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Nonlinear Differential Equations and Applications NoDEA

A priori estimates for a class of degenerate elliptic equations

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Abstract. In this paper we investigate the regularity of solutions for the following degenerate partial differential equation

$$\begin{cases} -\Delta_p u + u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

when $f \in L^q(\Omega), p > 2$ and $q \geq 2$. If u is a weak solution in $W^{1,p}(\Omega)$, we obtain estimates for u in the Nikolskii space $\mathcal{N}^{1+2/r,r}(\Omega)$, where $r = q(p-2)+2$, in terms of the L^q norm of f. In particular, due to embedding theorems of Nikolskii spaces into Sobolev spaces, we conclude that $\|u\|^r_{W^{1+2/r-\epsilon,r}(\Omega)} \leq C(\|f\|^q_{L^q(\Omega)} + \|f\|^r_{L^q(\Omega)} + \|f\|^{2r/p}_{L^q(\Omega)})$ for every $\epsilon > 0$ sufficiently small. Moreover, we prove that the resolvent operator is continuous and compact in $W^{1,r}(\Omega)$.

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1. Introduction

The aim of this paper is to discuss the regularity of solutions for the p-Laplace equation with Neumann boundary condition, namely

$$\begin{cases} -\Delta_p u + u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

when $\Omega \subset \mathbb{R}^N$ is an open bounded smooth domain, $N \geq 2, f \in L^q(\Omega), p > 2, q \geq 2$. Indeed, we obtain a result concerning the L^q regularity for the p-Laplacian in spaces of fractional order of smoothness.

The main result is stated as follows:

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Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^N, N \geq 2$, is an open bounded smooth domain, $p > 2, q \geq 2$ and $f \in L^q(\Omega)$. Let $u \in W^{1,p}(\Omega)$ be the weak solution of (1.1).*

Then $u \in \mathcal{N}^{1+2/r,r}(\Omega)$, where $r = q(p-2)+2$.

Moreover, there exists a constant $C = C(N,p,q,\Omega) > 0$ such that

$$\|u\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} \leq C\left(\|f\|^q_{L^q(\Omega)} + \|f\|^r_{L^q(\Omega)} + \|f\|^{2r/p}_{L^q(\Omega)}\right). \tag{1.2}$$

Mainly, we will work with the so-called Nikolskii spaces $\mathcal{N}^{s,r}(\Omega)$, for $1 < s < 2$ and $r \geq 2$. For the reader's convenience, their definition is given below.

$$\mathcal{N}^{s,r}(\Omega) = \Bigg\{ u \in L^r(\Omega) : \sum_{i=1}^N \sup_{h\neq 0}\left(\int_{\Omega_{|h|}} \frac{\left|\partial_{x_i}u(x+h) - \partial_{x_i}u(x)\right|^r}{|h|^{\sigma r}}\right)^{1/r} < +\infty \Bigg\},$$

with norm

$$\|u\|_{\mathcal{N}^{s,r}} = \sum_{i=1}^N \sup_{h\neq 0}\left(\int_{\Omega_{|h|}} \frac{\left|\partial_{x_i}u(x+h) - \partial_{x_i}u(x)\right|^r}{|h|^{\sigma r}}\right)^{1/r} + \|u\|_{L^r(\Omega)}, \tag{1.3}$$

where

$$1+\sigma = s \quad \text{and} \quad \Omega_{|h|} = \{x \in \Omega : d(x,\partial\Omega) > |h|\}, \text{ for } |h| > 0. \tag{1.4}$$

We stress that Nikolskii spaces $\mathcal{N}^{s,r}(\Omega)$ are larger than Sobolev-Slobodeckii spaces $W^{s,r}(\Omega)$, although with a small increase of regularity for $\mathcal{N}^{s,r}(\Omega)$ the inclusion holds in the opposite direction. Indeed, there holds the following continuous imbeddings involving $\mathcal{N}^{s,r}(\Omega)$ and the Sobolev-Slobodeckii spaces

$$\mathcal{N}^{s+\epsilon,r}(\Omega) \hookrightarrow W^{s,r}(\Omega) \hookrightarrow \mathcal{N}^{s,r}(\Omega), \quad \forall\epsilon > 0 \text{ sufficiently small.} \tag{1.5}$$

A proof for this fact can be found in [11] Lemma 2.1. For further details concerning Nikolskii and Sobolev-Slobodeckii spaces, see [10–12].

