

A wave equation with structural damping and nonlinear memory

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Abstract. In this paper, we obtain the *critical exponent* for a wave equation with structural damping and nonlinear memory:

$$u_{tt} - \Delta u + \mu (-\Delta)^{\frac{1}{2}} u_t = \int_0^t (t-s)^{-\gamma} |u(s, \cdot)|^p ds,$$

where $\mu > 0$. In the supercritical case, we prove the existence of small data global solutions, whereas, in the subcritical case, we prove the nonexistence of global solutions for suitable arbitrarily small data, in the special case $\mu = 2$.

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1. Introduction

We study the global existence of small data solutions to

$$\begin{cases} u_{tt} - \Delta u + \mu (-\Delta)^{\frac{1}{2}} u_t = F(t, u), \\ u(0, x) = u_0(x), \\ u_t(0, x) = u_1(x), \end{cases} \quad (1.1)$$

where $\mu > 0$, and the term

$$F(t, u) := \int_0^t (t-s)^{-\gamma} |u(s, \cdot)|^p ds, \quad (1.2)$$

for some $\gamma \in (0, 1)$ and $p > 1$, represents a nonlinear memory.

Recently, fractional PDEs have been deeply investigated, since they are particularly interesting for the real world applications. According to [23], fractional derivatives provide an excellent instrument for the description of memory and hereditary process. The advantage of fractional derivatives become

apparent in modelling mechanical and electrical properties of real materials and in many other fields.

In particular, the linear part of the equation in (1.1), i.e.

$$u_{tt} - \Delta u + \mu(-\Delta)^{\frac{1}{2}}u_t = 0, \tag{1.3}$$

is frequently used in the determination of lifespan for primary or rechargeable batteries [10]. The model in (1.3) is a special case of a wave equation with *structural* damping:

$$u_{tt} - \Delta u + \mu(-\Delta)^\sigma u_t = 0, \quad \sigma > 0. \tag{1.4}$$

Different kind of estimates in Sobolev spaces for (1.4) with $\sigma \in (0, 1)$ have been recently studied in [1, 14, 22]. In the limit case $\sigma = 1$, the damping is also called *visco-elastic* and it has been studied in [18, 26] and, in abstract setting, in [13, 16]. Some $L^2 - L^2$ estimates have been derived for (1.4) in the case with time-dependent damping $b(t)(-\Delta)^\sigma u_t$ in [19, 24]. Smoothing effects for (1.4) are studied in [12], including the case $\sigma \geq 1$.

The nonlinear term $F(t, u)$ in (1.2) may be written as

$$F(t, u) = \Gamma(1 - \gamma)J_{0|t}^{1-\gamma}(|u|^p),$$

where Γ is the Euler Gamma function, and $J_{0|t}^{1-\gamma}(|u|^p)$ is the fractional Riemann–Liouville integral of $|u(\tau, \cdot)|^p$ in $[0, t]$. Therefore, it is reasonable to expect relations with the case of a power nonlinearity $F(u) = |u|^p$, as $\gamma \rightarrow 1$. It has been recently proved [8] that small data global solutions to

$$\begin{cases} u_{tt} - \Delta u + \mu(-\Delta)^{\frac{1}{2}}u_t = |u|^p, \\ u(0, x) = u_0(x), \\ u_t(0, x) = u_1(x), \end{cases} \tag{1.5}$$

exist if $p > 1 + 2/(n - 1)$, and that this exponent is *optimal*. On the other hand, the *optimal critical exponent* for global small data solutions to

$$\begin{cases} u_{tt} - \Delta u + \mu u_t = F(t, u), \\ u(0, x) = u_0(x), \\ u_t(0, x) = u_1(x), \end{cases} \tag{1.6}$$

is given by (see [4, 11])

$$\max \{ \tilde{p}_\gamma(n), \gamma^{-1} \}, \quad \tilde{p}_\gamma(n) := 1 + \frac{2(2 - \gamma)}{[n - 2(1 - \gamma)]_+}. \tag{1.7}$$

From the point of view of the global existence argument, the transition from the exponent $\tilde{p}_\gamma(n)$ to the exponent γ^{-1} , which occurs at $\gamma = (n - 2)/n$, is deeply related to the profile of the decay estimates for the solution to (1.6).

It is natural to ask in which way the *critical exponent* is influenced if we replace the power nonlinearity in (1.5) by a nonlinear memory term.

The purpose of this paper is to give an answer to this question. We study into details the model in (1.3), stating only partial results for the general case (1.4) (see Sect. 5), for two reasons.

- The special structure of (1.3) allows us to derive a global existence result for (1.1) in any space dimension $n \geq 2$ (see later, Remark 3.1).
- It is possible to use the test function method to prove a counterpart result of nonexistence of global solutions thanks to a maximum principle (see later, Lemma 2.10).

We will prove that the *critical exponent* for (1.1) is

$$\bar{p}(n, \gamma) = \max\{p_\gamma(n), \gamma^{-1}\}, \quad p_\gamma(n) := 1 + \frac{3 - \gamma}{n + \gamma - 2}, \tag{1.8}$$

for any $n \geq 2$. We notice that

$$\bar{p}(2, \gamma) = p_\gamma(2) = 3\gamma^{-1},$$

whereas for any $n \geq 3$ it holds

$$\bar{p}(n, \gamma) = \begin{cases} p_\gamma(n) & \text{if } (n - 2)/n \leq \gamma < 1, \\ \gamma^{-1} & \text{if } 0 < \gamma \leq (n - 2)/n. \end{cases}$$

In particular, the threshold $(n - 2)/n$ for γ , which denotes the transition to the *critical exponent* γ^{-1} is the same for both models (1.1) and (1.6). We notice that $\tilde{p}_\gamma(n) < p_\gamma(n)$ if, and only if, $\gamma \in ((n - 2)/n, 1)$. This suggests that if the nonlinear memory term is sufficiently *strong* in space dimension $n \geq 3$, namely $\gamma \in (0, (n - 2)/n]$, then the influence from classical and structural damping is the same. Otherwise, the *critical exponent* is larger in the structural damping case, as it happens for the power nonlinearity $|u|^p$.

2. Results

In each statement of this paper, we will briefly denote the normed space of initial data of the Cauchy problem as \mathcal{A} . Therefore, the definition of \mathcal{A} will be related to the statement in which it appears. By the letter $\varepsilon > 0$ we will denote the smallness of initial data in the norm of \mathcal{A} . On the other hand, in the following, we will respectively denote sufficiently small and sufficiently large positive constants by $\epsilon > 0$ and $C > 0$.

In the following, we generalize the nonlinear term in (1.2) to

$$F(t, u) = \int_0^t (t - s)^{-\gamma} g(u(s, \cdot)) ds, \tag{2.1}$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function satisfying

$$g(0) = 0, \quad |g(u) - g(v)| \lesssim |u - v| (|u|^{p-1} + |v|^{p-1}), \quad \text{for some } p > 1. \tag{2.2}$$

Definition 2.1. For any $q \in [1, \infty)$, we define $q^\sharp = q^\sharp(n, q)$ as

$$q^\sharp := \max \left\{ \frac{nq}{n + q}, 1 \right\}. \tag{2.3}$$

If $q = \infty$, we put $\infty^\sharp = n$.

We notice that q^\sharp is the Sobolev conjugate of q if $q \geq n/(n - 1)$.

Definition 2.2. For any function $u \in \mathcal{C}([0, T], H^1) \cap \mathcal{C}^1([0, T], L^2)$ we define

$$\mathcal{E}[u](t) := \|\nabla u(t, \cdot)\|_{L^2} + \|u_t(t, \cdot)\|_{L^2},$$

which is equivalent to the square root of the classical energy of u .

We are now ready to state our results. For the sake of clarity, we first deal with space dimension $n \geq 3$.

Theorem 2.3. Let $n \geq 3$, $\gamma \in (0, 1)$ and $p > \bar{p}(n, \gamma)$. Let

$$r \geq \max \left\{ n(p - 1), \frac{2n}{n - 2} \right\}, \tag{2.4}$$

and r^\sharp be defined as in (2.3). Then there exists $\varepsilon > 0$ such that for any initial data

$$(u_0, u_1) \in \mathcal{A} := (L^1 \cap H^1 \cap L^r) \times (L^1 \cap L^{r^\sharp}) \quad \text{with } \|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon, \tag{2.5}$$

there exists a $\mathcal{C}([0, \infty), L^1 \cap H^1 \cap L^r) \cap \mathcal{C}^1([0, \infty), L^2)$ solution to (1.1). Moreover, the solution satisfies the estimates

$$\|u(t, \cdot)\|_{L^q} \leq \begin{cases} C(1+t)^{-\gamma+2-n(1-\frac{1}{q})} \|(u_0, u_1)\|_{\mathcal{A}} & \text{if } q \in [1, n/(n-2)), \\ C(1+t)^{-\gamma} \ln(e+t) \|(u_0, u_1)\|_{\mathcal{A}} & \text{if } q = n/(n-2), \\ C(1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}} & \text{if } q \in (n/(n-2), r], \end{cases} \tag{2.6}$$

$$\mathcal{E}[u](t) \leq C(1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}. \tag{2.7}$$

If we set $r = \infty$ in Theorem 2.7, we obtain a statement for any $p > \bar{p}(n, \gamma)$.

