

# Hardy inequalities resulted from nonlinear problems dealing with $A$ -Laplacian

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**Abstract.** We derive Hardy inequalities in weighted Sobolev spaces via anticoercive partial differential inequalities of elliptic type involving  $A$ -Laplacian  $-\Delta_A u = -\operatorname{div} A(\nabla u) \geq \Phi$ , where  $\Phi$  is a given locally integrable function and  $u$  is defined on an open subset  $\Omega \subseteq \mathbb{R}^n$ . Knowing solutions we derive Caccioppoli inequalities for  $u$ . As a consequence we obtain Hardy inequalities for compactly supported Lipschitz functions involving certain measures, having the form

$$\int_{\Omega} F_{\bar{A}}(|\xi|)\mu_1(dx) \leq \int_{\Omega} \bar{A}(|\nabla \xi|)\mu_2(dx),$$

where  $\bar{A}(t)$  is a Young function related to  $A$  and satisfying  $\Delta'$ -condition, while  $F_{\bar{A}}(t) = 1/(\bar{A}(1/t))$ . Examples involving  $\bar{A}(t) = t^p \log^\alpha(2+t)$ ,  $p \geq 1$ ,  $\alpha \geq 0$  are given. The work extends our previous work (Skrzypczaki, in *Nonlinear Anal TMA* 93:30–50, 2013), where we dealt with inequality  $-\Delta_p u \geq \Phi$ , leading to Hardy and Hardy–Poincaré inequalities with the best constants.

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## 1. Introduction

In this paper we derive Hardy–Sobolev inequalities having the form

$$\int_{\Omega} F_{\bar{A}}(|\xi|)\mu_1(dx) \leq \int_{\Omega} \bar{A}(|\nabla \xi|)\mu_2(dx), \quad (1.1)$$

where  $\xi : \Omega \rightarrow \mathbb{R}$  is compactly supported Lipschitz function,  $\Omega$  is an open subset of  $\mathbb{R}^n$  not necessarily bounded,  $\bar{A}(t)$  is an  $N$ -function satisfying  $\Delta'$ -condition and  $F_{\bar{A}}(t) = 1/(\bar{A}(1/t))$ . The involved measures  $\mu_1(dx)$ ,  $\mu_2(dx)$

depend on  $u$ —a nonnegative weak solution to the anticoercive partial differential inequality of elliptic type involving  $A$ -Laplacian:

$$-\Delta_A u = -\operatorname{div} A(\nabla u) = -\operatorname{div} \left( \frac{\bar{A}(|\nabla u|)}{|\nabla u|^2} \nabla u \right) \geq \Phi \quad \text{in } \Omega, \tag{1.2}$$

with a locally integrable function  $\Phi$ . Rather general function  $\Phi$  is allowed. It can be even negative or sign changing if only there exists  $\sigma \in \mathbb{R}$  such that

$$\Phi + \sigma \frac{\bar{A}(|\nabla u|)}{g(u)} \chi_{\{\nabla u \neq 0\}} \geq 0 \quad \text{a.e.}, \tag{1.3}$$

where nonnegative function  $g$  satisfies some compatibility conditions (see Assumption  $(\Psi)$ , i.e. (2.11) and (2.12)).

Operators related to this type of  $\bar{A}$  are considered in regularity theory [10,31,32,58,62].

The motivation to consider Hardy–Sobolev-type inequalities (1.1) is clear. They are widely spread in various fields of analysis playing significant role among others in functional analysis, harmonic analysis, probability theory, and PDEs. In the theory of PDEs they are used to obtain a priori estimates, existence, regularity results, and to study qualitative properties of solutions and their asymptotic behaviour [3,6,12,13,33,34,57,72]. Hardy inequalities are applied to derivation of embedding theorems, Gagliardo–Nirenberg interpolation inequalities and in real interpolation theory [21,22,29,38–40,45]. Moreover, functions achieving the best constants in Hardy–Sobolev-type inequalities satisfy some nonlinear eigenvalue problems [16, Chapter 5]. Weighted Sobolev spaces related to inequalities with various weights are investigated e.g. in books [51,63] and papers [48,49,64]. Furthermore, Orlicz–Sobolev spaces are considered e.g. in [2,5,20,27,35].

Hardy–Sobolev-type inequalities are also interesting on their own [52–54,61]. Many authors consider generalized versions of the inequalities with remainder terms [1,4,28] as well as those expressed in Orlicz setting [15,17,44,46]. Recently, Hardy-type inequalities are investigated also on Riemannian manifolds [26].

Hardy-type inequalities in Orlicz framework are considered in particular in [19]. We find there balance conditions for  $A, B$ —a pair of (not necessarily equal) Young functions for the validity of inequalities of the form

$$\left\| \frac{\xi(x)}{d^{1+\theta}(x)} \right\|_{L^B(\Omega)} \leq \left\| \frac{\nabla \xi(x)}{d^\theta(x)} \right\|_{L^A(\Omega)}, \tag{1.4}$$

where  $d$  is distance from the boundary of open and bounded domain  $\Omega$  with Lipschitz-continuous boundary and  $\xi$  is sufficiently smooth function with compact support in  $\Omega$ . We note that in (1.4), unlike in our inequality (1.1),  $B$  is the function of  $\xi/d^{1+\theta}(x)$  and  $A$  is the function of  $(\nabla \xi)/d^\theta(x)$ .

Similar results dealing with conditions for validity of inequality

$$\int_\Omega \Psi \left( \frac{|\xi(x)|}{d^{1+\theta}(x)} \right) dx \leq C \int_\Omega \Psi \left( \frac{|\nabla \xi(x)|}{d^\theta(x)} \right) dx, \tag{1.5}$$

where  $d(x)$  is distance from the boundary of open and bounded  $\Omega$  (sufficiently regular) and  $\xi$  is sufficiently smooth function with compact support in  $\Omega$ , we find in the recent paper [14]. The conditions are expressed in the terms of capacities. Note that the left-hand side of (1.1) is of similar, but different type than left hand side in (1.5).

Our purpose is to give the constructive method of derivation of Hardy–Sobolev inequalities on the basis of nonlinear problems. We find such an approach in papers by Barbatis, Filippas, and Tertikas [7, 8], where Hardy–Sobolev inequalities are derived on a domain where certain power of distance function is  $p$ -superharmonic. Furthermore, in papers of D’Ambrosio [23–25] the author derives inequality related to (1.1) involving  $\bar{A}(\lambda) = F_{\bar{A}}(\lambda) = \lambda^p$  as the consequence of inequality  $-\Delta_p(u^\alpha) \geq 0$  with certain constant  $\alpha$ .

Our considerations are based on the methods from [47] developed further in [68]. The idea is as follows. In [47] the authors investigate nonexistence of nontrivial nonnegative weak solutions to the  $A$ -harmonic problem

$$-\Delta_A u \geq \Phi(u) \quad \text{on } \mathbb{R}^n, \tag{1.6}$$

where  $\Phi$  is a nonnegative function. Among other results, the authors derive Caccioppoli-type estimate for nonnegative weak solutions to (1.6).

In [68] we considered the case when

$$-\Delta_p u \geq \Phi \quad \text{in } \Omega, \tag{1.7}$$

with a locally integrable function  $\Phi$  satisfying less restrictive conditions than in [47]. As it is shown in [68], the certain substitution in the derived Caccioppoli-type inequality for solutions, implies the family of Hardy-type inequalities having the form

$$\int_{\Omega} |\xi|^p \mu_{1,\beta}(dx) \leq \int_{\Omega} |\nabla \xi|^p \mu_{2,\beta}(dx),$$

where  $1 < p < \infty$ ,  $\xi : \Omega \rightarrow \mathbb{R}$  is compactly supported Lipschitz function, and  $\Omega$  is an open subset of  $\mathbb{R}^n$ . The involved measures  $\mu_{1,\beta}(dx), \mu_{2,\beta}(dx)$  depend on a certain parameter  $\beta$  and on  $u$ —a nonnegative weak solution to (1.7). It leads among other results to classical Hardy and Hardy–Poincaré inequalities with optimal constants (see [68, 69], respectively). We retrieve this results as the special case here and therefore we confirm all the examples from [68, 69].

Our goal now is to extend techniques from [68] to the more general situation when we deal with (1.2) instead of (1.7). We supply new weighted power-logarithmic Hardy–Sobolev inequalities of the form

$$\int_{\Omega} |\xi|^p \log^{-\alpha}(2 + 1/|\xi|) \mu_1(dx) \leq \tilde{C} \int_{\Omega} |\nabla \xi|^p \log^{\alpha}(2 + |\nabla \xi|) \mu_2(dx)$$

for compactly supported Lipschitz functions  $\xi$ , which result from the methods introduced in this paper.

The method may be used not only to construct new inequalities. We indicate also estimates for constants in the inequalities, which can be useful in investigating existence, as well as regularity in theory of partial differential equations in weighted Sobolev and Orlicz–Sobolev spaces.

## 2. Preliminaries

### Notation

In the sequel we assume that  $\Omega \subseteq \mathbb{R}^n$  is an open subset not necessarily bounded.

By  $A$ -harmonic problems we understand those, which involve  $A$ -Laplace operator  $\Delta_A u = \operatorname{div}(A(\nabla u))$ , understood in the weak sense, where  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a  $C^1$ -function. Choosing  $A(\lambda) = |\lambda|^{p-2}\lambda$  we deal with the usual  $p$ -Laplacian.

We restrict ourselves to  $A$ 's such that  $A(\lambda) = B(|\lambda|)\lambda$ ,  $\lambda \in \mathbb{R}^n$ , and we set

$$\bar{A}(s) = B(s)s^2, \quad \text{where } s \in [0, \infty). \tag{2.1}$$

We assume that  $\bar{A}$  is a Young function, i.e. it is an integral of nonzero, nonnegative nondecreasing function. In particular it is increasing, convex, and  $\lim_{s \rightarrow 0} \bar{A}(s) = 0$ . We refer to the monographs [55,66] for basic properties of Orlicz spaces. By  $\bar{A}^*$  we denote the Legendre transform of  $\bar{A}$ , i.e.  $\bar{A}^* = \sup_{t>0}(st - \bar{A}(t))$ .

