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Global existence of a weak solution to 3d stochastic Navier–Stokes equations in an exterior domain

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Abstract. In this paper we consider the existence of a weak solution to a 3d stochastic Navier–Stokes equation perturbed by a noise $g(X(t))dW$, where $W(t)$ is a cylindrical Wiener process, in an exterior domain D:

$$dX(t) = [-AX(t) + B\,(X(t))]dt + g(X(t))dW(t),$$

where $A = -P_2\Delta$ is the Stokes operator and g satisfies some conditions.

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1. Introduction

In this paper we consider the existence of a weak solution to a $3d$ stochastic Navier–Stokes equation in an exterior domain D with smooth boundary Γ :

$$\begin{cases} dX(t) = [\Delta X(t) + (X(t), \nabla)X(t) - \nabla p]dt + g(X(t))dW(t), \\ div\ X\ = 0 \quad \text{in} \quad [0, \infty) \times D, \\ \ X(t, x) = 0 \quad\ \text{on} \quad [0, \infty) \times \Gamma,\ X(0) = \phi, \end{cases}$$

where X is the velocity field of the fluid, p the pressure, ϕ the initial velocity field, $g(X(t))$ the intensity of the external noise.

In the last years the existence of the mild solutions and weak solutions, and martingale solutions to the two dimensional or three dimensional stochastic Navier–Stokes equations have been extensively investigated. We emphasize that the stochastic Navier–Stokes equations on a bounded domain or R^d $(d = 2, 3)$ have been well discussed by many authors(Capinski and Gatarek [1], Da Prato and Zabczyk [8], Flandoli and Gatarek [10], Flandoli [11], Mikulevicius and Rozovskii [19] and references therein). But it is seemed that there

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exist a few papers about 2d stochastic Navier–Stokes equations in unbounded domain (see e.g. Sritharan and Sunder [24], Brzezniak and Li [1], Taniguchi [27] and references therein). Very recently global existence of a strong solution to a 3-dimensional stochastic Navier–Stokes equation with an additive noise in an exterior domain was considered in [26]. See also Brzezniak and Motyl [2], Brzezniak and Motyl [3], Capinski and Peszat [6].

On the other hand one can find many papers about the existence of weak solutions to the deterministic Navier–Stokes equations. For the case on a bounded domain see e.g. Fujita and Kato [13], Temam [29], Sohr [25], and references therein. For the case on an exterior domain see e.g. Miyakawa [20], Miyakawa and Sohr [14] and therein references. For the case on R^d see e.g. Kato [17].

In this paper we investigate the existence of weak solutions to the 3d stochastic Navier–Stokes equation perturbed by a additive noise and a multiplicative noise in an exterior domain D following Miyakawa and Sohr [14].

The Miyakawa and Sohr Theorem (Miyakawa and Sohr[14]). *Let $a \in L^2_\sigma(D)$ and $f \in L^2(0,T;L^2_\sigma(D))$ for all $T > 0$. Then there exists a weak solution u to a Navier–Stokes equation:*

$$\begin{aligned} u_t - \Delta u + (u,\nabla)u + \nabla p = f \quad &\text{in } D\times(0,\infty)\\ \nabla\cdot u = 0 \quad &\text{in } D\times(0,\infty)\\ u = 0 \quad &\text{on } \partial D\times(0,\infty),\\ u(x,0) = a(x), \end{aligned}$$

where a weak solution u is defined by $u \in L^\infty(0,T;L^2_\sigma(D)) \cap L^2(0,T;D(A^{\frac{1}{2}}))$ and

$$\begin{aligned} &-\int_0^T (u(t),v)h'(t)dt + \int_0^T (\nabla u(t),\nabla v)h(t)dt + \int_0^T ((u(t),\nabla)u(t),v)h(t)dt\\ &= h(0)(a,v) + \int_0^T (f(t),v)h(t)dt \end{aligned}$$

for all $v \in D(A^{\frac{1}{2}})$ and all $h \in C^1([0,T];R)$ with $h(T) = 0$.

The contents of this paper are as follows: In Sect. 2 we give preliminaries and we set the 3d stochastic Navier–Stokes equation (2.1) in a functional analysis setting by using the Stokes operator $A = -P_2\Delta$. In Sect. 3 we collect Lemmas used in this paper. In Sect. 4 the existence of a local weak solution to a 3d stochastic NS equation with an additive noise is considered following Miyakawa and Sohr [14] and we also consider global existence of solutions. In Sect. 5 the existence of a weak solution to a stochastic Navier–Stokes equation driven by a multiplicative noise is considered. We omit offen the notation $\omega \in \Omega$ if no confusion arises. Let C, c, C_T and c_T denote positive constants which change from a line to a line.

2. Preliminaries

In this paper we use the following Banach spaces:
Let the spaces $C^\infty_{0,\sigma}(D)$ and $L^q_\sigma(D),\ q>1$ be defined as follows:

$$C^\infty_{0,\sigma}(D) := \text{the space of } \varphi \in C^\infty_0(D) \text{ with div } \varphi = 0 \text{ in } D,$$

$$L^q_\sigma(D) := \text{the closure of } C^\infty_{0,\sigma}(D) \text{ with respect to the } L^q(D) - \text{norm } \left|\cdot\right|_q,$$

where

$$\left|u\right|_q = \left(\int_D |u|^q\, dx\right)^{\frac{1}{q}}.$$

Define the forms B and b by

$$B(u,v) = \sum_{i=1}^{3} u^i(x)\frac{\partial v}{\partial x_i}(x)dx,$$

$$b(u,v,\varphi) = \int_D ((u,\nabla)v,\varphi).$$

We also set

$$B(u) := B(u,u).$$

Next let

$$G^q(D) := \left\{\nabla p \in L^q(D); p \in L^q_{loc}(\bar{D})\right\}.$$

Then it is well known that the Helmholtz decomposition:

$$L^q(D) = L^q_\sigma(D) \oplus G^q(D)$$

holds. Let P_q denote the projection operator from $L^q(D)$ onto $L^q_\sigma(D)$ and let the operator A_q on $L^q_\sigma(D)$ be defined by $A_q u = -P_q\Delta u,\ u \in D(A_q)$ with the domain

$$D(A_q) = W^{2,q}(D) \cap W^{1,q}_0(D) \cap L^q_\sigma(D).$$

The operator A_2 is a self-adjoint in the real Hilbert space $L^2_\sigma(D)$. Let $A_2 = A$. $\left|A^{\frac{1}{2}}u\right|_2 = \left|\nabla u\right|_2$ for $u \in D(A^{\frac{1}{2}})$. See p.122, [20] in detail.

Let K, H be two separable Hilbert spaces. Let $L(K,H)$ denote the space of all bounded linear operators from K to H. Let $Q \in L(K,K)$ be a nonnegative self-adjoint operator. $L^0_2(K,H)$ denotes the space of all $\xi \in L(K,H)$ such that $\xi\sqrt{Q}$ is a Hilbert–Schmidt operator. The norm is given by

$$|\xi|^2_{L^0_2} := \left|\xi\sqrt{Q}\right|^2_{HS} = tr(\xi Q\xi^*) < \infty.$$

Then ξ is called a $Q-$Hilbert–Schmidt operator from K to H .

Let $(\Omega, P, \mathcal{F})$ be a complete probability space on which an increasing and right continuous family $(\mathcal{F}_t)_{t\in[0,\infty]}$ of complete sub-σ-algebra of $\mathcal{F}$ is defined. $\mathcal{F}_0$ contains all the null sets of $\mathcal{F}$. Let $\beta_j(t)$ $(j = 1, 2, 3, \ldots)$ be a sequence of real valued one-dimensional standard Brownian motions mutually independent on $(\Omega, P, \mathcal{F})$.

In this paper we set

$$W(t)=\sum_{n=1}^{\infty}\sqrt{\sigma_n}\beta_n(t)e_n, t\geq 0$$

where $\sigma_n \geq 0$ $(n=1,2,3\ldots)$ are nonnegative real numbers and $\{e_n\}$ $(n=1,2,3,\ldots)$ is a complete orthonormal basis in the real and separable Hilbert space K. Let $Q \in L(K,K)$ be a operator defined by $Qe_n=\sigma_n e_n$. We assume that Q is the trace operator, that is, $trace\ Q=\Sigma\sigma_n<\infty$.

We consider the following stochastic Naver–Stokes equation:

$$\begin{cases} dX(t)=[-AX(t)+B\,(X(t))]dt+g(X(t))dW(t), \\ X(t,x)=0,\ (t,x)\in[0,T]\times\Gamma\ \text{(boundary condition)}, \\ \qquad X(0)=\phi\ \ \text{(initial condition)}, \end{cases} \tag{2.1}$$

where ϕ is an $\mathcal{F}_0-$measurable function, $\phi\in D(A^{\frac{1}{2}})$ and $g: L^2_\sigma(D)\to L^0_2(K, L^2_\sigma(D))$ is a continuous function. Let $S(t)$ denote an analytic semigroup generated by $-A$.

Definition 1. A stochastic process $X(t)$ is called a mild solution to (2.1) on $[0,T]$, if the following conditions are satisfied:

(a) $X(t)$ is a progressively measurable process such that $X\in L^\infty(0,T; L^2_\sigma(D))\cap L^2(0,T;D(A^{\frac{1}{2}}))$, almost surely,

(b) The process $X(t)$ satisfies

$$X(t)=S(t)\phi+\int_0^t S(t-s)B(X(s))ds+\int_0^t S(t-s)g(X(s))dW(s)$$

on $[0,T]$, almost surely.

Definition 2. A stochastic process $X(t)$ is called a weak solution to (2.1) on $[0,T]$, if the following conditions are satisfied:

(c) $X(t)$ is a progressively measurable process such that $X\in L^\infty(0,T; L^2_\sigma(D))\cap L^2(0,T;D(A^{\frac{1}{2}}))$, almost surely,

(d) For any fixed $\varphi\in C_0^\infty(D)$, the process $(X(t),\varphi)$ is continuous,

(e) the process $X(t)$ satisfies

$$(X(t),\varphi)=(X(0)-\int_0^t AX(s)ds+\int_0^t B(X(s))ds+\int_0^t g(X(s))dW(s),\varphi) \tag{2.2}$$

on $[0,T]$, almost surely.

3. Lemmas

It is known that $-A_q$ generates a uniformly bounded analytic semigroup $\{S_q(t): t>0\}$ on $L^q_\sigma(D)$. There exists an $M>0$ and a $\beta>0$ such that $|S(t)|\leq Me^{\beta t}$, $t\geq 0$.