Theorem 1.1 can be viewed as a counterpart for problem (1.1) of the classical Calderón-Zygmund estimate

$$\|u\|_{W^{2,q}(\Omega)} \leq \|f\|_{L^q(\Omega)},$$

for weak solutions of $-\Delta u = f$, see for instance [9] Section 9.4. Indeed, if $f \in L^q(\Omega)$, by (1.5) one obtains

$$\|u\|^r_{W^{(1+2/r)-\epsilon,r}(\Omega)} \leq C\left(\|f\|^q_{L^q(\Omega)} + \|f\|^r_{L^q(\Omega)} + \|f\|^{2r/p}_{L^q(\Omega)}\right),$$

provided that u is a weak solution of (1.1) belonging to $W^{1,p}(\Omega)$. We observe that the estimate (1.2) also generalizes a previous result for $f \in L^2(\Omega)$ investigated in [5]. Despite that the techniques employed to prove Theorem 1.1 still hold for $p = 2$, the present paper does not cover this classical case which follows from the Calderón-Zygmund inequality and may be found for instance in [10], Chapter 2.

As a consequence of Theorem 1.1, we obtain some information on the resolvent operator of (1.1).

Corollary 1.2. *Under the hypotheses of Theorem 1.1, consider*

$$S : L^q(\Omega) \longrightarrow W^{1,r}(\Omega)$$

such that $S(f) = u$ if and only if u satisfies (1.1). Then S is well defined, continuous and compact.

Observe that Corollary 1.2 implies continuity and compactness for S in $W^{1,r}(\Omega) \hookrightarrow W^{1,p}(\Omega)$, since $r > p$ if $q \geq 1$ and $p > 2$, conditions which include the case $q = p$, when $p > 2$. This result indicates that, under appropriate conditions, degenerate equations have compact resolvent operators which are more regular than previous results had shown (see [4]).

It is relevant to mention other results concerning the regularity for solutions of problems related to (1.1) which have been investigated during the past years. For results addressing the $C^{1,\alpha}$ regularity for solutions of degenerate equations, we cite [2,13,16]. According to these works, the solution u belongs to $C^{1,\alpha}$ even if the data f is in C^∞. Further, in [1,3] local Hölder regularity for the solutions is established when the data f is in the Lorentz space $L^{N,1}(\Omega)$. Besov regularity for solutions of degenerated equations was addressed in [8,14,15]. There are also results which investigate how the regularity of the solution can be improved when we have more differentiability of the data f, see [6,7] for more details. Finally, we remark that a priori bounds in Lebesgue spaces for a class of degenerate equations similar to (1.1) were investigated in [4].

The plan for the paper is the following. In Sect. 2 we state the basic notation used in the approximation scheme and then prove some lemmata regarding the control of nonlinear boundary data and preliminary energy estimates. Section 3 is reserved for the proof of our main results.

2. Notation and preliminary results

The proof of the main result will be based on an approximation technique. We proceed to introduce a differential operator which is "close" enough to the p-Laplacian

$$\Delta_p^n u = \operatorname{div}\left(\left||\nabla u|^2 + \frac{1}{n}\right|^{(p-2)/2} \nabla u\right), \quad n \in \mathbb{N}.$$

Formally, $\Delta_p^n \to \Delta_p$ when $n \to +\infty$, so that we can recover (1.1) from its approximate version, namely

$$-\Delta_p^n u + u = f \quad \text{in } \Omega.$$

In order to visually simplify the calculations, the subsequent notations will be introduced. Firstly, we define

$$a(x) = |x|^{(p-2)}, \quad a_n(x) = \left(|x|^2 + \frac{1}{n}\right)^{(p-2)/2},$$

and further

$$D_e^h f \overset{\text{def}}{=} \frac{f(x+eh)-f(x)}{h}, \tag{2.1}$$
$$T_e^h f \overset{\text{def}}{=} f(x+eh), \text{ for } |h|>0,$$

where $e \in \mathbb{R}^N$ is such that $|e|=1$.

Moreover, we stress that in this paper r denotes

$$r = q(p-2)+2.$$

We are now able to prove the basic results of this section.

As a first step, we concentrate in obtaining an estimation for certain boundary terms which play a key role for the analysis of (1.1).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N, N \geq 2$ be an open and bounded smooth domain. Then given $0<\epsilon<1/2r$ and $n \in \mathbb{N}$, there exists a constant $C>0$ such that*

$$\begin{aligned}
&\left|\sum_{j,k=1}^{N}\int_{\partial\Omega} a_n(|\nabla u|)\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right)\nu_k\right| \\
&\quad\leq \epsilon\left(\|u\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} + \frac{1}{n^{(r-2)/2}}\|u\|^2_{\mathcal{N}^{1+2/r,r}(\Omega)}\right) \\
&\qquad + C\left(\|u\|^r_{L^p(\Omega)} + \frac{1}{n^{(r-2)/2}}\|u\|^2_{L^p(\Omega)}\right),
\end{aligned} \tag{2.2}$$

for every $u \in C^3(\overline{\Omega})$ such that $\dfrac{\partial u}{\partial \nu}=0$, where $C=C(N,p,q,\Omega,\epsilon)$.

Proof. We rewrite

$$\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right)\nu_k$$

in local coordinates of $\partial\Omega$ and then use this identity in order to obtain global bounds for the boundary terms.

