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An elliptic problem with an indefinite nonlinearity and a parameter in the boundary condition

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Abstract. We establish the existence of solutions of nonlinear elliptic boundary value problems involving a positive parameter on the boundary. We also examine a profile of solutions of problem (1.2) when a parameter λ tends to 0.

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1. Introduction

The aim of this paper is to investigate the solvability of the following boundary value problem

$$\begin{cases} \Delta u = a(x)u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega, \quad u \geq 0, \quad u \not\equiv 0, \text{ on } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with a smooth boundary $\partial\Omega$, ν denotes an outward normal to $\partial\Omega$ and $\lambda > 0$ is a parameter. It is assumed that $0 < p \leq 2^* - 1$. In fact, we consider three cases: (i) $1 < p < 2^* - 1$, (ii) $p = 2^* - 1$ and (iii) $0 < p < 1$. Here 2^* denotes the critical Sobolev exponent, that is, $2^* = \frac{2N}{N-2}$, $N \geq 3$. In the case (iii) we consider a modified problem (1.1) by introducing the parameter λ into the equation, that is, we will look for solutions of the following problem

$$\begin{cases} \Delta u = \lambda a(x)u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega, \quad u \geq 0, \quad u \not\equiv 0, \text{ on } \Omega. \end{cases} \tag{1.2}$$

In all these three cases we assume that the coefficient $a(x)$ changes sign. Further assumptions on $a(x)$ will be formulated later. In this paper we aim to establish the existence of weak solutions of problems (1.1) and

(1.2). We show that there exist intervals $(0, \lambda^*)$ and $(0, \tilde{\lambda})$ such problems (1.1) and (1.2) admit a least two solutions in the respective intervals for a parameter λ. One solution is obtained by applying the mountain-pass principle [5] and a second one comes from a local minimizer with the aid of the Ekeland variational principle ([11]) of the corresponding variational functional.

Problem (1.1) with $a(x) \geq 0$ and $1 < p < 2^* - 1$, under some regularity assumptions on $a(x)$ and $\partial\Omega$ and $1 < p < 2^* - 1$, has been investigated in papers [13,14]. In particular, if $a(x) > 0$ on $\bar{\Omega}$ and $0 < p < 2^* - 1$, then problem (1.1) has a classical solution. If $1 < p < 2^* - 1$, then this solution is unique. If $0 < p < 1$, then there exists a constant $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ problem (1.1) has a unique solution. If $a(x)$ vanishes on a smooth sub-domain $\Omega_0 \subset \Omega$ with $\partial\Omega_0 \cap \partial\Omega \neq \emptyset$, then a solution to problem (1.1) with $1 < p < 2^* - 1$ exists for λ belonging to a bounded interval $(0, \sigma_1)$. The constant σ_1 is a principal eigenvalue for the Laplace equation $\Delta u = 0$ in Ω_0 with mixed boundary conditions. For the description of the configuration of Ω_0 and the definition of σ_1 we refer to papers [13,14]. Also, the authors of these papers derived a number of interesting asymptotic properties of solutions of (1.1) when λ converges to one of the ends of the interval $(0, \sigma_1)$ in the case when $a(x)$ is positive.

The paper is organized as follows. In Sect. 2 we recall some basic definitions. In Sect. 3 we discuss the mountain-pass structure of variational functionals corresponding to problems (1.1) and (1.2). These functionals are denoted by J_λ and $\bar{J}_\lambda$, respectively. Our approach is based on a qualitative property of the embedding of the Sobolev space $H^1(\Omega$ into the Lebesgue space $L^q(\Omega)$, $1 \leq q < 2^*$ (see property (**P**) in this section). It appears that this property has been used for the first time in paper [6]. Section 4 is devoted to the boundedness of Palais–Smale sequences of the functionals J_λ and $\bar{J}_\lambda$. The existence results for the case $0 < p < 2^* - 1$ are given in Sect. 5. The existence of solutions for the critical case $p + 1 = 2^*$ is discussed in Sect. 6. The main ingredient in our approach is the P.L. Lions' concentration–compactness principle [15]. We note that the shape of the graph of the coefficient $a(x)$ plays an important role (see also paper [10]). We only consider the existence of solutions for $\lambda > 0$. In the critical case with $\lambda = 0$ some existence results have been obtained in paper [9]. The results of this paper can be easily extended to the case $1 < p < 2^* - 1$. Problem (1.2) with $\lambda = 0$ is a linear problem so it makes sense to consider the behavior of solutions of problem (1.2) when $\lambda \to 0$. Profiles of local minimizers and mountain-pass solutions when $\lambda \to 0$ in the case $0 < p < 1$ are presented in Sect. 7. Finally, in Appendix we give a sketch of the proof of the fact that solutions constructed in the case $1 < p \leq 2^* - 1$ are regular up to the boundary, that is, they belong to $C^{1,\beta}(\bar{\Omega})$ for some $0 < \beta < 1$. We point out that the continuity of a local minimizer in the case $p = 2^* - 1$ has been used in the proof of Proposition 6.2.

In this paper we use standard notations. In a given Banach space we denote by "$\to$" strong convergence and by "$\rightharpoonup$" weak convergence. The norms in the Lebesgue spaces $L^p(\Omega)$, $1 \leq p \leq \infty$, are denoted by $\|\cdot\|_p$.

2. Preliminaries

Throughout this paper we assume that the coefficient $a(x)$ is continuous on $\bar{\Omega}$ and changes sign with a dominant positive part $a^+(x) = \max(a(x), 0)$, that is,

(A) $\int_\Omega a(x)\,dx > 0$.

At the end of this section we give a short justification for this assumption. Solutions of problem (1.1) are sought in a Sobolev space $H^1(\Omega)$. We recall that $H^1(\Omega)$ is the usual Sobolev space equipped with norm

$$\|u\|^2 = \int_\Omega (|\nabla u|^2 + u^2)\,dx.$$

We frequently use the decomposition of the space $H^1(\Omega)$:

$$H^1(\Omega) = V \oplus \mathbb{R},\ V = \left\{ v \in H^1(\Omega);\ \int_\Omega v(x)\,dx = 0 \right\}.$$

This decomposition yields the following equivalent norm on $H^1(\Omega)$

$$\|u\|_V^2 = \||\nabla v|\|_2^2 + t^2.$$

We also use a norm $\|\cdot\|_*$ given by

$$\|u\|_*^2 = \int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} u^2\,dS_x.$$

Obviously these three norms on $H^1(\Omega)$ are equivalent.

By $J_\lambda(u)$ we denote a variational functional associated with problem (1.1), that is,

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{2}\int_{\partial\Omega} u^2\,dS_x + \frac{1}{p+1}\int_\Omega a(x)|u|^{p+1}\,dx.$$

A critical point u of J_λ is a weak solution of problem (1.1), that is,

$$\int_\Omega \nabla u \nabla \phi\,dx - \lambda \int_{\partial\Omega} u\phi\,dS_x + \int_\Omega a(x)|u|^{p-1}u\phi\,dx = 0 \tag{2.1}$$

for every $\phi \in H^1(\Omega)$. The functional J_λ is of class C^1 on $H^1(\Omega)$.

The variational functional for problem (1.2) is denoted by $\bar{J}_\lambda$ and is given by

$$\bar{J}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{2}\int_{\partial\Omega} u^2\,dS_x + \frac{\lambda}{p+1}\int_\Omega a(x)|u|^{p+1}\,dx$$

and critical points of $\bar{J}_\lambda$ are solutions to problem (1.2).

Let us assume that problem (1.1) [or (1.2)] has a solution u. Testing (2.1) with $\phi = (u^2+\epsilon^2)^{-\frac{p}{2}}$ we get

$$-p\int_\Omega |\nabla u|^2 u(u^2+\epsilon^2)^{-\frac{p}{2}-1}\,dx - \lambda\int_{\partial\Omega} u(u^2+\epsilon^2)^{-\frac{p}{2}}\,dS_x + \int_\Omega a(x)u^p(u^2+\epsilon^2)^{-\frac{p}{2}}\,dx = 0.$$

Hence

$$\int_\Omega a(x)u^p(u^2+\epsilon^2)^{-\frac{p}{2}}\,dx > 0.$$

Letting $\epsilon \to 0$ we obtain $\int_\Omega a(x)\,dx > 0$ provided $u(x) > 0$ on Ω. Thus condition (**A**) is necessary for the existence of a solution to problem (1.2).

If $\lambda > 0$, then the quadratic parts of both functionals J_λ and $\bar{J}_\lambda$ change sign. On the other hand if $\lambda < 0$, then the quadratic parts are positively definite, and the corresponding problem requires a separate treatment. For the present paper we opted for the case $\lambda > 0$ that we find more interesting.