Lemma 1. *The following estimate holds: for* $0 < \alpha < 1$

$$\left|A_q^\alpha S_q(t)\right|_q \leq M_{\alpha q} t^{-\alpha}, \; M_{\alpha q} \geq 1, \; t > 0.$$

Let $M_\alpha := M_{\alpha 2}$. In this paper we use the Yosida approximation

$$J_\lambda := (1 + \lambda^{-1} A)^{-1}.$$

Let $\phi_\lambda := J_\lambda \phi$, $f_\lambda := J_\lambda f$ and

$$B_\lambda(u, v) := P(J_\lambda u \cdot \nabla) v = B(J_\lambda u, v), \; B_\lambda(u) := B_\lambda(u, u).$$

Lemma 2. ([14], p. 462) *For the Yosida approximation the following hold:*

$$|J_\lambda w|_2 \leq |w|_2 \quad \textit{for } w \in L^2_\sigma(D).$$

$$|J_\lambda w - w|_2 \to 0 \quad \textit{as } \lambda \to \infty,$$

$$|J_\lambda w|_\infty \leq C(\lambda) \, |w|_2 \quad \textit{for } w \in L^2_\sigma(D).$$

$$\begin{aligned}(B_\lambda(u, v), w) &= b(J_\lambda u, v, w) \leq |J_\lambda u|_\infty \, |\nabla v|_2 \, |w|_2 \\ &\leq C(\lambda) \, |u|_2 \, |\nabla v|_2 \, |w|_2 \, .\end{aligned}$$

The following lemmas are crucial in this paper. See also Farwig and Komo [9].

Lemma 3. (Theorem 2.7, p. 81, Giga and Sohr [12]). *Let* $1 < q < \infty, \; 1 < r < \infty, \; 0 < T \leq \infty$. *Then for every* $f \in L^r(0, T; L^q_\sigma(D))$ *and* $a \in D_q^{1-1/r, r}$ *there exists a unique solution of the Stokes equation*

$$u' + A_q u = f$$

for a.e. $t \in (0, T), \; u(0) = a$ *satisfying*

$$\int_0^T |u'(t)|_q^r \, dt + \int_0^T |A_q u(t)|_q^r \, dt \leq C \left(\int_0^T |f(t)|_q^r \, dt + |a|^r_{D_q^{1-1/r,r}} \right),$$

where $C = C(q, r, D) > 0$ *independent of* a, f *and* T, *and the definition of the space* $D_q^{1-1/r, r}$ *is given as follows:*

$$D_q^{1-1/r, r} := \left\{ v \in L^r_\sigma(D); |v|_q + \left(\int_0^\infty |AS(t) v|_q^r \, dt \right)^{\frac{1}{r}} < \infty \right\}.$$

Remark 1. The general definition of the space $D_q^{\alpha, r}$, $0 < \alpha < 1$, is given in (p.77, [12]).

4. The existence of a weak solution to (4.2)

Let $\Phi : [0, T] \times \Omega \to L_2^0(K, L^2_\sigma(D))$ be a progressively measurable process. Set

$$W_A(t) := \int_0^t S(t - s) \Phi(s) dW(s).$$

Condition 1. There exist positive constants $\Phi_{2,T}$ and $\Phi_{4,T} > 0$ depending on $T > 0$ such that

$$E\left[\sup_{0\le s\le T}|\Phi(s)|^2_{L^2_0}\right] \vee E\left[\sup_{0\le s\le T}\left|A^{\frac{1}{2}}\Phi(s)\right|^2_{L^2_0}\right] < \Phi_{2,T},$$
$$E\left[\sup_{0\le s\le T}|\Phi(s)|^4_{L^2_0}\right] \vee E\left[\sup_{0\le s\le T}\left|A^{\frac{1}{2}}\Phi(s)\right|^4_{L^2_0}\right] < \Phi_{4,T}.$$

We need the following lemma.

Lemma 4. *If, $W_A(t)(\omega)$ and $A^{\frac{1}{2}}W_A(t)(\omega)$ are continuous, then for a.e. $\omega \in \Omega$, for any given $R_\omega > 0$ there exists a sufficiently small $T_0 = T_0(\omega) > 0$ such that*

$$\sup_{0\le t\le T_0}|W_A(t)(\omega)| \vee \sup_{0\le t\le T_0}\left|A^{\frac{1}{2}}W_A(t)(\omega)\right| < R_\omega. \tag{4.1}$$

In this section we consider the existence of a weak solution to a stochastic Navier–Stokes equation with an additive noise:

$$\begin{cases} dX(t) = [-AX(t) + B\left(X(t)\right)]dt + \Phi(t)dW(t),\\ X(t,x) = 0,\ (t,x) \in [0,\infty)\times\Gamma\ \text{(boundary condition)},\\ X(0) = \phi\ \text{(initial condition)}. \end{cases} \tag{4.2}$$

To this end we first consider the following stochastic equation:

$$X(t) = S(t)\phi + \int_0^t S(t-s)B(X(s))ds + \int_0^t S(t-s)\Phi(s)dW(s) \tag{4.3}$$

for an initial value ϕ with $E\,|\phi|_2^2 < \infty$ and $E\left|A^{\frac{1}{2}}\phi\right|_2^2 < \infty$. Set

$$Y(t) = X(t) - W_A(t). \tag{4.4}$$

Then it follows that

$$Y(t) = S(t)\phi + \int_0^t S(t-s)B\left(Y(s) + W_A(s)\right)ds.$$

We consider the following equation

$$X(t) = S(t)\phi + \int_0^t S(t-s)B_\lambda\left(X(s)\right)ds + \int_0^t S(t-s)\Phi(s)dW(s). \tag{4.5}$$

In other word we ask if there exists a solution $Y(t)(\omega)$ to

$$Y(t) = S(t)\phi + \int_0^t S(t-s)B_\lambda\left(Y(s) + W_A(s)\right)ds,\ a.e.\ \omega\in\Omega. \tag{4.6}$$

For any fixed ϕ with $E\,|\phi|_2^2 < \infty$ and $E\left|A^{\frac{1}{2}}\phi\right|_2^2 < \infty$, we can take an $R_\omega > 0$, for *a.e.* $\omega \in \Omega$, such that $|\phi(\omega)|_2 + \left|A^{\frac{1}{2}}\phi(\omega)\right|_2 \le \frac{1}{M}R_\omega$, where $|S(t)| \le M,\ M \ge 1,\ t \ge 0$. Let $v : [0,T]\times\Omega \to L^2_\sigma(D)\cap D(A^{\frac{1}{2}})$ and let $\|v(\omega)\|_T := \sup_{0\le s\le T}\left(|v(s)(\omega)|_2 + \left|A^{\frac{1}{2}}v(s)(\omega)\right|_2\right)$.

Definition 3. Denote by $PP([0,T];L^2_\sigma(D)\cap D(A^{\frac{1}{2}}))$, the set of all $L^2_\sigma(D)\cap D(A^{\frac{1}{2}})-$ valued, $\mathcal{F}_t-$ adapted and continuous processes $v(t)$ on $[0,T]$. Then for a.e. $\omega\in\Omega$, define the space

$$S(\phi,T,\omega,R_\omega) := \left\{ \begin{array}{c} v\in PP([0,T];L^2_\sigma(D)\cap D(A^{\frac{1}{2}})); \\ v(0)(\omega)=\phi(\omega), \|v(\omega)\|_T \le 2R_\omega, |\phi(\omega)|_2 + \left|A^{\frac{1}{2}}\phi(\omega)\right|_2 \le \frac{1}{M}R_\omega. \end{array} \right\}$$

First we show the existence of a solution local in time to (4.5).

Lemma 5. *Let ϕ be $\mathcal{F}_0-$ measurable with $E|\phi|_2^2<\infty$ and $E\left|A^{\frac{1}{2}}\phi\right|_2^2<\infty$. Assume that Condition 1 is satisfied. Then there exists a local solution X_λ to (4.5).*

Proof. If Condition 1 is satisfied, then we have that

$$\begin{aligned}
&E\left|A^{\frac{1}{2}}W_A(t)-A^{\frac{1}{2}}W_A(\tau)\right|_2^4 \\
&= E\left|\int_0^t S(t-s)A^{\frac{1}{2}}\Phi(s)dW(s)-\int_0^\tau S(\tau-s)A^{\frac{1}{2}}\Phi(s)dW(s)\right|_2^4 \\
&= E\left|\int_\tau^t S(t-s)A^{\frac{1}{2}}\Phi(s)dW(s)+\int_0^\tau \left(S(t-s)-S(\tau-s)\right)A^{\frac{1}{2}}\Phi(s)dW(s)\right|_2^4 \\
&\le 8E\left|\int_\tau^t S(t-s)A^{\frac{1}{2}}\Phi(s)dW(s)\right|_2^4 \\
&\quad +8E\left|\int_0^\tau \left(S(t-s)-S(\tau-s)\right)A^{\frac{1}{2}}\Phi(s)dW(s)\right|_2^4.
\end{aligned}$$

Thus by the Burkholder inequality we obtain that

$$\begin{aligned}
&E\left|A^{\frac{1}{2}}W_A(t)-A^{\frac{1}{2}}W_A(\tau)\right|_2^4 \\
&\le E\left(\int_\tau^t \left|S(t-s)A^{\frac{1}{2}}\Phi(s)\right|_{L_2^0}^2 ds\right)^2 \\
&\quad +8E\left(\int_0^\tau \left|\left(S(t-s)-S(\tau-s)\right)A^{\frac{1}{2}}\Phi(s)\right|_{L_2^0}^2 ds\right)^2 \\
&:= I_1+I_2.
\end{aligned}$$

Then

$$\begin{aligned}
I_1 &\le 8M^4\exp(4\beta T)E\left(\sup_{0\le t\le T}\left|A^{\frac{1}{2}}\Phi(s)\right|_{L_2^0}^2\int_\tau^t 1ds\right)^2 \\
&\le 24M^4\exp(4\beta T)\Phi_{4,T}(t-\tau)^2,
\end{aligned}$$

Let γ be any fixed real number such that $\gamma \in (\frac{1}{4}, \frac{1}{2})$. By the similar method as in (p.133 [7])

$$
\begin{aligned}
I_2 &\leq 8E\left(\int_0^\tau \left|\int_{\tau-s}^{t-s} \frac{d}{d\rho} S(\rho) A^{\frac{1}{2}}\Phi(s)d\rho\right|_{L_2^0}^2 ds\right)^2 \\
&\leq 8E\left(\int_0^\tau \left|\int_{\tau-s}^{t-s} AS(\rho) A^{\frac{1}{2}}\Phi(s)d\rho\right|_{L_2^0}^2 ds\right)^2 \\
&\leq 8\left(M_1^2 \int_0^\tau \left|\int_{\tau-s}^{t-s} \frac{d\rho}{\rho}\right|^2 ds\right)^2 E\left(\sup\left|A^{\frac{1}{2}}\Phi(s)\right|^2\right)^2 \\
&\leq 8\left(\frac{M_1^2 T^{1-2\gamma}}{\gamma^2(1-2\gamma)}\right)^2 \Phi_{4,T}(t-\tau)^{4\gamma},
\end{aligned}
$$

where we assumed $|t-\tau| < 1$. Therefore

$$
E\left|A^{\frac{1}{2}}W_A(t) - A^{\frac{1}{2}}W_A(\tau)\right|_2^4 \leq c\,|t-\tau|^{4\gamma}.
$$