Indeed, let $\tau_1, \ldots, \tau_{N-1}$ be the associated tangent vectors for a given family of $N-1$ curves which are orthogonal at P and contained in a sufficiently small neighborhood of P. In addition, denote by $s_1, \ldots, s_{N-1}$ their arc length.

Clearly, $s_1, \ldots, s_{N-1}$ is a local parametrization for $\partial\Omega$ in a sufficiently small neighborhood of P and $\tau_1, \ldots, \tau_{N-1}, \nu$ form a system of coordinates for $\mathbb{R}^N$.

Thence, by writing in local coordinates we have that

$$\begin{aligned}
&(v\cdot\nabla)\left(a_n(|v|)^{q-1}v\right) \\
&\quad = \sum_{l=1}^{N-1} v_l \frac{\partial}{\partial s_l}\left(a_n(|v|)^{q-1}v\right) + v_\nu\frac{\partial}{\partial \nu}\left(a_n(|v|)^{q-1}v\right),
\end{aligned}$$

for every $v \in \left(C^2(\overline{\Omega})\right)^N$, where v_l for $l = 1, \dots, N-1$ denotes the tangential components and v_ν denotes the normal component. Moreover, since

$$\frac{\partial}{\partial s_l}\left(a_n(|v|)^{q-1}v\right) = \sum_{m=1}^{N-1}\frac{\partial}{\partial s_l}\left(a_n(|v|)^{q-1}v_m\tau_m\right) + \frac{\partial}{\partial s_l}\left(a_n(|v|)^{q-1}(v\cdot\nu)\nu\right)$$

we obtain that

$$(v\cdot\nabla)\left(a_n(|v|)^{q-1}v\right) = \textstyle\sum_{l,m=1}^{N-1} v_l\frac{\partial}{\partial s_l}\left(a_n(|v|)^{q-1}v_m\tau_m\right) + \textstyle\sum_{l=1}^{N-1} v_l\frac{\partial}{\partial s_l}\left(a_n(|v|)^{q-1}(v\cdot\nu)\nu\right) + v_\nu\frac{\partial}{\partial\nu}\left(a_n(|v|)^{q-1}v\right).$$

Thus, if for instance $v\cdot\nu = 0$, from the last equation we infer that

$$(v\cdot\nabla)\left(a_n(|v|)^{q-1}v\right)\cdot\nu = \sum_{l,m=1}^{N-1} a_n(|v|)^{q-1}v_l v_m\,\frac{\partial\tau_m}{\partial s_l}\cdot\nu, \tag{2.3}$$

the aforementioned identity in local coordinates.
However, observe that

$$\sum_{j,k=1}^{N} a_n(|\nabla u|)\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right)\nu_k = a_n(|\nabla u|)(\nabla u\cdot\nabla)\left(a_n(|\nabla u|)^{q-1}\nabla u\right)\cdot\nu.$$

In this way, by setting $v = \nabla u$ in (2.3), we obtain

$$\begin{aligned}\Bigg|\sum_{j,k=1}^{N} a_n(|\nabla u|)\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right)\nu_k\Bigg| &\le \sum_{l,m=1}^{N-1} a_n(|\nabla_T u|)^q\left|\frac{\partial u}{\partial s_l}\frac{\partial u}{\partial s_m}\frac{\partial\tau_m}{\partial s_l}\cdot\nu\right| \\ &\le C_1\left(|\nabla_T u|^r + \frac{1}{n^{(r-2)/2}}|\nabla_T u|^2\right),\end{aligned}$$

on $\partial\Omega$, where $C_1 = C_1(N,p,q,\Omega) > 0$ is going to be fixed for the rest of this proof.

But observe that the last inequality holds globally in $\partial\Omega$. This implies that

$$\begin{aligned}&\Bigg|\sum_{i,j=1}^{N}\int_{\partial\Omega}\sum_{j,k=1}^{N} a_n(|\nabla u|)\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right)\nu_k d\sigma\Bigg| \\ &\quad\le C_1\int_{\partial\Omega}|\nabla_T u|^r + \frac{1}{n^{(r-2)/2}}|\nabla_T u|^2 d\sigma.\end{aligned} \tag{2.4}$$

Now we focus on controlling the tangential derivatives of u on $\partial\Omega$.

On one hand, for $0 < \epsilon < 1/2r$, recall that there exists a continuous trace operator

$$W^{1+2/r-\epsilon,r}(\Omega) \to W^{1,r}(\partial\Omega) \hookrightarrow W^{1,2}(\partial\Omega) \tag{2.5}$$

see Theorem 1.5.1.2 [10, p. 37].

