition $A(0) := A_0$ but also to choose a specific feedback control law that substituted in place of $U(t)$ makes the system behave in a desired way.

In this section, we show that the system introduced is the solution of an infinite time horizon optimal control problem, using the Hamilton–Jacobi–Bellman approach.

Theorem 4.1. *Consider the following deterministic optimal control problem over an infinite horizon:*

$$\begin{aligned} &\min_U \int_0^{+\infty} \mathrm{trace}\left(([A^T, A]_{du})^T([A^T, A]_{du})\right) + \mathrm{trace}(U^T U)\, ds, \\ &\quad \text{subject to } \frac{dA}{dt} = [U, A], \end{aligned} \tag{4.2}$$

where $U(t)$ is a sufficiently smooth function taking value in the Lie algebra of upper triangular matrices. Then the optimal value function is given by $V(A) = \mathrm{trace}(A^T A)$ and the optimal feedback is given by $U = [A^T, A]_{du}$, i.e the flow (2.1) is the solution of this infinite horizon optimal control problem.

Before proving Theorem 4.1, observe that U is not assumed to have zero trace, but just to be upper triangular. The fact that U has zero trace is then a consequence of the form of the optimal solution.

Proof. The Hamilton–Jacobi–Bellman equation which determines the optimal $U(t)$ for the problem above is given by

$$\min_U \left[\mathrm{trace}\left(([A^T, A]_{du})^T([A^T, A]_{du}) + U^T U\right) + \frac{d}{dt}V(A)\right] = 0. \tag{4.3}$$

If the value function is smooth, the fulfillment of the above equation is a sufficient condition for optimality (see for instance [16]). With $V(A) = \mathrm{trace}(A^T A)$ and after some straightforward manipulations, Eq. (4.3) reads

$$\min_U \left[\mathrm{trace}\left(([A^T, A]_{du})^T([A^T, A]_{du}) + U^T U - 2[A^T, A]U\right)\right] = 0.$$

Notice that since U is upper triangular, we have in the previous equation that $\mathrm{trace}(2[A^T, A]U) = \mathrm{trace}(2[A^T, A]_{dl}U) = \mathrm{trace}(2U^T[A^T, A]_{du})$, since $([A^T, A]_{dl})^T = [A^T, A]_{du}$. Therefore, the Hamilton–Jacobi–Bellman equation becomes

$$\min_U \left[\mathrm{trace}\left(([A^T, A]_{du})^T([A^T, A]_{du}) + U^T U - 2[A^T, A]_{du}U^T\right)\right] = 0.$$

To find the optimal U we just take the gradient with respect to U of

$$\operatorname{trace}\left(([A^T, A]_{du})^T([A^T, A]_{du}) + U^TU - 2[A^T, A]_{du}U^T\right),$$

and set it to zero. Since

$$\frac{d}{dt}\operatorname{trace}(U^TU) = 2\operatorname{trace}\left(U^T\frac{dU}{dt}\right) = \left\langle \nabla_U \operatorname{trace}(U^TU), \frac{dU}{dt}\right\rangle,$$

where $\langle A, B\rangle$ is the usual Riemannian metric $\operatorname{trace}(A^TB)$, using the definition of gradient we have that

$$\nabla_U \operatorname{trace}(U^TU) = 2U.$$

Analogously, one finds in a similar manner

$$\nabla_U \left(2\operatorname{trace}([A^T, A]_{du}U^T\right) = 2[A^T, A]_{du}.$$

Therefore the optimal U upper triangular is given by $[A^T, A]_{du}$ and it is indeed a minimum since the expression

$$\operatorname{trace}(([A^T, A]_{du})^T([A^T, A]_{du}) + U^TU - 2[A^T, A]_{du}U^T)$$

is quadratic in U and convex. Substituting $U = [A^T, A]_{du}$ in the above expression yields zero identically and consequently $V(A) = \operatorname{trace}(A^TA)$ is the value function and U is the optimal feedback control. □

5. Some applications and simulations

In this section we present some applications of the flow introduced and some simulations partly illustrating the convergence and also extending the scope of applicability of the flow. Simulations are implemented using `MatLab`™ ODE solvers `ode15s` and `ode23s`.