Thus $A^{\frac{1}{2}}W_A(t)$ has a continuous modification in t and similarly $W_A(t)$ has also a continuous modification in t. Let $Z(\omega) \in S(\phi, T, \omega, R_\omega)$ and hence $\|Z(\omega)\|_T = \sup_{0\leq t\leq T}\left(|Z(t)(\omega)|_2 + \left|A^{\frac{1}{2}}Z(t)(\omega)\right|_2\right) \leq 2R_\omega$. Define a mapping Γ on the space $S(\phi, T, \omega, R_\omega)$ by

$$
\begin{aligned}
\Gamma(Z) &:= S(t)\phi + \int_0^t S(t-s)B_\lambda\left(Z(s) + W_A(s)\right)ds \\
&= S(t)\phi + \int_0^t S(t-s)B(J_\lambda(Z(s)+W_A(s)), Z(s)+W_A(s))ds.
\end{aligned}
$$

We show that $\Gamma: S(\phi, T, \omega, R_\omega) \to S(\phi, T, \omega, R_\omega)$ is well defined. By lemmas 1 and 2

$$
\begin{aligned}
&\|\Gamma(Z)\|_T \\
&\leq \|S(t)\phi\|_T + \left\|\int_0^t S(t-s)B(J_\lambda(Z(s)+W_A(s)), Z(s)+W_A(s))ds\right\|_T \\
&\leq \|S(t)\phi\|_T + \sup_{0\leq t\leq T}\int_0^t |S(t-s)B(J_\lambda(Z(s)+W_A(s)), Z(s)+W_A(s))|_2\, ds \\
&\quad + \sup_{0\leq t\leq T}\int_0^t \left|A^{\frac{1}{2}}S(t-s)B(J_\lambda(Z(s)+W_A(s)), Z(s)+W_A(s))\right|_2 ds \\
&\leq \|S(t)\phi\|_T + C(\lambda)\sup_{0\leq t\leq T}\int_0^t |(Z(s)+W_A(s))|_2 \left|A^{\frac{1}{2}}(Z(s)+W_A(s))\right|_2 ds \\
&\quad + C(\lambda)\sup_{0\leq t\leq T}\int_0^t \left|A^{\frac{1}{2}}S(t-s)\right| |(Z(s)+W_A(s))|_2 \\
&\quad \times \left|A^{\frac{1}{2}}(Z(s)+W_A(s))\right|_2 ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sup_{0\leq t\leq T}\left(|S(t)\phi(\omega)|_2+\left|A^{\frac{1}{2}}S(t)\phi(\omega)\right|_2\right)\\
&\quad +C(\lambda)\sup_{0\leq t\leq T}\int_0^t (1+M_{\frac{1}{2}}(t-s)^{-\frac{1}{2}})\left(|Z(s)(\omega)|_2+|W_A(s))(\omega)|_2\right)\\
&\quad \times\left(\left|A^{\frac{1}{2}}Z(s)(\omega)\right|_2+\left|A^{\frac{1}{2}}W_A(s)(\omega)\right|_2\right)ds.
\end{aligned}
$$

Since thanks to Auxiliary lemma 1, we have a sufficiently small $T_0>0$ such that (4.1) holds and so we can choose a sufficiently small $T\in(0,T_0)>0$ such that

$$
\begin{aligned}
\|\Gamma(Z)(\omega)\|_T &\leq R_\omega+9C(\lambda)\sup_{0\leq t\leq T}\int_0^t(1+M_{\frac{1}{2}}(t-s)^{-\frac{1}{2}})R_\omega^2 ds\\
&\leq R_\omega+9C(\lambda)M_{\frac{1}{2}}(T+2\sqrt{T})R_\omega^2\\
&< 2R_\omega.
\end{aligned}
$$

Thus we have that

$$
\Gamma(Z)\in S(\phi,T,\omega,R_\omega).
$$

Next we show that the mapping Γ is a contractive mapping if T is small enough.

$$
\begin{aligned}
\|\Gamma Z_1-\Gamma Z_2\|_T &\leq \left\|\int_0^t S(t-s)B(J_\lambda(Z_1(s)-Z_2(s)),Z_1(s)+W_A(s))\right.\\
&\quad + S(t-s)\left(B(J_\lambda(Z_2(s)+W_A(s)),Z_1(s)-Z_2(s))\right)ds\|_T\\
&\leq \left\|\int_0^t S(t-s)B(J_\lambda(Z_1(s)-Z_2(s)),Z_1(s)+W_A(s))ds\right\|_T\\
&\quad +\left\|\int_0^t S(t-s)B(J_\lambda(Z_2(s)+W_A(s)),Z_1(s)-Z_2(s))ds\right\|_T.
\end{aligned}
$$

Then using Lemmas 2 and 1 we have that

$$
\begin{aligned}
&\int_0^t |S(t-s)|\,|B(J_\lambda(Z_1(s)-Z_2(s)),Z_1(s)+W_A(s))|_2\,ds\\
&\quad\leq M\int_0^t |J_\lambda(Z_1(s)-Z_2(s))|_\infty\left|A^{\frac{1}{2}}(Z_1(s)+W_A(s))\right|_2 ds\\
&\quad\leq MC(\lambda)\int_0^t |Z_1(s)-Z_2(s)|_2\left|A^{\frac{1}{2}}(Z_1(s)+W_A(s))\right|_2 ds\\
&\quad\leq 3MC(\lambda)R_\omega\int_0^t |Z_1(s)-Z_2(s)|_2\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t \left|A^{\frac{1}{2}}S(t-s)\right|\,|B(J_\lambda(Z_1(s)-Z_2(s)),Z_1(s)+W_A(s))|_2\,ds\\
&\quad\leq M_{\frac{1}{2}}\int_0^t \left|A^{\frac{1}{2}}S(t-s)\right|\,|J_\lambda(Z_1(s)-Z_2(s))|_\infty\left|A^{\frac{1}{2}}(Z_1(s)+W_A(s))\right|_2 ds
\end{aligned}
$$

$$\leq M_{\frac{1}{2}} C(\lambda) \int_0^t \left| A^{\frac{1}{2}} S(t-s) \right| |Z_1(s) - Z_2(s)|_2 \left| A^{\frac{1}{2}} (Z_1(s) + W_A(s)) \right|_2 ds$$
$$\leq 3M_{\frac{1}{2}} C(\lambda) R_\omega \int_0^t (t-s)^{-\frac{1}{2}} |Z_1(s) - Z_2(s)|_2 \, ds.$$

Similarly

$$\int_0^t |S(t-s)| \, |B(J_\lambda(Z_2(s) + W_A(s)), Z_1(s) - Z_2(s))|_2 \, ds$$
$$+ \int_0^t \left| A^{\frac{1}{2}} S(t-s) \right| |B(J_\lambda(Z_2(s) + W_A(s)), Z_1(s) - Z_2(s))|_2 \, ds$$
$$\leq 3C_0 C(\lambda) R_\omega \int_0^t (1 + (t-s)^{-\frac{1}{2}}) \, \|Z_1 - Z_2\|_2 \, ds,$$

where $C_0 = (M + M_{\frac{1}{2}})$. Therefore

$$\|\Gamma Z_1 - \Gamma Z_2\|_T \leq 6C_0 C(\lambda) R_\omega \sup_{0 \leq t \leq T} \int_0^t (1 + (t-s)^{-\frac{1}{2}}) \, \|Z_1 - Z_2\|_T \, ds$$
$$\leq 12 C_0 C(\lambda) R_\omega (T + T^{\frac{1}{2}}) \, \|Z_1 - Z_2\|_T \, .$$

Thus by taking a sufficiently small $T > 0$, the mapping Γ is a contraction mapping. Thus there exists a fixed point $Y_\lambda \in S(\phi, T, \omega, R_\omega)$. Set $X_\lambda(s) = Y_\lambda(s) + W_A(s)$. Then we have that $X_\lambda(t)$ satisfies

$$X_\lambda(t) = S(t)\phi + \int_0^t S(t-s) B_\lambda(X_\lambda(s)) \, ds + \int_0^t S(t-s)\Phi(s) dW(s). \tag{4.7}$$

Thus the proof of the existence of a local solution is complete. □

Next we show the existence of a global weak solution.

Theorem 1. *Assume that all the conditions of Lemma 5 and $E|\phi|_2^4 < \infty$ are satisfied. Then there exists a global weak solution $X_\lambda(t)$ to (4.5).*

Proof. One can prove that a local solution $X_\lambda(t)$ to (4.7) is a weak solution by the standard method using the Fubini theorem [2]. That is, $X_\lambda(t)$ satisfies for any fixed $\varphi \in C_0^\infty(D)$,

$$(X_\lambda(t), \nabla\varphi) = \left(\phi - \int_0^t A X_\lambda(s) ds + \int_0^t B_\lambda(X_\lambda(s)) \, ds + \int_0^t \Phi(s) dW(s), \nabla\varphi \right).$$

Thus, assume that the solution $X_\lambda(s)$ exists on $0 \leq s \leq t \leq t_0$, $0 < t_0 < T$. Then the proof of the theorem is complete if $E|X_\lambda(t_0)|_2^2$ and $E\left|A^{\frac{1}{2}} X_\lambda(t_0)\right|_2^2$ are bounded. Since it is clear that $E|X_\lambda(t_0)|_2^2$ is bounded, we only show that $E\left|A^{\frac{1}{2}} X_\lambda(t_0)\right|_2^2$ is bounded. Applying the Ito formula to $|X_\lambda(t)|_2^4$, we have that

$$
\begin{aligned}
|X_\lambda(t)|_2^4 &= |\phi|_2^4 + 4\int_0^t \Big(X_\lambda(s)\,|X_\lambda(s)|_2^2\,, -AX_\lambda(s) + B_\lambda(X_\lambda(s))\Big)\,ds \\
&\quad +6\int_0^t |\Phi(s)|_{L_2^0}^2\,|X_\lambda(s)|_2^2\,ds + 4\int_0^t \Big(|X_\lambda(s)|_2^2\,X_\lambda(s), \Phi(s)dW(s)\Big) \\
&\le |\phi|_2^4 - 4\int_0^t \left|A^{\frac12}X_\lambda(s)\right|_2^2 |X_\lambda(s)|_2^2\,ds \\
&\quad +3\int_0^t |\Phi(s)|_{L_2^0}^4 + 3\int_0^t |X_\lambda(s)|_2^4\,ds + 4\int_0^t \Big(|X_\lambda(s)|_2^2\,X_\lambda(s), \Phi(s)dW(s)\Big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&E\,|X_\lambda(t)|_2^4 + 4\int_0^t E\left|A^{\frac12}X_\lambda(s)\right|_2^2 |X_\lambda(s)|_2^2\,ds \\
&\quad \le E\,|\phi|_2^4 + 3\left[\int_0^t E\,|\Phi(s)|_{L_2^0}^4\,ds + \int_0^t E\,|X_\lambda(s)|_2^4\,ds\right].
\end{aligned}
$$

Thus we have a $C_T > 0$ such that

$$
E\,|X_\lambda(t)|_2^4 + 4\int_0^t E\left|A^{\frac12}X_\lambda(s)\right|_2^2 |X_\lambda(s)|_2^2\,ds \le C_T. \tag{4.8}
$$