Remark 2.4. If (2.4) holds for some $p > 1$ and $n \geq 3$, then it follows that, in particular, $r \geq 2p$. Indeed, if $p \leq n/(n-2)$ the statement trivially follows from $r \geq 2n/(n-2)$. Otherwise, if $p > n/(n-2)$ then it holds

$$r \geq n(p - 1) = 2p + (n - 2)p - n > 2p.$$

The bound $r \geq 2n/(n-2)$ in Theorem 2.3 is a consequence of the assumption $u_0 \in H^1$, since $H^1 \subset L^{\frac{2n}{n-2}}$ for $n \geq 3$. Indeed, if we set $\bar{r} = 2n/(n-2)$, then $\bar{r}^\sharp = 2$ and we obtain the following.

Corollary 2.5. Let $n \geq 3$, $\gamma \in ((n-2)/n, 1)$ and $p_\gamma(n) < p \leq n/(n-2)$. Then there exists $\varepsilon > 0$ such that for any initial data

$$(u_0, u_1) \in \mathcal{A} := (L^1 \cap H^1) \times (L^1 \cap L^2) \quad \text{with } \|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon, \tag{2.8}$$

there exists a $\mathcal{C}([0, \infty), L^1 \cap H^1) \cap \mathcal{C}^1([0, \infty), L^2)$ solution to (1.1). Moreover, the solution satisfies estimates (2.6) with $r = 2n/(n-2)$, and estimate (2.7).

The restriction $p \leq n/(n-2)$ in Corollary 2.5 is equivalent to assumption (2.4) for $r = 2n/(n-2)$, in Theorem 2.3. Moreover, the restriction $p \leq n/(n-2)$ implies that the interval $(\bar{p}(n, \gamma), n/(n-2)]$ is nonempty if, and only if, $\gamma \in ((n-2)/n, 1)$. This gives $\bar{p}(n, \gamma) = p_\gamma(n)$.

If $n = 2$, we have the following.

Theorem 2.6. *Let $n = 2$, $\gamma \in (0, 1)$ and $p > 3\gamma^{-1}$. Then there exists $\varepsilon > 0$ such that for any initial data as in (2.8), there exists a $\mathcal{C}([0, \infty), L^1 \cap H^1) \cap C^1([0, \infty), L^2)$ solution to (1.1). Moreover, the solution satisfies the estimates*

$$\|u(t, \cdot)\|_{L^q} \leq C(1+t)^{-\gamma+2/q} \|(u_0, u_1)\|_{\mathcal{A}} \quad \text{for any } q \in [1, \infty), \tag{2.9}$$

$$\mathcal{E}[u](t) \leq C(1+t)^{-\gamma} \ln(e+t) \|(u_0, u_1)\|_{\mathcal{A}}. \tag{2.10}$$

If we also assume that $u_0 \in L^\infty$ and $\|u_0\|_{L^\infty} \leq \varepsilon$, with ε as in (2.8), then the solution u is also continuous from $[0, \infty)$ into L^∞ and it satisfies the following estimate:

$$\|u(t, \cdot)\|_{L^\infty} \leq C(1+t)^{-\gamma} \ln(e+t) \|(u_0, u_1)\|_{\mathcal{A}}. \tag{2.11}$$

If we are not interested in energy solutions, we may drop the assumption $u_0 \in H^1$ in Theorem 2.3. In this case, we say that $u \in \mathcal{C}([0, \infty), L^1 \cap L^r)$ is a global solution to (1.1) if it solves (1.1) in a weak sense. We may now state the following.

Theorem 2.7. *Let $n \geq 2$ and $\gamma \in (0, 1)$. Let $p > \bar{p}(n, \gamma)$, $r \geq n(p-1)$ and r^\sharp as in (2.3). Then there exists $\varepsilon > 0$ such that for any initial data*

$$(u_0, u_1) \in \mathcal{A} := (L^1 \cap L^r) \times (L^1 \cap L^{r^\sharp}) \quad \text{with } \|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon, \tag{2.12}$$

there exists a weak $\mathcal{C}([0, \infty), L^1 \cap L^r)$ solution to (1.1). Moreover, the solution satisfies estimate (2.6) if $n \geq 3$, and estimate

$$\|u(t, \cdot)\|_{L^q} \leq C(1+t)^{-\gamma+2/q} \|(u_0, u_1)\|_{\mathcal{A}} \quad \text{for any } q \in (1, r], \tag{2.13}$$

if $n = 2$ and $r < \infty$, or estimates (2.9) and (2.11), if $n = 2$ and $r = \infty$.

The Sobolev exponent r in Theorem 2.7 may be now lesser than $2n/(n-2)$, since we dropped the hypothesis $u_0 \in H^1$. However, restriction $r \geq n(p-1)$ implies that $r > 2n/(n-1)$ for any $\gamma \in (0, 1)$ and $p > \bar{p}(n, \gamma)$. In particular, the interval $(n/(n-2), r]$ in estimate (2.6) is never empty if $n \geq 3$.

Remark 2.8. Assumptions $r \geq n(p-1)$ and $p > \bar{p}(n, \gamma)$ imply that $r > p+1$ for any $\gamma \in (0, 1)$. Indeed, $\bar{p}(n, \gamma) > 1 + 2/(n-1)$ for any $\gamma \in (0, 1)$, hence

$$r \geq p-1 + (n-1)(p-1) > p-1 + (n-1)(\bar{p}(n, \gamma) - 1) > p+1.$$

Remark 2.9. In estimates (2.6), (2.7), (2.9), (2.10), (2.11) and (2.13) it does not appear a decay rate better than $(1+t)^{-\gamma}$. This restriction comes from the influence of the nonlinear memory term (see Lemma 3.2). In particular, if $q < n/(n-2)$, a loss of decay $(1+t)^{1-\gamma}$ with respect to the corresponding linear estimates [see (3.3)] appears in estimate (2.6). This loss is larger if $q > n/(n-2)$. This phenomenon has been described in [4].

2.1. Optimality of the critical exponent

The fractional Laplacian is a nonlocal operator, thus it is not easy, in general, to prove the optimality of $\bar{p}(n, \gamma)$, by using the test function method. We are able to do that if $\mu = 2$ and $u_0 \equiv 0$ in (1.1), thanks to a maximum principle for (1.3).

Lemma 2.10. *The solution to*

$$\begin{cases} u_{tt} - \Delta u + 2(-\Delta)^{\frac{1}{2}}u_t = 0, \\ u(0, x) = 0, \\ u_t(0, x) = u_1(x) \geq 0, \end{cases} \tag{2.14}$$

is nonnegative, and it is explicitly given by

$$u(t, x) = c_n \frac{t}{(|x|^2 + t^2)^{\frac{n+1}{2}}} \ast(x) u_1(x),$$

where c_n depends on the space dimension [22].

Thanks to Lemma 2.10, we can state the following.

Theorem 2.11. *Let us consider the Cauchy problem*

$$\begin{cases} u_{tt} - \Delta u + 2(-\Delta)^{\frac{1}{2}}u_t = \int_0^t (t-s)^{-\gamma} u(s, x)^p ds, \\ u(0, x) = 0, \\ u_t(0, x) = u_1(x) \geq 0, \end{cases} \tag{2.15}$$

with $\gamma \in (0, 1)$, $p > 1$ and $u_1 \in L^1_{loc}$, non trivial. Then there exists no global weak solution to (2.15) in the following four cases:

- *if $n = 1$, for any $\gamma \in (0, 1)$ and $p > 1$,*
- *if $n \geq 2$, for any $\gamma \in [(n - 2)/n, 1)$ and $p \in (1, p_\gamma(n)]$,*
- *if $n \geq 3$, for any $\gamma \in (0, (n - 2)/n)$ and $p \in (1, \gamma^{-1})$.*

In view of the previously stated results of global existence, Theorem 2.11 says that the *critical exponent* $\bar{p}(n, \gamma)$ is *optimal*, in general. It remains open to study what happens if $\gamma \in (0, (n - 2)/n)$ and $p = \gamma^{-1}$, since this case is not included in our global existence statements, neither in Theorem 2.11.