3. The mountain-pass geometry for functionals J_λ and $\bar{J}_\lambda$

To apply the mountain-pass principle we need the following qualitative property:

(P) there exists a constant $\eta > 0$ such that for every $t \in \mathbb{R}$ and $v \in V$ the inequality

$$\left(\int_\Omega |\nabla v|^2\,dx \right)^{\frac{1}{2}} \le \eta |t|$$

yields

$$\int_\Omega a(x)|v+t|^{p+1}\,dx \ge \frac{|t|^{p+1}}{2} \int_\Omega a(x)\,dx.$$

This property follows from the continuity of the embedding of the space $H^1(\Omega)$ into the Lebesgue space $L^{p+1}(\Omega)$ with $0 < p \le 2^* - 1$ (see [6], Lemma 9). Indeed, arguing by contradiction, suppose that (**P**) does not hold. Then for every $n \in \mathbb{N}$ there exist $t_n \in \mathbb{R}$ and $v_n \in V$ such that

$$\left(\int_\Omega |\nabla v_n|^2\,dx\right)^{\frac{1}{2}} \le \frac{1}{n}|t_n|$$

and

$$(*) \qquad \int_\Omega a(x)|v_n + t_n|^{p+1}\,dx < \frac{|t_n|^{p+1}}{2} \int_\Omega a(x)\,dx.$$

Setting $w_n = \frac{v_n}{t_n}$ we see that $\nabla w_n \to 0$ in $L^2(\Omega)$ and, by continuous embedding of V into $L^{p+1}(\Omega)$, $w_n \to 0$ in $L^{p+1}(\Omega)$. Dividing (*) by $|t_n|^{p+1}$ and letting $n \to \infty$ we derive

$$\int_\Omega a(x)\,dx \le \frac{1}{2}\int_\Omega a(x)\,dx,$$

that is $\int_\Omega a(x)\,dx \le 0$, and this contradicts assumption (**A**).

We commence with the functional J_λ.

Proposition 3.1. *Let $1 < p \le 2^* - 1$. Then there exist constants $\lambda^* > 0$, $\kappa > 0$ and $\rho > 0$ such that*

$$J_\lambda(u) \ge \kappa$$

for $0 < \lambda < \lambda^*$ and $\|u\|_V = \rho$.

Proof. We distinguish two cases: (i) $\||\nabla v|\|_2 \le \eta|t|$ and (ii) $\||\nabla v|\|_2 > \eta|t|$, where $\eta > 0$ is a constant from property (**P**). If (i) holds and $\||\nabla v|\|_2^2 + t^2 = \rho^2$, then $t^2 \ge \frac{\rho^2}{1+\eta^2}$ and by property (**P**) we have

$$\int_\Omega a(x)|u|^{p+1}\,dx \geq \frac{|t|^{p+1}}{2}\int_\Omega a(x)\,dx = \frac{|t|^{p+1}}{2}\beta,$$

where $\beta = \int_\Omega a(x)\,dx > 0$. From this we derive that

$$J_\lambda(u) \geq \frac{\beta\rho^{p+1}}{2(p+1)(1+\eta^2)^{\frac{p+1}{2}}} - \frac{\lambda}{2}\int_{\partial\Omega} u^2\,dS_x. \tag{3.1}$$

In case (ii) we have

$$\|u\|_V \leq \||\nabla v|\|_2 \left(1+\frac{1}{\eta^2}\right)^{\frac{1}{2}}. \tag{3.2}$$

From the Sobolev embedding theorem we get

$$\left|\int_\Omega a(x)|u|^{p+1}\,dx\right| \leq C_1\|u\|_V^{p+1} \leq C_1\||\nabla v|\|_2^{p+1}\left(1+\frac{1}{\eta^2}\right)^{\frac{p+1}{2}},$$

where $C_1 > 0$ is a constant independent of u. Thus

$$J_\lambda(u) \geq \frac{1}{2}\||\nabla v|\|_2^2 - \frac{C_1\||\nabla v|\|_2^{p+1}}{p+1}\left(1+\frac{1}{\eta^2}\right)^{\frac{p+1}{2}} - \frac{\lambda}{2}\int_{\partial\Omega} u^2\,dS_x.$$

Taking $\||\nabla v|\|_2 \leq \rho$ small enough we derive from this inequality that (observe that $2 < p+1$)

$$J_\lambda(u) \geq \frac{1}{4}\||\nabla v|\|_2^2 - \frac{\lambda}{2}\int_{\partial\Omega} u^2\,dS_x.$$

If $\|u\|_V = \rho$, then by (3.2) we get

$$J_\lambda(u) \geq \frac{\rho^2\eta^2}{4\,(1+\eta^2)} - \frac{\lambda}{2}\int_{\partial\Omega} u^2\,dS_x. \tag{3.3}$$

Both cases (3.1) and (3.3) lead to the estimate

$$J_\lambda(u) \geq \min\left(\frac{\rho^2\eta^2}{4\,(1+\eta^2)}, \frac{\beta\rho^{p+1}}{2(p+1)\,(1+\eta^2)^{\frac{p+1}{2}}}\right) - \frac{\lambda}{2}\int_{\partial\Omega} u^2\,dS_x.$$

Since norms $\|\cdot\|_V$ and $\|\cdot\|_*$ are equivalent we get

$$J_\lambda(u) \geq \min\left(\frac{\rho^2\eta^2}{4\,(1+\eta^2)}, \frac{\beta\rho^{p+1}}{2(p+1)\,(1+\eta^2)^{\frac{p+1}{2}}}\right) - \lambda C_2\rho^2$$

for $\|u\|_V = \rho$, where $C_2 > 0$ is a constant independent of u and ρ. Choosing

$$\lambda^* = \frac{1}{2C_2\rho^2}\min\left(\frac{\rho^2\eta^2}{4\,(1+\eta^2)}, \frac{\beta\rho^{p+1}}{2(p+1)\,(1+\eta^2)^{\frac{p+1}{2}}}\right)$$

the result follows with κ given by

$$\kappa = \frac{1}{2}\min\left(\frac{\rho^2\eta^2}{4\,(1+\eta^2)}, \frac{\beta\rho^{p+1}}{2(p+1)\,(1+\eta^2)^{\frac{p+1}{2}}}\right).$$

$\square$

Proposition 3.2. *Let* $0 < p < 1$. *Then there exist constants* $\lambda^{**} > 0$ *and* $\rho_0 > 0$ *such that for every* $0 < \lambda < \lambda^{**}$ *there exists a constant* $\kappa = \kappa(\lambda) > 0$ *with property*

$$\bar{J}_\lambda(u) \geq \kappa$$

for $\|u\|_V = \rho_0$.

Proof. As in the proof of Proposition 3.1 we distinguish two cases (i) and (ii). If (i) holds, we get

$$\bar{J}_\lambda(u) \geq \lambda\left(\frac{\beta\rho^{p+1}}{2(p+1)\,(1+\eta^2)^{\frac{p+1}{2}}} - C_2\rho^2\right).$$

Since $p+1 < 2$, we can choose $\rho_0 > 0$ and $\alpha > 0$ such that

$$\bar{J}_\lambda(u) \geq \alpha\lambda \tag{3.4}$$

for $\|u\|_V = \rho_0$ and for all $\lambda > 0$. In the case (ii) we have

$$\rho = \|u\|_V \leq \||\nabla v|\|_2 \left(1 + \frac{1}{\eta^2}\right)^{\frac{1}{2}}.$$

Thus by the Sobolev embedding theorem we have

$$\begin{aligned}\left|\int_\Omega a(x)|u|^{p+1}\,dS_x\right| &\leq C_1\|u\|_V^{p+1} \leq C_1\||\nabla v|\|_2^{p+1}\left(1+\frac{1}{\eta^2}\right)^{\frac{p+1}{2}} \\ &\leq C_1\rho^{p+1}\left(1+\frac{1}{\eta^2}\right)^{\frac{p+1}{2}}.\end{aligned}$$

Thus

$$\bar{J}_\lambda(u) \geq \frac{\rho^2\eta^2}{2(1+\eta^2)} - \lambda\left[\frac{C_1\rho^{p+1}}{p+1}\left(1+\frac{1}{\eta^2}\right)^{\frac{p+1}{2}} + C_2\rho^2\right].$$

We now apply this inequality with $\rho = \rho_0$ and choose $\lambda^{**} > 0$ and $\alpha_1 > 0$ so that

$$\bar{J}_\lambda(u) \geq \alpha_1 \tag{3.5}$$

for $\|u\|_V = \rho_0$ and $0 < \lambda < \lambda^{**}$. Therefore (3.4) and (3.5) yield

$$\bar{J}_\lambda(u) \geq \min(\lambda\alpha, \alpha_1)$$

for $\|u\|_V = \rho_0$ and $0 < \lambda < \lambda^{**}$. □

If $1 < p \leq 2^* - 1$, testing $\bar{J}_\lambda(u)$ with a constant function $u = t$ we get

$$J_\lambda(t) = \frac{t^{p+1}}{p+1}\int_\Omega a(x)\,dx - \frac{\lambda t^2}{2}|\partial\Omega| = t^2\left(\frac{t^{p-1}}{p+1}\int_\Omega a(x)\,dx - \frac{\lambda}{2}|\partial\Omega|\right) < 0$$

for $t > 0$ sufficiently small.