By the Ito formula

$$
\begin{aligned}
\left|A^{\frac12}X_\lambda(t)\right|_2^2 &= \left|A^{\frac12}\phi\right|_2^2 - 2\int_0^t |AX_\lambda(s)|_2^2\,ds + 2\int_0^t (A^{\frac12}X_\lambda(s), A^{\frac12}B_\lambda\,(X_\lambda(s))\,ds \\
&\quad + \int_0^t \left|A^{\frac12}\Phi(s)\right|_{L_0^2}^2 ds + 2\int_0^t \Big(A^{\frac12}X_\lambda(s), A^{\frac12}\Phi(s)dW(s)\Big)
\end{aligned}
$$

and hence

$$
\begin{aligned}
&E\left|A^{\frac12}X_\lambda(t)\right|_2^2 + 2\int_0^t E\,|AX_\lambda(t)|_2^2\,ds \\
&\quad \le E\left|A^{\frac12}\phi\right|_2^2 + E\int_0^t \left|A^{\frac12}\Phi(s)\right|_{L_2^0}^2 ds \\
&\qquad +2E\int_0^t |J_kX_\lambda(s)|_\infty\,|\nabla X_\lambda(s)|_2\,|AX_\lambda(s)|_2\,ds.
\end{aligned}
$$

Then by Lemma 2 and (4.8) we have a $C_T > 0$ such that

$$
\begin{aligned}
&2E\int_0^t |J_kX_\lambda(s)|_\infty\,|\nabla X_\lambda(s)|_2\,|AX_\lambda(s)|_2\,ds \\
&\le 2C(\lambda)E\int_0^t |X_\lambda(s)|_2\,|\nabla X_\lambda(s)|_2\,|AX_\lambda(s)|_2\,ds \\
&\le E\int_0^t |AX_\lambda(s)|_2^2\,ds + C^2(\lambda)E\int_0^t |X_\lambda(s)|_2^2\left|A^{\frac12}X_\lambda(s)\right|_2^2 ds \\
&\le E\int_0^t |AX_\lambda(s)|_2^2\,ds + C^2(\lambda)C_T.
\end{aligned}
$$

Thus by Condition 1

$$\begin{aligned} &E\left|A^{\frac{1}{2}}X_\lambda(t_0)\right|_2^2+\int_0^{t_0}E\left|AX_\lambda(t)\right|_2^2ds\\ &\le E\left|A^{\frac{1}{2}}\phi\right|_2^2+E\int_0^{t_0}\left|A^{\frac{1}{2}}\Phi(s)\right|_{L_2^0}^2ds+C^2(\lambda)C_T\\ &\le E\left|A^{\frac{1}{2}}\phi\right|_2^2+T\Phi_{2,T}+C^2(\lambda)C_T. \end{aligned}$$

Thus $E\left|A^{\frac{1}{2}}X_\lambda(t_0)\right|_2^2$ is bounded. This implies that the solution $X_\lambda(t)$ exists on $[0,T]$. The proof of the theorem is complete. □

5. Existence of a weak solution to (2.1)

In this section we first prove the existence of a weak solution X_λ to

$$X(t)=\phi-\int_0^t AX(s)ds+\int_0^t B_\lambda\left(X(s)\right)ds+\int_0^t g(X(s))dW(s). \quad (5.1)$$

Since a function $g:L_\sigma^2(D)\to L_2^0(K,L_\sigma^2(D))$ is continuous, for any $L_\sigma^2(D)-$ valued progressively measurable process ξ, we have that $g(\xi):[0,T]\times\Omega\to L_2^0(K,L_\sigma^2(D))$ is also a progressively measurable process. We need the following condition on g.

Condition 2. The function $g:L_\sigma^2(D)\to L_2^0(K;L_\sigma^2(D))$ satisfies the Lipschitz condition with $g(0)=0$. In other word, there exists an $L_g>0$ such that

$$|g(u)-g(v)|_{L_2^0}^2\le L_g\,|u-v|_2^2$$

for $u,v\in L_\sigma^2(D)$, and furthermore assume that $g:D(A^{\frac{1}{2}})\to L_2^0(K;D(A^{\frac{1}{2}}))$ and there exists a positive constant L_G such that

$$\left|A^{\frac{1}{2}}g(u)-A^{\frac{1}{2}}g(v)\right|_{L_2^0}^2\le L_G\left|A^{\frac{1}{2}}u-A^{\frac{1}{2}}v\right|_2^2$$

for $u,v\in D(A^{\frac{1}{2}})$.

Condition 3. Assume that an initial value ϕ is $\mathcal{F}_0-$ measurable and the following conditions are satisfied:

$$E\left|\phi\right|_2^2<\infty \quad\text{and}\quad E\left|A^{\frac{1}{2}}\phi\right|_2^2<\infty.$$

$$E\left|\phi\right|_2^4<\infty \quad\text{and}\quad E\left|A^{\frac{1}{2}}\phi\right|_2^4<\infty.$$

Assume that ϕ satisfies Condition 3 and g satisfies Condition 2. Consider the sequence of the processes.

$$\begin{cases} X_\lambda^0(t):=\phi,\\ X_\lambda^{n+1}(t)=\phi+\int_0^t\left[-AX_\lambda^{n+1}(s)+B_\lambda(X_\lambda^{n+1}(s))\right]ds\\ \quad+\int_0^t g(X_\lambda^n(s))dW(s),\ n\ge 0. \end{cases} \quad (5.2)$$

We have that if the process $X^n_\lambda(t)$ $(n \geq 1)$ is an $L^2_\sigma(D)-$progressively measurable process, then $g(X^n_\lambda(t))$ is also a progressively measurable process and so $\int_0^t g(X^n_\lambda(s))dW(s)$ is well defined. If $E\sup|X^n_\lambda(s)|_2^4$ and $E\sup\left|A^{\frac{1}{2}}X^n_\lambda(s)\right|_2^2$ are both bounded, then by Conditions 2 and 3, Theorem 1 and Lemma 7 below the process $X^{n+1}_\lambda(t)$ is well defined from (5.2), which is an $\mathcal{F}_t-$ measurable and continuous process and hence it is progressively measurable. Thus the sequence $\{X^n_\lambda(t)\}$ $(n = 1, 2, 3, , ,)$ is well defined.

Lemma 6. *Assume that $a_0, a_1 > 0$ and $f^0(t) := \beta > 0$ are positive constants. Let $f^n(t) \geq 0,\ n \geq 1$. If it holds that for all $n \geq 0$*

$$f^{n+1}(t) \leq a_0 + a_1 \int_0^t f^n(s)ds, \quad 0 \leq t \leq T,$$

then there exists an $M_\xi > 0$ such that for all $n \geq 0$

$$f^{n+1}(t) \leq M_\xi \exp(a_1 T).$$

Proof. Let $M_\xi := \max\{a_0, \beta\}$. It holds that $f^1(t) \leq M_\xi(1+a_1 t)$. Assume that the following inequality holds:

$$f^n(t) \leq M_\xi \sum_{k=0}^{k=n} \frac{(a_1 t)^k}{k!},\ n \geq 1.$$

Then it holds that

$$\begin{aligned} f^{n+1}(t) &\leq a_0 + a_1 \int_0^t \left(M_\xi \sum_{k=0}^{k=n} \frac{(a_1 s)^k}{k!} \right) ds \\ &= a_0 + M_\xi \left(\sum_{k=0}^{k=n} \frac{a_1^{k+1}(t)^{k+1}}{(k+1)!} \right) \\ &\leq M_\xi \sum_{k=0}^{k=n+1} \frac{(a_1 t)^k}{k!}. \end{aligned}$$

This means that it holds that for all $n \geq 0$

$$f^{n+1}(t) \leq M_\xi \exp(a_1 t), \quad t \geq 0.$$

This completes the proof of the lemma. □

Lemma 7. *Let $p \geq 2$ and let ϕ be $\mathcal{F}_0-$measurable with $E|\phi|_2^p < \infty$ and $E\left|A^{\frac{1}{2}}\phi\right|_2^2 < \infty$. Assume that Condition 2 is satisfied. Then there exists a constant $B_e > 0$ such that*

$$E\left[\sup_{0 \leq s \leq T} \left|X^{n+1}_\lambda(s)\right|_2^p\right] < B_e \text{ and } E\left[\sup_{0 \leq s \leq T} \left|A^{\frac{1}{2}}X^{n+1}_\lambda(s)\right|_2^2\right] < B_e$$

uniformly in all $n \geq 0$.

Proof. Let $X^0_\lambda(s) = \phi,\ 0 \le s \le T$. By (5.2), applying the Ito formula to $\left|X^{n+1}_\lambda(t)\right|_2^p$, we have that

$$\begin{aligned}
\left|X^{n+1}_\lambda(t)\right|_2^p &= |\phi|_2^p \\
&\quad + p\int_0^t \left(X^{n+1}_\lambda(s)\left|X^{n+1}_\lambda(s)\right|_2^{p-2}, -AX^{n+1}_\lambda(s) + B_\lambda(X^{n+1}_\lambda(s))\right) ds \\
&\quad + \frac{1}{2}p(p-1)\int_0^t |g(X^n_\lambda(s))|^2_{L_2^0}\left|X^{n+1}_\lambda(s)\right|_2^{p-2} ds \\
&\quad + p\int_0^t \left(\left|X^{n+1}_\lambda(s)\right|_2^{p-2} X^{n+1}_\lambda(s), g(X^n_\lambda(s))dW(s)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&E\left[\sup_{0\le\tau\le t}\left|X^{n+1}_\lambda(\tau)\right|_2^p\right] + p\int_0^t E\left|A^{\frac{1}{2}}X^{n+1}_\lambda(s)\right|_2^2\left|X^{n+1}_\lambda(s)\right|_2^{p-2} ds \\
&\quad \le E|\phi|_2^p + \frac{1}{2}p(p-1)E\left[\int_0^t |g(X^n_\lambda(s))|^2_{L_2^0}\left|X^{n+1}_\lambda(s)\right|_2^{p-2} ds\right] \\
&\qquad + pE\left[\sup_{0\le\tau\le t}\left|\int_0^\tau \left(\left|X^{n+1}_\lambda(s)\right|_2^{p-2} X^{n+1}_\lambda(s), g(X^n_\lambda(s))dW(s)\right)\right|\right] \\
&\quad = E|\phi|_2^p + J_1 + J_2.
\end{aligned}$$

Thus we have a constants $p_1,\, p_2 > 0$ such that

$$\begin{aligned}
J_1 &\le \frac{1}{2}p(p-1)L_g E\left[\int_0^t |X^n_\lambda(s)|_2^2\left|X^{n+1}_\lambda(s)\right|_2^{p-2} ds\right] \\
&\le p_1\int_0^t E|X^n_\lambda(s)|_2^p\, ds + p_2\int_0^t E\left|X^{n+1}_\lambda(s)\right|_2^p ds.
\end{aligned}$$