As usual,  $C^k(\Omega)$  (respectively  $C_0^k(\Omega)$ ) denotes functions of class  $C^k$  defined on an open set  $\Omega \subset \mathbb{R}^n$  (respectively  $C^k$ -functions on  $\Omega$  with compact support). If  $f$  is defined on  $\Omega$ , by  $f\chi_\Omega$  we understand function  $f$  extended by 0 outside  $\Omega$ . When  $V \subseteq \mathbb{R}^n$ , by  $|V|$  we denote its Lebesgue's measure. Having an arbitrary  $u \in L^1_{loc}(\Omega)$  it is possible to define its value at every point by the formula

$$u(x) := \limsup_{r \rightarrow 0} \int_{B(x,r) \cap \Omega} u(y)dy. \tag{2.2}$$

We write  $f \sim g$  if function  $f$  is comparable with function  $g$ , i.e. if there exist positive constants  $c_1, c_2$  such that for every  $x$

$$c_1g(x) \leq f(x) \leq c_2g(x).$$

We deal with  $\Delta_2$  and  $\Delta'$  conditions defined below.

**Definition 2.1.** We say that the function  $F : [0, \infty) \rightarrow [0, \infty)$  satisfies the  $\Delta_2$ -condition (denoted  $F \in \Delta_2$ ), if there exists a constant  $\bar{C}_F > 0$  such that for every  $s > 0$  we have

$$F(2s) \leq \bar{C}_F F(s). \tag{2.3}$$

**Definition 2.2.** We say that the function  $F : [0, \infty) \rightarrow [0, \infty)$  satisfies the  $\Delta'$ -condition (denoted  $F \in \Delta'$ ), if there exists a constant  $C_F > 0$  such that for every  $s_1, s_2 > 0$  we have

$$F(s_1s_2) \leq C_F F(s_1)F(s_2). \tag{2.4}$$

**Remark 2.1.** Let us note that the  $\Delta'$ -condition is stronger than the  $\Delta_2$ -condition.

Typical examples of functions satisfying the  $\Delta'$ -condition can be found among Zygmund-type logarithmic functions. Their construction is based on the following easy observation.

**Fact 2.1.** [42] *The family of functions satisfying  $\Delta'$ -condition is invariant under multiplications and compositions.*

*Example 2.1.* [42] The following functions satisfy  $\Delta'$ -condition:

1.  $F(s) = s^p, 1 \leq p < \infty,$
2.  $M_{p,\alpha}(s) = s^p(\log(2 + s))^\alpha, 1 \leq p < \infty, \alpha \geq 0,$
3.  $M_{p,\alpha}^1(s) = s^p(\log(1 + s))^\alpha, 1 \leq p < \infty, \alpha \geq 0,$
4.  $F(s) = M_{p_1,\alpha_1} \circ M_{p_2,\alpha_2} \circ \dots \circ M_{p_k,\alpha_k}(s), \alpha_1, \dots, \alpha_k \geq 0, p_i \geq 1$  for  $i = 1, \dots, k.$

Let us state some useful facts and lemmas.

**Fact 2.2.** *Let  $F(s) = s^p \log^\alpha(b + s), b > 1, p \geq 1, \alpha > 0.$  Then  $F$  satisfies the  $\Delta'$ -condition (see Definition 2.2) and  $C_F \leq \left(\frac{2}{\log b}\right)^\alpha.$*

*Proof.* Suppose  $0 < s_1 \leq s_2.$  Then

$$\begin{aligned} \log(b + s_1 s_2) &\leq \log(b + s_2^2) \leq \log(b + s_2)^2 \\ &= 2 \log(b + s_2) \leq 2 \log(b + s_2) \cdot \frac{\log(b + s_1)}{\log b} \end{aligned}$$

and

$$\begin{aligned} F(s_1 s_2) &= (s_1 s_2)^p \log^\alpha(b + s_1 s_2) \leq \left(\frac{2}{\log b}\right)^\alpha s_1^p s_2^p \log^\alpha(b + s_1) \log^\alpha(b + s_2) \\ &= \left(\frac{2}{\log b}\right)^\alpha F(s_1) F(s_2). \end{aligned}$$

□

The following lemma comes from [44] (Lemma 4.2 therein), where the authors require  $F$  to be an  $N$ -function, whereas its proof holds for every Young function as well.

**Lemma 2.1.** *Suppose that  $F$  is a differentiable Young function satisfying  $\Delta_2$ -condition. Then there exists constants  $1 \leq d_F \leq D_F,$  such that for every  $r > 0$*

$$d_F \frac{F(r)}{r} \leq F'(r) \leq D_F \frac{F(r)}{r}. \tag{2.5}$$

Moreover for every  $r, s > 0$  the following estimate holds true

$$\frac{F(r)}{r} s \leq \frac{D_F - 1}{d_F} F(r) + \frac{1}{d_F} F(s). \tag{2.6}$$

**Remark 2.2.** We have the following observations.

1. When  $F(r) = r^p, \frac{1}{p} + \frac{1}{p'} = 1,$  we get  $r^{p-1} s \leq \frac{1}{p'} r^p + \frac{1}{p} s^p,$  equivalent to Young inequality  $qs \leq \frac{q^{p'}}{p'} + \frac{s^p}{p}.$
2. For general convex function  $F$  the latter inequality in (2.5) with finite constant  $D_F$  is equivalent to  $F \in \Delta_2,$  while the condition  $d_F > 1$  is equivalent to  $F^* \in \Delta_2$  (see [55], Theorem 4.3, or [43], Proposition 4.1). If  $d_F$  and  $D_F$  are the best possible in (2.5), they are called Simonenko lower and upper index of  $F,$  respectively (see e.g. [11, 30, 37, 67]) for definition and discussion of their properties.

**Fact 2.3.** *Let  $F(s) = s^p \log^\alpha(b + s)$ ,  $b > 1$ ,  $p \geq 1$ ,  $\alpha \geq 0$ . Then the constants in (2.5) satisfy  $D_F \leq p + \frac{\alpha}{\log b}$  and  $d_F = p$ .*

*Proof.*  $F'(s) = (s^p \log^\alpha(b + s))' = ps^{p-1} \log^\alpha(b + s) + \alpha \frac{s^p}{b+s} \log^{\alpha-1}(b + s)$   
 $= s^{p-1} \log^\alpha(b + s) \left( p + \alpha \frac{s}{(b+s) \log(b+s)} \right) \leq \frac{F(s)}{s} \left( p + \frac{\alpha}{\log(b+s)} \right) \leq \left( p + \frac{\alpha}{\log b} \right) \frac{F(s)}{s}$ ,  
 thus  $D_F \leq p + \frac{\alpha}{\log b}$ . Moreover,  $F'(s) \geq d_F \frac{F(s)}{s}$ , with  
 $d_F = \inf_{s>0} \left( p + \alpha \frac{s}{(b+s) \log(b+s)} \right)$ . □

**Orlicz–Sobolev spaces**

By  $W^{1,\bar{A}}(\Omega)$  we mean the completion of the set

$$\{u \in C^\infty(\Omega) : \|u\|_{W^{1,\bar{A}}(\Omega)} := \|u\|_{L^{\bar{A}}(\Omega)} + \|\nabla u\|_{L^{\bar{A}}(\Omega)} < \infty\},$$

under the Luxemburg norm

$$\|f\|_{L^{\bar{A}}(\Omega)} = \inf \left\{ K > 0 : \int_\Omega \bar{A} \left( \frac{|f(x)|}{K} \right) dx \leq 1 \right\}$$

(in the sequel we assume that  $\inf \emptyset = +\infty$ ). By  $W_{loc}^{1,\bar{A}}(\Omega)$  we denote such functions  $u : \Omega \rightarrow \mathbb{R}$  that  $u\phi \in W^{1,\bar{A}}(\Omega)$  for every  $\phi \in C_0^1(\Omega)$  (analogous notation is used for local Orlicz spaces  $L_{loc}^{\bar{A}}(\Omega)$ ). Observe that we always have  $W_{loc}^{1,\bar{A}}(\Omega) \subseteq W_{loc}^{1,1}(\Omega)$ . By  $W_0^{1,\bar{A}}(\Omega)$  we denote the completion of smooth compactly supported functions in  $W^{1,\bar{A}}(\Omega)$ .

The following fact holds true.

**Fact 2.4.** ([47], Fact 2.3) *If  $\bar{A}$  is a Young function and  $u \in W_{loc}^{1,\bar{A}}(\Omega)$ , then*

$$B(|\nabla u|)\nabla u = \frac{\bar{A}(|\nabla u|)}{|\nabla u|} \chi_{\{|\nabla u| \neq 0\}} \in L_{loc}^{\bar{A}^*}(\Omega, \mathbb{R}^n),$$

where  $B$  and  $\bar{A}$  are the same as in (2.1).

**Remark 2.3.** Despite the formulation given in [47] involves  $N$ -functions instead of Young functions, the proof therein works for Young functions as well.

Let  $u \in W_{loc}^{1,\bar{A}}(\Omega)$ . For  $w \in W^{1,\bar{A}}(\Omega)$  with compact support we define

$$\langle \Delta_A u, w \rangle := - \int_\Omega B(|\nabla u|) \langle \nabla u, \nabla w \rangle dx. \tag{2.7}$$

According to Fact 2.4 the right-hand side in (2.7) is well defined.

**Differential inequality**

The differential inequality we want to analyze is given by the following definition.

**Definition 2.3.** Let  $\Omega$  be any open subset of  $\mathbb{R}^n$  and  $\Phi$  be the locally integrable function defined in  $\Omega$ , such that for every nonnegative compactly supported  $w \in W^{1,\bar{A}}(\Omega)$

$$\left| \int_\Omega \Phi w dx \right| < \infty. \tag{2.8}$$

Let  $u \in W_{loc}^{1,\bar{A}}(\Omega)$ . We will say that

$$-\Delta_A u \geq \Phi \tag{2.9}$$

if for every nonnegative compactly supported  $w \in W^{1,\bar{A}}(\Omega)$  we have

$$\langle -\Delta_A u, w \rangle = \int_{\Omega} B(|\nabla u|) \langle \nabla u, \nabla w \rangle dx \geq \int_{\Omega} \Phi w dx. \tag{2.10}$$

**Remark 2.4.** We may choose  $\Phi = \Phi(x, u, \nabla u)$ .

**Set of assumptions.**

In the sequel we will consider functions satisfying the following assumptions.

( $\bar{A}$ )  $\bar{A}$  is a Young function satisfying  $\Delta'$ -condition;

( $\Psi$ ) there exists a function  $\Psi : [0, \infty) \rightarrow [0, \infty)$ , which is nonnegative, belongs to  $C^1((0, \infty))$ , and satisfies the following conditions

(i) inequality

$$g(t)\Psi'(t) \leq -C\Psi(t) \tag{2.11}$$

holds for all  $t > 0$  with  $C > 0$  independent of  $t$  and certain continuous function  $g : (0, \infty) \rightarrow (0, \infty)$ , such that  $\Psi(t)/g(t)$  is nonincreasing.