If $0 < p < 1$, taking a function $\phi \in H^1(\Omega)$ with $\operatorname{supp} \phi \subset \{x \in \bar{\Omega},\, a(x) < 0\}$ we obtain

$$\bar{J}_\lambda(t\phi) = t^{p+1}\Bigg(\frac{t^{1-p}}{2}\int_\Omega |\nabla \phi|^2\, dx - \frac{t^{1-p}\lambda}{2}\int_{\partial\Omega} \phi^2\, dS_x \\ -\frac{\lambda}{p+1}\int_\Omega a^-(x)|\phi|^{p+1}\, dS_x\Bigg) < 0$$

for $t > 0$ sufficiently small. Therefore we have

$$\inf_{\|u\|_V \le \rho} J_\lambda(u) < 0 \quad \text{for} \quad 0 < \lambda < \lambda^* \tag{3.6}$$

and

$$\inf_{\|u\|_V \le \rho_0} \bar{J}_\lambda(u) < 0 \quad \text{for} \quad 0 < \lambda < \lambda^{**}. \tag{3.7}$$

4. Palais–Smale sequences

In this section we show that Palais–Smale sequences are bounded in $H^1(\Omega)$ in both cases $1 < p \le 2^*-1$ and $0 < p < 1$. We begin with the case $1 < p \le 2^*-1$.

Proposition 4.1. *Let $1 < p \le 2^* - 1$ and $\lambda > 0$. Assume that $|\{x \in \bar{\Omega} : a(x) = 0\}| = 0$. Then every Palais–Smale sequence for J_λ is bounded in $H^1(\Omega)$.*

Proof. Let $J_\lambda(u_n) \to c \in \mathbb{R}$ and $J'_\lambda(u_n) \to 0$ in $H^{-1}(\Omega)$. Arguing by contradiction assume $\|u_n\| \to \infty$ as $n \to \infty$. We have

$$\begin{aligned}\frac{1}{2}\langle J'_\lambda(u_n), u_n\rangle - J_\lambda(u_n) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega a(x)|u_n|^{p+1}\, dx \\ &= \epsilon_n\|u_n\| + c + o(1),\end{aligned} \tag{4.1}$$

where $\epsilon_n \to 0$. On the other hand we have

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2\, dx - \frac{\lambda}{2}\int_{\partial\Omega} u_n^2\, dS_x + \frac{1}{p+1}\int_\Omega a(x)|u_n|^{p+1}\, dx = c + o(1). \tag{4.2}$$

We derive from (4.1) and (4.2) that $\int_{\partial\Omega} u_n^2\, dS_x \to \infty$ as $n \to \infty$. We put

$$v_n = \frac{u_n}{\left(\int_{\partial\Omega} u_n^2\, dS_x\right)^{\frac{1}{2}}}.$$

It then follows from (4.1) and (4.2) that

$$\frac{1}{2}\int_\Omega |\nabla v_n|^2\, dx - \frac{\lambda}{2} \le C_1\frac{\|u_n\|}{\|u_n\|^2_{L^2(\partial\Omega)}} + o(1) \le C_1\frac{\|v_n\|}{\|u_n\|_{L^2(\partial\Omega)}} + o(1). \tag{4.3}$$

This shows that $\{v_n\}$ is bounded in $H^1(\Omega)$. We may assume that $v_n \rightharpoonup v$ in $H^1(\Omega)$, $v_n \to v$ in $L^2(\partial\Omega)$ and $v_n \to v$ in $L^q(\Omega)$ for $1 \le q < 2^*$. For every $\phi \in H^1(\Omega)$ we have

$$\int_\Omega \nabla v_n \nabla \phi\, dx - \lambda\int_{\partial\Omega} v_n\phi\, dS_x = \|u_n\|^{p-1}_{L^2(\partial\Omega)}\int_\Omega a(x)|v_n|^{p-1}v_n\phi\, dx + o(1).$$

This yields (since $1 < p$)

$$\int_\Omega a(x)|v|^{p-1}v\phi\, dx = 0$$

for every $\phi \in H^1(\Omega)$. Since $a(x)$ vanishes on a set of measure 0 we must have $v = 0$ a.e. on Ω and this contradicts the fact that $\int_{\partial\Omega} v^2\, dS_x = 1$. Hence $\{u_n\}$ is bounded in $H^1(\Omega)$. □

We now turn our attention to the case $0 < p < 1$. We need some information about the following eigenvalue problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

The smallest (principal) eigenvalue is obviously equal to 0 and the corresponding eigenfunctions are constant functions. All eigenfunctions corresponding to positive eigenvalues are orthogonal in $L^2(\partial\Omega)$ to 1. We denote by λ_2 the first positive eigenvalue to problem (4.4).

Proposition 4.2. *Let $0 < p < 1$. Then there exists $0 < \tilde{\lambda} \le \min(\lambda^{**}, \lambda_2)$ such that Palais–Smale sequences for $\bar{J}_\lambda$ with $0 < \lambda < \tilde{\lambda}$ are bounded in $H^1(\Omega)$.*

Proof. Let $\{u_n\} \subset H^1(\Omega)$ be such that $\bar{J}_\lambda(u_n) \to c$ and $\bar{J}_\lambda'(u_n) \to 0$ in $H^{-1}(\Omega$. We use the decomposition $u_n = v_n + t_n$, $v_n \in V$ and $t_n \in \mathbb{R}$. We claim that $\{t_n\}$ is a bounded sequence. Arguing by contradiction assume that $t_n \to \infty$ (the case $t_n \to -\infty$ can be treated in a similar way). We have

$$\begin{aligned}\int_\Omega |\nabla v_n|^2\, dx &= 2c + \lambda \int_{\partial\Omega} (v_n + t_n)^2\, dS_x - \frac{2\lambda}{p+1}\int_\Omega a(x)|v_n + t_n|^{p+1}\, dx + o(1) \\ &\le 2c + 2\lambda C_1 \int_\Omega |\nabla v_n|^2\, dx + 2\lambda t_n^2 |\partial\Omega| \\ &\quad + C_2\|a\|_\infty \left(\int_\Omega |\nabla v_n|^2\, dx + t_n^2\right)^{\frac{p+1}{2}} + o(1).\end{aligned}$$

Here we have used a trace embedding theorem for the space V. We put $w_n = \frac{v_n}{t_n}$. We derive from the above inequality that

$$\begin{aligned}&(1 - 2\lambda C_1)\int_\Omega |\nabla w_n|^2\, dx \\ &\quad \le \frac{2c}{t_n^2} + 2\lambda|\partial\Omega| + C_2\|a\|_\infty t_n^{p-1}\lambda\left(\int_\Omega |\nabla w_n|^2\, dx + 1\right)^{\frac{p+1}{2}} + o(1).\end{aligned}$$

Choosing $2\tilde{\lambda}C_1 < 1$ we deduce from this inequality that $\{w_n\}$ is bounded in $H^1(\Omega)$. Hence we may assume that $\nabla w_n \rightharpoonup \nabla w$ in $L^2(\Omega)$ and $w_n \to w$ in $L^2(\partial\Omega)$. On the other hand for every $\phi \in H^1(\Omega)$ we have

$$\begin{aligned}&\int_\Omega \nabla w_n \nabla\phi\, dx - \lambda\int_{\partial\Omega}(w_n + 1)\phi\, dS_x + t_n^{p-1}\lambda\int_\Omega a(x)|w_n + 1|^{p-1}(w_n + 1)\phi\, dx \\ &\quad = \frac{\epsilon_n}{t_n}\|u_n\| \to 0,\end{aligned} \tag{4.5}$$

where $\epsilon_n \to 0$. Letting $t_n \to \infty$ we derive from (4.5) that

$$\int_\Omega \nabla w \nabla \phi \, dx - \lambda \int_{\partial\Omega} (w+1)\phi \, dx = 0$$

for every $\phi \in H^1(\Omega)$. If $w = 0$ on Ω we get a contradiction. So $w \neq 0$ and the function w cannot be constant as $w \in V$. Hence $w + 1$ is a nonzero solution of (4.4) with $0 < \lambda < \lambda_2$. Since λ_2 is the first positive eigenvalue of (4.4) we have arrived at a contradiction. Therefore $\{t_n\}$ is bounded. This yields the boundedness of $\{\nabla u_n\}$ in $L^2(\Omega)$ and the result follows. $\square$

5. Existence of solutions (subcritical cases)

We prove in both cases $0 < p < 1$ and $1 < p < 2^* - 1$ the existence of at least two solutions.