3. Proof of the global existence results

In [22] it has been proved that the solution to the linear problem

$$\begin{cases} v_{tt} - \Delta v + \mu(-\Delta)^{\frac{1}{2}}v_t = 0, \\ v(0, x) = v_0(x), \\ v_t(0, x) = v_1(x). \end{cases} \tag{3.1}$$

satisfies the estimate

$$\|v(t, \cdot)\|_{L^q} \leq t^{-n(\frac{1}{q_0} - \frac{1}{q})} \|v_0\|_{L^{q_0}} + t^{1-n(\frac{1}{q_1} - \frac{1}{q})} \|v_1\|_{L^{q_1}}, \tag{3.2}$$

for any $q_0, q_1 \geq 1$ and $q \geq \max\{q_0, q_1\}$. In particular, setting $q_0 = q_1 = 1$ for $t \geq 1$, and $q_0 = q, q_1 = q^\sharp$ for $t \in [0, 1]$, we may derive

$$\|v(t, \cdot)\|_{L^q} \leq (1+t)^{1-n(1-\frac{1}{q})} (\|v_0\|_{L^1 \cap L^q} + \|v_1\|_{L^1 \cap L^{q^\sharp}}), \tag{3.3}$$

where q^\sharp is as in (2.3). Indeed,

$$1 - n \left(\frac{1}{q^\sharp} - \frac{1}{q} \right) \geq 1 - n \left(\frac{n+q}{nq} - \frac{1}{q} \right) = 0.$$

The solution to (3.1) satisfies the energy estimate (see, for instance, [8]):

$$\mathcal{E}[v](t) \leq (1+t)^{-\frac{\alpha}{2}} (\|v_0\|_{L^1 \cap H^1} + \|v_1\|_{L^1 \cap L^2}). \tag{3.4}$$

Remark 3.1. An important difference between linear estimates (3.2) for (1.1) and the corresponding linear estimates for (1.6) is that, in this latter case, a loss of regularity appears in the estimate for $\|u(t, \cdot)\|_{L^q}$ when $q \neq 2$ (see, for instance, [21]). This loss of regularity increases as the space dimension n grows, imposing an heavy restriction to the applicability of these estimates to prove global existence in high space dimension. This restriction does not appear in (1.1), so we may obtain a global existence result in any space dimension $n \geq 2$, and this motivated us to focus on this model.

We remark that in the case of the classical damped wave equation with power nonlinearity $|u|^p$ this problem has been solved in [15,27] by using weighted energy estimates for the nonlinear problem.

A fundamental role will be played by the following.

Lemma 3.2. *Let $\alpha \in \mathbb{R}$, $\beta > 1$ and $\gamma \in (0, 1)$. Then it holds*

$$\int_0^t (1+t-\tau)^{-\alpha} \int_0^\tau (\tau-s)^{-\gamma} (1+s)^{-\beta} ds d\tau \lesssim \begin{cases} (1+t)^{1-\alpha-\gamma} & \text{if } \alpha < 1, \\ (1+t)^{-\gamma} \ell(t) & \text{if } \alpha = 1, \\ (1+t)^{-\gamma} & \text{if } \alpha > 1, \end{cases}$$

where $\ell(t) := \ln(e+t)$.

Proof. First let us consider the interior integral. Since $\gamma \in (0, 1)$ and $\beta > 1$, we may estimate (see, for instance, Lemma 4.1 in [3])

$$\int_0^\tau (\tau-s)^{-\gamma} (1+s)^{-\beta} ds \lesssim (1+\tau)^{-\gamma}.$$

Therefore,

$$\int_0^t (1+t-\tau)^{-\alpha} \int_0^\tau (\tau-s)^{-\gamma} (1+s)^{-\beta} ds d\tau \lesssim \int_0^t (1+t-\tau)^{-\alpha} (1+\tau)^{-\gamma} d\tau,$$

and the proof immediately follows (see, for instance, [9,20]). □

We also recall Gagliardo–Nirenberg inequality:

$$\|u\|_{L^q} \leq C \|u\|_{L^2}^{1-\eta} \|\nabla u\|_{L^2}^\eta, \quad \eta = n \left(\frac{1}{2} - \frac{1}{q} \right), \tag{3.5}$$

for any $q \in [2, \infty]$ if $n = 1$, for any $q \in [2, \infty)$ if $n = 2$, and for any $q \in [2, 2n/(n-2)]$ if $n \geq 3$.

We are now ready to prove our statements.

Proof of Theorem 2.3. For any $T > 0$, we introduce the space

$$X(T) := \mathcal{C}([0, T], L^1 \cap H^1 \cap L^r) \cap \mathcal{C}^1([0, T], L^2), \tag{3.6}$$

with norm given by

$$\|w\|_{X(T)} := \max_{t \in [0, T]} \left\{ (1+t)^{-(2-\gamma)} \|w(t, \cdot)\|_{L^1} + (1+t)^\gamma (\ell(t)^{-1} \|w(t, \cdot)\|_{L^{\bar{q}}} + \|w(t, \cdot)\|_{L^r} + \mathcal{E}[w](t)) \right\}, \quad (3.7)$$

where

$$\bar{q} = \bar{q}(n) := \frac{n}{n-2}, \quad (3.8)$$

and $\ell(t) := \ln(e+t)$ here and in the following. If $w \in X(T)$ then we derive, by interpolation, that

$$\|w(t, \cdot)\|_{L^q} \leq (1+t)^{-\gamma + [2-n(1-1/q)]^+} \ell(t) \|w\|_{X(T)}, \quad \text{for any } q \in [1, r]. \quad (3.9)$$

We consider the operator N defined by

$$Nw := u^{\text{lin}} + Gw, \quad \text{with } Gw := \int_0^t E(t-\tau, x) *_{(x)} F(\tau, w) \, d\tau, \quad (3.10)$$

where $u^{\text{lin}}(t, x)$ is the solution to (3.1) with $(v_0, v_1) = (u_0, u_1)$, and $E(t, x)$ is the fundamental solution to (3.1) for $v_0 \equiv 0$ and $v_1 = \delta$. A function $w \in X(T)$ is a solution to (1.1) for any $t \in [0, T]$ if, and only if, $w = Nw$ in $X(T)$. If we prove that

$$\|Nw\|_{X(T)} \leq C_1 \|(u_0, u_1)\|_{\mathcal{A}} + C_2 \|w\|_{X(T)}^p, \quad (3.11)$$

$$\|Nw - N\tilde{w}\|_{X(T)} \leq C \|w - \tilde{w}\|_{X(T)} (\|w\|_{X(T)}^{p-1} + \|\tilde{w}\|_{X(T)}^{p-1}), \quad (3.12)$$

where C_1, C_2 and C do not depend on T , by standard arguments (see, for instance, [7]), we may derive the existence of a unique fixed point of N in $X(T)$, and then the existence of small data global solutions to (1.1), satisfying

$$\|u\|_{X(T)} \leq \tilde{C} \|(u_0, u_1)\|_{\mathcal{A}}, \quad \text{uniformly for any } T > 0. \quad (3.13)$$

Recalling the definition of \mathcal{A} and setting $u_0 = v_0$ and $u_1 = v_1$ in estimates (3.3) and (3.4), we immediately derive that

$$\|u^{\text{lin}}\|_{X(T)} \leq C_1 \|(u_0, u_1)\|_{\mathcal{A}},$$

thus (3.11) follows once we prove that

$$\|Gw\|_{X(T)} \leq C \|w\|_{X(T)}^p. \quad (3.14)$$

On the other hand, it is clear that we may rewrite (3.12) as

$$\|Gw - G\tilde{w}\|_{X(T)} \leq C \|w - \tilde{w}\|_{X(T)} \left(\|w\|_{X(T)}^{p-1} + \|\tilde{w}\|_{X(T)}^{p-1} \right). \quad (3.15)$$

We first prove (3.14). Recalling the definition of the operator G in (3.10) and (2.2), thanks to (3.3) and (3.4), we obtain

$$\begin{aligned} \|Gw(t, \cdot)\|_{L^q} &\leq C \int_0^t (1+t-\tau)^{1-n(1-\frac{1}{q})} \|F(\tau, w)\|_{L^1 \cap L^{q^\sharp}} d\tau \\ &\leq C_1 \int_0^t (1+t-\tau)^{1-n(1-\frac{1}{q})} \int_0^\tau (\tau-s)^{-\gamma} \|w(\tau, \cdot)\|_{L^p \cap L^{pq^\sharp}}^p ds d\tau \\ \mathcal{E}[Gw](t) &\leq C \int_0^t (1+t-\tau)^{-\frac{n}{2}} \|F(\tau, w)\|_{L^1 \cap L^2} d\tau \\ &\leq C_1 \int_0^t (1+t-\tau)^{-\frac{n}{2}} \int_0^\tau (\tau-s)^{-\gamma} \|w(\tau, \cdot)\|_{L^p \cap L^{2p}}^p ds d\tau. \end{aligned}$$

We can set $q = p$ in (3.9) (see Remark 2.4), obtaining

$$\|w(t, \cdot)\|_{L^p} \leq (1+t)^{-\gamma+[2-n(1-1/p)]^+} \ell(t) \|w\|_{X(T)}.$$

If, and only if, $p > \bar{p}(n, \gamma)$, then from the estimate above we may deduce that there exists $\beta(n, \gamma, p) > 1$ such that

$$\|w(t, \cdot)\|_{L^p}^p \leq C(1+t)^{-\beta(n, \gamma, p)} \|w\|_{X(T)}^p, \tag{3.16}$$

where C is independent of t and T . Having in mind (3.9), we see that $\|w(t, \cdot)\|_{L^{pq^\sharp}}^p$ also satisfies (3.16), provided that $pq^\sharp \leq r$,

for any $q \in [1, r]$. This latter holds if, and only if, $r \geq n(p-1)$.