Moreover, from Theorem 1.4.3.3 [10, p. 26], there holds that

$$\|u\|^r_{W^{s'',r}(\Omega)} \leq \frac{\epsilon}{(C_1+1)}\|u\|^r_{W^{s',r}(\Omega)} + C\|u\|^r_{L^r(\Omega)}$$

and that

$$\|u\|^2_{W^{s'',r}(\Omega)} \leq \frac{\epsilon}{(C_1+1)}\|u\|^2_{W^{s',r}(\Omega)} + C\|u\|^2_{L^r(\Omega)},$$

where

$$C = C(N,p,q,\Omega,\epsilon) > 0, \quad s' = 1+2/r - \frac{\epsilon}{2}, \quad s'' = 1+2/r-\epsilon \quad \text{and} \quad s''' = 0.$$

Thus, from (2.4) and (2.5) there holds

$$\begin{aligned}\Bigg|\sum_{i,j=1}^{N}\int_{\partial\Omega}\sum_{j,k=1}^{N} a_n(|\nabla u|)\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right)\nu_k d\sigma\Bigg| \\ \leq \epsilon\|u\|^r_{W^{s',r}(\Omega)} + C\|u\|^r_{L^r(\Omega)} + \frac{\epsilon}{n^{(r-2)/2}}\|u\|^2_{W^{s',r}(\Omega)} + C\|u\|^2_{L^r(\Omega)}.\end{aligned} \tag{2.6}$$

On the other hand, recall that from (1.5) and by Theorem 1.4.3.2 [10, p. 26], there follows that

$$\mathcal{N}^{1+2/r,r}(\Omega) \hookrightarrow W^{1+2/r-\epsilon/2,r}(\Omega) \hookrightarrow\hookrightarrow W^{1,r}(\Omega), \tag{2.7}$$

where $\hookrightarrow$ means continuous embedding and $\hookrightarrow\hookrightarrow$ compact embedding.

Since $r > p$, a simple interpolation yields

$$\|u\|^r_{W^{s',r}(\Omega)} \leq \|u\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} + C\|u\|^r_{L^p(\Omega)} \tag{2.8}$$

and

$$\|u\|^2_{W^{s',r}(\Omega)} \leq \|u\|^2_{\mathcal{N}^{1+2/r,r}(\Omega)} + C\|u\|^2_{L^p(\Omega)}. \tag{2.9}$$

Therefore, by combining (2.6) with (2.8) and (2.9) we obtain

$$\begin{aligned}\Bigg|\sum_{i,j=1}^{N}\int_{\partial\Omega}\sum_{j,k=1}^{N} a_n(|\nabla u|)\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right)\nu_k d\sigma\Bigg| \\ \leq \epsilon\left(\|u\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} + \frac{1}{n^{(r-2)/2}}\|u\|^2_{\mathcal{N}^{1+2/r,r}(\Omega)}\right) + C\left(\|u\|^r_{L^p(\Omega)} + \|u\|^2_{L^p(\Omega)}\right)\end{aligned}$$

and the result follows. □

The next lemma gives an estimation for the $\mathcal{N}^{1+2/r,r}(\Omega)$ norm in terms of a singular integral. Its proof is a straightforward adaptation of Lemma 4 of [5], combined with a simple interpolation.

Lemma 2.2. *Suppose that* $\Omega \subset \mathbb{R}^N$, $N \geq 2$, *is an open bounded smooth domain, that* $p > 2$ *and* $q \geq 2$. *Then, there exists a constant* $C > 0$ *such that*

$$\|u\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} \leq C\bigg(\int_\Omega |\nabla u|^{r-2}|D^2u|^2 + \|u\|^r_{L^p(\Omega)}\bigg), \tag{2.10}$$

for all $u \in C^2(\overline{\Omega})$, *where* $C = C(N, p, q, \Omega)$.

Proof. Initially, we claim that

$$|||v||| = \sup_{h>0}\left(\int_{\Omega_{|h|}} \frac{|T^h\nabla u - \nabla u|^r}{|h|^2}\right)^{1/r} + \|v\|_{L^p(\Omega)}$$

is an equivalent norm for $\mathcal{N}^{1+2/r,r}(\Omega)$.

Indeed, by using Gagliardo-Nirenberg's inequality and the embedding $\mathcal{N}^{1+2/r,r}(\Omega) \hookrightarrow W^{1,r}(\Omega)$, it is straightforward to check that

$$\|v\|_{L^r(\Omega)} \leq C\|v\|_{L^p(\Omega)} + \epsilon\|v\|_{N^{1+2/r,r}(\Omega)},$$

for every $v \in \mathcal{N}^{1+2/r,r}(\Omega)$ and $\epsilon > 0$ sufficiently small, where

$$C = C(N, p, q, \Omega, \epsilon) > 0.$$

Then, by the latter inequality and (1.3), there follows that

$$(1-\epsilon)\|v\|_{\mathcal{N}^{1+2/r,r}(\Omega)} \leq C\Bigg(\sup_{h>0}\left(\int_{\Omega_{|h|}} \frac{|T^h\nabla u - \nabla u|^r}{|h|^2}\right)^{1/r} + \|v\|_{L^p(\Omega)}\Bigg),$$

what proves our claim.