Theorem 5.1. *Let $1 < p < 2^* - 1$ and $0 < \lambda < \lambda^*$. Assume that $|\{x \in \bar{\Omega}, a(x) = 0\}| = 0$. Then problem (1.1) has a solution u with $J_\lambda(u) > 0$ and a solution v with $J_\lambda(v) < 0$.*

Proof. Let $0 \neq w \in H^1(\Omega)$ such that supp $w \subset \{x \in \bar{\Omega},\, a(x) < 0\}$. Then

$$J_\lambda(tw) = \frac{t^2}{2}\left(\int_\Omega |\nabla w|^2 \, dx - \lambda \int_{\partial\Omega} w^2 \, dS_x\right) + \frac{t^{p+1}}{p+1}\int_\Omega a(x)|w|^{p+1} \, dx < 0$$

for $t > 0$ sufficiently large since $p > 1$. We set

$$0 < c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], H^1(\Omega)), \gamma(0) = 0,\ \gamma(1) = tw\},$$

where $t > 0$ is so large that $\|tw\| > \rho$ and $J_\lambda(tw) < 0$. Here ρ is a constant from Proposition 3.1. These observations show that the functional J_λ has a mountain-pass structure in $H^1(\Omega)$. Therefore there exists a sequence $\{u_n\} \subset H^1(\Omega)$ such that $J_\lambda(u_n) \to c_\lambda$ and $J'_\lambda(u_n) \to 0$ in $H^{-1}(\Omega)$. By Proposition 4.1 the sequence $\{u_n\}$ is bounded in $H^1(\Omega)$. Hence we may assume that, up to a subsequence, $u_n \rightharpoonup u$ in $H^1(\Omega)$, $u_n \to u$ in $L^q(\Omega)$ for $1 \leq q < 2^*$ and $u_n \to u$ in $L^2(\partial\Omega)$. Since $J'_\lambda(u_n) \to 0$ in $H^{-1}(\Omega)$ we get for $m > n$

$$\int_\Omega |\nabla u_m - \nabla u_n|^2 \, dx - \lambda \int_{\partial\Omega} (u_m - u_n)^2 \, dS_x + \int_\Omega a(x)(|u_m|^{p-1}u_m - |u_n|^{p-1}u_n)(u_m - u_n) \, dx = o(1).$$

The second and third integrals tend to 0 as $m, n \to \infty$. This yields $u_n \to u$ in $H^1(\Omega)$ and u is a solution of (1.1) with a positive energy. Applying Theorem 10 in [6] we may assume that u is nonnegative on Ω. Since $\inf_{\|u\| \leq \rho} J_\lambda(u) < 0$ for some $\rho > 0$, a solution with a negative energy is obtained by the Ekeland variational principle. If v is a local minimizer then $|v|$ is also local minimizer so we may assume that v is nonnegative. By the elliptic regularity theory both solutions u and v belong to $C^{2,\alpha}_{\mathrm{loc}}(\Omega)$ (see Appendix B in [17]). It then follows from the Harnack inequality (Theorem 8.20 in [12]) that both solutions u and v are positive on Ω. One can also show that these solution belong to $C^{1,\beta}(\bar{\Omega})$ for some $0 < \beta < 1$ (see Appendix). □

We now consider the case $0 < p < 1$. Testing $\bar{J}_\lambda$ with a constant function $u = t$ we get

$$\bar{J}_\lambda(t) = -\frac{\lambda t^2}{2}|\partial\Omega| + \frac{\lambda t^{p+1}}{p+1}\int_\Omega a(x)\,dx < 0$$

for $t > 0$ large. So we can choose $w = t$ so that $\|w\| > \rho$ and $\bar{J}_\lambda(t) < 0$ for a given λ belonging to the interval $(0, \tilde{\lambda})$ (see Propositions 3.2 and 4.2).

We can now formulate the following existence result.

Theorem 5.2. *Let $0 < p < 1$. Then for every $0 < \lambda < \tilde{\lambda}$ problem (1.2) has a solution u with $\bar{J}_\lambda(u) > 0$ and a solution v with $\bar{J}_\lambda(v) < 0$.*

The proof is similar to that of Theorem 5.1 and is omitted. The boundedness of Palais–Smale sequences follows from Proposition 4.2. In this case both solutions are nonnegative and nonzero. The positivity argument employed in the proof of Theorem 5.1 cannot be extended to this case.

6. Existence of solutions (critical case)

In this section we investigate the solvability of problem (1.1) with $p = 2^* - 1$. According to Proposition 3.1 the functional J_λ has a mountain-pass structure for $0 < \lambda < \lambda^*$. Moreover, $\inf_{\|u\|\le\rho} J_\lambda(u) < 0$. It is easy to show, using the Ekeland variational principle (see [11]), that the infimum of J_λ on the ball $\|u\| < \rho$ is attained by a function v, that is, $J_\lambda(v) = \inf_{\|u\|\le\rho} J_\lambda(u)$. Repeating the argument from the previous section we can assume that $v > 0$ on Ω.

In what follows we assume that $a(x)$ is negative somewhere on $\partial\Omega$. We put

$$A_m = \max_{x\in\partial\Omega}(-a(x)) \quad \text{and} \quad A_M = \max_{x\in\bar{\Omega}}(-a(x)).$$

These constants play an important role in finding sufficient conditions for the existence of solution of problem (1.1) in critical case. By S we denote the best Sobolev constant for the continuous embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$, that is,

$$S = \inf_{u\in H_0^1(\Omega)-\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{\left(\int_\Omega |u|^{2^*}\,dx\right)^{\frac{2}{2^*}}}.$$

We recall that the best Sobolev constant S is attained only if $\Omega = \mathbb{R}^N$. The minimizers are given by a family of functions

$$U_{\epsilon,y}(x) = \epsilon^{-\frac{N-2}{2}}U\left(\frac{x-y}{\epsilon}\right), \quad \epsilon > 0, \quad y \in \mathbb{R}^N,$$

where

$$U(x)=\left(\frac{N(N-2)}{N(N-2)+|x|^2}\right)^{\frac{N-2}{2}}.$$

The function U, called an instanton, satisfies the equation

$$-\Delta U=U^{2^*-1} \text{ in } \mathbb{R}^N.$$

We also have (see [17, 18])

$$\int_{\mathbb{R}^N}|\nabla U_{\epsilon,y}|^2\,dx=\int_{\mathbb{R}^N}U_{\epsilon,y}^{2^*}\,dx=S^{\frac{N}{2}}.$$

Let $H(y)$ denote the mean curvature of $\partial\Omega$ at $y\in\partial\Omega$. It is well-known (see [3, 4]) that

$$\frac{\int_\Omega|\nabla U_{\epsilon,y}|^2\,dx}{\left(\int_\Omega U_{\epsilon,y}^{2^*}\,dx\right)^{\frac{2}{2^*}}}=\begin{cases}2^{-\frac{2}{N}}S-A_NH(y)\epsilon\log\frac{1}{\epsilon}+O(\epsilon), & N=3,\\ 2^{-\frac{2}{N}}S-A_NH(y)\epsilon+O\left(\epsilon^2\right)\log\frac{1}{\epsilon}, & N=4,\\ 2^{-\frac{2}{N}}S-A_NH(y)\epsilon+O\left(\epsilon^2\right), & N\geq 5,\end{cases} \tag{6.1}$$

where A_N is a positive constant depending only on N (see [1, 2, 19]).

Proposition 6.1. *Let $p=2^*-1$ and $0<\lambda<\lambda^*$. Assume that $|\{x\in\bar{\Omega},\ a(x)=0\}|=0$ and that $a(x)$ is negative somewhere on $\partial\Omega$. Let $u_n\rightharpoonup u$ in $H^1(\Omega)$. If*

$$J_\lambda(u_n)\to c<J_\lambda(u)+\frac{1}{N}\min\left(\frac{S^{\frac{N}{2}}}{A_M^{\frac{N-2}{2}}},\frac{S^{\frac{N}{2}}}{2A_m^{\frac{N-2}{2}}}\right) \quad \textit{and} \quad J'_\lambda(u_n)\to 0 \textit{ in } H^{-1}(\Omega),$$

then $u_n\to u$ in $H^1(\Omega)$.