(ii) function

$$s \mapsto \Theta(s) := \frac{\bar{A}(g(s))\Psi(s)}{g(s)} \tag{2.12}$$

is nonincreasing or bounded in certain neighbourhood of 0.

(u)  $u \in W_{loc}^{1,\bar{A}}(\Omega)$  is a given nonnegative solution to (2.9) which is nontrivial, i.e.  $u \not\equiv const$ , and there exists  $\sigma \in \mathbb{R}$  such that

$$\Phi + \sigma \frac{\bar{A}(|\nabla u|)}{g(u)} \chi_{\{\nabla u \neq 0\}} \geq 0 \quad \text{a.e.} \tag{2.13}$$

We define

$$\sigma_0 = \inf\{\sigma \in \mathbb{R} : (2.13) \text{ is satisfied}\}, \tag{2.14}$$

recalling that  $\inf \emptyset = +\infty$ .

**Remark 2.5.** Examples when those conditions are satisfied in the case  $\bar{A}(s) = s^p$ ,  $p > 1$ ,  $g(s) = s$ ,  $\Psi(s) = s^{-\beta}$ ,  $\beta > 0$  can be found in [68, 69].

**Remark 2.6.** Let us discuss the assumption ( $\Psi$ ) i). It implies that  $\Psi$  is decreasing. Elementary calculation confirms that pairs of  $\Psi$  and  $g$  from Table 1 satisfy condition  $g(t)\Psi'(t) \leq -C\Psi(t)$  a.e.

To ensure that additionally  $\Psi(t)/g(t)$  is nonincreasing we have to assume that  $g'(t) \geq -C$  with  $C$  as in (2.11). Indeed,  $\Psi/g$  is nonincreasing because

$$\begin{aligned} \left(\frac{\Psi(t)}{g(t)}\right)' &= \frac{\Psi'(t)g(t) - \Psi(t)g'(t)}{g^2(t)} \leq \frac{-C\Psi(t) - \Psi(t)g'(t)}{g^2(t)} \\ &= -\frac{\Psi(t)}{g^2(t)}(C + g'(t)) \leq 0, \end{aligned}$$

This condition is also satisfied by pairs from Table 1.

TABLE 1. Good pairs of  $\Psi$  and  $g$

$\Psi(t)$	$g(t)$	$C$	Remarks
$t^{-\alpha}$	$t$	$\alpha$	$\alpha > 0$
$(t \log(a + t))^{-1}$	$t \log(a + t)$	$\log a$	$a > 1$
$e^{-t}$	Bounded by $C, g' \geq -C$	$C$	$C > 0$
$e^{-t}/t$	$t/(1 + t)$	1	–
$e^{\frac{1}{2} \log^2(t)}$	$t/ \log t $	1	Considered on $(0, 1)$

### 3. Caccioppoli estimates for solutions to $-\Delta_A u \geq \Phi$

Our main goal in this section is to obtain the following result.

**Theorem 3.1.** *Let  $u \in W_{loc}^{1, \bar{A}}(\Omega)$  be a nonnegative solution to PDI  $-\Delta_A u \geq \Phi$ , in the sense of Definition 2.3, where  $\Phi$  is locally integrable and assumptions  $(\bar{A}), (\Psi), (u)$  are satisfied with  $C > 0$  and  $\sigma \in [\sigma_0, C)$ , where  $\sigma_0$  is given by (2.14). Let  $C_{\bar{A}} > 0$  be a constant coming from  $\Delta'$ -condition for  $\bar{A}$  (see Definition 2.2) and  $D_{\bar{A}} \geq d_{\bar{A}} \geq 1, D_{\bar{A}} > 1$  be constants coming from (2.5) applied to  $\bar{A}$ .*

*Then the inequality*

$$\int_{\Omega} \left( \Phi + \sigma \frac{\bar{A}(|\nabla u|)}{g(u)} \chi_{\{\nabla u \neq 0\}} \right) \Psi(u) \phi \, dx \leq K \int_{\Omega \cap \{\nabla u \neq 0\} \cap \text{supp} \phi} \frac{\bar{A}(g(u)) \Psi(u)}{g(u)} \cdot \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx, \tag{3.1}$$

*holds for every nonnegative Lipschitz function  $\phi$  with compact support in  $\Omega$ , such that the integral  $\int_{\text{supp} \phi \cap \{\nabla u \neq 0\}} \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx$  is finite and  $K = (C - \sigma) \bar{A} \left( \frac{D_{\bar{A}} - 1}{(C - \sigma) d_{\bar{A}}} \right) \frac{C_{\bar{A}}^2}{D_{\bar{A}} - 1}$ .*

We call (3.1) Caccioppoli inequality because it involves  $\nabla u$  on the left-hand side and only  $u$  on the right-hand side (see e.g. [18, 41]).

We note that we do not assume that the right-hand side in (3.1) is finite.

The proof is based on the idea of the proof of Theorem 3.1 from [68] inspired by the proof of Proposition 3.1 from [47].

*Proof of Theorem 3.1* The proof follows by three steps.

STEP 1. DERIVATION OF LOCAL INEQUALITY.

We obtain the following lemma.

**Lemma 3.1.** *Let  $u \in W_{loc}^{1, \bar{A}}(\Omega)$  be a nonnegative solution to PDI  $-\Delta_A u \geq \Phi$ , in the sense of Definition 2.3, where  $\Phi$  is locally integrable and assumptions  $(\bar{A}), (\Psi), (u)$  are satisfied with  $C > 0$  and  $\sigma \in [\sigma_0, C)$ , where  $\sigma_0$  is given by (2.14). Let  $K$  be the constant from Theorem 3.1.*



Then for every  $0 < \delta < R$  and every nonnegative Lipschitz function  $\phi$  with compact support in  $\Omega$  the inequality

$$\begin{aligned} & \int_{\{u \leq R-\delta\}} \left( \Phi + \sigma \frac{\bar{A}(|\nabla u|)}{g(u+\delta)} \chi_{\{\nabla u \neq 0\}} \right) \Psi(u+\delta) \phi \, dx \tag{3.2} \\ & \leq K \int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} \Theta(u+\delta) \cdot \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx + \tilde{C}(\delta, R), \end{aligned}$$

holds with  $\Theta(u)$  given by (2.12) and

$$\tilde{C}(\delta, R) := \Psi(R) \left[ \int_{\Omega \cap \{\nabla u \neq 0, u > R-\delta\}} B(|\nabla u|) \langle \nabla u, \nabla \phi \rangle \, dx - \int_{\Omega \cap \{u > R-\delta\}} \Phi \phi \, dx \right]. \tag{3.3}$$

Before we prove the theorem let us formulate the following facts.

**Fact 3.1.** [47] For  $u, \phi$  as in the assumptions of Theorem 3.1 we fix  $0 < \delta < R$  and denote

$$u_{\delta,R}(x) := \min(u(x) + \delta, R), \quad G(x) := \Psi(u_{\delta,R}(x))\phi(x). \tag{3.4}$$

Then  $u_{\delta,R} \in W_{loc}^{1,\bar{A}}(\Omega)$  and  $G \in W_0^{1,\bar{A}}(\Omega) \subseteq W^{1,\bar{A}}(\Omega)$ .

**Fact 3.2.** [47] Let  $u \in W_{loc}^{1,1}(\Omega)$  be defined everywhere by the formula (2.2) and let  $t \in \mathbb{R}$ . Then

$$\{x \in \Omega : u(x) = t\} \subseteq \{x \in \Omega : \nabla u(x) = 0\} \cup N, \tag{3.5}$$

where  $|N| = 0$ .

*Proof of Lemma 3.1.*

Let us introduce some notation

$$\begin{aligned} \tilde{A}(\delta, R) &= \int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} \bar{A}(|\nabla u|) \Psi'(u+\delta) \phi \, dx, \\ \tilde{A}_1(\delta, R) &= \int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} \bar{A}(|\nabla u|) \left( \frac{\Psi(u+\delta)}{g(u+\delta)} \right) \phi \, dx, \\ \tilde{B}(\delta, R) &= \int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} B(|\nabla u|) \langle \nabla u, \nabla \phi \rangle \Psi(u+\delta) \, dx, \\ \tilde{C}_1(\delta, R) &= \Psi(R) \int_{\Omega \cap \{u > R-\delta\}} \Phi \phi \, dx, \tag{3.6} \end{aligned}$$

$$\tilde{C}_2(\delta, R) = \Psi(R) \int_{\Omega \cap \{\nabla u \neq 0, u > R-\delta\}} B(|\nabla u|) \langle \nabla u, \nabla \phi \rangle \, dx, \tag{3.7}$$

$$\tilde{D}(\bar{\epsilon}, \delta, R) = \bar{\epsilon} \bar{A} \left( \frac{1}{\bar{\epsilon}} \right) \frac{C_{\bar{A}}^2}{d_{\bar{A}}} \int_{\text{supp } \phi \cap \{\nabla u \neq 0, u \leq R-\delta\}} \Theta(u+\delta) \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx,$$

where  $\Theta(u)$  is given by (2.12). Let us consider  $u_{\delta,R}$  and  $G$  defined by (3.4).

We note that

$$\begin{aligned}
 I &:= \int_{\Omega} \Phi G \, dx = \int_{\Omega} \Phi \Psi(u_{\delta,R}) \phi \, dx \\
 &= \int_{\Omega \cap \{u \leq R-\delta\}} \Phi \Psi(u + \delta) \phi \, dx + \Psi(R) \int_{\Omega \cap \{u > R-\delta\}} \Phi \phi \, dx \\
 &= \int_{\Omega \cap \{u \leq R-\delta\}} \Phi \Psi(u + \delta) \phi \, dx + \tilde{C}_1(\delta, R).
 \end{aligned} \tag{3.8}$$

On the other hand, inequality (2.9) implies

$$\begin{aligned}
 I &:= \int_{\Omega} \Phi G \, dx \leq \langle -\Delta_A u, G \rangle = \int_{\Omega \cap \{\nabla u \neq 0\}} B(|\nabla u|) \langle \nabla u, \nabla G \rangle \, dx \\
 &= \int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} \bar{A}(|\nabla u|) \Psi'(u + \delta) \phi \, dx \\
 &\quad + \int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} B(|\nabla u|) \langle \nabla u, \nabla \phi \rangle \Psi(u + \delta) \, dx \\
 &\quad + \Psi(R) \int_{\Omega \cap \{\nabla u \neq 0, u > R-\delta\}} B(|\nabla u|) \langle \nabla u, \nabla \phi \rangle \, dx \\
 &= \tilde{A}(\delta, R) + \tilde{B}(\delta, R) + \tilde{C}_2(\delta, R).
 \end{aligned} \tag{3.9}$$