Further, recalling (1.4) and (2.1), standard arguments allow us to show that

$$\begin{aligned}\int_{\Omega_{|h|}} |D^h_e(|\nabla u|^{r-2}\nabla u)|^2 &\leq C\int_\Omega |\nabla(|\nabla u|^{r-2}\nabla u)|^2 \\ &\leq C\int_\Omega |\nabla u|^{r-2}|D^2u|^2 \text{ for } C = C(N, p, q, \Omega) > 0,\end{aligned} \tag{2.11}$$

since $D^h_e(|\nabla u|^{r-2}\nabla u)$ is a difference quotient in $|\nabla u|^{r-2}\nabla u$. We point out that by combining (2.11) and an auxiliary inequality we find a bound for (1.3). In fact, we take advantage of the inequality

$$|x-y|^r \leq C\big|x|x|^{r-2} - y|y|^{r-2}\big|^2, \quad \forall\ x \text{ and } y \in \mathbb{R}^N$$

(see [7], inequalities (30) and (31)) to infer that

$$\int_{\Omega_{|h|}} \frac{|T^h_e\nabla u - \nabla u|^r}{|h|^2} \leq C\int_{\Omega_{|h|}} \big|D^h_e(\nabla u|\nabla u|^{r-2})\big|^2.$$

Then

$$\int_{\Omega_{|h|}} \frac{|T^h_e\nabla u - \nabla u|^r}{|h|^2} \leq C\int_\Omega |\nabla u|^{r-2}|D^2u|^2,$$

where $C = C(N, p, q, \Omega) > 0$.

Thus, we conclude that

$$\sup_{h>0}\left(\int_{\Omega_{|h|}}\frac{|T_e^h\nabla u-\nabla u|^r}{|h|^2}\right)^{1/r}\leq C\left(\int_\Omega|\nabla u|^{r-2}|D^2u|^2\right)^{1/r}. \qquad (2.12)$$

Therefore, from (2.12) and the previous claim, we prove (2.10). □

Our purpose now is to discuss certain a priori estimates related to (1.1) which are part of the crucial contributions of this work. Actually, the control of the fractional norms for solutions of (1.1) follows from the combination between these estimates with the previous lemmata.

Lemma 2.3. *Suppose that $p>2$ and that $\Omega\subset\mathbb{R}^N, N\geq 2$, is an open, bounded smooth domain. Given $0<\epsilon<1/2r$ there exists a constant $C>0$ such that*

$$\begin{aligned}&\int_\Omega \Delta_p^n u\ \mathrm{div}(a_n(|\nabla u|)^{q-1}\nabla u)\\&\geq\left(\int_\Omega a_n(|\nabla u|)^q|D^2u|^2-\epsilon\|u\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)}-\frac{\epsilon}{n^{(r-2)/2}}\|u\|^2_{\mathcal{N}^{1+2/r,r}(\Omega)}\right.\\&\quad\left.-C\left(\|u\|^r_{L^p(\Omega)}+\frac{1}{n^{(r-2)/2}}\|u\|^2_{L^p(\Omega)}\right)\right),\quad\forall\ n\in\mathbb{N},\end{aligned}$$

for all $u\in C^3(\overline{\Omega})$ such that $\dfrac{\partial u}{\partial\nu}=0$ on $\partial\Omega$, where $C=C(N,p,q,\Omega,\epsilon)$.

Proof. The idea is to use integration by parts in order to obtain the integral terms which control the $\mathcal{N}^{1+2/r,r}(\Omega)$ norms of such functions.

In fact, by integrating by parts and interchanging the order of the derivatives, we obtain

$$\begin{aligned}&\int_\Omega\Delta_p^n u\ \mathrm{div}(a_n(|\nabla u|)^{q-1}\nabla u)\\&\quad=-\sum_{j,k=1}^N\int_\Omega a_n(|\nabla u|)\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right)\right)\\&\qquad+\sum_{j,k=1}^N\int_{\partial\Omega}a_n(|\nabla u|)\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_k}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right)\nu_j.\end{aligned}$$