Proof. We may also assume that $|\nabla u_n|^2\rightharpoonup\mu$ and $|u_n|^{2^*}\rightharpoonup\nu$ in the sense of measures. Then by P.L.Lions' concentration - compactness principle (see [15]) there exist an at most countable set J and sequences $\{x_j\}\subset\bar{\Omega}$, $\{\mu_j\},\{\nu_j\}\subset(0,\infty)$, $j\in J$, such that

$$|\nabla u_n|^2\rightharpoonup\mu\geq|\nabla u|^2+\sum_{j\in J}\mu_j\delta_{x_j}$$

and

$$|u_n|^{2^*}\rightharpoonup\nu=|u|^{2^*}+\sum_{j\in J}\nu_j\delta_{x_j},$$

where δ_{x_j} are the Dirac measures assigned to points x_j. Moreover, we have

$$S\nu_j^{\frac{2}{2^*}}\leq\mu_j \text{ if } x_j\in\Omega \tag{6.2}$$

and

$$\frac{S}{2^{\frac{2}{N}}}\nu_j^{\frac{2}{2^*}}\leq\mu_j \text{ if } x_j\in\partial\Omega. \tag{6.3}$$

Testing $J'_\lambda(u_n)\to 0$ in $H^{-1}(\Omega)$ with a family $\{\phi_\delta\}$, $\delta>0$, of C^1-functions concentrating at 0 as $\delta\to 0$ we easily derive that

$$\mu_j\leq -a(x_j)\nu_j \text{ for } j\in J. \tag{6.4}$$

This means that a concentration can only occur at points x_j with $a(x_j) < 0$. We now observe that if $\nu_j > 0$ for some $j \in J$ then by (6.2) and (6.4) we get

$$\mu_j \geq \frac{S^{\frac{N}{2}}}{(-a(x_j))^{\frac{N-2}{2}}} \quad \text{if} \quad x_j \in \Omega \tag{6.5}$$

and by (6.2) and (6.3) we get

$$\mu_j \geq \frac{S^{\frac{N}{2}}}{2(-a(x_j))^{\frac{N-2}{2}}} \quad \text{if} \quad x_j \in \partial\Omega. \tag{6.6}$$

We set $v_n = u_n - u$. It follows from the Brezis–Lieb lemma (see [7]) that

$$\int_\Omega a(x)|u_n|^{p+1}\,dx = \int_\Omega a(x)|u|^{p+1}\,dx + \int_\Omega a(x)|v_n|^{p+1}\,dx + o(1).$$

We also have

$$\int_\Omega |\nabla u_n|^2\,dx = \int_\Omega |\nabla u|^2\,dx + \int_\Omega |\nabla v_n|^2\,dx + o(1).$$

Since $u_n \to u$ in $L^2(\partial\Omega)$ we obtain

$$J_\lambda(u) + \frac{1}{2}\int_\Omega |\nabla v_n|^2\,dx + \frac{1}{p+1}\int_\Omega a(x)|v_n|^{p+1}\,dx = c + o(1) \tag{6.7}$$

and

$$\int_\Omega |\nabla u|^2\,dx - \lambda\int_{\partial\Omega} u^2\,dS_x + \int_\Omega a(x)|u|^{p+1}\,dx \\ + \int_\Omega |\nabla v_n|^2\,dx + \int_\Omega a(x)|v_n|^{p+1}\,dx = o(1). \tag{6.8}$$

Since u is a weak solution of (1.1) we deduce from (6.8) that

$$\int_\Omega |\nabla v_n|^2\,dx + \int_\Omega a(x)|v_n|^{p+1}\,dx = o(1).$$

Substituting this into (6.7) we obtain

$$J_\lambda(u) + \frac{1}{N}\int_\Omega |\nabla v_n|^2\,dx = c + o(1). \tag{6.9}$$

Assume that $\mu_{j_0} > 0$ for some $j_0 \in J$. Then by (6.5), if $x_{j_0} \in \Omega$,

$$c \geq J_\lambda(u) + \frac{S^{\frac{N}{2}}}{N(-a(x_{j_0}))} \geq \frac{S^{\frac{N}{2}}}{NA_M^{\frac{N-2}{2}}}$$

and if $x_{j_0} \in \partial\Omega$, then (6.6) yields

$$c \geq J_\lambda(u) + \frac{S^{\frac{N}{2}}}{2NA_m^{\frac{N-2}{2}}}$$

which is impossible and this contradiction completes the proof. □

In what follows we use notation $A(x) = -a(x)$.

We choose a point $y \in \partial\Omega$ such that $A(y) = A_m$. Let ϕ be a C^1-function on $\mathbb{R}^N$ such that $\phi(x) = 1$ on $B(y, \frac{\delta}{2})$, $\phi(x) = 0$ on $\mathbb{R}^N \backslash B(0, \delta)$ and $0 \leq \phi(x) \leq 1$ on $\mathbb{R}^N$. We put $w_{y,\epsilon}(x) = \phi(x) U_{\epsilon,y}(x)$. The radius of the ball $B(0, \delta)$ is chosen so that $\Omega \cap B(0, \delta) \subset \{x \in \bar{\Omega}, a(x) < 0\}$.

We now prove the existence of a second solution of problem (1.1) in the case

$$\frac{1}{N} \min\left(\frac{S^{\frac{N}{2}}}{A_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2A_m^{\frac{N-2}{2}}} \right) = \frac{S^{\frac{N}{2}}}{2NA_m^{\frac{N-2}{2}}},$$

that is, when $A_M \leq 2^{\frac{2}{N-2}} A_m$. The case $A_M > 2^{\frac{2}{N-2}} A_m$ seems to be more difficult and we were unable to find conditions on the coefficient $a(x)$ guaranteeing the existence of a second solution.

Proposition 6.2. *Let $p = 2^* - 1$, $0 < \lambda < 2^*$ and $A_M \leq 2^{\frac{2}{N-2}} A_m$ and $N \geq 5$. Assume that $a(x)$ is negative somewhere on $\partial\Omega$. Furthermore assume that there exists a point $y \in \partial\Omega$ such that $A(y) = A_m$, $H(y) > 0$ and*

$$|A(y) - A(x)| = o(|x - y|) \quad \text{for } x \text{ near } y. \tag{6.10}$$

If v is a local minimizer for J_λ then

$$\max_{0 \leq t} J_\lambda(v + tw_{\epsilon,y}) < J_\lambda(v) + \frac{S^{\frac{N}{2}}}{2NA_m^{\frac{N-2}{2}}}.$$

Proof. We follow some ideas from papers [8,16]. We need the following inequality (see [16]): given $q > 2$ and $\kappa \in (1, q-1)$ there exists a constant $C > 0$ such that

$$(s+t)^q \geq s^q + t^q + qs^{q-1}t + qst^{q-1} - Ct^\kappa s^{q-\kappa} \tag{6.11}$$

for $s, t \geq 0$. We have

$$\begin{aligned} J_\lambda(v + tw_{y,\epsilon}) = &\frac{1}{2}\int_\Omega |\nabla v|^2\,dx + \frac{t^2}{2}\int_\Omega |\nabla w_{y,\epsilon}|^2\,dx + t\int_\Omega \nabla v \nabla w_{y,\epsilon}\,dx \\ &- \frac{\lambda t^2}{2}\int_{\partial\Omega} w_{y,\epsilon}^2\,dS_x - t\lambda\int_{\partial\Omega} w_{y,\epsilon} v\,dS_x - \frac{\lambda}{2}\int_{\partial\Omega} v^2\,dS_x \\ &- \frac{1}{2^*}\int_\Omega A(x)|v + tw_{y,\epsilon}|^{2^*}\,dx. \end{aligned}$$

Using (6.11) we get (with $q = 2^*$ and $\kappa = \frac{N+1}{N-2}$)

$$\begin{aligned} J_\lambda(v + tw_{y,\epsilon}) &= J_\lambda(v) + \frac{1}{2^*}\int_\Omega A(x)|v|^{2^*}\,dx + t\int_\Omega A(x)v^{2^*-1}w_{y,\epsilon}\,dx \\ &\quad + \frac{t^2}{2}\int_\Omega |\nabla w_{y,\epsilon}|^2\,dx - \frac{\lambda t^2}{2}\int_{\partial\Omega} w_{y,\epsilon}^2\,dS_x \\ &\quad - \frac{1}{2^*}\int_\Omega A(x)|v + tw_{y,\epsilon}|^{2^*}\,dx \\ &\leq J_\lambda(v) + \frac{t^2}{2}\int_\Omega |\nabla w_{\epsilon,y}|^2\,dx - \frac{\lambda t^2}{2}\int_{\partial\Omega} w_{y,\epsilon}^2\,dS_x \end{aligned}$$

$$
\begin{aligned}
&-\frac{t^{2^*}}{2^*}\int_\Omega A(x)w_{y,\epsilon}^{2^*}\,dx + t\int_\Omega A(x)v^{2^*-1}w_{y,\epsilon}\,dx\\
&+Ct^{\frac{N+1}{N-2}}\int_\Omega A(x)v^{\frac{N-1}{N-2}}w_{y,\epsilon}^{\frac{N+1}{N-2}}\,dx - t\int_\Omega A(x)w_{y,\epsilon}v^{2^*-1}\,dx\\
&-t^{2^*-1}\int_\Omega A(x)w_{y,\epsilon}^{2^*-1}v\,dx\\
\le{}& J_\lambda(v) + \frac{t^2}{2}\int_\Omega |\nabla w_{y,\epsilon}|^2\,dx - \frac{t^{2^*}}{2^*}\int_\Omega A(x)w_{y,\epsilon}^{2^*}\,dx\\
&+Ct^{\frac{N+1}{N-2}}\int_\Omega A(x)v^{\frac{N-1}{N-2}}w_{y,\epsilon}^{\frac{N+1}{N-2}}\,dx. \qquad (6.12)
\end{aligned}
$$