Note that all integrals above are finite, what follows from Fact 2.4 (for  $0 \leq u \leq R - \delta$  we have  $\delta \leq u + \delta \leq R$ ). Using assumption  $(\Psi)$  we get

$$\tilde{A}(\delta, R) \leq -C \int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} \bar{A}(|\nabla u|) \left( \frac{\Psi(u + \delta)}{g(u + \delta)} \right) \phi \, dx = -C \tilde{A}_1(\delta, R). \tag{3.10}$$

Moreover, for an arbitrary  $\bar{\epsilon} > 0$ ,

$$\begin{aligned}
 &\tilde{B}(\delta, R) \\
 &\leq \int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} B(|\nabla u|) |\nabla u| |\nabla \phi| \Psi(u + \delta) \, dx \\
 &= \bar{\epsilon} \int_{\text{supp } \phi \cap \{\nabla u \neq 0, u \leq R-\delta\}} (B(|\nabla u|) |\nabla u|) \cdot \left( \frac{|\nabla \phi|}{\phi} \frac{g(u + \delta)}{\bar{\epsilon}} \right) \left( \frac{\Psi(u + \delta)}{g(u + \delta)} \phi \right) \, dx.
 \end{aligned}$$

As  $B(|\nabla u|) |\nabla u| = \frac{\bar{A}(|\nabla u|)}{|\nabla u|}$ , we can apply (2.6) for the Young function  $\bar{A}$  with  $r = |\nabla u|$ ,  $s = \left( \frac{|\nabla \phi|}{\phi} \frac{g(u + \delta)}{\bar{\epsilon}} \right)$  to get

$$\begin{aligned}
 \tilde{B}(\delta, R) &\leq \frac{\bar{\epsilon} D_{\bar{A}} - 1}{d_{\bar{A}}} \int_{\text{supp } \phi \cap \{\nabla u \neq 0, u \leq R-\delta\}} \bar{A}(|\nabla u|) \frac{\Psi(u + \delta)}{g(u + \delta)} \phi \, dx \\
 &\quad + \frac{\bar{\epsilon}}{d_{\bar{A}}} \int_{\text{supp } \phi \cap \{\nabla u \neq 0, u \leq R-\delta\}} \bar{A} \left( \frac{|\nabla \phi|}{\phi} \frac{g(u + \delta)}{\bar{\epsilon}} \right) \frac{\Psi(u + \delta)}{g(u + \delta)} \phi \, dx.
 \end{aligned}$$

Then, applying  $\Delta'$ -condition for  $\bar{A}$  twice in the second expression above, we obtain

$$\tilde{B}(\delta, R) \leq \bar{\epsilon} \frac{D_{\bar{A}} - 1}{d_{\bar{A}}} \tilde{A}_1(\delta, R) + \tilde{D}(\bar{\epsilon}, \delta, R). \tag{3.11}$$

Combining estimates (3.9), (3.10) and (3.11) we get

$$\begin{aligned} I &\leq -C \tilde{A}_1(\delta, R) + \tilde{B}(\delta, R) + \tilde{C}_2(\delta, R) \\ &\leq \left( -C + \bar{\epsilon} \frac{D_{\bar{A}} - 1}{d_{\bar{A}}} \right) \tilde{A}_1(\delta, R) + \tilde{D}(\bar{\epsilon}, \delta, R) + \tilde{C}_2(\delta, R). \end{aligned}$$

Moreover,  $\tilde{C}_1(\delta, R)$  and  $\tilde{A}_1(\delta, R)$  are finite (and  $\tilde{D}(\epsilon, \delta, R)$  is finite as well). This and (3.8) imply

$$\begin{aligned} &\int_{\Omega \cap \{u \leq R - \delta\}} \Phi \Psi(u + \delta) \phi \, dx + \left( C - \bar{\epsilon} \frac{D_{\bar{A}} - 1}{d_{\bar{A}}} \right) \tilde{A}_1(\delta, R) \\ &\leq \tilde{D}(\bar{\epsilon}, \delta, R) + (\tilde{C}_2(\delta, R) - \tilde{C}_1(\delta, R)). \end{aligned}$$

This is (3.2). Indeed, we have  $\tilde{C}(\delta, R) = \tilde{C}_2(\delta, R) - \tilde{C}_1(\delta, R)$ . Moreover, when we substitute  $\sigma := C - \bar{\epsilon} \frac{D_{\bar{A}} - 1}{d_{\bar{A}}}$  we get

$$\begin{aligned} \bar{\epsilon} \bar{A} \left( \frac{1}{\bar{\epsilon}} \right) \frac{C_{\bar{A}}^2}{d_{\bar{A}}} &= \frac{(C - \sigma) d_{\bar{A}}}{D_{\bar{A}} - 1} \bar{A} \left( \frac{D_{\bar{A}} - 1}{(C - \sigma) d_{\bar{A}}} \right) \frac{C_{\bar{A}}^2}{d_{\bar{A}}} \\ &= \frac{(C - \sigma)}{D_{\bar{A}} - 1} \bar{A} \left( \frac{D_{\bar{A}} - 1}{(C - \sigma) d_{\bar{A}}} \right) C_{\bar{A}}^2 = K. \end{aligned}$$

We notice that  $\bar{\epsilon} > 0$  is arbitrary and we may always choose  $0 < \bar{\epsilon} \leq \frac{(C - \sigma_0) d_{\bar{A}}}{D_{\bar{A}} - 1}$ , so that  $\sigma_0 \leq \sigma < C$ .  $\square$

We have to introduce parameters  $\delta$  and  $R$  to make sure that some quantities in the estimates, which we move to opposite sides of inequalities, are finite.

STEP 2. PASSING TO THE LIMIT WITH  $\delta \searrow 0$ .

In this step we show that when assumptions  $(\bar{A})$ ,  $(\Psi)$  and  $(\Phi)$  are satisfied with  $\epsilon > 0$ ,  $K$  is the constant from Theorem 3.1, then for any  $R > 0$  inequality

$$\begin{aligned} &\int_{\{u \leq R\}} \left( \Phi + \sigma \frac{\bar{A}(|\nabla u|)}{g(u)} \chi_{\{\nabla u \neq 0\}} \right) \Psi(u) \phi \, dx \tag{3.12} \\ &\leq K \int_{\{\nabla u \neq 0, u \leq R\}} \frac{\bar{A}(g(u)) \Psi(u)}{g(u)} \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx + \tilde{C}(R), \end{aligned}$$

where

$$\tilde{C}(R) = \Psi(R) \left[ \int_{\Omega \cap \{u \geq \frac{R}{2}\}} B(|\nabla u|) |\nabla u| \cdot |\nabla \phi| \, dx + \int_{\Omega \cap \{u \geq \frac{R}{2}\}} \Phi \phi \, dx \right] \tag{3.13}$$

holds for every nonnegative Lipschitz function  $\phi$  with compact support in  $\Omega$ , such that the integral  $\int_{\text{supp } \phi \cap \nabla u \neq 0} \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx$  is finite. Moreover, all quantities appearing in (3.12) are finite.

For this, we show first that under our assumptions when  $\delta \searrow 0$  we have

$$\begin{aligned} & \int_{\Omega \cap \{\nabla u \neq 0, u + \delta \leq R\}} \Theta(u + \delta) \cdot \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx \\ & \rightarrow \int_{\Omega \cap \{\nabla u \neq 0, u \leq R\}} \Theta(u) \cdot \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx. \end{aligned} \tag{3.14}$$

Note that  $\Theta(u + \delta)\chi_{u + \delta \leq R} \xrightarrow{\delta \rightarrow 0} \Theta(u)\chi_{u \leq R}$ , a.e. This follows from Lemma 3.2 (which gives that the sets  $\{u = 0, |\nabla u| \neq 0\}$  and  $\{u = R, |\nabla u| = 0\}$  are of measure zero) and the continuity outside zero of the involved functions.

We assumed in  $(\Theta)$  that  $\Theta$  is nonincreasing or bounded in the neighbourhood of zero. We start with the case when there exists  $\kappa > 0$  such that for  $\lambda < \kappa$  the function  $\Theta(\lambda)$  is nonincreasing. Without loss of generality we may consider  $\kappa \leq R$ .

We divide the domain of integration

$$\begin{aligned} & \int_{\Omega \cap \{\nabla u \neq 0, u + \delta \leq R\}} \Theta(u + \delta) \cdot \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx \\ & = \int_{E_\kappa} \Theta(u + \delta) \cdot \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx + \int_{F_\kappa} \Theta(u + \delta)\chi_{\{u + \delta \leq R\}} \cdot \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx, \end{aligned}$$

where

$$E_\kappa = \left\{ u < \frac{\kappa}{2}, \nabla u \neq 0 \right\} \cap \text{supp } \phi, \quad F_\kappa = \left\{ \frac{\kappa}{2} \leq u, \nabla u \neq 0 \right\} \cap \text{supp } \phi.$$

Let us begin with integral over  $E_\kappa$ . We consider  $\delta \rightarrow 0$ , so we may assume that  $\delta < \kappa/2$ . Then for  $x \in E_\kappa$  we have  $u + \delta < \kappa$ . As function  $\lambda \rightarrow \Theta(\lambda)$  is nonincreasing when  $\lambda < \kappa$ , thus for fixed  $u$  and  $\delta \searrow 0$  the function  $\delta \rightarrow \Theta(u + \delta)$  is nondecreasing and so convergent monotonically almost everywhere to  $\Theta(u)$ . Therefore, due to the Lebesgue's Monotone Convergence Theorem

$$\lim_{\delta \rightarrow 0} \int_{E_\kappa} \Theta(u + \delta) \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx = \int_{E_\kappa} \Theta(u) \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx.$$

In the case of  $F_\kappa$  we have  $\kappa/2 \leq u + \delta \leq R$ . Over this domain  $\Theta$  is a bounded function, so in particular on  $F_\kappa$

$$\Theta(u + \delta)\chi_{\{u + \delta \leq R\}} \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \leq \sup_{t \in [\kappa/2, R]} \Theta(t) \cdot \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \in L^1(F_\kappa).$$

We apply the Lebesgue's Dominated Convergence Theorem to deduce that

$$\lim_{\delta \rightarrow 0} \int_{F_\kappa} \Theta(u + \delta)\chi_{\{u + \delta \leq R\}} \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx = \int_{F_\kappa \cap \{u \leq R\}} \Theta(u) \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx.$$

This completes the case of  $\Theta$  nonincreasing in the neighbourhood of 0.

In the case when  $\Theta$  is bounded in the neighbourhood of 0, we note that  $\Theta$  is bounded also on every interval  $[0, R]$ , where  $R > 0$ . Hence, we can use previous computations dealing with  $F_\kappa$  in case  $\kappa = 0$ .