Then, by integrating by parts again

$$\begin{aligned}&\int_\Omega\Delta_p^n u\ \mathrm{div}(a_n(|\nabla u|)^{q-1}\nabla u)\\&\quad=\sum_{j,k=1}^N\int_\Omega\frac{\partial}{\partial x_k}\left(a_n(|\nabla u|)\frac{\partial u}{\partial x_j}\right)\times\frac{\partial}{\partial x_j}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right)\\&\qquad-\sum_{j,k=1}^N\int_{\partial\Omega}a_n(|\nabla u|)\frac{\partial u}{\partial x_j}\times\frac{\partial}{\partial x_j}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right)\nu_k\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j,k=1}^{N}\int_{\partial\Omega} a_n(|\nabla u|)\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_k}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right)\nu_j.\\
&= I - J + K, \qquad (2.13)
\end{aligned}
$$

where

$$
\begin{aligned}
I &= \sum_{j,k=1}^{N}\int_{\Omega}\frac{\partial}{\partial x_k}\left(a_n(|\nabla u|)\frac{\partial u}{\partial x_j}\right)\times\frac{\partial}{\partial x_j}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right),\\
J &= \sum_{j,k=1}^{N}\int_{\partial\Omega} a_n(|\nabla u|)\frac{\partial u}{\partial x_j}\times\frac{\partial}{\partial x_j}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right)\nu_k
\end{aligned}
\qquad (2.14)
$$

and

$$
K = \sum_{j,k=1}^{N}\int_{\partial\Omega} a_n(|\nabla u|)\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_k}\left(a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_k}\right)\nu_j.
$$

Notice that $K = 0$, since $\dfrac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. Then, consider I_i, for i from 1 to 4, defined as

$$
I_1 = \int_{\Omega} a_n(|\nabla u|)^q |D^2 u|^2,
$$

$$
I_2 = \sum_{j,k=1}^{N}\int_{\Omega} a_n(|\nabla u|)\times\frac{\partial}{\partial x_j}a_n(|\nabla u|)^{q-1}\frac{\partial^2 u}{\partial x_j\partial x_k}\frac{\partial u}{\partial x_k},
$$

$$
I_3 = \sum_{j,k=1}^{N}\int_{\Omega} \frac{\partial}{\partial x_k}a_n(|\nabla u|)\times a_n(|\nabla u|)^{q-1}\frac{\partial^2 u}{\partial x_j\partial x_k}\frac{\partial u}{\partial x_j},
$$

$$
I_4 = \sum_{j,k=1}^{N}\int_{\Omega} \frac{\partial}{\partial x_k}a_n(|\nabla u|)\times \frac{\partial}{\partial x_j}a_n(|\nabla u|)^{q-1}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k}.
$$

Therefore

$$
I = I_1 + I_2 + I_3 + I_4. \qquad (2.15)
$$

Now we investigate each term of (2.15).

First, recall that by definition

$$
a_n(x) = (|x|^2 + 1/n)^{(p-2)/2}.
$$

Further, notice that

$$
I_2 = \sum_{j,k=1}^{N}\int_{\Omega} a_n(|\nabla u)\times\frac{\partial}{\partial x_j}a_n(|\nabla u|)^{q-1}\frac{\partial^2 u}{\partial x_j\partial x_k}\frac{\partial u}{\partial x_k}
$$

$$= \sum_{j,k,l=1}^{N} \int_\Omega (q-1)(p-2)(|\nabla u|^2 + 1/n)^{(r-4)/2} \frac{\partial u}{\partial x_l} \frac{\partial^2 u}{\partial x_k \partial x_l} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_k \partial x_j}$$

$$= \int_\Omega (q-1)(p-2)(|\nabla u|^2 + 1/n)^{(r-4)/2} \sum_{k=1}^{N} \left(\sum_{j=1}^{N} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_k \partial x_j} \right)^2 \geq 0.$$

Analogously, we prove that $I_3 \geq 0$, and consequently

$$I_2 + I_3 \geq 0. \tag{2.16}$$

Moreover, observe that by applying the chain rule

$$I_4 = \sum_{j,k=1}^{N} \int_\Omega \frac{\partial}{\partial x_k} a_n(|\nabla u|) \times \frac{\partial}{\partial x_j} a_n(|\nabla u|)^{q-1} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k}.$$

$$= \sum_{i,j,k,l=1}^{N} \int_\Omega (p-2)^2 (q-1) \Bigg[\left(|\nabla u|^2 + \frac{1}{n} \right)^{((p-2)q-4)/2}$$

$$\times \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} \frac{\partial^2 u}{\partial x_k x_l} \frac{\partial u}{\partial x_l} \Bigg] \geq 0. \tag{2.17}$$

Then, by virtue of (2.15)–(2.17), we have

$$I \geq \int_\Omega a_n(|\nabla u|)^q |D^2 u|^2. \tag{2.18}$$

However, by the choice of J (see (2.14)) and by Lemma 2.1

$$|J| \leq \epsilon \left(\|u\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} + \frac{1}{n^{(r-2)/2}} \|u\|^2_{\mathcal{N}^{1+2/r,r}(\Omega)} \right)$$

$$+ C \left(\|u\|^r_{L^p(\Omega)} + \frac{1}{n^{(r-2)/2}} \|u\|^2_{L^p(\Omega)} \right). \tag{2.19}$$