We now observe that

$$\int_\Omega A(x)v^{\frac{N-1}{N-2}}w_{y,\epsilon}^{\frac{N+1}{N-2}}\,dx \le C\epsilon^{\frac{N-1}{2}},$$

where $C > 0$ is a constant independent of ϵ. To obtain this estimate we have used the fact that $v \in C(\bar{\Omega})$. From (6.12) we derive that

$$
\begin{aligned}
J_\lambda(v + tw_{y,\epsilon}) \le{}& J_\lambda(v) + \frac{t^2}{2}\int_\Omega |\nabla w_{y,\epsilon}|^2\,dx - \frac{t^{2^*}}{2^*}\int_\Omega A(x)w_{y,\epsilon}^{2^*}\,dx\\
&+Ct^{\frac{N+1}{N-2}}\epsilon^{\frac{N-1}{2}}\\
:={}& J_\lambda(v) + \frac{t^2}{2}\int_\Omega |\nabla w_{y,\epsilon}|^2\,dx - \frac{t^{2^*}}{2^*}\int_\Omega A(x)w_{y,\epsilon}^{2^*}\,dx + \Psi_\epsilon(t).
\end{aligned}
$$

As in paper [8] we show that for every $\epsilon > 0$ (small) there exists $t_\epsilon > 0$ such that

$$
\begin{aligned}
&\max_{t\ge 0}\left[\frac{t^2}{2}\int_\Omega |\nabla w_{y,\epsilon}|^2\,dx - \frac{t^{2^*}}{2^*}\int_\Omega A(x)w_{y,\epsilon}^{2^*}\,dx + \Psi_\epsilon(t)\right]\\
&= \frac{t_\epsilon^2}{2}\int_\Omega |\nabla w_{y,\epsilon}|^2\,dx - \frac{t_\epsilon^{2^*}}{2^*}\int_\Omega A(x)w_{y,\epsilon}^{2^*}\,dx + \Psi_\epsilon(t_\epsilon) < \infty
\end{aligned}
$$

and there exist constants $0 < T_1 < T_2 < \infty$ such that $T_1 \le t_\epsilon \le T_2$. Hence, using the expansion

$$\int_\Omega A(x)w_{y,\epsilon}(x)^{2^*}\,dx = A_m\int_\Omega w_{y,\epsilon}^{2^*}\,dx + o(1),$$

we obtain

$$\max_{t\le 0} J_\lambda(v + tw_{y,\epsilon}) \le J_\lambda(v) + \frac{1}{N}\left(\frac{\int_\Omega |\nabla w_{y,\epsilon}|^2\,dx}{\left(A_m\int_\Omega w_{y,\epsilon}^{2^*}\,dx + o(\epsilon)\right)^{\frac{2}{2^*}}}\right)^{\frac{N}{2}} + T_1^{\frac{N+1}{N-2}}\epsilon^{\frac{N-1}{2}}.$$

The asymptotic estimates (6.1) remain true for the truncated instantons $w_{y,\epsilon}$. So combining them with the above inequality we get

$$\max_{t\ge 0} J_\lambda(v + tw_{y,\epsilon}) < J_\lambda(v) + \frac{S^{\frac{N}{2}}}{2NA_m^{\frac{N-2}{2}}}$$

provided $N \ge 5$. □

We are now ready to prove the existence of a second solution.

Theorem 6.3. *Let $p = 2^* - 1$, $0 < \lambda < \lambda^*$ and $A_M \leq 2^{\frac{2}{N-2}} A_m$ and $N \geq 5$. Assume that $|\{x \in \bar{\Omega},\ a(x) = 0\}| = 0$ and that $a(x)$ is negative somewhere on the boundary. Furthermore assume that there exists a point $y \in \partial\Omega$ such that $A(y) = A_m$, $H(y) > 0$ and that (6.10) holds. Then problem (1.1) has a second solution u with $J_\lambda(u) > 0$.*

Proof. Since v is a local minimizer on the ball $\{\|u\| \leq \rho\}$ (see (3.6) in Section 3), we can define a mountain-pass level around v

$$0 < d_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\Omega)),\ \gamma(0) = v \text{ and } J(\gamma(1)) < 0\}$. It follows from Proposition 6.2 that

$$0 < d_\lambda < J_\lambda(v) + \frac{S^{\frac{N}{2}}}{2NA_m^{\frac{N-2}{2}}} < \frac{S^{\frac{N}{2}}}{2NA_m^{\frac{N-2}{2}}}.$$

Let $\{u_n\} \subset H^1(\Omega)$ be such that $J_\lambda(u_n) \to d_\lambda$ and $J'_\lambda(u_n) \to 0$ in $H^1(\Omega)$. Since $\{u_n\}$ is bounded in $H^1(\Omega)$ we may assume that $u_n \rightharpoonup u$. If $u = 0$, then by Proposition 6.1 $u_n \to 0$ in $H^1(\Omega)$. Since $J_\lambda(u_n) \to d_\lambda > 0$ we get a contradiction. Hence $u \neq 0$. If $u \neq v$ we are done. So it remains to consider the case $u_n \to v$. Again by Proposition 6.1 $u_n \to v$ in $H^1(\Omega)$ and $J_\lambda(u_n) \to J_\lambda(v) = d_\lambda$. Since $J_\lambda(v) < 0$ we have arrived at a contradiction. Hence u is a nontrivial solution of (1.1) distinct from v and this completes the proof. As in Sect. 5 we can assume that the mountain-pass solution u is positive on Ω. □

7. Behavior of solutions in the case $0 < p < 1$

First we consider the collection of the mountain-pass solutions of problem (1.2). Let $\{w_\lambda\} \subset H^1(\Omega)$ be the mountain-pass solutions of problem (1.2). We begin by showing that the mountain-pass level around v for $\bar{J}_\lambda$ behaves like $O(\lambda)$ when $\lambda \to 0$. We need the following lemma.

Lemma 7.1. *Let $0 < p < 2^* - 1$. Then there exists a constant $C > 0$ such that*

$$\int_{\partial\Omega} u^2 \, dS_x \leq C \left[\int_\Omega |\nabla u|^2 \, dx + \left| \int_\Omega a(x)|u|^{p+1} \, dx \right|^{\frac{2}{p+1}} \right] \tag{7.1}$$

for every $u \in H^1(\Omega)$.

Proof. Arguing by contradiction assume that there exists a sequence $\{u_n\} \subset H^1(\Omega)$ such that $\int_{\partial\Omega} u_n^2 \, dS_x = 1$ for each $n \in \mathbb{N}$, $\int_\Omega |\nabla u_n|^2 \, dx \to 0$ and $\int_\Omega a(x)|u_n|^{p+1} \, dx \to 0$ as $n \to \infty$. Note that $\{u_n\}$ is bounded in $H^1(\Omega)$. We may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$. By the lower semi-continuity of norm with respect to a weak convergence we get that $\int_\Omega |\nabla u|^2 \, dx = 0$. Thus u is a constant, say $u = t$. Since $H^1(\Omega)$ is compactly embedded into $L^{p+1}(\Omega)$ we derive that $|t|^{p+1} \int_\Omega a(x) \, dx = 0$. Since $\int_\Omega a(x) \, dx > 0$ we must have $t = 0$. Thus by the compact embedding of $H^1(\Omega)$ into $L^2(\partial\Omega)$ we get that $\int_{\partial\Omega} u_n^2 \, dS_x \to 0$ which is impossible. □

Remark 7.2. The square root of the right side of (7.1) defines a quasi-norm on $H^1(\Omega)$ (an equivalent norm if $p \geq 1$).

In what follows we denote by C a positive constant which can change from line to line and which is independent of λ. The functional $\bar{J}_\lambda$ is unbounded from below on $H^1(\Omega)$ and by Proposition 3.2 it has a mountain-pass structure. For $0 < \lambda < \tilde{\lambda}$ we denote by $\bar{d}_\lambda$ a mountain-pass level for $\bar{J}_\lambda$.