To finish the proof of this step we note that (3.14) says that when  $\delta \searrow 0$  the first integral on the right-hand side of (3.2) is convergent to the first integral on the right-hand side of (3.12). To deal with the second expression we note that for  $\delta \leq \frac{R}{2}$

$$|\tilde{C}(\delta, R)| \leq |\tilde{C}_2(\delta, R)| + |\tilde{C}_1(\delta, R)| \leq \tilde{C}(R),$$

where  $\tilde{C}(\delta, R)$ ,  $\tilde{C}_2(\delta, R)$ ,  $\tilde{C}_1(\delta, R)$ ,  $\tilde{C}(R)$  are given by (3.3), (3.6), (3.7), (3.13), respectively.

We can pass to the limit with  $\delta \rightarrow 0$  on the left-hand side of (3.2) due to the Lebesgue’s Monotone Convergence Theorem as an expression in brackets is nonnegative by (2.13) and the whole integrand therein is nonincreasing by assumption  $(\Psi)$ .

STEP 3. WE LET  $R \rightarrow \infty$  AND FINISH THE PROOF.

We are going to let  $R \rightarrow \infty$  in (3.12). Without loss of generality we can assume that the integral in the right-hand side of (3.1) is finite, as otherwise the inequality follows trivially. Note that as  $B(|\nabla u|)\langle \nabla u, \nabla \phi \rangle$  and  $\Phi \phi$  are integrable, we have  $\lim_{R \rightarrow \infty} \tilde{C}(R) = 0$ . Therefore, (3.1) follows from (3.12) by the Lebesgue’s Monotone Convergence Theorem.  $\square$

### 4. Hardy type inequalities

Our most general conclusion resulting from Theorem 3.1 reads as follows.

**Theorem 4.1.** *Let  $u \in W_{loc}^{1,\bar{A}}(\Omega)$  be a nonnegative solution to PDI  $-\Delta_{\bar{A}}u \geq \Phi$ , in the sense of Definition 2.3, where  $\Phi$  is locally integrable and assumptions  $(\bar{A})$ ,  $(\Psi)$ ,  $(u)$  are satisfied with  $C > 0$  and  $\sigma \in [\sigma_0, C)$ , where  $\sigma_0$  is given by (2.14) (in the case  $\sigma_0 = -\infty$  we assume  $\sigma \in (-\infty, C)$ ). Set*

$$F_{\bar{A}}(\lambda) = \frac{1}{\bar{A}(1/\lambda)}, \text{ when } \lambda > 0 \text{ and } F_{\bar{A}}(0) = 0. \tag{4.1}$$

Then for every Lipschitz function  $\xi$  with compact support in  $\Omega$  we have

$$\int_{\Omega} F_{\bar{A}}(|\xi|)\mu_1(dx) \leq \tilde{C} \int_{\Omega} \bar{A}(|\nabla \xi|)\mu_2(dx). \tag{4.2}$$

where

$$\mu_1(dx) = \Psi(u) \left[ \Phi + \sigma \frac{\bar{A}(|\nabla u|)}{g(u)} \right] \chi_{\{u>0\}} dx, \tag{4.3}$$

$$\mu_2(dx) = \frac{\bar{A}(g(u))\Psi(u)}{g(u)} \chi_{\{\nabla u \neq 0\}} dx, \tag{4.4}$$

$$\tilde{C} = (C - \sigma)\bar{A} \left( \frac{D_{\bar{A}} - 1}{(C - \sigma)d_{\bar{A}}} \right) \frac{\bar{A}(D_{\bar{A}})C_{\bar{A}}^4}{D_{\bar{A}} - 1}. \tag{4.5}$$

with the constants  $C_{\bar{A}} > 0$  coming from  $\Delta'$ -condition for  $\bar{A}$  (see Definition 2.2) and  $D_{\bar{A}} \geq d_{\bar{A}} \geq 1$ ,  $D_{\bar{A}} > 1$  coming from (2.5) applied to  $\bar{A}$ .

*Proof.* Let  $\xi$  be a compactly supported Lipschitz function. We define  $\phi = F_{\bar{A}}(\xi)$  and apply Theorem 3.1. For this we have to verify that  $\phi$  is compactly supported Lipschitz function and  $\int_{\Omega \cap \text{supp } \phi} \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx < \infty$ . We observe that  $\phi$  is compactly supported, because  $F_{\bar{A}}(t)$  is continuous at 0. Indeed,

$$\lim_{t \rightarrow 0} F_{\bar{A}}(t) = \lim_{t \rightarrow 0} \frac{1}{\bar{A}(1/t)} = \lim_{s \rightarrow \infty} \frac{1}{\bar{A}(s)} = 0,$$

which ensures that  $\text{supp } \phi = \text{supp } \xi$ . This holds because as a Young function  $\bar{A}$  is superlinear and we have  $\lim_{s \rightarrow \infty} \bar{A}(s) = \infty$ .

Furthermore,  $F_{\bar{A}}(t)$  is a locally Lipschitz function. We obtain it from Lemma 2.1 which implies

$$F'_{\bar{A}}(t) = \left( \frac{1}{\bar{A}(1/t)} \right)' \sim \frac{1}{t \bar{A}(1/t)}.$$

The last term above is bounded for bounded  $t$ . Indeed, when  $t \leq C_0$ , we have  $\frac{1}{t} \geq \frac{1}{C_0}$ , and so

$$\frac{\bar{A}(1/t)}{1/t} \geq \frac{\bar{A}(1/C_0)}{1/C_0}.$$

Therefore,  $F_{\bar{A}}(t)$  is locally Lipschitz. The composition of locally Lipschitz function  $F_{\bar{A}}(t)$  with Lipschitz and bounded  $\xi$ , i.e.  $F_{\bar{A}}(\xi) = \phi$ , is Lipschitz.

We verify that  $\int_{\Omega \cap \text{supp } \phi} \bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx < \infty$ . Note that for every compactly supported Lipschitz function  $\xi$  we have  $\int_{\Omega} \bar{A}(|\nabla \xi|) \, dx < \infty$ . Therefore, it suffices to prove that

$$\bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \leq C_{\bar{A}}^2 \bar{A}(D_{\bar{A}}) \bar{A}(|\nabla \xi|). \tag{4.6}$$

Let  $\bar{A}^{-1}$  be the inverse function of  $\bar{A}$ . As  $\bar{A} \in \Delta'$  we note that for each pair of  $x, y \geq 0$  we have

$$\begin{aligned} \bar{A}(x)y &= \bar{A} \left( \frac{x}{\bar{A}^{-1}(\frac{1}{y})} \bar{A}^{-1} \left( \frac{1}{y} \right) \right) y \\ &\leq C_{\bar{A}} \bar{A} \left( \frac{x}{\bar{A}^{-1}(\frac{1}{y})} \right) \bar{A} \left( \bar{A}^{-1} \left( \frac{1}{y} \right) \right) y = C_{\bar{A}} \bar{A} \left( \frac{x}{\bar{A}^{-1}(\frac{1}{y})} \right). \end{aligned} \tag{4.7}$$

Hence, taking  $x = \frac{|\nabla \phi|}{\phi}$  and  $y = \phi$ , we obtain from (4.7)

$$\bar{A} \left( \frac{|\nabla \phi|}{\phi} \right) \phi \leq C_{\bar{A}} \bar{A} \left( \frac{|\nabla \phi|}{\phi} \frac{1}{\bar{A}^{-1}(\frac{1}{\phi})} \right) \tag{4.8}$$

on every  $x$  where  $\phi(x) > 0$ .

Now we show that on every  $x$  where  $\phi(x) > 0$  we have

$$\frac{|\nabla \phi(x)|}{\phi(x)} \frac{1}{\bar{A}^{-1} \left( \frac{1}{\phi(x)} \right)} \leq D_{\bar{A}} |\nabla \xi(x)|. \tag{4.9}$$

Indeed, we have  $\phi = \frac{1}{\bar{A}(1/\xi)}$ , so that

$$\nabla\phi = F'_{\bar{A}}(\xi) = -\frac{1}{\bar{A}^2\left(\frac{1}{\xi}\right)}\bar{A}'\left(\frac{1}{\xi}\right)\left(-\frac{1}{\xi^2}\right)\nabla\xi.$$

Applying (2.5) to  $\bar{A} \in \Delta_2$  we have  $\bar{A}'(\lambda) \leq D_{\bar{A}}\frac{\bar{A}(\lambda)}{\lambda}$  with the constant  $D_{\bar{A}}$ . Therefore

$$|\nabla\phi| \leq \frac{1}{\bar{A}^2\left(\frac{1}{\xi}\right)}D_{\bar{A}}\bar{A}\left(\frac{1}{\xi}\right)\frac{|\nabla\xi|}{\xi} = D_{\bar{A}}\phi\frac{|\nabla\xi|}{\xi}.$$

Hence, we have  $\frac{|\nabla\phi|}{\phi}\xi \leq D_{\bar{A}}|\nabla\xi|$ , which is exactly (4.9).

Summing up the estimates (4.8) and (4.9) we obtain (4.6)

$$\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi \leq C_{\bar{A}}\bar{A}\left(\frac{|\nabla\phi|}{\phi}\frac{1}{\bar{A}^{-1}\left(\frac{1}{\phi}\right)}\right) \leq C_{\bar{A}}\bar{A}(D_{\bar{A}}|\nabla\xi|) \leq C_{\bar{A}}^2\bar{A}(D_{\bar{A}})\bar{A}(|\nabla\xi|).$$

Thus the assumptions of Theorem 3.1 are satisfied and we obtain (3.1). The substitution  $\phi = F_{\bar{A}}(\xi)$ , equivalently taking

$$\xi(x) = \begin{cases} \frac{1}{\bar{A}^{-1}(1/\phi(x))}, & \text{when } \phi(x) \neq 0, \\ 0, & \text{when } \phi(x) = 0, \end{cases}$$

transforms the left-hand side of (3.1) into the left-hand side of (4.2). What remains to show is that the right-hand side in (3.1) is estimated as follows

$$\begin{aligned} & \int_{\{\nabla u \neq 0\}} \frac{\bar{A}(g(u))\Psi(u)}{g(u)}\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi \, dx \\ & \leq C_{\bar{A}}^2\bar{A}(D_{\bar{A}})\int_{\{\nabla u \neq 0\}} \frac{\bar{A}(g(u))\Psi(u)}{g(u)}\bar{A}(|\nabla\xi|) \, dx. \end{aligned}$$

This is a direct consequence of (4.6). The proof is complete. □

Examples dealing with various  $\bar{A}$ ,  $\Psi$  and  $g$  are given in the following sections.