Thus, from (2.13), (2.15), (2.18) and (2.19) we obtain that

$$\int_\Omega \Delta_p^n u \, \mathrm{div}(a_n(|\nabla u|)^{q-1} \nabla u)$$

$$\geq \int_\Omega |a_n(|\nabla u|)^q |D^2 u|^2 - \epsilon \left(\|u\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} + \frac{1}{n^{(r-2)/2}} \|u\|^2_{\mathcal{N}^{1+2/r,r}(\Omega)} \right)$$

$$- C \left(\|u\|^r_{L^p(\Omega)} + \frac{1}{n^{(r-2)/2}} \|u\|^2_{L^p(\Omega)} \right),$$

and the result follows. □

The next result concerns the existence and regularity of approximate solutions for (1.1). Its proof is a direct consequence of standard results, so that it will be omitted here. For instance, we refer the reader to the proof of Theorem 1 step 1 in [1] p. 119 and subsequent commentaries for further details.

Proposition 2.4. *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded smooth domain and that $p > 2$. Let $g \in C^\infty(\Omega)$. Then, there exists a unique $u_n \in C^3(\overline{\Omega})$ solution of*

$$\begin{cases} -\Delta_p^n u_n + u_n = g & a.e.\ \Omega, \\ \dfrac{\partial u_n}{\partial \nu} = 0 & on\ \partial\Omega. \end{cases} \tag{2.20}$$

3. Proof of the main results

We are finally able to obtain the fractional order a priori estimates for the solution of (1.1).

Proof of Theorem 1.1.
Consider u_n, a solution of the following approximate version of (1.1):

$$\begin{cases} -\Delta_p^n u_n + u_n = f_n & \text{a.e. } \Omega \\ \dfrac{\partial u_n}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_n \in C^\infty(\Omega)$ is such that $f_n \to f$ in $L^q(\Omega)$. Observe that the existence of u_n is guaranteed by Proposition 2.4.

Then, given $C_1 > 1$, take $n_0 = n_0(f, C_1) \in \mathbb{N}$ such that

$$\|f_n\|_{L^q}^q \leq C_1 \|f\|_{L^q}^q, \forall n \geq n_0.$$

From now on, **let $\mathbf{n} \geq \mathbf{n_0}$**.

Our main goal is to obtain energy estimates for u_n with respect to its $\mathcal{N}^{1+2/r,r}(\Omega)$ norm. First, we obviously focus on lower order estimates.

Clearly, there holds that

$$\begin{aligned} \|u_n\|_{W^{1,p}(\Omega)} &\leq C\left(\|f_n\|_{L^2(\Omega)} + \|f_n\|_{L^2(\Omega)}^{2/p} \right) \\ &\leq C\left(\|f\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)}^{2/p} \right), \end{aligned} \tag{3.1}$$

where $C = C(N, p, q, \Omega) > 0$. Next, we are going to exploit the preliminary a priori bounds, given by Lemma 2.3, in order to obtain higher order estimates.

Indeed, by multiplying $-\operatorname{div}(a_n(|\nabla u|)^{q-1}\nabla u)$ in (2.20) and then by integrating over Ω, we find

$$\begin{aligned} &\int_\Omega \Delta_p^n u_n \operatorname{div}(a_n(|\nabla u_n|)^{q-1}\nabla u_n) \\ &\quad \leq \int_\Omega \left| f_n \operatorname{div}(a_n(|\nabla u_n|)^{q-1}\nabla u_n) \right|, \\ &\quad \leq \int_\Omega \left| f_n \operatorname{div}(a_n(|\nabla u_n|)^{q-1}\nabla u_n) \right|, \quad \text{where } C = C(N, p, q, \Omega) > 0, \end{aligned}$$

where for the sake of simplicity, we denoted the product CC_1 by C. From now on, any fixed constant, like C_1, will be included within the standard general constant C.

But notice that

$$\left| f_n \ \mathrm{div}(a_n(|\nabla u_n|)^{q-1}\nabla u_n)\right| \leq C|f_n| \ a_n(|\nabla u_n|)^{q-1}|D^2u_n|,$$

where $C = C(N, p, q) > 0$.

It is clear from the above inequalities that

$$\int_\Omega \Delta_p^n u_n \ \mathrm{div}(a_n(|\nabla u_n|)^{q-1}\nabla u_n) \leq C\int_\Omega |f_n| \ a_n(|\nabla u_n|)^{q-1}|D^2u_n|. \tag{3.2}$$

At this point, we balance the right-hand and left-hand sides of (3.2) by the use of Lemma 2.3.