Similarly, $\|w(t, \cdot)\|_{L^{2p}}^p$ also satisfies (3.16) due to $2p \leq r$ (see Remark 2.4).

We may now apply Lemma 3.2, since

$$\|Gw(t, \cdot)\|_{L^q} \leq C \|w\|_{X(T)}^p \int_0^t (1+t-\tau)^{1-n(1-\frac{1}{q})} I_\gamma(\tau) d\tau, \tag{3.17}$$

$$\mathcal{E}[Gw](t) \leq C \|w\|_{X(T)}^p \int_0^t (1+t-\tau)^{-\frac{n}{2}} I_\gamma(\tau) d\tau, \tag{3.18}$$

where $I_\gamma(\tau) = \int_0^\tau (\tau-s)^{-\gamma} (1+s)^{-\beta} ds,$

for some $\beta > 1$. Recalling that $1-n(1-1/q) < 1$ if, and only if, $q < \bar{q}(n)$, we obtain, in particular,

$$\|Gw(t, \cdot)\|_{L^1} \leq C(1+t)^{2-\gamma} \|w\|_{X(T)}^p, \tag{3.19}$$

$$\|Gw(t, \cdot)\|_{L^{\bar{q}(n)}} \leq C(1+t)^{-\gamma} \ell(t) \|w\|_{X(T)}^p, \tag{3.20}$$

$$\|Gw(t, \cdot)\|_{L^r} \leq C(1+t)^{-\gamma} \|w\|_{X(T)}^p, \tag{3.21}$$

$$\mathcal{E}[Gw](t) \leq C(1+t)^{-\gamma} \|w\|_{X(T)}^p, \tag{3.22}$$

and this concludes the proof of (3.14).

To prove (3.15), it is sufficient to use (2.2) and Hölder inequality, to estimate

$$\|g(w) - g(\tilde{w})\|_{L^1 \cap L^{q^\sharp}} \leq \|w - \tilde{w}\|_{L^p \cap L^{pq^\sharp}} \left(\|w\|_{L^p \cap L^{pq^\sharp}}^{p-1} + \|\tilde{w}\|_{L^p \cap L^{pq^\sharp}}^{p-1} \right),$$

and similarly for $\|g(w) - g(\tilde{w})\|_{L^1 \cap L^2}$, then we proceed as in the proof of (3.14).

It remains to prove that the solution satisfies (2.6) and (2.7). For any fixed t we may write $u = Nu = u^{\text{lin}} + Gu$. The estimates for u^{lin} directly

follows from (3.3)–(3.4) with $(v_0, v_1) = (u_0, u_1)$. Therefore, it is sufficient to come back to (3.17) and to use again Lemma 3.2, to obtain

$$\begin{aligned} \|Gu(t, \cdot)\|_{L^q} &\leq C \|u\|_{X(t)}^p \int_0^t (1+t-\tau)^{1-n(1-\frac{1}{q})} I_\gamma(\tau) d\tau \\ &\leq C (1+t)^{-\gamma+[2-n(1-\frac{1}{q})]^+} \|u\|_{X(t)}^p, \end{aligned}$$

for any $t \geq 0$ and $q \neq \bar{q}(n)$. The desired estimate follows thanks to (3.13) and the smallness of $\|(u_0, u_1)\|_{\mathcal{A}}$. Similarly, we derive the estimate in (2.6) for $q = \bar{q}(n)$ and estimate (2.7), taking into account of (3.20) and (3.22). \square

Proof of Theorem 2.6. We proceed as in the proof of Theorem 2.3, but we replace the definitions (3.6) and (3.7) of $X(T)$ and its norm by

$$X(T) := \mathcal{C}([0, T], L^1 \cap H^1) \cap \mathcal{C}^1([0, T], L^2), \tag{3.23}$$

$$\|w\|_{X(T)} := \max_{t \in [0, T]} \left\{ (1+t)^{-(2-\gamma)} \|w(t, \cdot)\|_{L^1} + (1+t)^\gamma \ell(t)^{-1} \mathcal{E}[w](t) \right\}. \tag{3.24}$$

By interpolation if $q \in [1, 2]$, or by Gagliardo–Nirenberg inequality (3.5) if $q > 2$, we obtain

$$\|w(t, \cdot)\|_{L^q} \leq (1+t)^{-\gamma+2/q} \ell(t) \|w\|_{X(T)}, \quad \text{for any } q \in [1, \infty). \tag{3.25}$$

This allows us to prove (3.14) and (3.15). In particular, it is clear that $pq^\sharp < \infty$ for any $q \in (1, \infty)$; hence, from (3.25) we obtain that

$$\|w(t, \cdot)\|_{L^p \cap L^{pq^\sharp}} \leq (1+t)^{-(n(1-\frac{1}{p})+\gamma-2)} \ell(t) \|w\|_{X(T)}.$$

The global existence of the solution follows, then we prove the desired estimates, as in the proof of Theorem 2.3.

If we also assume $\|u_0\|_{L^\infty} \leq \varepsilon$, then we modify (3.23)–(3.24) accordingly,

$$X(T) := \mathcal{C}([0, T], L^1 \cap H^1 \cap L^\infty) \cap \mathcal{C}^1([0, T], L^2), \tag{3.26}$$

$$\begin{aligned} \|w\|_{X(T)} &:= \max_{t \in [0, T]} \left\{ (1+t)^{-(2-\gamma)} \|w(t, \cdot)\|_{L^1} \right. \\ &\quad \left. + (1+t)^\gamma \ell(t)^{-1} (\|w(t, \cdot)\|_{L^\infty} + \mathcal{E}[w](t)) \right\}, \end{aligned} \tag{3.27}$$

and we follow the proof of Theorem 2.3. \square

Proof of Theorem 2.7. The proof of Theorem 2.7 is analogous to the proof of Theorem 2.3, once we replace the definitions (3.6) and (3.7) of $X(T)$ and its norm by

$$\begin{aligned} X(T) &:= \mathcal{C}([0, T], L^1 \cap L^r), \\ \|w\|_{X(T)} &:= \max_{t \in [0, T]} \left\{ (1+t)^{-(2-\gamma)} \|w(t, \cdot)\|_{L^1} \right. \\ &\quad \left. + (1+t)^\gamma \ell(t)^{-1} \|w(t, \cdot)\|_{L^{\bar{q}}} \right. \\ &\quad \left. + (1+t)^\gamma \|w(t, \cdot)\|_{L^r} \right\} \quad \text{if } n \geq 3, \end{aligned}$$

$$\begin{aligned} \|w\|_{X(T)} &:= \max_{t \in [0, T]} \left\{ (1+t)^{-(2-\gamma)} \|w(t, \cdot)\|_{L^1} \right. \\ &\quad \left. + (1+t)^\gamma (\ell(t))^{-1} \|w(t, \cdot)\|_{L^\infty} \right\} \quad \text{if } (n, r) = (2, \infty), \\ \|w\|_{X(T)} &:= \max_{t \in [0, T]} \left\{ (1+t)^{-(2-\gamma)} \|w(t, \cdot)\|_{L^1} \right. \\ &\quad \left. + (1+t)^{\gamma-2/r} \|w(t, \cdot)\|_{L^r} \right\} \quad \text{if } n = 2 \text{ and } r < \infty. \end{aligned}$$

The proof follows as in Theorem 2.3. □

4. Proof of Theorem 2.11

Following [11], for any $\alpha \in (0, 1)$ and for a fixed $T > 0$, we introduce the fractional integral and differential operators $J_{0|t}, D_{0|t}, J_{t|T}, D_{t|T}$ defined by:

$$\begin{aligned} J_{0|t}^\alpha f(t) &:= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{-(1-\alpha)} f(s) ds, & D_{0|t}^\alpha &:= \partial_t J_{0|t}^{1-\alpha}, \\ J_{t|T}^\alpha f(t) &:= \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{-(1-\alpha)} f(s) ds, & D_{t|T}^\alpha &:= -\partial_t J_{t|T}^{1-\alpha}. \end{aligned}$$

We have the following properties (see (2.64), p. 46 in [25] and (2.106) in [23]):

$$\int_0^T (D_{0|t}^\alpha f)(t) g(t) dt = \int_0^T f(t) (D_{t|T}^\alpha g)(t) dt, \tag{4.1}$$

$$D_{0|t}^\alpha J_{0|t}^\alpha f(t) = f(t). \tag{4.2}$$

Let us define

$$\omega(t) := \begin{cases} (1-t/T) & \text{if } t \in [0, T], \\ 0 & \text{if } t > T. \end{cases} \tag{4.3}$$

It follows that $\text{supp } \omega = [0, T]$ and $\omega(t)^\beta \in \mathcal{C}_c^k([0, \infty))$, $k \geq 0$, for any $\beta > k$. Moreover, we have the following.