Lemma 7.3. *Let $0 < p < 1$. Then there exists a constant $C > 0$ such that the mountain-pass level $\bar{d}_\lambda$ for the functional $\bar{J}_\lambda$ satisfies $0 < \bar{d}_\lambda \leq C\lambda$ for $0 < \lambda < \tilde{\lambda}$.*

Proof. First note that $u = t > 0$ (a constant) is an admissible test function for $\bar{d}_\lambda$. Since $0 < p < 1$ we get

$$\bar{J}_\lambda(t) = \lambda\left(\frac{t^{p+1}}{p+1}\int_\Omega a(x)\,dx - \frac{t^2}{2}|\partial\Omega|\right) \to -\infty \quad \text{as } t \to \infty.$$

Therefore $\bar{d}_\lambda \leq \max_{t\geq 0} \bar{J}_\lambda(t) = C\lambda$, where

$$C = \max_{t\geq 0}\left(\frac{t^{p+1}}{p+1}\int_\Omega a(x)\,dx - \frac{t^2}{2}|\partial\Omega|\right) > 0.$$

□

Proposition 7.4. *Let $0 < p < 1$ and let $\{w_\lambda\}$, $0 < \lambda < \tilde{\lambda}$, be mountain-pass solutions of problem (1.2). Then*

$$w_\lambda \to \left(\frac{1}{|\partial\Omega|}\int_\Omega a(x)\,dx\right)^{\frac{1}{1-p}} \quad \text{in } H^1(\Omega) \text{ as } \lambda \to 0. \tag{7.2}$$

Proof. Since

$$\bar{d}_\lambda = \bar{J}_\lambda(w_\lambda) = \bar{J}_\lambda(w_\lambda) - \frac{1}{2}\langle \bar{J}_\lambda(w_\lambda), w_\lambda\rangle = \left(\frac{1}{p+1} - \frac{1}{2}\right)\lambda\int_\Omega a(x)w_\lambda^{p+1}\,dx,$$

by Lemma 7.3 we obtain that

$$\left|\int_\Omega a(x)w_\lambda^{p+1}\,dx\right| \leq C$$

for $0 < \lambda < \tilde{\lambda}$. It then follows from Lemma 7.1 that

$$\begin{aligned}\int_\Omega |\nabla w_\lambda|^2\,dx &\leq \lambda\left(\int_{\partial\Omega} w_\lambda^2\,dS_x + \left|\int_\Omega a(x)w_\lambda^{p+1}\,dx\right|\right)\\ &\leq C\lambda\left(\int_\Omega |\nabla w_\lambda|^2\,dx + \left|\int_\Omega a(x)w_\lambda^{p+1}\,dx\right| + \left|\int_\Omega a(x)w_\lambda^{p+1}\,dx\right|^{\frac{2}{p+1}}\right)\end{aligned}$$

for $0 < \lambda < \tilde{\lambda}$. Taking $\tilde{\lambda}$ smaller if necessary, we derive from this

$$\int_\Omega |\nabla w_\lambda|^2\,dx \leq C\lambda \tag{7.3}$$

for $0 < \lambda < \tilde{\lambda}$. Combining this with (7.1) we see that $\{w_\lambda\}$ is bounded in $H^1(\Omega)$. By (7.3) $|\nabla w_\lambda| \to 0$ in $L^2(\Omega)$. Therefore we may assume that $w_\lambda \to t \geq 0$ (a constant) in $H^1(\Omega)$. We now observe that

$$\int_{\partial\Omega} w_\lambda \, dS_x = \int_\Omega a(x) w_\lambda^p \, dx,$$

and letting $\lambda \to 0$ we derive from this that

$$t|\partial\Omega| = t^p \int_\Omega a(x) \, dx.$$

Hence either $t = 0$ or $t = \left(\frac{1}{|\partial\Omega|} \int_\Omega a(x) \, dx \right)^{\frac{1}{1-p}}$. Assuming that $w_\lambda \to 0$ in $H^1(\Omega)$ and observing that

$$\begin{aligned} \|\nabla w_\lambda\|^2 &\leq \lambda \left(\int_{\partial\Omega} w_\lambda^2 \, dS_x + \left| \int_\Omega a(x) w_\lambda^{p+1} \, dx \right| \right) \\ &\leq C\lambda \left(\|w_\lambda\|^2 + \|w_\lambda\|^{p+1} \right) \end{aligned}$$

we check that $\bar{J}_\lambda(w_\lambda) = o(\lambda)$ as $\lambda \to 0$. On the other hand by Proposition 3.2, we see that $\bar{J}_\lambda(w_\lambda) \geq C\lambda$ for $0 < \lambda < \tilde{\lambda}$ and we have arrived at a contradiction. This shows that relation (7.2) holds. □

We point out here that the limiting value in (7.2) has already appeared in paper [13] where problem (1.1) with $a(x) > 0$ on $\bar{\Omega}$ and $0 < p < 1$ has been investigated. The authors of [13] showed that if $\{u_\lambda\}$ is a collection of positive classical solutions for $\lambda > 0$ then

$$\lim_{\lambda \to 0} \lambda^{\frac{1}{1-p}} u_\lambda = \left(\frac{1}{|\partial\Omega|} \int_\Omega a(x) \, dS_x \right)^{\frac{1}{1-p}} \quad \text{in} \quad C^{2,\alpha}(\bar{\Omega}).$$

We notice that in formula (7.2) the factor $\lambda^{\frac{1}{1-p}}$ does not appear. This is due to the fact that $\{w_\lambda\}$ are solutions of problem (1.2) where the parameter λ appears on the right hand side of the equation.

We now turn our attention to local minimizers $\{u_\lambda\}$, $0 < \lambda < \tilde{\lambda}$ of the functional $\bar{J}_\lambda$ which are solutions of problem (1.2). Below we will frequently use the decomposition $u_\lambda = v_\lambda + t_\lambda$, $v_\lambda \in V$ and $t_\lambda \in \mathbb{R}$.

Lemma 7.5. *Local minimizers of $\bar{J}_\lambda$ satisfy*

$$\|u_\lambda\| \leq C\lambda^{\frac{1}{1-p}}$$

for $0 < \lambda < \tilde{\lambda}$, where $C > 0$ is a constant independent of λ.

Proof. It follows from

$$\bar{J}_\lambda(u_\lambda) - \frac{1}{2}\langle \bar{J}'_\lambda(u_\lambda), u_\lambda \rangle < 0,$$

that

$$\left(\frac{1}{p+1} - \frac{1}{2} \right) \int_\Omega a(x) u_\lambda^{p+1} \, dx < 0.$$

Hence

$$\int_\Omega a(x)u_\lambda^{p+1}\,dx < 0.$$

We now use the following inequality

$$\left|a(x)|1+w|^{p+1} - a(x)\right| \le C\left(|w|^{p+1} + |w|\right)$$

for every $w \in \mathbb{R}$, where $C > 0$ is a constant independent of w. Letting $w = \frac{v_\lambda}{t_\lambda}$ and assuming that $t_\lambda \neq 0$ we obtain

$$\left|a(x)|t_\lambda + v_\lambda|^{p+1} - |t_\lambda|^{p+1}a(x)\right| \le C\left(|v_\lambda|^{p+1} + |t_\lambda|^p|v_\lambda|\right) \tag{7.4}$$

Inequality (7.4) remains true if $t_\lambda = 0$. Integrating we get

$$\begin{aligned}|t_\lambda|^{p+1}\int_\Omega a(x)\,dx &\le |t_\lambda|^{p+1}\int_\Omega a(x)\,dx - \int_\Omega a(x)|v_\lambda + t_\lambda|^{p+1}\,dx\\ &= \int_\Omega a(x)\left[|t_\lambda|^{p+1} - |v_\lambda + t_\lambda|^{p+1}\right]\,dx\\ &\le C\left(\int_\Omega |v_\lambda|^{p+1}\,dx + |t_\lambda|^p\int_\Omega |v_\lambda|\,dx\right)\end{aligned}$$

By the Sobolev embeddings of the space V we derive from this that

$$\begin{aligned}|t_\lambda|^{p+1}\int_\Omega a(x)\,dx &\le C(\||\nabla v_\lambda|\|_2^{p+1} + |t_\lambda|^p\||\nabla v_\lambda|\|_2)\\ &\le \epsilon|t_\lambda|^{p+1} + C_\epsilon\||\nabla v_\lambda|\|_2^{p+1}.\end{aligned}$$

Hence

$$|t_\lambda| \le C\||\nabla v_\lambda|\|_2 = C\||\nabla u_\lambda|\|_2. \tag{7.5}$$

On the other hand from $\langle \bar{J}'_\lambda(u_\lambda), u_\lambda\rangle = 0$, using (7.5), we derive that

$$\begin{aligned}\|u_\lambda\|_V^2 &\le \||\nabla u_\lambda|\|_2^2 \le C\lambda(\|u_\lambda\|_V^{p+1} + \|u_\lambda\|_V^2)\\ &\le C\lambda(1 + \|u_\lambda\|_V^{1-p})\|u_\lambda\|_V^{p+1} \le C\lambda\|u_\lambda\|_V^{p+1}.\end{aligned}$$

This yields

$$\|u_\lambda\|_V \le C\lambda^{\frac{1}{1-p}}.$$

The result follows from the equivalence of norms $\|\cdot\|$ and $\|\cdot\|_V$. □

Lemma 7.6. *Let $0 < p < 1$ and let $\{u_\lambda\}$, $0 < \lambda < \tilde{\lambda}$, be local minimizers of $\bar{J}_\lambda$. Then there exists a constant $c < 0$ such that*

$$\bar{J}_\lambda(u_\lambda) \le c\lambda^{\frac{2}{1-p}}$$

for $0 < \lambda < \tilde{\lambda}$.