We use the notation introduced in [17] in which two row vectors are used to describe tridiagonal symmetric matrices, one for the diagonal entries, the other for codiagonals. First we examine some simulations illustrating the convergence properties of the system introduced.

We show that one can indeed generate an arbitrary sign pattern in the codiagonal elements, choosing the same sign pattern in the lower diagonal elements of A_0. Suppose A_0 is a 7×7 lower bidiagonal matrix with diagonal entries given by $[1, 2, 3, -4, , 5, -6, 7]$ and with lower diagonal entries given by $[-10, -10, -10, 10, -10, 10]$.

Using our flow, a numerical approximation of the corresponding ω-limit point is a symmetric tridiagonal matrix $\Omega(A_0)$ with diagonal entries given by $[2.1623, 1.5243, -0.9848,\ 1.9942, -0.8570, 1.6415, 2.5195]$ and with codiagonal entries given by $[-1.0552, -3.6127,\ -3.4461, 3.2230, -4.4945, 1.5824]$. The spectrum $\sigma(\Omega(A_0)) = \{-6.0002, -3.9988, 1.0004,\ 1.9994, 2.9993, 4.9992, 7.0007\}$ is within the third decimal digit from the spectrum of A_0. It can be seen that the sign pattern has been faithfully reproduced.

Secondly, let us observe that instead of using an admissible initial condition A_0 for the flow as previously defined, one can initialize the flow with an initial condition that is the Frobenius companion matrix associated to the characteristic polynomial. Indeed, it could be useful to construct a Jacobi matrix by specifying the eigenvalues (inverse problem for Jacobi matrices) or the characteristic polynomial without computing the eigenvalues (but assuming that the roots of the polynomial are simple and real). For instance, suppose we want to construct an 8×8 symmetric tridiagonal with the following characteristic polynomial:

$$p(x) = x^8 + 2x^7 - \frac{67}{2}x^6 - 52x^5 + \frac{5689}{16}x^4 + \frac{3019}{8}x^3 - \frac{21549}{16}x^2 - \frac{5769}{8}x + \frac{2832}{2}.$$

The Frobenius companion matrix (which is automatically in upper Hessenberg form) of this characteristic polynomial is easily computed. From what we proved previously we know that $\Omega(A_0)$ corresponds to a Jacobi matrix with the given spectrum. Indeed the simulation shows that a numerical approximation to $\Omega(A_0)$ has diagonal entries given by $[0.5871, -1.0067, 0.9614, -1.4554, 1.2097, -1.7632, 1.3971, -1.9301]$ and with codiagonal entries given by $[1.0132, 0.7670, 1.9364, 1.1970, 2.7854, 1.2024, 3.4866]$. Notice that using the companion matrix there are no additional parameters involved in the determination of A_0.

Finally we give an example to show that the scope of applicability of this flow might be extended beyond what has been proved in the main part of the paper. We construct an even dimensional *real skew-symmetric tridiagonal matrices* with given simple imaginary spectrum and with given sign pattern for the codiagonal elements. Suppose we want to construct such a matrix with dimension $n = 8$ and with a sign pattern for subdiagonal elements given by $\{+, -, +, +, +, -, -\}$ and with spectrum $\{\pm i, \pm 3i, \pm\sqrt{2}i, \pm 4i\}$. Then we consider as initial condition for the flow the following tridiagonal non-symmetric matrix A. The entries of the upper codiagonal are given by $[-1, 0, -3, 0, -1.4142, 0, 4]$. The entries of the main diagonal are all zero and the entries of the lower codiagonal are given by $[1, -1, 3, 1, 1.4142, -1, -4]$. Now it is immediate to see that A_0 has the specified spectrum, because it is block-diagonal. Moreover, the sign pattern for subdiagonal elements in A_0 is indeed given by $\{+, -, +, +, +, -, -\}$. Using the flow (2.1) we obtain the following skew-symmetric tridiagonal matrix $\Omega(A_0)$ with lower codiagonal entries given by $[1.0827, -0.7826, 2.7384, 0.8289, 1.4986, -1.0929, -3.8196]$. This has the right sign pattern for codiagonal elements and has spectrum given by $\{\pm 1, 000i, \pm 1.4142i, \pm 3.000i, \pm 4.000i\}$.