Lemma 4.1. *For any $\alpha \in (0, 1)$, it follows that*

$$D_{t|T}^\alpha \omega(t)^\beta = C(\alpha, \beta) T^{-\alpha} \omega(t)^{\beta-\alpha}, \quad \text{for any } \beta > \alpha, \tag{4.4}$$

where

$$C(\alpha, \beta) = \frac{\Gamma(\beta + 1)}{(\beta + 2 - \alpha)\Gamma(\beta - \alpha)}.$$

Proof. By the change of variables $\tau := (s-t)/(T-t)$, we get (see [23])

$$\begin{aligned} \int_t^T (s-t)^{-\alpha} (1-t/T)^\beta ds &= (T-t)^{-\alpha+\beta+1} T^{-\beta} \int_0^1 \tau^{-\alpha} (1-\tau)^\beta d\tau \\ &= \omega(t)^{-\alpha+\beta+1} T^{1-\alpha} \frac{\Gamma(1-\alpha)\Gamma(\beta+1)}{\Gamma(-\alpha+\beta+2)}. \end{aligned}$$

Therefore,

$$\begin{aligned}
 J_{t|T}^{1-\alpha} \omega(t)^\beta &= \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 2 - \alpha)} T^{1-\alpha} \omega(t)^{\beta+1-\alpha}, \\
 D_{t|T}^\alpha \omega(t)^\beta &= \frac{\Gamma(\beta + 1)}{(\beta + 2 - \alpha)\Gamma(\beta - \alpha)} T^{-\alpha} \omega(t)^{\beta-\alpha}.
 \end{aligned}$$

□

Now we are ready to prove Theorem 2.11.

Proof of Theorem 2.11. We put $\alpha := 1 - \gamma$. By contradiction, let u be a global weak solution to (1.1). Let $\Psi \in C_c^\infty$ be a radial test function, such that:

- $\text{supp } \Psi = B_1$;
- $\Psi(x) = 1$, for any $x \in B_{1/2}$;
- $\Psi(x_1) \geq \Psi(x_2)$ if $|x_1| \leq |x_2|$.

For any $R \geq 1$, we denote $\Psi_R(t, x) := \Psi(x/R)$. Let us fix

$$\beta > (\alpha + 2)p', \quad \text{and} \quad \ell > p', \tag{4.5}$$

where $p' := p/(p - 1)$ is the Hölder conjugate of p , and let

$$\Phi_R(t, x) := \omega(t)^\beta \Psi_R(x)^\ell, \quad \varphi := D_{t|T}^\alpha \Phi_R(t, x).$$

Then, $\text{supp } \varphi \subset [0, T] \times B_R$, for any $T, R \geq 1$, but $\text{supp } (-\Delta)^{\frac{1}{2}} \varphi \subset [0, T] \times \mathbb{R}^n$, in general, since the fractional Laplacian is a nonlocal operator. We may write

$$\begin{aligned}
 &\int_0^T \int_{B_R} u(\varphi_{tt} - \Delta \varphi) \, dx \, dt - 2 \int_0^T \int_{\mathbb{R}^n} u(-\Delta)^{\frac{1}{2}} \varphi_t \, dx \, dt \\
 &= \int_{B_R} u_1(x) \varphi(0, x) \, dx + \Gamma(\alpha) \int_0^T \int_{B_R} J_{0|t}^\alpha(u^p) \varphi \, dx \, dt.
 \end{aligned}$$

We remark that $\varphi \geq 0$ thanks to (4.4). By virtue of (4.1) and (4.2), we get

$$\int_0^T \int_{B_R} J_{0|t}^\alpha(u^p) \varphi \, dx \, dt = \Gamma(\alpha) \int_0^T \int_{B_R} u^p \Phi_R \, dx \, dt,$$

whereas, being u_1 and φ nonnegative, we obtain

$$- \int_{B_R} u_1(x) \varphi(0, x) \, dx \leq 0.$$

Now we use Young inequality to estimate

$$\int_0^T \int_{B_R} u \varphi_{tt} \, dx \, dt \leq \epsilon \int_0^T \int_{B_R} u^p \Phi_R \, dx \, dt + C_\epsilon \int_0^T \int_{B_R} \varphi_{tt}^{p'} \Phi_R^{-\frac{1}{p-1}} \, dx \, dt,$$

and, thanks to

$$\begin{aligned}
 \partial_t^2 D_{t|T}^\alpha \omega(t)^\beta &= \frac{\Gamma(\beta + 1)}{(\beta + 2 - \alpha)\Gamma(\beta - \alpha)} T^{-\alpha} \partial_t^2 \omega(t)^{\beta-\alpha} \\
 &= \frac{\Gamma(\beta + 1)}{(\beta + 2 - \alpha)\Gamma(\beta - \alpha - 2)} T^{-(\alpha+2)} \omega(t)^{\beta-\alpha-2},
 \end{aligned}$$

and

$$\text{meas}([0, T]) = T, \quad \text{meas}(B_R) \approx R^n, \tag{4.6}$$

we obtain

$$\begin{aligned} & \int_0^T \int_{B_R} \varphi_{tt}^{p'} \Phi_R^{-\frac{1}{p-1}} dx dt \\ &= CT^{-(\alpha+2)p'} \int_0^T \int_{B_R} \omega(t)^{(\beta-\alpha-2)p' - \frac{\beta}{p-1}} \Psi_R(x)^\ell dx dt \\ &\lesssim T^{-(\alpha+2)p'+1} R^n. \end{aligned}$$

We remark that the power of $\omega(t)$ in the integral above is positive, by virtue of (4.5). Let us recall that

$$(-\Delta)^\theta \Psi^\ell \leq \ell \Psi^{\ell-1} (-\Delta)^\theta \Psi, \tag{4.7}$$

holds for any $\theta \in (0, 1]$, $\ell > 1$, and for any nonnegative test function Ψ (see [2, 17]). Then, since u and Ψ are nonnegative, we may estimate

$$u(-\Delta)^\theta \Psi_R(x)^\ell \leq \ell u \Psi_R(x)^{\ell-1} ((-\Delta)^\theta \Psi_R)(x).$$

It is clear that

$$((-\Delta)^\theta \Psi_R)(x) = R^{-2\theta} ((-\Delta)^\theta \Psi)(x/R),$$

and that $(-\Delta)^\theta \Psi \in \mathcal{S}$, is a bounded function. Therefore, we obtain

$$\begin{aligned} & \int_0^T \int_{B_R} u(-\Delta)^\theta \varphi dx dt \\ &\leq \ell \int_0^T \int_{B_R} u D_{t|T}^\alpha \omega(t)^\beta \Psi_R(x)^{\ell-1} (-\Delta \Psi_R)(x) dx dt \\ &\leq \epsilon \int_0^T \int_{B_R} u^p \Phi_R dx dt \\ &\quad + C_\epsilon T^{-\alpha p'} R^{-2p'} \int_0^T \int_{B_R} \left(D_{t|T}^\alpha \omega(t)^\beta \Psi_R(x)^{\ell-1} \right)^{p'} \Phi_R^{-\frac{1}{p-1}} dx dt. \end{aligned}$$

Recalling (4.4) and (4.6), we may estimate the last term by

$$\begin{aligned} & T^{-\alpha p'} R^{-2p'} \int_0^T \int_{B_R} \omega(t)^{(\beta-\alpha)p' - \frac{\beta}{p-1}} \Psi_R(x)^{(\ell-1)p' - \frac{\ell}{p-1}} dx dt \\ &\leq CT^{-\alpha p'+1} R^{-2p'+n}. \end{aligned}$$

We notice that the powers of $\beta(t)$ and $\Psi_R(x)$ in the integral above are positive, by virtue of β . Similarly, since u and $-\varphi_t$ are nonnegative, we derive

$$\begin{aligned}
 & - \int_0^T \int_{\mathbb{R}^n} u(-\Delta)^{\frac{1}{2}} \varphi_t \, dx \, dt \\
 & \leq \ell \int_0^T \int_{B_R} u \left(-\partial_t D_{t|T}^\alpha \omega(t)^\beta \right) \Psi_R^{\ell-1} \left((-\Delta)^{\frac{1}{2}} \Psi_R \right) (x) \, dx \, dt \\
 & \leq \epsilon \int_0^T \int_{B_R} u^p \Phi_R \, dx \, dt + C_\epsilon T^{-(\alpha+1)p'+1} R^{-p'+n}.
 \end{aligned}$$

We remark that the first passage above is the key point of our approach, since we may restrict the positive part of the integral over \mathbb{R}^n to an integral on B_R thanks to properties (4.7) and $\text{supp } \Psi_R = B_R$. In other words, even if the fractional Laplacian is a nonlocal operator, we have

$$\text{supp } [(-\Delta)^\theta \Psi(x)^\ell]^+ \subset \text{supp } \Psi, \quad \text{if } \ell > 1.$$

This strategy motivated the use of a maximum principle for our problem. Summarizing, we obtained:

$$\begin{aligned}
 (1 - 3\epsilon) \int_0^T \int_{B_R} u^p \Phi_R \, dx \, dt \\
 \leq \bar{C} T^{-\alpha p'+1} R^n \left(T^{-2p'} + R^{-2p'} + T^{-p'} R^{-p'} \right). \tag{4.8}
 \end{aligned}$$

In the following, we fix $\epsilon \in (0, 1/3)$. If $p < p_\gamma(n)$, it is sufficient to set $R = T$ in (4.8) to derive

$$\int_0^T \int_{B_T} u^p \Phi_T \, dx \, dt \lesssim T^{-(\alpha+2)p'+n+1}.$$

We notice that $p < p_\gamma(n)$ if, and only if, $(\alpha + 2)p' > n + 1$. By Beppo Levi’s theorem on monotone convergence, being $\Phi_T \nearrow 1$ as $T \rightarrow \infty$, we derive

$$\int_0^\infty \int_{\mathbb{R}^n} u^p \, dx \, dt = \lim_{T \rightarrow \infty} \int_0^\infty \int_{\mathbb{R}^n} u^p \Phi_T \, dx \, dt = 0;$$

hence $u \equiv 0$.