Proof. We choose a nonnegative function $w \in H^1(\Omega)$ such that $\int_\Omega a(x)w^{p+1}\,dx < 0$. For $s > 0$ we have

$$\begin{aligned}\bar{J}_\lambda(sw) &= \frac{s^2}{2}\int_\Omega |\nabla w|^2\,dx + \lambda\frac{s^{p+1}}{p+1}\int_\Omega a(x)w^{p+1}\,dx - \frac{\lambda s^2}{2}\int_{\partial\Omega} w^2\,dS_x\\ &\le \frac{s^2}{2}\int_\Omega |\nabla w|^2\,dx + \lambda\frac{s^{p+1}}{p+1}\int_\Omega a(x)w^{p+1}\,dx := g(s).\end{aligned}$$

The function $g(s)$ attains its minimum at

$$\bar{s} = \left(\frac{\int_\Omega |\nabla w|^2 \, dx}{-\lambda \int_\Omega a(x) w^{p+1} \, dx} \right)^{\frac{1}{p-1}}$$

and

$$g(\bar{s}) = \frac{p-1}{2(p+1)} \frac{\left(\int_\Omega |\nabla w|^2 \, dx\right)^{\frac{p+1}{p-1}}}{\left(-\lambda \int_\Omega a(x) w^{p+1} \, dx\right)^{\frac{2}{p-1}}} := c\lambda^{\frac{2}{1-p}}$$

with $c < 0$ (we recall that $\int_\Omega a(x) w^{p+1} \, dx < 0$). Hence

$$\bar{J}_\lambda(\bar{s}w) \le c\lambda^{\frac{2}{1-p}}.$$

Taking $\tilde{\lambda} > 0$ smaller, if necessary, we may assume that $\bar{s}w \in \{u; \, \|u\| \le \rho\}$ (see Proposition 3.2). Hence we get

$$\bar{J}_\lambda(u_\lambda) \le \bar{J}_\lambda(\bar{s}w) \le c\lambda^{\frac{2}{1-p}}$$

for $0 < \lambda < \tilde{\lambda}$. □

Proposition 7.7. *Let $0 < p < 1$. Then every sequence $\lambda_j \to 0$ has a subsequence such that $\lambda_j^{-\frac{1}{1-p}} u_{\lambda_j} \to v_0 \neq 0$ in $H^1(\Omega)$ and v_0 is a solution of the following problem*

$$\begin{cases} -\Delta v_0 + a(x) v_0^p = 0 & \text{in } \Omega, \\ \frac{\partial v_0}{\partial \nu} = 0 & \text{on } \partial\Omega, \quad v_0 \ge 0, \quad v_0 \not\equiv 0, \text{ on } \Omega. \end{cases} \tag{7.6}$$

Proof. We put $w_\lambda = \lambda^{-\frac{1}{1-p}} u_\lambda$. It then follows from Lemma 7.5 that w_λ is bounded in $H^1(\Omega)$. Therefore we can select a sequence $\lambda_j \to 0$ such that $w_{\lambda_j} \rightharpoonup v_0$ in $H^1(\Omega)$. Obviously v_0 is a solution of problem (7.6). By Lemma 7.6 we have

$$\bar{J}_{\lambda_j}(u_{\lambda_j}) = \bar{J}_{\lambda_j}\left(\lambda_j^{\frac{1}{1-p}} w_{\lambda_j}\right) \le c\lambda_j^{\frac{2}{1-p}},$$

where $c < 0$ is a constant independent of λ_j. Multiplying by $\lambda_j^{-\frac{2}{1-p}}$ we get

$$\frac{1}{2} \||\nabla w_{\lambda_j}|\|_2^2 + \frac{1}{p+1} \int_\Omega a(x) w_{\lambda_j}^{p+1} \, dx = c + \frac{\lambda_j}{2} \int_{\partial\Omega} w_{\lambda_j}^2 \, dS_x \le \frac{c}{2} \tag{7.7}$$

for j sufficiently large. From (7.7) we deduce that

$$\frac{1}{2} \||\nabla v_0|\|_2^2 + \frac{1}{p+1} \int_\Omega a(x) v_0^{p+1} \, dx \le \frac{c}{2} < 0.$$

This implies that $v_0 \neq 0$. Finally, to show that $w_{\lambda_j} \to v_0$ in $H^1(\Omega)$ we set

$$J_0(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx + \frac{1}{p+1} \int_\Omega a(x) |v|^{p+1} \, dx.$$

We check that $J_0'(w_{\lambda_j}) \to 0$ in $H^{-1}(\Omega)$. Indeed, for $\phi \in H^1(\Omega)$ we have

$$\begin{aligned} 0 &= \langle \bar{J}_{\lambda_j}\big(\lambda_j^{\frac{1}{1-p}} w_{\lambda_j}\big), \phi\rangle \\ &= \lambda_j^{\frac{1}{1-p}} \left[\int_\Omega \nabla w_{\lambda_j} \nabla \phi \, dx + \int_\Omega a(x) w_{\lambda_j}^p \phi \, dx - \lambda_j \int_{\partial\Omega} w_{\lambda_j} \phi \, dS_x\right]. \end{aligned}$$

This can be rewritten as

$$\int_\Omega \nabla w_{\lambda_j} \nabla \phi \, dx + \int_\Omega a(x) w_{\lambda_j}^p \phi \, dx = \lambda_j \int_\Omega w_{\lambda_j} \phi \, dS_x.$$

Hence

$$\left|\int_\Omega \nabla w_{\lambda_j} \nabla \phi \, dx + \int_\Omega a(x) w_{\lambda_j}^p \phi \, dx\right| \le \lambda_j \|w_{\lambda_j}\| \|\phi\| \to 0$$

as $j \to \infty$ uniformly in ϕ on a unit ball in $H^1(\Omega)$. Thus $J_0'(w_{\lambda_j}) \to 0$ in $H^{-1}(\Omega)$. Since $w_{\lambda_j} \rightharpoonup v_0$ in $H^1(\Omega)$ and $w_{\lambda_j} \to v_0$ in $L^{p+1}(\Omega)$, we derive from this that $w_{\lambda_j} \to v_0$ in $H^1(\Omega)$. □

Appendix: regularity up to the boundary

Below we sketch the proof of the fact that solutions of problem (1.1), with $1 < p \le 2^* - 1$ belong to $C^{1,\beta}(\bar{\Omega})$ for some $0 < \beta < 1$.

First we prove that solutions to problem (1.1) belong to $L^t(\Omega)$ for every $t \ge 1$. This follows from Lemma 5.1 in [19]:

Lemma 7.8. *Suppose that $\partial\Omega \in C^1$ and that $u \in H^1(\Omega)$ is a weak solution of the Neumann problem*

$$\begin{cases} -\Delta u = A(x) u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \alpha(x) u & \text{on } \partial\Omega, \end{cases}$$

where $A \in L^{\frac{N}{2}}(\Omega)$, $\alpha \in L^\infty(\Omega)$. Then $u \in L^t(\Omega)$ for every $t \ge 1$.

We apply this lemma to a solution to our problem (1.1) with $A(x) = -a(x)u^{p-1} \in L^{\frac{N}{2}}(\Omega)$ and $\alpha(x) = \lambda$ (we may assume that $u \ge 0$ and $\ne 0$).

In the next step we follow argument of Proposition 5 in [14] (page 506). We consider the auxiliary problem

$$\begin{cases} -\Delta v + Mv = f(x) & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \lambda v & \text{on } \partial\Omega, \end{cases} \tag{7.8}$$

where $f(x) = -a(x)u^p + Mu$ and $M > 0$ is a constant. By Lemma 7.8, $f \in L^t(\Omega)$ for every $t \ge 1$. Let $h(x)$ be a C^2 extension to $\bar{\Omega}$ of the distance function $\operatorname{dist}(x, \partial\Omega)$ in a neighbourhood of $\partial\Omega$. Problem (7.8) can be transformed into the Neumann problem by introducing a new unknown function $w = e^{-h(x)}v$. The new problem (transformed) is uniquely solvable in $W^{2,t}(\Omega)$ if M is sufficiently large. Hence $u \in W^{2,t}(\Omega)$ for every $t \ge 1$ and so $u \in C^{1,\beta}(\bar{\Omega})$ for some $0 < \beta < 1$. Applying the strong maximum principle we see that $u > 0$ on $\bar{\Omega}$.

In the case $0 < p < 1$ one can also prove that solution of problem (1.2) belong to $C^{1,\beta}(\bar{\Omega})$ for some $0 < \beta < 1$. This follows by adopting with some obvious modifications argument used in the proof of Lemma 7 in [13].

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