By the Burkholder–Davis–Gundy lemma, Condition 2 and the Young inequality, there exist positive constants $c,\, k > 0$ such that

$$\begin{aligned}
J_2 &\le 4cE\left[\left(\int_0^t \left|X^{n+1}_\lambda(s)\right|_2^{2(p-2)}\left|X^{n+1}_\lambda(s)\right|_2^2 |g(X^n_\lambda(s))|^2_{L_2^0} ds\right)^{\frac{1}{2}}\right] \\
&\le 4cL_g^{\frac{1}{2}}E\left[\left(\int_0^t \left|X^{n+1}_\lambda(s)\right|_2^{2(p-1)}|X^n_\lambda(s)|_2^2\, ds\right)^{\frac{1}{2}}\right] \\
&\le 4cL_g^{\frac{1}{2}}E\left[\sup_{0\le s\le t}\left|X^{n+1}_\lambda(s)\right|_2^{p-1}\left(\int_0^t |X^n_\lambda(s)|^2\, ds\right)^{\frac{1}{2}}\right] \\
&\le \frac{1}{2}E\left[\sup_{0\le\tau\le t}\left|X^{n+1}_\lambda(\tau)\right|_2^p\right] + \kappa T^{\frac{p-2}{2}}E\left(\int_0^t |X^n_\lambda(s)|_2^p\, ds\right)^2.
\end{aligned}$$

Thus

$$\begin{aligned}
&E\left[\sup_{0\le\tau\le t}\left|X_\lambda^{n+1}(\tau)\right|_2^p\right]+2p\int_0^t E\left|A^{\frac12}X_\lambda^{n+1}(s)\right|_2^2\left|X_\lambda^{n+1}(s)\right|_2^{p-2}ds\\
&\le 2E\left|\phi\right|_2^p+2p_1\int_0^t E\left|X_\lambda^n(s)\right|_2^p ds+2p_2\int_0^t E\left|X_\lambda^{n+1}(s)\right|_2^p ds\\
&\quad+2\kappa T^{\frac{p-2}{2}}\int_0^t E\left[\sup_{0\le\tau\le s}\left|X_\lambda^n(\tau)\right|_2^p\right]ds.
\end{aligned}$$

By Lemma 6 there exists a constant $B_c>0$ such that for all $n\ge 0$

$$E\left[\sup_{0\le s\le T}\left|X_\lambda^{n+1}(s)\right|_2^p\right]<B_c.$$

We also have a constant $C_T>0$ such that

$$\int_0^t E\left|A^{\frac12}X_\lambda^{n+1}(s)\right|_2^2\left|X_\lambda^{n+1}(s)\right|_2^2 ds<C_T,\ 0\le t\le T. \tag{5.3}$$

Next applying the Ito formula to $\left|A^{\frac12}X_\lambda^{n+1}(t)\right|_2^2$, we have that

$$\begin{aligned}
&\left|A^{\frac12}X_\lambda^{n+1}(t)\right|_2^2=\left|A^{\frac12}\phi\right|_2^2\\
&\quad+2\int_0^t\left(A^{\frac12}X_\lambda^{n+1}(s),-A^{\frac12}AX_\lambda^{n+1}(s)+A^{\frac12}B_\lambda(X_\lambda^{n+1}(s))\right)ds\\
&\quad+\int_0^t\left|A^{\frac12}g(X_\lambda^n(s))\right|_{L_2^0}^2 ds\\
&\quad+2\int_0^t\left(A^{\frac12}X_\lambda^{n+1}(s),A^{\frac12}g(X_\lambda^n(s))dW(s)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&E\left[\sup_{0\le\tau\le t}\left|A^{\frac12}X_\lambda^{n+1}(\tau)\right|_2^2\right]+2\int_0^t E\left|AX_\lambda^{n+1}(s)\right|_2^2 ds\\
&\le E\left|A^{\frac12}\phi\right|_2^2+E\left[\int_0^t\left|A^{\frac12}g(X_\lambda^n(s))\right|_{L_2^0}^2 ds\right]\\
&\quad+2\int_0^t E\left|J_kX_\lambda^{n+1}\right|_\infty\left|\nabla X_\lambda^{n+1}\right|_2\left|AX_\lambda^{n+1}\right|_2 ds\\
&\quad+2E\left[\sup_{0\le\tau\le t}\left|\int_0^\tau\left(A^{\frac12}X_\lambda^{n+1}(s),A^{\frac12}g(X_\lambda^n(s))dW(s)\right)\right|\right]\\
&=E\left|A^{\frac12}\phi\right|_2^2+J_3+J_4+J_5.
\end{aligned}$$

Thus by Condition 2

$$J_3\le L_GE\left[\int_0^t\left|A^{\frac12}X_\lambda^n(s)\right|_2^2 ds\right].$$

By the Young inequality and (5.3) we have a positive constant $\theta > 0$ such that

$$\begin{aligned}J_4 &\leq 2C(\lambda) \int_0^t E \left|X_\lambda^{n+1}\right|_2 \left|A^{\frac{1}{2}} X_\lambda^{n+1}\right|_2 \left|AX_\lambda^{n+1}\right|_2 ds \\ &\leq \theta \int_0^t E \left|X_\lambda^{n+1}(s)\right|_2^2 \left|A^{\frac{1}{2}} X_\lambda^{n+1}(s)\right|_2^2 ds \\ &\quad + \int_0^t E \left|AX_\lambda^{n+1}(s)\right|_2^2 ds \\ &\leq \theta C_T + \int_0^t E \left|AX_\lambda^{n+1}(s)\right|_2^2 ds.\end{aligned}$$

By the Burkholder-Davis-Gundy lemma and the Young inequality, there exist constants c, $k > 0$ such that

$$\begin{aligned}J_5 &\leq 2cE \left[\left(\int_0^t \left|A^{\frac{1}{2}} X_\lambda^{n+1}(s)\right|_2^2 \left|A^{\frac{1}{2}} g(X_\lambda^n(s))\right|_{L_2^0}^2 ds \right)^{\frac{1}{2}} \right] \\ &\leq 2cL_G^{\frac{1}{2}} E \left[\sup_{0 \leq \tau \leq t} \left|A^{\frac{1}{2}} X_\lambda^{n+1}(\tau)\right|_2 \left(\int_0^t \left|A^{\frac{1}{2}} X_\lambda^n(s)\right|_2^2 ds \right)^{\frac{1}{2}} \right] \\ &\leq \frac{1}{2} E \left[\sup_{0 \leq \tau \leq t} \left|A^{\frac{1}{2}} X_\lambda^{n+1}(\tau)\right|_2^2 \right] + \kappa E \left(\int_0^t \left|A^{\frac{1}{2}} X_\lambda^n(s)\right|_2^2 ds \right).\end{aligned}$$

Thus

$$\begin{aligned}&E \left[\sup_{0 \leq \tau \leq t} \left|A^{\frac{1}{2}} X_\lambda^{n+1}(\tau)\right|_2^2 \right] + 2 \int_0^t E \left|AX_\lambda^{n+1}(s)\right|_2^2 ds \\ &\quad \leq 2E \left|A^{\frac{1}{2}} \phi\right|_2^2 + 2L_G \int_0^t E \left[\sup_{0 \leq \tau \leq s} \left|A^{\frac{1}{2}} X_\lambda^n(\tau)\right|_2^2 \right] ds \\ &\quad\quad + 2\kappa \int_0^t E \left[\sup_{0 \leq \tau \leq s} \left|A^{\frac{1}{2}} X_\lambda^n(\tau)\right|_2^2 \right] ds + 2\theta C_T.\end{aligned}$$

By Lemma 6 there exists a constant $B_d > 0$ such that all $n \geq 0$

$$E \left[\sup_{0 \leq s \leq T} \left|A^{\frac{1}{2}} X_\lambda^{n+1}(s)\right|_2^2 \right] < B_d.$$

Thus, the proof of the lemma is complete. □

In what follows we prove that there exists a $X_\lambda(t)$ such that the sequence $\{X_\lambda^n(t)\}$ converges to the $X_\lambda(t)$, which is a weak solution to (5.1).

Lemma 8. *Assume that Conditions 2 and 3 are satisfied. Then there exists a weak solution $X_\lambda(t)$ to (5.1), that is*

$$(X_\lambda(t), \varphi) = \left(\phi + \int_0^t \left[-AX_\lambda(s) + B_\lambda(X_\lambda(s)) \right] ds + \int_0^t g(X_\lambda(s)) dW(s), \varphi \right)$$

for any $\varphi \in C_0^\infty(D)$ *with* $X_\lambda \in L^2(\Omega, C(0,T; L^2_\sigma(D))) \cap L^2(\Omega \times [0,T]; D(A^{\frac{1}{2}}))$. *Furthermore* $X_\lambda(t)$ *satisfies that*

$$X_\lambda(t) = S(t)\phi + \int_0^t S(t-s)B_\lambda\left(X_\lambda(s)\right)ds + \int_0^t S(t-s)g(X_\lambda(s))dW(s).$$

Proof. From (5.2)

$$\begin{aligned} X_\lambda^{n+1}(t) - X_\lambda^n(t) = &\int_0^t -\left[AX_\lambda^{n+1}(s) - AX_\lambda^n(s)\right]ds \\ &+ \int_0^t \left[B_\lambda(X_\lambda^{n+1}(s)) - B_\lambda(X_\lambda^n(s))\right]ds \\ &+ \int_0^t [g(X_\lambda^n(s)) - g(X_\lambda^{n-1}(s))]dW(s). \end{aligned}$$

By applying the Ito formula to $\left|X_\lambda^{n+1}(t) - X_\lambda^n(t)\right|^2$,

$$\begin{cases} \left|X_\lambda^{n+1}(t) - X_\lambda^n(t)\right|_2^2 + \int_0^t \left|A^{\frac{1}{2}}(X_\lambda^{n+1}(s) - X_\lambda^n(s))\right|_2^2 ds \\ \quad = 2\int_0^t \left(X_\lambda^{n+1}(s) - X_\lambda^n(s), B_\lambda(X_\lambda^{n+1}(s)) - B_\lambda(X_\lambda^n(s))\right)ds \\ \qquad + \int_0^t \left|g(X_\lambda^n(s)) - g(X_\lambda^{n-1}(s))\right|_{L_2^0}^2 ds \\ \qquad +2\int_0^t \left(X_\lambda^{n+1}(s) - X_\lambda^n(s), (g(X_\lambda^n(s)) - g(X_\lambda^{n-1}(s)))dW(s)\right). \end{cases} \tag{5.4}$$