We plan to further address this possible application, studying the flow on complex matrices.

Acknowledgments

A. Arsie expresses his gratitude to the Institute for Systems Theory and Automatic Control of the University of Stuttgart for the kind hospitality, the stimulating environment and for providing a pleasant accommodation while this work was started. A. Arsie also acknowledges partial support through selected funds from the University of Toledo. C. Ebenbauer acknowledges support by the German Research Foundation (DFG), Emmy Noether grant EB 425/2-1. The authors would like to thank the anonymous referee for help provided in improving to a great extent the presentation and the results of the paper.

References

[1] Lax, P.: Integrals of nonlinear equations of evolution and solitary waves Comm. Pure Appl. Math. **21**, 467–490 (1968)

[2] Flaschka, H.: The Toda lattice, I. Phys. Rev. B **9**, 1924–1925 (1974)

[3] Moser, J.: Finitely many mass points on the line under the influence of an exponential potential an integrable system. In: Dynamical Systems, Theory and Applications (Rencontres, Battelle Res. Inst., Seattle, Wash., 1974), pp. 467–497. Lecture Notes in Physics, vol. 38. Springer, Berlin (1975)

[4] Deift, P., Li, L.-C., Tomei, C.: Matrix factorizations and integrable systems. Commun. Pure Appl. Math. **42**(4), 443–521 (1989)

[5] Symes, W.: Hamiltonian group actions and integrable systems. Phys. D **1**, 339–374 (1980)

[6] Perelomov, A.M.: Integrable Systems of Classical Mechanics and Lie Algebras, vol. 1. Birkhäuser, Basel (1990)

[7] Brockett, R.: Dynamical systems that sort lists, diagonalize matrices and solve linear programming problems. Linear Algebra Appl. **146**, 79–91 (1991)

[8] Chu, M.: Linear algebra algorithms as dynamical systems. Acta Numer. **17**, 1–86 (2008)

[9] Helmke, U., Moore, J.: Optimization and Dynamical Systems. Springer, Berlin (1994)

[10] Tomei, C.: The topology of manifolds of isospectral tridiagonal matrices. Duke Math. J. **51**, 981–996 (1984)

[11] Leite, R., Saldanha, N., Tomei, C.: An atlas for tridiagonal isospectral manifolds. Linear Algebra Appl. **429**(1), 387–402 (2008)

[12] Ebenbauer, C., Arsie, A.: On an eigenflow equation and its structure preserving properties. Conference on Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. In: Proceedings of the 48th IEEE conference on publication, pp. 7491–7496 (2009)

[13] Lax, P.: Linear Algebra and Its Applications, Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts (2007)

[14] Horn, R., Johnson, C.: Matrix Analysis. Cambridge University Press, Cambridge (1985)

[15] Ryashko, L.B., Shnol, E.E.: On exponentially attracting invariant manifolds of ODEs. Nonlinearity **16**, 147–160 (2003)

[16] Vinter, R.: Optimal Control. Modern Birkhäuser Classics (2010)

[17] Demmel, J.W.: Applied Numerical Linear Algebra. SIAM, Philadelphia (1997)

Alessandro Arsie
Department of Mathematics and Statistics
The University of Toledo
Toledo, OH 43606, USA
e-mail: alessandro.arsie@utoledo.edu

Christian Ebenbauer
Institute for Systems Theory and Automatic Control
University of Stuttgart
70550 Stuttgart, Baden-Württemberg, Germany
e-mail: ce@ist.uni-stuttgart.de

Received: 1 September 2012.
Accepted: 26 April 2014.