### 5. Links with existing results

In this section we present how our result is related to several other ones.

#### Results of Cianchi [19]

In paper by Cianchi [19] one finds necessary and sufficient conditions for the Hardy inequality

$$\left\| \frac{\xi(x)}{d^{1+\theta}(x)} \right\|_{L^B(\Omega)} \leq C \left\| \frac{\nabla\xi(x)}{d^\theta(x)} \right\|_{L^A(\Omega)}, \tag{5.1}$$

where  $d = \text{dist}(x, \partial\Omega)$  is distance from the boundary of bounded Lipschitz-continuous boundary domain  $\Omega$  and  $\xi$  is sufficiently smooth function with compact support in  $\Omega$ . Note that (5.1) involves the internal measures:  $1/d^{1+\theta}$  and  $1/d^\theta$ , while in our Theorem 4.1 the internal measures are trivial. On the other hand, in Cianchi's inequality (5.1) the external measure on the both

sides of the inequality is the Lebesgue’s measure, while in our case the external measures on the both sides of the inequality are different and nontrivial. It would be interesting to generalize both issues to the inequality involving the nontrivial external and internal measures.

**Results of Buckley and Hurri–Syrjänen [14] and Buckley and Koskela [15]**

In [14] the authors consider inequality

$$\int_{\Omega} \Psi \left( \frac{|\xi(x)|}{d^{1+\theta}(x)} \right) dx \leq C \int_{\Omega} \Psi \left( \frac{|\nabla \xi(x)|}{d^{\theta}(x)} \right) dx, \tag{5.2}$$

where  $d(x)$  is distance from the boundary of Lipschitz domain  $\Omega$  and  $\xi$  is sufficiently smooth function with compact support in  $\Omega \subseteq \mathbb{R}^n$ . It is assumed that complement of  $\Omega$  satisfies the fatness condition, while  $\Psi$  belongs to the class of functions called  $G(p, q, C)$ . The typical representants of such functions are power-logarithmic type functions like e.g.  $\Psi(t) = t^p \log_+^{\alpha} t$ ,  $\Psi_a(t) = t^p \log^{\alpha}(a + t)$ ,  $p, a > 1, \alpha \geq 0$ .

The case of (5.2) with  $\theta = 1$  is considered in [15].

Let us mention that when one considers Theorem 4.1 with

$$\bar{A}(t) = \Psi_a(t) = t^p \log^{\alpha}(a + t), \quad p > 1, a > e, \alpha > 0,$$

we get  $F_{\bar{A}}(t) = t^p \log^{-\alpha}(a + 1/t) \not\sim \bar{A}(t)$ . Thus our inequality (4.2) cannot be compared directly with (5.2). Furthermore, we deal also with the weight functions outside  $F_{\bar{A}}$ .

**Stein LLogL-type results [71]**

When we consider  $\bar{A}(t) = t \log^{\alpha}(e + t)$ , we obtain the following corollary.

**Corollary 5.1.** *Suppose  $\bar{A}(t) = t \log^{\alpha}(a + t)$ ,  $\alpha > 0$ ,  $a > 1$ . Let  $u \in W_{loc}^{1, \bar{A}}(\Omega)$  be a nonnegative solution to PDI  $-\Delta_A u \geq \Phi$ , in the sense of Definition 2.3, where  $\Phi$  is locally integrable and assumptions  $(\Psi)$ ,  $(u)$  are satisfied with  $C > 0$  and  $\sigma \in [\sigma_0, C)$ , where  $\sigma_0$  is given by (2.14) (in the case  $\sigma_0 = -\infty$  we assume  $\sigma \in (-\infty, C)$ ).*

*Then for every Lipschitz function  $\xi$  with compact support in  $\Omega$  we have*

$$\int_{\Omega} \frac{|\xi|}{\log^{\alpha}(a + 1/|\xi|)} \mu_1(dx) \leq \tilde{C} \int_{\Omega} |\nabla \xi| \log^{\alpha}(a + |\nabla \xi|) \mu_2(dx). \tag{5.3}$$

where

$$\mu_1(dx) = \Psi(u) \left[ \Phi + \sigma \frac{|\nabla u| \log^{\alpha}(e + |\nabla u|)}{g(u)} \right] \chi_{\{u>0\}} dx, \tag{5.4}$$

$$\mu_2(dx) = \log^{\alpha}(e + g(u)) \Psi(u) \chi_{\{\nabla u \neq 0\}} dx \tag{5.5}$$

*Proof.* We apply Theorem 4.1 with  $\bar{A}(t) = t \log^{\alpha}(a + t)$ . We note that according to (4.1) we have

$$F_{\bar{A}}(\lambda) = \frac{\lambda}{\log^{\alpha}(a + 1/\lambda)}, \text{ when } \lambda > 0 \text{ and } F_{\bar{A}}(0) = 0. \tag{5.6}$$

For the constant estimate (4.5) we note that if  $\bar{A}(t) = t \log^{\alpha}(a + t)$ ,  $\alpha > 0$ ,  $a > 1$ , then  $C_{\bar{A}} \leq (2/\log a)^{\alpha}$ ,  $d_{\bar{A}} = 1$  and  $D_{\bar{A}} \leq 1 + \alpha/\log a$ .  $\square$



The reverse Stein inequality has the form [50, 70]

$$\int_B (M\phi)(x)dx \leq c_\phi + c \int_{\mathbb{R}^n} |\phi| \log^+ |\phi| dx,$$

and deals with measurable functions  $\phi$  supported in some ball  $B \subseteq \mathbb{R}^n$ , where  $\log^+(\lambda) = \log(\lambda)\chi_{\lambda>1}$ ,  $M\phi(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |\phi(y)| dy$ , where supremum is taken over all balls containing  $x$ , is the Hardy–Littlewood maximal function. In the one-dimensional case one can deduce from the above inequality the following Hardy inequality

$$\int_{(0,R)} \left( \frac{1}{x} \int_0^x |\phi(t)| dt \right) dx \leq c_\phi + c \int_{\mathbb{R}_+} |\phi| \log^+ |\phi| dx,$$

where  $R > 0$  and  $\phi$  has bounded support in  $\mathbb{R}_+$ , as we have  $\frac{1}{x} \int_0^x |\phi(t)| dt \leq M\phi(x)$ . The substitution of  $\xi(x) = \int_0^x |\phi(t)| dt$  allows to interpret the above inequality as

$$\int_{(0,R)} |\xi(x)| \frac{1}{x} dx \leq c_\xi + c \int_{\mathbb{R}} |\xi'| \log^+ |\xi'| dx. \tag{5.7}$$

When in our inequality (5.3) we consider  $\bar{A}(t) = t \log(a + t)$  where  $a > 1$ , we obtain inequalities having the form:

$$\int_{\Omega} F_{\bar{A}}(|\xi|) \mu_1(dx) \leq \tilde{C} \int_{\Omega} |\nabla \xi| \log(a + |\nabla \xi|) \mu_2(dx).$$

As for big arguments  $\lambda$  we have  $F_{\bar{A}}(\lambda) \sim \lambda$ , (5.3) is similar to (5.7). That is why we call it Stein-type inequality.

Below we derive inequalities which are similar to (5.7) and come as a consequence of Corollary 5.1. For the related results we refer e.g to [56, 59, 60] and their references.

**Theorem 5.1.** *Let  $\bar{A}(\lambda) = t \log(a + t)$ ,  $F_{\bar{A}}(\lambda) = t \log^{-1}(a + 1/t)$ ,  $a > 1$  and  $\alpha \in (0, 1)$ . For every Lipschitz function  $\xi$  with compact support we have*

$$\begin{aligned} C_t^{-1} \int_{(1,\infty) \cap \{|\xi| \geq t\}} \left( \frac{|\xi(x)|}{x} \right) e^{-x} dx &\leq \int_{(1,\infty)} \left( \frac{F_{\bar{A}}(|\xi(x)|)}{x} \right) e^{-x} dx \\ &\leq C \int_{(1,\infty)} \bar{A}(|\xi'(x)|) e^{-x} dx, \end{aligned} \tag{5.8}$$

$$\begin{aligned} B_t^{-1} \int_{(0,1) \cap \{|\xi| \geq t\}} \left( \frac{|\xi(x)|}{x} \right) x^{-\alpha} dx &\leq \int_{(0,1)} \left( \frac{F_{\bar{A}}(|\xi(x)|)}{x} \right) x^{-\alpha} dx \\ &\leq C \int_{(0,1)} \bar{A}(|\xi'(x)|) x^{-\alpha} dx, \end{aligned} \tag{5.9}$$

where the constants  $C_t, C, B_t, B$  are independent of  $\xi$ .

*Proof.* We apply Corollary 5.1 with  $\Psi(t) = \frac{1}{\bar{A}(t)}$  and  $g(t) = \bar{A}(t)$ , which satisfy assumption  $(\Psi)$ . Indeed, easy computations shows that  $g(t)\Psi'(t) \leq -\log a \cdot \Psi$  and  $\Theta(t) = \log(a + t \log(a + t))/t \log(a + t)$  is decreasing in the neighbourhood of zero.

To obtain (5.8) we observe that  $u(x) = e^x$  solves the equation

$$-\Delta_A u = -\frac{e^x}{a + e^x} = \Phi.$$

Due to Corollary 5.1 (5.3) holds with

$$\begin{aligned} \mu_1(dx) &= \left(2 - \frac{e^x}{a + e^x}\right) \frac{1}{e^x \log(a + e^x)} dx \succ \frac{1}{xe^x} dx \text{ on } (1, \infty), \\ \mu_2(dx) &= \log(a + e^x \log(a + e^x)) \frac{1}{e^x \log(a + e^x)} dx \prec \frac{1}{e^x} dx \text{ on } (1, \infty), \end{aligned}$$

because  $\sigma = 2$  is admissible. The estimates  $\succ, \prec$  hold up to the constant. Thus, we have (5.8).

To obtain (5.9) we observe that  $u(x) = x^\alpha$  solves the equation

$$-\Delta_{\bar{A}} u = \frac{\alpha(1 - \alpha)x^{\alpha-2}}{a - \alpha x^{\alpha-1}} = \Phi > 0.$$

Due to Corollary 5.1 (5.3) holds with

$$\begin{aligned} \mu_1(dx) &= \alpha(1 - \alpha) \frac{x^{\alpha-2}}{a + x^{\alpha-1}} \frac{1}{x^\alpha \log(a + x^\alpha)} \succ \frac{1}{x^{\alpha+1}} \text{ on } (1, \infty), \\ \mu_2(dx) &= \frac{\log(a + x^\alpha \log(a + x^\alpha))}{x^\alpha \log(a + x^\alpha)} \prec \frac{1}{x^\alpha} \text{ on } (1, \infty). \end{aligned}$$

Thus, we have (5.9). □

**Retrieving author’s results from [68]**

Let us mention the following result of [68], which is the special case of our Theorem 4.1, when we apply  $\bar{A}(t) = t^p = F_{\bar{A}}(t)$ ,  $g(t) = t$ ,  $\Psi(t) = t^{-\beta}$ ,  $C = \beta > 0$ .