Let N be any fixed natural integer. Define the stopping time τ_N as follows:

$$\tau_N^n := \inf\left\{0 \le t \le T; |X_\lambda^n(t)|_2 \vee \left|A^{\frac{1}{2}}X_\lambda^n(s)\right|_2 \ge N\right\},$$

$$\tau_N := \inf_{n \ge N} \tau_N^n.$$

We have that

$$\begin{aligned} &E\left[\sup_{0\le\tau\le t\wedge\tau_N} \left|X_\lambda^{n+1}(\tau) - X_\lambda^n(\tau)\right|_2^2\right] \\ &\quad \le 2E\left[\sup_{0\le\tau\le t\wedge\tau_N}\left|\int_0^\tau \left(X_\lambda^{n+1}(s) - X_\lambda^n(s), B_\lambda(X_\lambda^{n+1}(s)) - B_\lambda(X_\lambda^n(s))\right)ds\right|\right] \\ &\qquad +E\left[\sup_{0\le\tau\le t\wedge\tau_N}\left|\int_0^\tau \left|g(X_\lambda^n(s)) - g(X_\lambda^{n-1}(s))\right|_{L_2^0}^2 ds\right|\right] \\ &\qquad +2E\left[\sup_{0\le\tau\le t\wedge\tau_N}\left|\int_0^\tau \left(X_\lambda^{n+1}(s) - X_\lambda^n(s), (g(X_\lambda^n(s)) - g(X_\lambda^{n-1}(s)))dW(s)\right)\right|\right] \\ &\quad := J_1 + J_2 + J_3. \end{aligned}$$

By using Lemma 2, we have that

$$\begin{aligned} &(B_\lambda(u,u) - B_\lambda(v,v), u - v) \\ &\quad = (B(J_\lambda u - J_\lambda v, u), u - v) \\ &\quad \le C(\lambda)\left|\nabla u\right|_2 \left|u - v\right|_2^2. \end{aligned}$$

Let $n \geq N$. Then

$$J_1 \leq 2C(\lambda)E\int_0^t \left|A^{\frac{1}{2}}X_\lambda^{n+1}(s\wedge\tau_N)\right|_2 \left|X_\lambda^{n+1}(s\wedge\tau_N)-X_\lambda^n(s\wedge\tau_N)\right|_2^2 ds$$
$$\leq 2C(\lambda)N\int_0^t E\left|X_\lambda^{n+1}(s\wedge\tau_N)-X_\lambda^n(s\wedge\tau_N)\right|_2^2 ds.$$

By the Burkholder-Davis-Gundy lemma, we have a $\kappa > 0$ such that

$$J_3 \leq \frac{1}{2}E\left[\sup_{0\leq s\leq t}\left|X_\lambda^{n+1}(s\wedge\tau_N)-X_\lambda^n(s\wedge\tau_N)\right|_2^2\right]$$
$$+\kappa\int_0^t E\left|g(X_\lambda^n(s\wedge\tau_N))-g(X_\lambda^{n-1}(s\wedge\tau_N))\right|_{L_2^0}^2 ds.$$

Thus by Condition 2 we have a $\gamma > 0$ such that for any fixed $n \geq N$,

$$\left\{\begin{array}{l} E\left[\sup_{0\leq s\leq t}\left|X_\lambda^{n+1}(s\wedge\tau_N)-X_\lambda^n(s\wedge\tau_N)\right|_2^2\right] \\ \quad\leq 4C(\lambda)N\int_0^t E\left[\sup_{0\leq\tau\leq s}\left|X_\lambda^{n+1}(\tau\wedge\tau_N)-X_\lambda^n(\tau\wedge\tau_N)\right|_2^2\right] ds \\ +\gamma\int_0^t E\sup_{0\leq\tau\leq s}\left|X_\lambda^n(\tau\wedge\tau_N)-X_\lambda^{n-1}(\tau\wedge\tau_N)\right|_2^2 ds. \end{array}\right. \tag{5.5}$$

Thus by the Gronwall lemma

$$\left\{\begin{array}{l} E\left[\sup_{0\leq s\leq t}\left|X_\lambda^{n+1}(s\wedge\tau_N)-X_\lambda^n(s\wedge\tau_N)\right|_2^2\right] \\ \quad\leq \gamma L_N\int_0^t E\left[\sup_{0\leq\tau\leq s}\left|X_\lambda^n(\tau\wedge\tau_N)-X_\lambda^{n-1}(\tau\wedge\tau_N)\right|_2^2\right] ds, \end{array}\right.$$

where $L_N := e^{4C(\lambda)NT}$. Set

$$\Psi^n(t) := E\left[\sup_{0\leq s\leq t}\left|X_\lambda^{n+1}(s\wedge\tau_N)-X_\lambda^n(s\wedge\tau_N)\right|_2^2\right].$$

Then we have a $\gamma_0 > 0$ such that

$$\Psi^n(t) \leq \gamma_0\int_0^t \Psi^{n-1}(s)ds,\ 0\leq t\leq T.$$

Thus for any fixed $N > 0$, we obtain that the sequence $\{X_\lambda^n(t\wedge\tau_N)\}$ is a Cauchy sequence in $L^2(\Omega, L^\infty(0,T;L_\sigma^2(D)))$. Next by the same method from (5.4) we have that the sequence $\{X_\lambda^n(t\wedge\tau_N)\}$ is a Cauchy sequence in $L^2(\Omega\times[0,T];D(A^{\frac{1}{2}}))$. On the other hand by the Chebyshev inequality and Lemma 7 we have that

$$P(\tau_N < T) = P\left(\sup_{0\leq s\leq T}|X_\lambda^n(s)|_2 \vee \sup_{0\leq s\leq T}\left|A^{\frac{1}{2}}X_\lambda^n(s)\right|_2 > N\right) \leq \frac{2B_e}{N^2}.$$

Thus it follows that $P(\tau_N < T) \to 0$ as $N \to \infty$. Thus the sequence $\{X_\lambda^n(t)\}$ is a Cauchy sequence in $L^2(\Omega;L^\infty(0,T;L_\sigma^2(D)))\cap L^2(\Omega\times[0,T];D(A^{\frac{1}{2}}))$. And hence there exists a $X_\lambda(t) \in L^2(\Omega, L^\infty(0,T;L_\sigma^2(D)))\cap L^2(\Omega\times[0,T];D(A^{\frac{1}{2}}))$ such that $X_\lambda^n(t)$ converges strongly to $X_\lambda(t)$. From (5.2) for any fixed $\varphi \in$

$C_0^\infty(D)$, $X_\lambda^n(t)$ satisfies that

$$(X_\lambda^{n+1}(t),\varphi)=\left(\phi+\int_0^t\left[-AX_\lambda^{n+1}(s)+B_\lambda(X_\lambda^{n+1}(s))\right]ds\right.$$
$$\left.+\int_0^t g(X_\lambda^n(s))dW(s),\varphi\right).$$

Furthermore since

$$\begin{aligned}&(B_\lambda(u,u)-B_\lambda(v,v),\varphi)\\&\quad=(B(J_\lambda u-J_\lambda v,u)+B(J_\lambda v,u-v),\varphi)\\&\quad\le C(\lambda)\left|u-v\right|_2\left|\nabla u\right|_2\left|\varphi\right|_2+C(\lambda)\left|u-v\right|_2\left|v\right|_2\left|\nabla\varphi\right|_2,\end{aligned}$$

we have that

$$\begin{aligned}&E\int_0^t(B_\lambda(X_\lambda^{n+1}(s))-B_\lambda(X_\lambda(s)),\varphi)ds\\&\le C(\lambda)E\int_0^t\left|X_\lambda^{n+1}(s)-X_\lambda(s)\right|_2\left|\nabla X_\lambda^{n+1}(s)\right|_2\left|\varphi\right|_2ds\\&\quad+C(\lambda)E\int_0^t\left|X_\lambda^{n+1}(s)-X_\lambda(s)\right|_2\left|X_\lambda(s)\right|_2\left|\nabla\varphi\right|_2ds\\&\le C(\lambda)\left(E\int_0^t\left|X_\lambda^{n+1}(s)-X_\lambda(s)\right|_2^2ds\right)^{\frac12}\left(E\int_0^t\left|A^{\frac12}X_\lambda^{n+1}(s)\right|_2^2ds\right)^{\frac12}\left|\varphi\right|_2\\&\quad+C(\lambda)\left(E\int_0^t\left|(X_\lambda^{n+1}(s)-X_\lambda(s))\right|_2^2ds\right)^{\frac12}\left(E\int_0^t\left|X_\lambda(s)\right|_2^2ds\right)^{\frac12}\left|\nabla\varphi\right|_2\\&\to0\quad\text{as }n\to\infty\end{aligned}$$

by the Hölder inequality and Lemma 7. Therefore the process $X_\lambda(t)$ satisfies that for $a.e.\ \omega\in\Omega$,

$$(X_\lambda(t),\varphi)=\left(\phi+\int_0^t\left[-AX_\lambda(s)+B_\lambda(X_\lambda(s))\right]ds+\int_0^t g(X_\lambda(s))dW(s),\varphi\right)$$

for $0\le t\le T$. Thus $X_\lambda(t)$ is a weak solution to (5.1). The proof of the lemma is complete. □

Finally we show that there exists a weak solution $X(t)$ to (2.1).

Theorem 2. *Assume that Conditions 2 and 3 are satisfied. Let $E\left|\phi\right|_r<\infty$ and $E\left|A^{\frac12}\phi\right|_r<\infty$ with $r=\frac54$. Then there exists a weak solution $X(t)$ to (2.1) with*

$$X\in L^\infty(0,T;L_\sigma^2(D))\cap L^2(0,T;D(A^{\frac12})),$$
$$X\varphi\in C(0,T;L_\sigma^r(D))\ \textit{for any fixed}\ \varphi\in C_0^\infty(D).$$

Proof. By lemma 8 there exists a weak solution $X_\lambda(t)$ to (5.1). Thus by the Ito formula

$$|X_\lambda(t)|_2^2 = |\phi|_2^2 - 2\int_0^t \left|A^{\frac{1}{2}}X_\lambda(s)\right|_2^2 ds + \int_0^t |g(X_\lambda(s))|_{L_0^2}^2 ds + 2\int_0^t (X_\lambda(s), g(X_\lambda(s))dW(s)).$$

By the Burkholder inequality and Condition 2 there exists a $\kappa > 0$ such that

$$\frac{1}{2}E\sup_{0\le s\le t}|X_\lambda(s)|_2^2 + 2E\int_0^t \left|A^{\frac{1}{2}}X_\lambda(s)\right|_2^2 ds \le E\,|\phi|_2^2 + (1+\kappa)L_g E\int_0^t |X_\lambda(s)|_2^2\, ds.$$

Thus by the Gronwall lemma and Lemma 2 we have that

$$E\sup_{0\le s\le T}|X_\lambda(s)|_2^2 \le 2E\,|\phi|_2^2\exp(2(1+\kappa)L_gT), \tag{5.6}$$

$$2E\int_0^T \left|A^{\frac{1}{2}}X_\lambda(s)\right|_2^2 ds \le E\,|\phi|_2^2 + M_\phi T, \tag{5.7}$$

where $M_\phi := (1+\kappa)L_g\left(2E\,|\phi|_2^2\exp(2(1+\kappa)L_gT)\right)$. Since $M_T := 2E\,|\phi|_2^2\exp(2(1+\kappa)L_gT) + E\,|\phi|_2^2 + M_\phi T$ are independent of λ and

$$E\left[\sup_{0\le s\le T}|X_\lambda(s)|_2^2 + 2\int_0^T \left|A^{\frac{1}{2}}X_\lambda(s)\right|_2^2 ds\right] \le M_T,$$

we obtain that

$$P\left(\cup_{M=1}^\infty \cap_{j=1}^\infty \cup_{\lambda=j}^\infty \left\{\sup_{0\le s\le T}|X_\lambda(s)|_2^2 + 2\int_0^T \left|A^{\frac{1}{2}}X_\lambda(s)\right|_2^2 ds \le M\right\}\right) = 1.$$