If $p = p_\gamma(n)$, we set $R = TK^{-1}$ for some fixed $K > 1$ and from (4.8) we get that

$$\int_0^T \int_{B_{T/K}} u^p \Phi_R \, dx \, dt \leq \bar{C} K^{-n} (1 + K^{p'} + K^{2p'}),$$

in particular

$$\int_0^T \int_{\mathbb{R}^n} u^p \, dx \, dt = \lim_{T \rightarrow \infty} \int_0^T \int_{B_{T/K}} u^p \Phi_{T/K} \, dx \, dt \leq C(K), \tag{4.9}$$

for some $C(K) > 0$, i.e. $u \in L^p$.

On the other hand, $\Delta \Psi_R(x)^\ell = 0$ for $|x| \leq R/2$, thus we may use Hölder inequality to estimate

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} u(-\Delta)\varphi \, dx \, dt &= \int_0^T \int_{|x|>T/(2K)} u(-\Delta)\varphi \, dx \, dt \\ &\leq \int_0^T \int_{|x|>T/(2K)} u |\Delta\varphi| \, dx \, dt \\ &\lesssim \left(\int_0^T \int_{|x|>T/(2K)} |\Delta\varphi|^{p'} (\Phi_R)^{-\frac{p'}{p}} \, dx \, dt \right)^{\frac{1}{p'}} \\ &\quad \times \left(\int_0^T \int_{|x|>T/(2K)} u^p \Phi_R \, dx \, dt \right)^{\frac{1}{p}}; \end{aligned}$$

hence, we may replace (4.8) by

$$\begin{aligned} (1 - 2\epsilon) \int_0^T \int_{\mathbb{R}^n} u^p \Phi_{T/K} \, dx \, dt &\leq \bar{C} K^{-n} (1 + K^{p'}) \\ &\quad + \bar{C} K^{-\frac{n}{p'}+2} \left(\int_0^T \int_{|x|>T/(2K)} u^p \Phi_{T/K} \, dx \, dt \right)^{\frac{1}{p'}}. \end{aligned} \tag{4.10}$$

Thanks to $u \in L^p$ [see (4.9)], for any fixed $K > 1$, it holds

$$\lim_{T \rightarrow \infty} \int_0^T \int_{|x|>T/(2K)} u^p \Phi_{T/K} \, dx \, dt = 0;$$

hence, from (4.10) we derive that

$$\int_0^\infty \int_{\mathbb{R}^n} u^p \, dx \, dt \leq \bar{C} K^{-n} (1 + K^{p'}),$$

which is an arbitrarily small quantity, since

$$p' = \frac{n + 1}{\alpha + 2} < n.$$

Therefore, $u \equiv 0$.

If $p < \gamma^{-1}$, i.e. $p'\alpha > 1$, we set $R = \ln T$ in (4.8) and we apply Beppo Levi's convergence theorem, obtaining

$$\int_0^\infty \int_{\mathbb{R}^n} u^p \, dx \, dt = C \lim_{T \rightarrow \infty} T^{-\alpha p' + 1} (\ln T)^n \left(T^{-2p'} + (T \ln T)^{-p'} + (\ln T)^{-2p'} \right),$$

so that $u \equiv 0$ again.

Since $u \equiv 0$ contradicts the assumption of nontrivial data u_1 , we proved our statement. \square

Remark 4.2. It remains open the question about nonexistence of the global solution only if $p = \gamma^{-1}$ and $\gamma < (n - 2)/n$. Indeed, in this case, (4.8) gives

$$\int_0^\infty \int_{\mathbb{R}^n} u^p \Phi_R \, dx \, dt \leq \bar{C} R^{n-2p'},$$

but $n > 2p'$. In other words, the test function method does not work even if we try to apply it to

$$-\Delta u = \int_0^t (t-s)^{-\gamma} |u(s,x)|^{\frac{1}{\gamma}} ds, \quad 0 < \gamma < \frac{n-2}{n}.$$

We expect that this effect is related to the fact that the test function method does not work for

$$-\Delta u = |u|^p, \quad \text{if } p > \frac{n}{n-2}.$$

5. The general case $\sigma \in (0, 1]$

It is natural to ask which *critical exponent* one may expect for small data solutions to

$$\begin{cases} u_{tt} - \Delta u + \mu(-\Delta)^\sigma u_t = F(t, u), & \sigma \in (0, 1/2), \\ u(0, x) = u_0(x), \\ u_t(0, x) = u_1(x), \end{cases} \tag{5.1}$$

taking into account that the *critical exponent* is given by (1.7) and (1.8) in the two limit cases $\sigma = 0$ and $\sigma = 1/2$.

The linear estimates for (5.1) suggest that the *critical exponent* should be

$$\bar{p}(n, \sigma, \gamma) = \max \{ p_\gamma(n, \sigma), \gamma^{-1} \}, \tag{5.2}$$

$$p_\gamma(n, \sigma) := 1 + \frac{2(2 - \sigma - \gamma(1 - \sigma))}{[n - 2 + 2\gamma(1 - \sigma)]^+}. \tag{5.3}$$

For $\sigma \in (0, 1/2)$ the transition from $p_\gamma(n, \sigma)$ to γ^{-1} happens at $\gamma = (n-2)/n$, for $n \geq 3$, as in the two limit cases $\sigma = 0$ and $\sigma = 1/2$. In particular,

$$\bar{p}(2, \sigma, \gamma) = p_\gamma(2, \sigma) = \frac{2 - \sigma}{1 - \sigma} \gamma^{-1}.$$

To motivate our expectation, we extend Corollary 2.5 to the following.

Theorem 5.1. *Let us consider the following cases:*

- $n = 1$, $\sigma \in (0, 1/4)$ and $1/(2(1 - \sigma)) < \gamma < 1$,
- $n = 2$, $\sigma \in (0, 1/2)$ and $\gamma \in (0, 1)$.

Let $p > p_\gamma(n, \sigma)$. Then there exists $\varepsilon > 0$ such that for any initial data as in (2.8), there exists a $C([0, \infty), H^1) \cap C^1([0, \infty), L^2)$ solution to (5.1). Moreover, the solution satisfies the estimates

$$\begin{aligned} \|u(t, \cdot)\|_{L^2} &\leq C(1+t)^{1 - (\frac{n}{4} - \sigma)\frac{1}{1-\sigma} - \gamma} \|(u_0, u_1)\|_{\mathcal{A}}, \\ \|\nabla u(t, \cdot)\|_{L^2} &\leq \begin{cases} C(1+t)^{1 - (\frac{3}{4} - \sigma)\frac{1}{1-\sigma} - \gamma} \|(u_0, u_1)\|_{\mathcal{A}} & \text{if } n = 1, \\ C(1+t)^{-\gamma} \ln(e+t) \|(u_0, u_1)\|_{\mathcal{A}} & \text{if } n = 2, \end{cases} \\ \|u_t(t, \cdot)\|_{L^2} &\leq C(1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}. \end{aligned}$$

Remark 5.2. For the sake of brevity, in Theorem 5.1 we choose to use only estimates on L^2 basis. Indeed, linear $L^{q_1} - L^{q_2}$ estimates are more difficult to be derived for (1.4) if $\sigma \in (0, 1/2)$; see, for instance, [5, 6]. Unfortunately, our

choice imposes the restriction $p \geq 2$, hence we may prove global existence for any $p > \bar{p}(n, \sigma, \gamma)$ and for any $\gamma \in (0, 1)$ only in space dimension $n = 1, 2$.