**Theorem 5.2.** ([68], Theorem 4.1) *Assume that  $1 < p < \infty$  and  $u \in W_{loc}^{1,p}(\Omega)$  is a nonnegative solution to PDI  $-\Delta_p u \geq \Phi$ , in the sense of Definition 2.3, where  $\Phi$  is locally integrable and  $\sigma_0 := \inf \{ \sigma \in \mathbb{R} : \Phi \cdot u + \sigma |\nabla u|^p \geq 0 \text{ a.e. in } \Omega \} \in \mathbb{R}$ , where  $\inf \emptyset = +\infty$ . Assume further that  $\beta$  and  $\sigma$  are arbitrary numbers such that  $\beta > 0$  and  $\beta > \sigma \geq \sigma_0$ . Then for every Lipschitz function  $\xi$  with compact support in  $\Omega$  we have*

$$\int_{\Omega} |\xi|^p \mu_1(dx) \leq \int_{\Omega} |\nabla \xi|^p \mu_2(dx), \tag{5.10}$$

where

$$\begin{aligned} \mu_1(dx) &= \left(\frac{\beta - \sigma}{p - 1}\right)^{p-1} [\Phi \cdot u + \sigma |\nabla u|^p] \cdot u^{-\beta-1} \chi_{\{u>0\}} dx, \\ \mu_2(dx) &= u^{p-\beta-1} \chi_{\{|\nabla u| \neq 0\}} dx. \end{aligned}$$

In [68] we show that the above theorem implies classical Hardy inequality with the optimal constant, as well as various other weighted Hardy inequalities e.g. with radial and exponential weights.

Another special case of Theorem 4.1, specializing in Hardy–Poincaré inequalities, which have the form

$$\bar{C}_{\gamma,n,p} \int_{\mathbb{R}^n} |\xi|^p \left[ (1 + |x|^{\frac{p}{p-1}})^{p-1} \right]^{\gamma-1} dx \leq \int_{\mathbb{R}^n} |\nabla \xi|^p \left[ (1 + |x|^{\frac{p}{p-1}})^{p-1} \right]^{\gamma} dx,$$

with the analysis of constants is obtained in [69]. See also the related papers [9, 36].

Theorem 4.1 enables us to derive inequalities

$$\int_{\Omega} |\xi|^p \mu_{1,\beta}(dx) \leq \int_{\Omega} |\nabla \xi|^p \mu_{2,\beta}(dx),$$

with various measures, which can have more general form than those from [68]. In the construction of measures we are not restricted to the pair  $\Psi(t) = t^{-\beta}$ ,  $g(t) = t$  as in [68], but we can consider also other pairs of admitted functions satisfying assumption  $(\Psi)$ , e.g. pairs from Table 1.

### 6. New results. Inequalities with power-logarithmic functions

We collect here a few examples of the inequality (4.2) with several choices of  $\bar{A}$ .

#### Preliminary preparations

In this part we derive two lemmas which will be used in the sequel.

**Lemma 6.1.** *Suppose  $p \geq 1$ ,  $\alpha > 0$ ,  $\bar{A}(t) = t^p \log^\alpha(2 + t)$  and  $\Omega \subseteq \mathbb{R}^n$ ,  $n \geq 1$ .*

*Let  $u \in W_{loc}^{1,\bar{A}}(\Omega)$  be a nonnegative solution to PDI  $-\Delta_A u \geq \Phi$ , in the sense of Definition 2.3, where  $\Phi$  is locally integrable and assumptions  $(\Psi)$ ,  $(u)$  are satisfied with  $\sigma \in \mathbb{R}$  and  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ . Then there exists a constant  $\tilde{C} > 0$  such that for every Lipschitz function  $\xi$  with compact support in  $\Omega$  we have*

$$\int_{\Omega \cap \{\xi \neq 0\}} |\xi|^p \log^{-\alpha}(2 + 1/|\xi|) \mu_1(dx) \leq \tilde{C} \int_{\Omega} |\nabla \xi|^p \log^\alpha(2 + |\nabla \xi|) \mu_2(dx),$$

where

$$\mu_1(dx) = \Psi(u) \left( \Phi + \frac{\sigma}{g(u)} |\nabla u|^p \log^\alpha(2 + |\nabla u|) \right) \chi_{\{u>0\}} dx, \tag{6.1}$$

$$\mu_2(dx) = g^{p-1}(u) \log^\alpha(2 + g(u)) \Psi(u) \chi_{\{\nabla u \neq 0\}} dx, \tag{6.2}$$

*Proof.* We apply Theorem 4.1. We remark first that assumption  $(\bar{A})$  is satisfied as, according to Example 2.1,  $\bar{A} \in \Delta'$  if  $p \geq 1$ ,  $\alpha > 0$ . We notice, that

$$F_{\bar{A}}(t) = \frac{1}{\bar{A}(1/t)} = \frac{1}{(1/t)^p \log^\alpha(2 + 1/t)} = t^p \log^{-\alpha}(2 + 1/t), \quad F_{\bar{A}}(0) = 0. \tag{6.3}$$

□

**Lemma 6.2.** *Suppose  $p \geq 1$ ,  $\alpha > 0$ ,  $\beta \in (0, 1)$ ,  $\bar{A}(t) = t^p \log^\alpha(2 + t)$  and  $\Omega \subseteq \mathbb{R}_+$ . Assume further that assumption  $(\Psi)$  is satisfied with functions  $\Psi, g$  and  $(u)$  is satisfied with*

$$\sigma > -(1/\beta - 1)(p - 1) \inf_{x>0} g(x^\beta)x^{-\beta} = \sigma_0.$$

*Then there exists a constant  $\tilde{C} > 0$  such that for every Lipschitz function  $\xi$  with compact support in  $\Omega$  we have*

$$\int_{\Omega} |\xi|^p \log^{-\alpha}(2 + 1/|\xi|) \mu_1(dx) \leq \tilde{C} \int_{\Omega} |\xi'|^p \log^\alpha(2 + |\xi'|) \mu_2(dx), \quad (6.4)$$

where

$$\mu_1(dx) = \frac{\Psi(x^\beta)}{g(x^\beta)} x^{p(\beta-1)} \log^\alpha(2 + \beta x^{\beta-1}) dx, \quad (6.5)$$

$$\mu_2(dx) = \frac{\Psi(x^\beta)}{g(x^\beta)} g^p(x^\beta) \log^\alpha(2 + g(x^\beta)) dx. \quad (6.6)$$

Moreover,

$$\tilde{C} \leq \frac{\beta^{1-p}}{(1-\beta)(p-1) + \sigma\beta} (C - \sigma) \bar{A} \left( \frac{p + \frac{\alpha}{\log 2} - 1}{(C - \sigma)p} \right) \frac{\bar{A} \left( p + \frac{\alpha}{\log 2} \right) \left( \frac{2}{\log 2} \right)^{4\alpha}}{p - 1}. \quad (6.7)$$

*Proof.* We apply Lemma 6.1 with  $\bar{A}(t) = t^p \log^\alpha(2 + t)$ . The assumption  $(\bar{A})$  is satisfied as, according to Example 2.1,  $\bar{A} \in \Delta'$  for  $p \geq 1$ ,  $\alpha > 0$ . We notice, that  $F_{\bar{A}}(t) = t^p \log^{-\alpha}(2 + 1/t)$ , when  $t > 0$  and  $F_{\bar{A}}(0) = 0$  (see (6.3)). Moreover,  $u = u_\beta(x) = x^\beta$ , with  $\beta \in (0, 1)$ , is the solution to PDI  $-\Delta_A u \geq \Phi$ , where

$$\Phi = -(\beta - 1)\beta^{p-1}(p - 1)x^{p\beta-\beta-p} \log^\alpha(2 + \beta x^{\beta-1}). \quad (6.8)$$

Indeed, we have  $\nabla u = \beta x^{\beta-1}$ ,  $|\nabla u| = |\beta|x^{\beta-1}$  and we compute the function  $\Phi$

$$\begin{aligned} -\Delta_A u &= -\operatorname{div} \left( \frac{\bar{A}(|\nabla u|)}{|\nabla u|^2} \nabla u \right) = -\beta|\beta|^{p-2} \left( x^{(p-1)(\beta-1)} \log^\alpha(2 + |\beta|x^{\beta-1}) \right)' \\ &= -\beta|\beta|^{p-2} (\beta - 1) x^{(p-1)(\beta-1)-1} \log^{\alpha-1}(2 + |\beta|x^{\beta-1}) \\ &\quad \cdot \left( (p - 1) \log(2 + |\beta|x^{\beta-1}) + \alpha \frac{|\beta|x^{\beta-1}}{2 + |\beta|x^{\beta-1}} \right) \\ &\geq -\beta|\beta|^{p-2} (\beta - 1)(p - 1) x^{p\beta-p-\beta} \log^\alpha(2 + |\beta|x^{\beta-1}) \\ &= |\beta|^p (1/\beta - 1)(p - 1) x^{p\beta-p-\beta} \log^\alpha(2 + |\beta|x^{\beta-1}) = \Phi, \end{aligned}$$

where the inequality holds for  $\beta \in (0, 1)$ , thus we remove the absolute value of  $\beta$  and write (6.8).

Now let us verify assumption  $(u)$ .

We note first that  $\bar{A}(|\nabla u|) = \beta^p x^{p(\beta-1)} \log^\alpha(2 + \beta x^{\beta-1})$ . Therefore

$$\begin{aligned} g(u)\Phi + \sigma \bar{A}(|\nabla u|) \\ = \beta^p x^{p(\beta-1)} \log^\alpha(2 + \beta x^{\beta-1}) \left[ (1/\beta - 1)(p - 1)g(x^\beta)x^{-\beta} + \sigma \right] \end{aligned}$$

is positive for  $\sigma > -(1/\beta - 1)(p - 1) \inf_{x>0} g(x^\beta)x^{-\beta} = \sigma_0$ .

We reach the goal by computing the weights according to Lemma 6.1 and dividing both sides by the constant.