Thus there exist a subsequence $\{\alpha_k\}$ of $\{\lambda\}$ and some constant M_ω for each $\omega \in \Omega_1$, where Ω_1 is a subset of Ω with $P(\Omega\setminus\Omega_1) = 0$, such that for all $k \ge 1$

$$\left[\sup_{0\le t\le T}|X_{\alpha_k}(t)(\omega)|_2^4\right] \le M_\omega, \tag{5.8}$$

$$\int_0^T \left|A^{\frac{1}{2}}X_{\alpha_k}(t)(\omega)\right|_2^2 \le M_\omega. \tag{5.9}$$

(see Kim(p. 42, [18]). Thus it follows that for each $\omega \in \Omega_1$ there exist a $X(\omega)$ and a subsequence $\{\lambda_k\}$ of $\{\alpha_k\}$ such that as $k \to \infty$,

$$X_{\lambda_k}(\omega) \to X(\omega) \ \text{ weakly star in } L^\infty(0,T;L_\sigma^2(D)), \tag{5.10}$$

$$X_{\lambda_k}(\omega) \to X(\omega) \ \text{ weakly in } L^2(0,T;D(A^{\frac{1}{2}})). \tag{5.11}$$

Applying the Ito formula to $|X_{\lambda_k}(t)|_2^4$, we have that

$$\begin{aligned}|X_{\lambda_k}(t)|_2^4 &= |\phi|_2^4 + 4\int_0^t \left(X_{\lambda_k}(s)\,|X_{\lambda_k}(s)|_2^2, -AX_{\lambda_k}(s) + B_{\lambda_k}(X_{\lambda_k}(s))\right) ds \\ &\quad +6\int_0^t |g(X_{\lambda_k}(s))|_{L_2^0}^2 \,|X_{\lambda_k}(s)|_2^2\, ds \\ &\quad +4\int_0^t \left(|X_{\lambda_k}(s)|_2^2\, X_{\lambda_k}(s), g(X_{\lambda_k}(s))dW(s)\right) \\ &\le |\phi|_2^4 - 4\int_0^t \left|A^{\frac{1}{2}} X_{\lambda_k}(s)\right|_2^2 |X_{\lambda_k}(s)|_2^2\, ds \\ &\quad +6L_g\int_0^t |X_{\lambda_k}(s)|_2^4\, ds + 4\int_0^t \left(|X_{\lambda_k}(s)|_2^2\, X_{\lambda_k}(s), g(X_{\lambda_k}(s))dW(s)\right).\end{aligned}$$

Therefore, by the Burkfolder inequality and the Young inequality, there exists a positive constant c_T independent of λ_k such that

$$\begin{aligned}&\frac{1}{2}E\sup_{0\le\tau\le t}|X_{\lambda_k}(\tau)|_2^4 + 4\int_0^t E\left|A^{\frac{1}{2}}X_{\lambda_k}(s)\right|_2^2 |X_{\lambda_k}(s)|_2^2\, ds \\ &\le 2E\,|\phi|_2^4 + c_T\int_0^t E\sup_{0\le\tau\le s}|X_{\lambda_k}(\tau)|_2^4\, ds.\end{aligned}$$

Thus by the Gronwall lemma we have that

$$E\sup_{0\le t\le T}|X_{\lambda_k}(t)|_2^4 \le 4E\,|\phi|_2^4 \exp(2c_T T).$$

Let

$$Y_{\lambda_k}(t) := X_{\lambda_k}(t) - \int_0^t S(t-s)g(X_{\lambda_k}(s))dW(s).$$

Then

$$Y_{\lambda_k}(t) = S(t)\phi + \int_0^t S(t-s)B_{\lambda_k}\left(X_{\lambda_k}(s)\right) ds. \tag{5.12}$$

It follows that

$$\begin{aligned}E\int_0^T \left|A^{\frac{1}{2}}Y_{\lambda_k}(t)\right|_2^2 dt &\le 2E\int_0^T \left|A^{\frac{1}{2}}X_{\lambda_k}(t)\right|_2^2 dt \\ &\quad +2E\int_0^T \left|\int_0^t S(t-s)A^{\frac{1}{2}}g(X_{\lambda_k}(s))dW(s)\right|_2^2 dt \\ &\le 2E\int_0^T \left|A^{\frac{1}{2}}X_{\lambda_k}(t)\right|_2^2 dt \\ &\quad +2M^2E\int_0^T \left(\int_0^t \left|A^{\frac{1}{2}}g(X_{\lambda_k}(s))\right|_{L_2^0}^2 ds\right) dt \\ &\le 2E\int_0^T \left|A^{\frac{1}{2}}X_{\lambda_k}(t)\right|_2^2 dt + 2M^2L_GTE\int_0^T \left|A^{\frac{1}{2}}X_{\lambda_k}(s)\right|_2^2 ds\end{aligned}$$

Thus we have a $C_T > 0$ such that

$$E\int_0^T \left|A^{\frac{1}{2}}Y_{\lambda_k}(t)\right|_2^2 dt \le C_T. \tag{5.13}$$

Next we have a constant $\hat{c}_T > 0$ such that

$$\begin{aligned}
E\left[\sup_{0\le t\le T}|Y_{\lambda_k}(t)|_2^4\right] &\le 8E\left[\sup_{0\le t\le T}|X_{\lambda_k}(t)|_2^4\right] \\
&\quad +8E\left[\sup_{0\le t\le T}\left|\int_0^t S(t-s)g(X_{\lambda_k}(s))dW(s)\right|_2^4\right] \\
&\le 32E\,|\phi|_2^4\exp(2c_TT) + 8\hat{c}_TE\int_0^T |g(X_{\lambda_k}(s))|_{L_2^0}^4 ds \\
&\le C_T,
\end{aligned}$$

where $C_T := 32E\,|\phi|_2^4\exp(2c_TT)(1+\hat{c}_TL_g^2T)$. Thus we have that

$$P\left(\cup_{N=1}^{\infty}\cap_{j=1}^{\infty}\cup_{k=j}^{\infty}\left\{\sup_{0\le t\le T}|Y_{\lambda_k}(t)|_2^4 + \int_0^T\left|A^{\frac{1}{2}}Y_{\lambda_k}(t)(\omega)\right|_2^2 \le N\right\}\right) = 1.$$

And hence there exist some subsequence $\left\{\hat{\theta}_k\right\}$ of λ_k and some constant $N_\omega > 0$ for each $\omega \in \Omega_2$, where Ω_2 is a subset of Ω_1 with $P(\Omega\setminus\Omega_2) = 0$, such that for all $k \ge 1$

$$\left[\sup_{0\le t\le T}\left|Y_{\hat{\theta}_k}(t)(\omega)\right|_2^4\right] \le N_\omega, \tag{5.14}$$

$$\int_0^T\left|A^{\frac{1}{2}}Y_{\hat{\theta}_k}(t)(\omega)\right|_2^2 \le N_\omega \tag{5.15}$$

since C_T is independent of λ_k. And hence it follows that there exist a $Y(t)(\omega) \in L^\infty(0,T;L^2_\sigma(D))\cap L^2(0,T;D(A^{\frac{1}{2}}))$ and a subsequence $\{\theta_k\}$ of $\left\{\hat{\theta}_k\right\}$ such that

$$Y_{\theta_k}(t)(\omega) \to Y(t)(\omega) \text{ weakly star in } L^\infty(0,T;L^2_\sigma(D)), \tag{5.16}$$

$$Y_{\theta_k}(t)(\omega) \to Y(t)(\omega) \text{ weakly in } L^2(0,T;D(A^{\frac{1}{2}})). \tag{5.17}$$

Let $n = 3, r = \frac{n+2}{n+1}, \frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$ and $q = \frac{2(n+2)}{n}$. Then by the Hölder inequality, Lemma 2 and the Interpolation lemma,

$$\begin{aligned}
|B_{\theta_k}(X_{\theta_k}(t)(\omega))|_r &\le |J_{\theta_k}X_{\theta_k}(t)(\omega)|_q\left|A^{\frac{1}{2}}X_{\theta_k}(t)(\omega)\right|_2 \\
&\le C\,|J_{\theta_k}X_{\theta_k}(t)(\omega)|_2^{\frac{2}{n+2}}\,|J_{\theta_k}X_{\theta_k}(t)(\omega)|_{2^*}^{\frac{n}{n+2}}\left|A^{\frac{1}{2}}X_{\theta_k}(t)(\omega)\right|_2 \\
&\le C\,|X_{\theta_k}(t)(\omega)|_2^{\frac{2}{n+2}}\left|A^{\frac{1}{2}}X_{\theta_k}(t)(\omega)\right|_2^{\frac{2}{r}},
\end{aligned}$$

where we used Lemma 3.3 [14]. Since by (5.8) and (5.9) for each $\omega \in \Omega_2$, there exists a $c_0(\omega) > 0$ such that

$$\int_0^T |B_{\theta_k}(X_{\theta_k}(t)(\omega))|_r^r \, dt \le C \int_0^T |X_{\theta_k}(t)(\omega)|_2^{\frac{2r}{n+2}} \left| A^{\frac{1}{2}} X_{\theta_k}(t)(\omega) \right|_2^2 dt$$
$$< c_0(\omega).$$

Since $E\,|\phi|_{\frac{5}{4}}, E\left|A^{\frac{1}{2}}\phi\right|_{\frac{5}{4}} < \infty$, there exists an $a_\omega > 0$ for each $\omega \in \Omega_2$ such that $|\phi(\omega)|_{\frac{5}{4}} < a_\omega$ and $\left|A^{\frac{1}{2}}\phi(\omega)\right|_{\frac{5}{4}} < a_\omega$. Furthermore

$$\begin{aligned}
\int_0^\infty |AS(t)\phi(\omega)|_r^r \, dt &\le \int_0^1 |AS(t)\phi(\omega)|_r^r \, dt + \int_1^\infty |AS(t)\phi(\omega)|_r^r \, dt \\
&\le \int_0^1 \left|A^{\frac{1}{2}}S(t)\right|^r \left|A^{\frac{1}{2}}\phi(\omega)\right|_r^r dt + \int_1^\infty |AS(t)|^r \, |\phi(\omega)|_r^r \, dt \\
&\le M_{\frac{1}{2}}^r \int_0^1 t^{-\frac{r}{2}} dt \left|A^{\frac{1}{2}}\phi(\omega)\right|_r^r + M_1^r \int_1^\infty t^{-r} \, |\phi(\omega)|_r^r \, dt \\
&\le \frac{8M_{\frac{1}{2}}^r \left|A^{\frac{1}{2}}\phi(\omega)\right|_r^r}{3} + 4M_1^r \, |\phi(\omega)|_r^r \\
&< \infty, \qquad r = \frac{5}{4}.
\end{aligned}$$