The model's properties in (1.4) completely change if $\sigma \in (1/2, 1]$. Indeed, we can prove global existence of small data solutions to

$$\begin{cases} u_{tt} - \Delta u + \mu(-\Delta)^\sigma u_t = F(t, u), & \sigma \in (1/2, 1], \\ u(0, x) = u_0(x), \\ u_t(0, x) = u_1(x), \end{cases} \tag{5.4}$$

if $p > \bar{p}(n, \sigma, \gamma)$, with $\bar{p}(n, \sigma, \gamma)$ as in (5.2) and

$$p_\gamma(n, \sigma) := 1 + \frac{1 + 2\sigma(2 - \gamma)}{[n - 1 - 2\sigma(1 - \gamma)]^+}. \tag{5.5}$$

We remark that now the threshold for γ such that $p_\gamma(n, \sigma) = \gamma^{-1}$ is given by

$$\gamma = \frac{n - 1 - 2\sigma}{n}, \quad \text{for } n \geq 3 \text{ if } \sigma \in (1/2, 1) \text{ and for } n \geq 4 \text{ if } \sigma = 1.$$

Theorem 5.3. *Let us consider the following cases:*

- $n=2, \sigma \in (1/2, 1], \gamma \in ((2\sigma - 1)/(2\sigma), 1)$ and $p > p_\gamma(2, \sigma)$,
- $n=3, \sigma \in (1/2, 3/4), \gamma \in ((3 - 4\sigma)/3, 1)$ and $p \in (\bar{p}(3, \sigma, \gamma), 3/(3 - 4\sigma)]$,
- $n=3, \sigma \in [3/4, 1], \gamma \in (0, 1)$ and $p > \bar{p}(3, \sigma, \gamma)$,
- $n=4, \sigma = 1, \gamma \in (0, 1)$ and $p > \bar{p}(4, 1, \gamma)$.

Then there exists $\varepsilon > 0$ such that for any initial data

$$(u_0, u_1) \in \mathcal{A} := (L^1 \cap H^{2\sigma}) \times (L^1 \cap L^2) \quad \text{with } \|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon, \tag{5.6}$$

there exists a $\mathcal{C}([0, \infty), H^{2\sigma}) \cap \mathcal{C}^1([0, \infty), L^2)$ solution to (5.4). Moreover, the solution satisfies the estimates

$$\begin{aligned} \|u(t, \cdot)\|_{L^2} &\leq C(1+t)^{1-\frac{n-2}{4\sigma}-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, \\ \mathcal{E}[u](t) &\leq \begin{cases} C(1+t)^{1-\frac{1}{2\sigma}-\gamma} \|(u_0, u_1)\|_{\mathcal{A}} & \text{if } n = 2, \\ C(1+t)^{1-\frac{3}{4\sigma}-\gamma} \|(u_0, u_1)\|_{\mathcal{A}} & \text{if } n = 3, \sigma \in (3/4, 1), \\ C(1+t)^{-\gamma} \ln(e+t) \|(u_0, u_1)\|_{\mathcal{A}} & \text{if } n = 3, \sigma = 3/4, \\ C(1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}} & \text{if } n = 3, \sigma \in (1/2, 3/4), \\ C(1+t)^{-\gamma} \ln(e+t) \|(u_0, u_1)\|_{\mathcal{A}} & \text{if } n = 4 \text{ and } \sigma = 1, \end{cases} \\ \|u(t, \cdot)\|_{\dot{H}^{2\sigma}} &\leq \begin{cases} C(1+t)^{-\gamma} \ln(e+t) \|(u_0, u_1)\|_{\mathcal{A}} & \text{if } n = 2, \\ C(1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}} & \text{if } n \geq 3. \end{cases} \end{aligned}$$

The limit case $\sigma = 1$ has a particular interest, since in the damping term in (5.7) it appears an integer power of Laplacian; hence, we may use the test function method to state the following result.

Theorem 5.4. *Let us consider Cauchy problem*

$$\begin{cases} u_{tt} - \Delta u - \mu \Delta u_t = \int_0^t (t-s)^{-\gamma} |u(s,x)|^p ds, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \end{cases} \tag{5.7}$$

with $\mu > 0$, $\gamma \in (0, 1)$, $p > 1$ and initial data satisfying $-\Delta u_0 + u_1 \in L^1$ and

$$\int_{\mathbb{R}^n} (-\Delta u_0(x) + u_1(x)) dx > 0. \tag{5.8}$$

Then, there exists no global weak solution to (5.7) in the following four cases:

- if $n = 1$, for any $\gamma \in (0, 1)$ and $p > 1$,
- if $n \geq 2$, for any $\gamma \in [(n - 2)/n, 1)$ and $p \in (1, p_\gamma(n)]$,
- if $n \geq 3$, for any $\gamma \in (0, (n - 2)/n)$ and $p \in (1, \gamma^{-1})$.

We remark that the nonexistence exponent in Theorem 5.4 is the same obtained in the case $\sigma = 1/2$ in Theorem 2.11. Therefore, we did not prove the optimality of $\bar{p}(n, 1, \gamma)$, in general, since the nonempty interval

$$(\bar{p}(n, \gamma), \bar{p}(n, 1, \gamma)],$$

is not covered by Theorem 5.3 neither by Theorem 5.4, for $\gamma \in ([n - 3]^+ / n, 1)$.

However, if $n \geq 4$ and $\gamma \in (0, (n - 3)/n]$, it holds $\bar{p}(n, 1, \gamma) = \bar{p}(n, \gamma) = \gamma^{-1}$, that is, the influence of the nonlinear memory on (1.4) is so strong that the global existence exponent is the same for any $\sigma \in [0, 1]$. In particular, if $n = 4$ and $\gamma \in (0, 1/4]$, combining Theorems 5.3 and 5.4, we showed that the critical exponent γ^{-1} is optimal.

We now prove our statements.

Proof of Theorem 5.1. In [8] it has been proved that the solution to

$$\begin{cases} v_{tt} - \Delta v + \mu(-\Delta)^\sigma v_t = 0, \quad \sigma \in (0, 1/2), \\ v(0,x) = v_0(x), \\ v_t(0,x) = v_1(x), \end{cases} \tag{5.9}$$

satisfies the estimates

$$\begin{aligned} \|v(t, \cdot)\|_{L^2} &\leq (1+t)^{-\left(\frac{n}{4}-\sigma\right)\frac{1}{1-\sigma}} \|(v_0, v_1)\|_{L^1 \cap L^2}, \\ \|\nabla v(t, \cdot)\|_{L^2} &\leq (1+t)^{-\left(\frac{n+2}{4}-\sigma\right)\frac{1}{1-\sigma}} (\|v_0\|_{L^1 \cap H^1} + \|v_1\|_{L^1 \cap L^2}), \\ \|v_t(t, \cdot)\|_{L^2} &\leq (1+t)^{-\left(\frac{n}{4}-\sigma\right)\frac{1}{1-\sigma}-1} (\|v_0\|_{L^1 \cap H^1} + \|v_1\|_{L^1 \cap L^2}), \end{aligned}$$

provided that $\sigma \in (0, 1/4)$ if $n = 1$. We follow the proof of Theorem 2.3. For a fixed $T > 0$, let

$$X(T) := \mathcal{C}([0, T], H^1) \cap \mathcal{C}^1([0, T], L^2), \tag{5.10}$$

$$\begin{aligned} \|w\|_{X(T)} := \max_{t \in [0, T]} & \left\{ (1+t)^{-1 + (\frac{n}{4} - \sigma)\frac{1}{1-\sigma} + \gamma} \|w(t, \cdot)\|_{L^2} \right. \\ & \left. + f(t) \|\nabla w(t, \cdot)\|_{L^2} + (1+t)^\gamma \|w_t(t, \cdot)\|_{L^2} \right\}, \end{aligned} \tag{5.11}$$

where

$$f(t) = \begin{cases} (1+t)^{-1 + (\frac{3}{4} - \sigma)\frac{1}{1-\sigma} + \gamma} & \text{if } n = 1, \\ (1+t)^\gamma \ell(t)^{-1} & \text{if } n = 2. \end{cases}$$

We recall that $\ell(t) := \ln(e + t)$. Since $p > p_\gamma(n, \sigma) > 2$ for $n = 1, 2$, by Gagliardo–Nirenberg inequality (3.5), it follows from (5.11) that

$$\|w(t, \cdot)\|_{L^p \cap L^{2p}} \leq \begin{cases} C (1+t)^{1 - (\frac{1}{2}(1 - \frac{1}{p}) - \sigma)\frac{1}{1-\sigma} - \gamma} \|w\|_{X(T)} & \text{if } n = 1 \\ C (1+t)^{1 - (1 - \frac{1}{p} - \sigma)\frac{1}{1-\sigma} - \gamma} \ell(t) \|w\|_{X(T)} & \text{if } n = 2. \end{cases}$$