We notice that we can estimate the constant  $\tilde{C}$  as in (6.7). Indeed, due to the above method, we have

$$\tilde{C} \leq \frac{\beta^{1-p}}{(1-\beta)(p-1) + \sigma\beta} (C-\sigma)\bar{A} \left( \frac{D_{\bar{A}}-1}{(C-\sigma)d_{\bar{A}}} \right) \frac{\bar{A}(D_{\bar{A}})C_{\bar{A}}^4}{D_{\bar{A}}-1}.$$

Moreover, according to Facts 2.2 and 2.3,  $C_{\bar{A}} \leq (\frac{2}{\log 2})^\alpha$ ,  $d_{\bar{A}} = p \leq D_{\bar{A}} \leq p + \frac{\alpha}{\log 2}$ . □

**Inequalities on  $(0, \infty)$ .**

Applying  $\Psi(t) = t^{-C}$ ,  $g(t) = t$  in Lemma 6.2, we obtain the following result.

**Theorem 6.1.** (Power-logarithmic Hardy–Sobolev inequality on  $(0, \infty)$ ) *Let  $p \geq 1$ ,  $\alpha > 0$ ,  $\beta \in (0, 1)$ ,  $C > 0$ ,  $C > \sigma > -(1/\beta - 1)(p - 1)$ .*

*Then there exists  $c > 0$  such that for every compactly supported Lipschitz function  $\xi$  we have*

$$\int_0^\infty |\xi|^p \log^{-\alpha}(2 + 1/|\xi|) \mu_1(dx) \leq c \int_0^\infty |\xi'|^p \log^\alpha(2 + |\xi'|) \mu_2(dx),$$

where

$$\begin{aligned} \mu_1(dx) &= x^{\gamma-p} \log^\alpha \left( 2 + \frac{1}{x} \right) dx, \\ \mu_2(dx) &= x^\gamma \log^\alpha(2 + x) dx, \end{aligned}$$

with  $\gamma = -\beta(C + 1 - p)$  and the constant  $c$  is dependent on  $\bar{A}$ ,  $p$ ,  $C$ ,  $\beta$ ,  $\sigma$ .

*Proof.* We apply Lemma 6.2. It suffices now to check that the pair  $\Psi(t) = t^{-C}$ ,  $g(t) = t$  with  $C > 0$  satisfies the assumption  $(\Psi)$  i) and ii) and finally we compute the weights.

- (i) The mentioned  $\Psi, g$  are positive functions.  $\Psi$  is locally Lipschitz,  $\Psi/g$  is decreasing, moreover

$$\Psi'(t)g(t) = -Ct^{-C-1}g(t) = -Ct^{-C-1}t = -Ct^{-C-1+1} = -C\Psi(t).$$

- (ii) The function  $\Theta = t^{p-1-C} \log^\alpha(2 + t)$  (see (2.12)) is bounded in the neighbourhood of 0 when  $p - 1 - C \geq 0$  and decreasing when  $p - 1 - C < 0$ .

We note that

$$\begin{aligned} \sigma > -(1/\beta - 1)(p - 1) \inf_{0 < x} g(x^\beta)x^{-\beta} &= -(1/\beta - 1)(p - 1) \inf_{0 < x} x^\beta x^{-\beta} \\ &= -(1/\beta - 1)(p - 1) = \sigma_0. \end{aligned}$$

As  $\sigma_0 < 0$ , there exists  $\sigma \in [\sigma_0, C)$  for any  $C > 0$ .

We apply Lemma 6.2 and obtain the following measures in inequality (6.4)

$$\begin{aligned} \mu_1(dx) &= (x^\beta)^{-C-1} \beta^p x^{p(\beta-1)} \log^\alpha \left( 2 + \beta x^{\beta-1} \right) [(1/\beta - 1)(p - 1) + \sigma] dx \\ &\sim x^{-\beta(C+1-p)-p} \log^\alpha \left( 2 + \frac{1}{x} \right) dx, \\ \mu_2(dx) &\sim x^{-\beta(C+1-p)} \log^\alpha(2 + x) dx. \end{aligned}$$

Now it suffices to take  $\gamma = -\beta(C + 1 - p)$ . □

**Remark 6.1.** We may estimate  $c$  due to (6.7).

**Inequalities on  $(0, 1)$**

This kind of inequalities is important in the further derivation of Hardy inequalities on Lipschitz boundary domains with measures involving distance from the boundary. It can be done by deriving the related inequality on the cube, then using the suitable covering and change of variables. Similar issues can be found for example in [51], proof of Theorem 8.2, Sec. 8.

We are now to present another application of  $g(\lambda)$  different from the identity function. For this, it is convenient to consider the extension of previous results where we consider the restriction of  $\Psi$  to the codomain of  $u$ .

We have the following remark.

**Remark 6.2.** Theorems 3.1 and 4.1, and Lemma 6.2 are valid when instead of assumption  $(\Psi)$  we use weaker assumption  $(\Psi)_2$  enclosed below. As their proofs in this case are easy modifications of the proofs from previous sections, we leave them to the reader.

$(\Psi)_2$  for a given nonnegative  $u \in W_{loc}^{1,\bar{A}}(\Omega)$ , there exists a function  $\Psi : [0, \infty) \rightarrow [0, \infty)$ , which is nonnegative and belongs to  $C^1(u(\Omega) \setminus \{0\})$ , where  $u(\Omega) = \{u(x) : x \in \Omega\}$ . Furthermore, the following conditions are satisfied

(i) inequality

$$g(t)\Psi'(t) \leq -C\Psi(t),$$

holds for all  $t \in u(\Omega) \setminus \{0\}$  with  $C > 0$  independent of  $t$  and certain continuous function  $g : (0, \infty) \rightarrow (0, \infty)$ , such that  $\Psi(t)/g(t)$  is nonincreasing for  $t \in u(\Omega)$ . Moreover, we set  $\Psi(t) \equiv 0$  for  $t \notin u(\Omega)$ .

(ii) function  $\Theta(t)$  given by (2.12) is nonincreasing or bounded in the neighbourhood of 0.

Let us restrict ourselves to  $\Omega = (0, 1)$ ,  $u(\Omega) \subseteq (0, 1)$  and choose  $\Psi(t) = e^{\frac{1}{2} \log^2(t)}$ ,  $g(t) = t/|\log t|$ . They do not satisfy assumption  $(\Psi)$ , but only  $(\Psi)_2$ . In particular assumption  $(\Psi)$  i) requires  $\Psi$  to be a decreasing function, but it does not hold outside  $(0, 1)$ . This generalization applied to Lemma 6.2 leads to the following result.

**Theorem 6.2.** (Hardy–Sobolev inequality on  $(0, 1)$ ) *Let  $p \geq 1$ ,  $\alpha > 0$ ,  $\beta \in (0, 1)$  and  $\bar{A}(t) = t^p \log^\alpha(2 + t)$ .*

*Then there exists a constant  $c > 0$  such that for every Lipschitz function  $\xi$  with compact support in  $(0, 1)$  we have*

$$\int_0^1 |\xi|^p \log^{-\alpha}(2 + 1/|\xi|) \mu_1(dx) \leq c \int_0^1 \bar{A}(|\xi'|) \mu_2(dx),$$

where

$$\mu_1(dx) = e^{\frac{\beta^2}{2} \log^2(x)} |\log x| \frac{x^{(p-1)\beta}}{x^p} \log^\alpha\left(2 + \frac{1}{x}\right) dx, \tag{6.9}$$

$$\mu_2(dx) = e^{\frac{\beta^2}{2} \log^2(x)} |\log x| \frac{x^{(p-1)\beta}}{|\log x|^p} \log^\alpha(2 + x) dx. \tag{6.10}$$

*Proof.* We apply Lemma 6.2, where  $u = u_\beta(x) = x^\beta$  is considered, with assumption  $(\Psi)_2$  instead of  $(\Psi)$ . It suffices now to check that the pair  $\Psi(t) = e^{\frac{1}{2} \log^2(t)}$ ,  $g(t) = t/|\log t|$ , with  $C = 1$  (for  $t \in (0, 1)$ ) satisfies the assumption  $(\Psi)_2$  i) and ii).

(i) The functions  $\Psi, g$  are positive.  $\Psi$  is locally Lipschitz. Moreover

$$\begin{aligned} \Psi'(t)g(t) &= -\frac{t}{\log t} \cdot \frac{1}{2}(\log^2 t)' e^{\frac{1}{2} \log^2(t)} = -\frac{t}{\log t} \cdot \frac{1}{2} 2 \frac{\log t}{t} e^{\frac{1}{2} \log^2(t)} \\ &= -e^{\frac{1}{2} \log^2(t)} = -\Psi(t). \end{aligned}$$

As  $t \in (0, 1)$ , we have  $\log t < 0$ . Therefore

$$\begin{aligned} g'(t) &= \left(-\frac{t}{\log t}\right)' = -\frac{t' \log t - t \log' t}{\log^2 t} = -\frac{\log t - 1}{\log^2 t} \\ &= \frac{1 + |\log t|}{\log^2 t} \geq 0 > -1. \end{aligned}$$

According to Remark 2.6 it is enough to ensure that  $\Psi/g$  is nonincreasing.

(ii) The function  $\Theta(s) = \frac{\bar{A}(g(s))\Psi(s)}{g(s)} = \left(\frac{s}{|\log s|}\right)^{p-1} \log^\alpha\left(2 + \frac{s}{|\log s|}\right) e^{\frac{1}{2} \log^2(s)}$  is decreasing in the neighbourhood of 0. Indeed, it is easy to show that for sufficiently small positive  $s$  we have  $\Theta'(s) < 0$ .

We note that there exists  $\sigma \in [\sigma_0, C) = [0, 1)$ . Indeed, the only condition for  $\sigma$  is the following one

$$\begin{aligned} \sigma \geq \sigma_0 &= -(1/\beta - 1)(p - 1) \inf_{0 < x < 1} g(x^\beta)x^{-\beta} \\ &= -(1/\beta - 1)(p - 1) \inf_{0 < x < 1} x^\beta |\log x^\beta| x^{-\beta} \\ &= -(1/\beta - 1)(p - 1) \inf_{0 < x < 1} |\log x^\beta| = 0. \end{aligned}$$

We apply generalization of Lemma 6.2 (see Remark 6.2) and obtain the following measures in inequality (6.4)

$$\bar{\mu}_1(dx) = e^{\frac{1}{2} \log^2(x^\beta)} |\log(x^\beta)| x^{p\beta - \beta - p} \log^\alpha(2 + \beta x^{\beta-1}) dx \sim \mu_1,$$

$$\bar{\mu}_2(dx) = e^{\frac{1}{2} \log^2(x^\beta)} |\log(x^\beta)|^{-p+1} x^{p\beta - \beta} \log^\alpha\left(2 + \frac{x^\beta}{|\log x^\beta|}\right) dx \sim \mu_2. \quad \square$$

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