From (5.12), by Lemma 3, for $a.e.\ \omega \in \Omega$(we use this presentation notation if no cofusion arises) and each $\theta_k > 0$,

$$\frac{d}{dt}Y_{\theta_k}(t)(\omega) + A_r Y_{\theta_k}(t)(\omega) = B_{\theta_k}\left(X_{\theta_k}(t)(\omega)\right). \tag{5.18}$$

Thus from Lemma 3, there exists a $C_T(\omega) > 0$ such that

$$\int_0^T \left|\frac{d}{dt}Y_{\theta_k}(t)(\omega)\right|_r^r dt < C_T(\omega). \tag{5.19}$$

Thus for any fixed $\varphi \in C_0^\infty(D)$,

$$\int_0^T \left|\frac{d}{dt}(Y_{\theta_k}(t)(\omega)\varphi)\right|_r^r dt < |\varphi|_\infty^r \, C_T(\omega). \tag{5.20}$$

And let B_δ be a closed ball with a radius δ in $C([0,T]; L_\sigma^r(K(\varphi))$. Then $\{Y(t) \in B_\delta\} \cap \Omega_1$ is $\mathcal{F}_t$−adapted since it holds that

$$\begin{aligned}
&\{Y(t) \in B_\delta\} \cap \Omega_1 \\
&\quad = \Omega_1 \cap \cup_{L=1}^\infty \cap_{\nu=1}^\infty \cap_{j=1}^\infty \cup_{k=j}^\infty \left((Y_{\theta_k}(t) \in B_{\delta+\frac{1}{\nu}}) \cap \sup_{0\le s\le t} |Y_{\theta_k}(s)| \le L \right),
\end{aligned}$$

where let Ω_1 be a subset of Ω with $P(\Omega \setminus \Omega_1) = 0$. Thus by Proposition 1.13 [15], for every $B_\alpha \in Borel(L_\sigma^2(D))$,

$$\{(s,\omega) : 0 \le s \le t, (Y(s,\omega) \in B_\alpha\} \in B([0,t]) \times \mathcal{F}_t.$$

Since every closed ball of finite radius in $L^2_\sigma(K(\varphi))$ is closed in $L^r_\sigma(K(\varphi))$, it follows that $Y(\omega)\varphi$ is an $L^2_\sigma(K(\varphi))-$valued progressively measurable. By (5.18) and (5.19) it holds that $Y(t)(\omega)\varphi \in L^\infty(0,T;L^2_\sigma(K(\varphi))\cap L^2(0,T;D(A^{\frac{1}{2}}))$ and

$$Y_{\theta_k}(t)(\omega)\varphi \to Y(t)(\omega)\varphi \text{ weakly star in } L^\infty(0,T;L^2_\sigma(K(\varphi)),$$

$$Y_{\theta_k}(t)(\omega)\varphi \to Y(t)(\omega)\varphi \text{ weakly in } L^2(0,T;D(A^{\frac{1}{2}}))$$

as $k\to\infty$. Since

$$D(A^{\frac{1}{2}})\cap L^2_\sigma(K(\varphi)) \subset L^2_\sigma(K(\varphi)) \subset L^r_\sigma(K(\varphi)),$$

by using the compactness lemma (Theorem 2.1, [14]) and by (5.16) for $a.e.\omega\in\Omega$, there exists a subsequence $\{\xi_i\}$ of $\{\rho_k\}$ such that

$$Y_{\xi_j}(t)(\omega)\varphi \to\ Y(t)(\omega)\varphi \text{ strongly in } L^2(0,T;L^2_\sigma(K(\varphi))), \tag{5.21}$$

as $j\to\infty$. We have that

$$\begin{aligned}
&E\left|X_{\xi_j}(t)\varphi - X_{\xi_{j+1}}(t)\varphi\right|_2^2\\
&\quad\le E\left|\left(Y_{\xi_j}(t)\varphi + \varphi\int_0^t S(t-s)g(X_{\xi_j}(s))dW(s)\right)\right.\\
&\qquad\left.-\left(Y_{\xi_{j+1}}(t)\varphi + \varphi\int_0^t S(t-s)g(X_{\xi_{j+1}}(s))dW(s)\right)\right|_2^2\\
&\quad\le 2E\left|Y_{\xi_j}(t)\varphi - Y_{\xi_{j+1}}(t)\varphi\right|_2^2\\
&\qquad+2E\left|\varphi\int_0^t S(t-s)g(X_{\xi_j}(s))dW(s) - \varphi\int_0^t S(t-s)g(X_{\xi_{j+1}}(s))dW(s)\right|_2^2\\
&\quad\le 2E\left|Y_{\xi_j}(t)\varphi - Y_{\xi_{j+1}}(t)\varphi\right|_2^2 + 2M^2CL_g^2\int_0^t E\left|X_{\xi_j}(s)\varphi - X_{\xi_{j+1}}(s)\varphi\right|_2^2 ds.
\end{aligned}$$

Thus by the Gronwall lemma it follows that

$$\begin{aligned}
&E\int_0^T \left|X_{\xi_j}(t)\varphi - X_{\xi_{j+1}}(t)\varphi\right|_2^2 dt\\
&\quad\le 2E\int_0^T \left|Y_{\xi_j}(t)\varphi - Y_{\xi_{j+1}}(t)\varphi\right|_2^2 dt \exp\left(2M^2CL_g^2T\right)\\
&\quad\to 0 \quad \text{as } j\to\infty.
\end{aligned}$$

by using (5.21). It follows that the sequence $\left\{X_{\xi_j}(t)(\omega)\varphi\right\}$ is a Cauchy sequence in mean square. Thus there exists a subsequence $\{\rho_k\}$ of $\{\xi_j\}$ such that $X_{\rho_k}(t)(\omega)\varphi$ converges strongly to $X(t)(\omega)\varphi$ in $L^2(0,T;L^2_\sigma(D))$ as $k\to\infty$ since $\left\{X_{\xi_j}(t)(\omega)\varphi\right\}$ has a subsequence which converges weakly to $X(t)(\omega)\varphi$

by (5.10) and (5.11). We use the notation ρ instead of ρ_k for the simplicity.

$$\begin{aligned}&\left|\int_0^t (B_\rho(X_\rho(s)(\omega)) - B(X(s)(\omega)), \varphi) ds\right| \\ &\le \int_0^t |b(J_\rho X_\rho(s)(\omega) - X(s)(\omega), X_\rho(s)(\omega), \varphi)|\, ds \\ &\quad + \int_0^t |b(X(s)(\omega), X_\rho(s)(\omega) - X(s)(\omega), \varphi)|\, ds \\ &:= I_1 + I_2,\end{aligned}$$

and hence it follows that for a.e. $\omega \in \Omega$,

$$\begin{aligned}I_1 &\le \int_0^t \int_{K(\varphi)} |\varphi(J_\rho X_\rho(s)(\omega) - X(s)(\omega)|\, |\nabla X_\rho(s)(\omega)|\, ds \\ &\le |\varphi|_\infty \int_0^t |J_\rho X_\rho(s)(\omega) - X(s)(\omega)|_{2,K(\varphi)} |\nabla X_\rho(s)(\omega)|_{2,K(\varphi)}\, ds \\ &\le |\varphi|_\infty \left(\int_0^t |J_\rho X_\rho(s)(\omega) - X(s)(\omega)|^2_{2,K(\varphi)}\right)^{\frac{1}{2}} \left(\int_0^t |\nabla X_\rho(s)(\omega)|^2_{2,K(\varphi)}\, ds\right)^{\frac{1}{2}} \\ &\to 0 \quad \text{as} \quad \rho \to \infty,\end{aligned}$$

where $|f|^2_{2,K(\varphi)} := \int_{K(\varphi)} |f(x)|^2\, dx$ and we used that

$$\begin{aligned}|J_\rho X_\rho(s) - X(s)|_{2,K(\varphi)} &\le |J_\rho X_\rho(s) - J_\rho X(s)|_{2,K(\varphi)} + |J_\rho X(s) - X(s)|_{2,K(\varphi)} \\ &\le |X_\rho(s) - X(s)|_{2,K(\varphi)} + |J_\rho X(s) - X(s)|_{2,K(\varphi)}\end{aligned}$$

and $|\nabla X_\rho(s)(\omega)|_2 = \left|A^{\frac{1}{2}} X_\rho(s)(\omega)\right|_2$ (see[13]). Then we can choose a a suitable subsequence ρ_i of ρ. Let $\xi(x)$ be a positive function in C_0^∞ such that $K(\varphi) \subset K(\xi)$ and $\xi = 1$ on $K(\varphi)$, where $K(\varphi)$ and $K(\xi)$ denote the compact support of φ and ξ, respectively. We have that

$$\begin{aligned}|(X_{\rho_i}(s)(\omega) - X(s)(\omega))|_{2,K(\varphi)} &\le |(X_{\rho_i}(s)(\omega)\xi - X(s)(\omega)\xi)|_{2,K(\xi)} \\ &\to 0 \quad \text{as } i \to \infty.\end{aligned}$$

Hence

$$\begin{aligned}I_2 &\le \int_0^t \left|\int_{K(\varphi)} \nabla(\varphi X(s)(\omega))(X_\rho(s)(\omega) - X(s)(\omega)) dx\right| ds \\ &\le \int_0^t \left(\int_{K(\varphi)} |\nabla(\varphi X(s)(\omega))|^2\right)^{\frac{1}{2}} |(X_{\rho_i}(s)(\omega) - X(s)(\omega))|_{2,K(\varphi)}\, ds \\ &\le c \int_0^t |(X_{\rho_i}(s)(\omega) - X(s)(\omega))|_{2,K(\varphi)}\, ds \to 0 \quad \text{as} \quad i \to \infty.\end{aligned}$$

Thus we have that

$$\left|\int_0^t (B_{\rho_i}(X_\rho(s)(\omega)) - B(X(s)(\omega)), \varphi) ds\right| \to 0$$

as $i \to \infty$. Next we assume that $\varphi \ge 0$ without loss of generality

$$\begin{aligned}
&E\left|\int_0^t (g(X_\rho(s)) - g(X(s)), \varphi) dW(s)\right| \\
&= E\left|\int_0^t ((g(X_\rho(s)) - g(X(s)))dW(s), \varphi)\right| \\
&\le c\left(\int_0^t E\left|\varphi g(X_\rho(s)) - \varphi g(X(s))\right|_{L_2^0}^2 ds\right)^{\frac{1}{2}} \\
&\le cL_g^{\frac{1}{2}}\left(\int_0^t E\varphi\left|X_\rho(s) - X(s)\right|_2^2 ds\right)^{\frac{1}{2}} \\
&\le cL_g^{\frac{1}{2}}\left(E\int_0^t \left|\varphi X_\rho(s) - \varphi X(s)\right|_2^2 ds\right)^{\frac{1}{2}} \\
&\to 0.
\end{aligned}$$

Thus we obtain that almost surely

$$(X(t), \varphi) = (\phi - \int_0^t AX(s)ds + \int_0^t B\left(X(s)\right) ds + \int_0^t g(X(s))dW(s), \varphi).$$

Consequently $X(t)$ is a weak solution to (2.1). This completes the proof of the theorem. □

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