Therefore,

$$\|w(t, \cdot)\|_{L^p \cap L^{2p}}^p \leq (1+t)^{-\beta} \|w\|_{X(T)}^p,$$

for some $\beta > 1$ if, and only if, y

$$p \left(\left(\frac{n}{2} \left(1 - \frac{1}{p} \right) - \sigma \right) \frac{1}{1-\sigma} + \gamma - 1 \right) > 1,$$

that is, $p > p_\gamma(n, \sigma)$. By Duhamel’s principle, for $j + k = 0, 1$ we derive

$$\begin{aligned} & \|\nabla^j \partial_t^k Gw(t, \cdot)\|_{L^2} \\ & \leq C \|w\|_{X(T)} \int_0^t (1+t-\tau)^{-\left(\frac{n+2j}{4} - \sigma\right)\frac{1}{1-\sigma} - k} \int_0^\tau (\tau-s)^{-\gamma} (1+s)^{-\beta} ds d\tau. \end{aligned}$$

Applying Lemma 3.2, we prove (3.14). Similarly, we prove (3.15). The global existence of the solution follows and we may derive the desired estimates as in the proof of Theorem 2.3. \square

Proof of Theorem 5.3. The solution to

$$\begin{cases} v_{tt} - \Delta v + \mu(-\Delta)^\sigma v_t = 0, & \sigma \in (1/2, 1], \\ v(0, x) = v_0(x), \\ v_t(0, x) = v_1(x), \end{cases} \tag{5.12}$$

satisfies the following estimates (see [26] for $\sigma = 1$ and [8] for $\sigma \in (1/2, 1)$):

$$\begin{aligned} \|v(t, \cdot)\|_{L^2} & \leq (1+t)^{-\frac{n-2}{4\sigma}} \|(v_0, v_1)\|_{L^1 \cap L^2}, \\ \mathcal{E}[v](t) & \leq (1+t)^{-\frac{n}{4\sigma}} (\|v_0\|_{L^1 \cap H^1} + \|v_1\|_{L^1 \cap L^2}), \\ \|v(t, \cdot)\|_{\dot{H}^{2\sigma}} & \leq (1+t)^{-\frac{n-2}{4\sigma} - 1} (\|v_0\|_{L^1 \cap H^{2\sigma}} + \|v_1\|_{L^1 \cap L^2}). \end{aligned}$$

First, let $n = 2$. We may follow the proof of Theorem 5.1, setting

$$X(T) := \mathcal{C}([0, T], H^{2\sigma}) \cap \mathcal{C}^1([0, T], L^2), \tag{5.13}$$

$$\begin{aligned} \|w\|_{X(T)} := \max_{t \in [0, T]} & \left\{ (1+t)^{-1+\gamma} \|w(t, \cdot)\|_{L^2} \right. \\ & \left. + (1+t)^{-1+\frac{1}{2\sigma}+\gamma} \mathcal{E}[w](t) + (1+t)^\gamma \ell(t)^{-1} \|w(t, \cdot)\|_{\dot{H}^{2\sigma}} \right\}. \end{aligned}$$

Since $p > p_\gamma(2, \sigma) > 2$, we may use Gagliardo–Nirenberg inequality (3.5) to derive that

$$\|w(t, \cdot)\|_{L^p \cap L^{2p}} \leq C (1+t)^{1+\frac{1}{2\sigma}-\frac{1}{p}(1-\frac{1}{p})-\gamma} \ell(t) \|w\|_{X(T)}.$$

Therefore,

$$\|w(t, \cdot)\|_{L^p \cap L^{2p}}^p \leq (1+t)^{-\beta} \|w\|_{X(T)}^p,$$

for some $\beta > 1$ if, and only if, $p > p_\gamma(2, \sigma)$. The end of the proof follows.

If $n = 3, 4$, we should proceed in a different way, employing fractional Sobolev embeddings instead of Gagliardo–Nirenberg inequality (3.5), as in [4]. We notice that the solution to (5.12) satisfies the estimate:

$$\|v(t, \cdot)\|_{\dot{H}^k} \leq (1+t)^{-\frac{n+2k-2}{4\sigma}} (\|v_0\|_{L^1 \cap H^{2\sigma}} + \|v_1\|_{L^1 \cap L^2}), \quad \text{for any } k \in [0, 2\sigma].$$

Let $n = 3$ and let us denote $\kappa = \kappa(\sigma) := 2\sigma - 1/2$. Let $X(T)$ be as in (5.13), with norm given by

$$\begin{aligned} \|w\|_{X(T)} := \max_{t \in [0, T]} & \left\{ (1+t)^{-1+\frac{1}{4\sigma}+\gamma} \|w(t, \cdot)\|_{L^2} + f_\sigma(t) \mathcal{E}[w](t) \right. \\ & \left. + (1+t)^\gamma (\ell(t)^{-1} \|w(t, \cdot)\|_{\dot{H}^\kappa} + \|w(t, \cdot)\|_{\dot{H}^{2\sigma}}) \right\}, \end{aligned}$$

where

$$f_\sigma(t) = \begin{cases} (1+t)^\gamma & \text{if } \sigma \in (1/2, 3/4), \\ (1+t)^\gamma \ell(t)^{-1} & \text{if } \sigma = 3/4, \\ (1+t)^{-1+\frac{3}{4\sigma}+\gamma} & \text{if } \sigma \in (3/4, 1]. \end{cases}$$

For any $p \geq 2$ we may use the fractional Sobolev embedding

$$\|w(t, \cdot)\|_{L^p \cap L^{2p}} \leq C \|w(t, \cdot)\|_{\dot{H}^{k_1} \cap \dot{H}^{k_2}}, \quad \text{with } k_m := \frac{n}{2} \left(1 - \frac{1}{mp}\right);$$

hence, we derive

$$\|w(t, \cdot)\|_{L^p \cap L^{2p}} \leq C (1+t)^{-\gamma + [1+\frac{1}{2\sigma}-\frac{3}{2\sigma}(1-\frac{1}{p})]^+} \ell(t) \|w\|_{X(T)}.$$

for any $p > \bar{p}(3, \sigma, \gamma)$, provided that $p < 3/(3-4\sigma)$ if $\sigma \in (1/2, 3/4)$. The end of the proof follows.

In the last case, $n = 4$ and $\sigma = 1$, we proceed as for $n = 3$, but we take

$$\begin{aligned} \|w\|_{X(T)} := \max_{t \in [0, T]} & \left\{ (1+t)^{-\frac{1}{2}+\gamma} \|w(t, \cdot)\|_{L^2} \right. \\ & \left. + (1+t)^\gamma (\ell(t)^{-1} \mathcal{E}[w](t) + \|\nabla^2 w(t, \cdot)\|_{L^2}) \right\}. \end{aligned} \tag{5.14}$$

This concludes the proof. □

Proof of Theorem 5.4. The proof is similar to the proof of Theorem 2.11 but, due to the fact that the damping term does not contain fractional powers of the Laplacian, we do not need a maximum principle for (5.7).

We may write

$$\begin{aligned} & \int_0^T \int_{B_R} u(\varphi_{tt} - \Delta\varphi) \, dx \, dt + \mu \int_0^T \int_{B_R} u\Delta\varphi_t \, dx \, dt \\ &= \int_{B_R} (-\mu\Delta u_0(x) + u_1(x)) \varphi(0, x) \, dx + \Gamma(\alpha) \int_0^T \int_{B_R} J_{0|t}^\alpha(|u|^p)\varphi \, dx \, dt. \end{aligned}$$

Thanks to (5.8), we derive

$$- \int_{B_R} (-\Delta u_0(x) + u_1(x)) \varphi(0, x) \, dx \leq 0,$$

for any sufficiently large $R > 0$. We follow the proof of Theorem 2.11 to estimate the term containing $\varphi_{tt} - \Delta\varphi$. Indeed, it still holds

$$\int_0^T \int_{B_R} |u|\varphi_{tt} \, dx \, dt \leq \epsilon \int_0^T \int_{B_R} |u|^p \Phi_R \, dx \, dt + C_\epsilon T^{-(\alpha+2)p'+1} R^n,$$

as well as

$$\int_0^T \int_{B_R} |u| |\Delta\varphi| \, dx \, dt \leq \epsilon \int_0^T \int_{B_R} |u|^p \Phi_R \, dx \, dt + C_\epsilon T^{-\alpha p'+1} R^{-2p'+n},$$

whereas the contribution coming from the damping term may now be directly estimated as

$$\int_0^T \int_{\mathbb{R}^n} |u| |\Delta\varphi_t| \, dx \, dt \leq \epsilon \int_0^T \int_{B_R} |u|^p \Phi_R \, dx \, dt + C_\epsilon T^{-(\alpha+1)p'+1} R^{-2p'+n}.$$

Summarizing, we obtained

$$(1 - 3\epsilon) \int_0^T \int_{B_R} u^p \Phi_R \, dx \, dt \leq \bar{C} T^{-\alpha p'+1} R^n \left(T^{-2p'} + R^{-2p'} + T^{-p'} R^{-2p'} \right),$$

which is equivalent to (4.8). Therefore, by taking T and R as in the proof of Theorem 2.11, it follows $u \equiv 0$ for any $p < \bar{p}(n, \gamma)$. If $p = p_\gamma(n)$, we proceed again as in the proof of Theorem 2.11.

The proof of the statement follows from the fact that $u \equiv 0$ contradicts (5.8). □

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