For the right-hand side, observe that by Hölder's inequality applied to $q, 2q/(q-2)$ and 2, one obtains

$$\begin{aligned}&\int_\Omega |f_n| \ a_n(|\nabla u_n|)^{q/2-1}a_n(|\nabla u_n|)^{q/2}|D^2u_n| \\ &\quad \leq C\|f\|_{L^q(\Omega)}\big\|a_n(|\nabla u|)\big\|_{L^q(\Omega)}^{(q-2)/2}\bigg(\int_\Omega a_n(|\nabla u_n|)^q|D^2u_n|^2\bigg)^{1/2}.\end{aligned}$$

Moreover, consider $\eta > 0$ and $\delta > 0$ such that

$$\eta < \frac{q}{(q-2)2^{(p-2)q/2+1}}$$

and

$$\delta < \min\left\{1/4, 1/2r, \left(\|f\|_{L^q(\Omega)}^q + \|f\|_{L^q(\Omega)}^r + \|f\|_{L^q(\Omega)}^{2r/p}\right), \kappa\right\},$$

where $\kappa > 0$ will be fixed later.

From Young's inequality applied to q, $2q/(q-2)$ and 2,

$$\begin{aligned}&\|f\|_{L^q(\Omega)}\big\|a_n(|\nabla u|)\big\|_{L^q(\Omega)}^{(q-2)/2}\bigg(\int_\Omega a_n(|\nabla u_n|)^q|D^2u_n|^2\bigg)^{1/2} \\ &\leq C\|f\|_{L^q(\Omega)}^q + \eta\delta\frac{q-2}{q}\|a_n(|\nabla u_n|)\|_{L^q(\Omega)}^q + \delta\int_\Omega a_n(|\nabla u_n|)^q|D^2u_n|^2 \\ &\leq C\|f\|_{L^q(\Omega)}^q + \eta\delta 2^{q(p-2)/2}\frac{q-2}{q}\int_\Omega\left(|\nabla u_n|^{q(p-2)} + \frac{1}{n^{q(p-2)/2}}\right) \\ &\quad +\delta\int_\Omega a_n(|\nabla u_n|)^q|D^2u_n|^2 \\ &\leq C\|f\|_{L^q(\Omega)}^q + \delta\int_\Omega\left(|\nabla u_n|^{r-2} + \frac{1}{n^{(r-2)/2}}\right) + \delta\int_\Omega a_n(|\nabla u_n|)^q|D^2u_n|^2,\end{aligned}$$

where $C = C(N, p, q, \Omega, \delta) > 0$.

In this way,

$$\begin{aligned}&\int_\Omega |f_n| \ a_n(|\nabla u_n|)^{q/2-1}a_n(|\nabla u_n|)^{q/2}|D^2u_n| \\ &\leq C\bigg(\|f\|_{L^q(\Omega)}^q + \frac{1}{n^{(r-2)/2}}\bigg) + \delta\bigg(\int_\Omega |\nabla u_n|^{r-2} + \int_\Omega a_n(|\nabla u_n|)^q|D^2u_n|^2\bigg).\end{aligned} \tag{3.3}$$

However, since $\mathcal{N}^{1+2/r,r}(\Omega) \hookrightarrow W^{1,r}(\Omega)$, there exists a constant

$$C_2 = C_2(N, p, q, \Omega) > 0$$

such that

$$\int_\Omega |\nabla u_n|^{r-2} \leq C_2 \|u_n\|^{r-2}_{\mathcal{N}^{1+2/r,r}(\Omega)}.$$

Thus, by combining (3.2) and (3.3) we obtain

$$\int_\Omega \Delta_p^n u_n \,\operatorname{div}(a_n(|\nabla u_n|)^{q-1}\nabla u_n) \\ \leq C\left(\|f\|^q_{L^q(\Omega)} + \frac{1}{n^{(r-2)/2}}\right) + \delta\left(C_2\|u_n\|^{r-2}_{\mathcal{N}^{1+2/r,r}} + \int_\Omega a_n(|\nabla u_n|)^q |D^2 u_n|^2\right). \tag{3.4}$$

Finished the analysis of the right-hand side, we now work with left-hand side of (3.2).

Notice that in a view of Lemma 2.1 by setting $\epsilon = \delta$ in (2.2) and by using (3.1), we end up with

$$\begin{aligned} &\int_\Omega \Delta_p^n u_n \,\operatorname{div}(a_n(|\nabla u_n|)^{q-1}\nabla u_n) \\ &\quad\geq \int_\Omega a_n(|\nabla u_n|)^q|D^2u_n|^2 - \delta\|u_n\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} - \frac{\delta}{n^{(r-2)/2}}\|u_n\|^2_{\mathcal{N}^{1+2/r,r}(\Omega)} \\ &\qquad -C\left(\|u_n\|^r_{L^p(\Omega)} + \frac{1}{n^{(r-2)/2}}\|u_n\|^2_{L^p(\Omega)}\right) \\ &\quad\geq \int_\Omega a_n(|\nabla u_n|)^q|D^2u_n|^2 - \delta\|u_n\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} - \frac{\delta}{n^{(r-2)/2}}\|u_n\|^2_{\mathcal{N}^{1+2/r,r}(\Omega)} \\ &\qquad -C\left(\|f\|^r_{L^q(\Omega)} + \|f\|^{2r/p}_{L^q(\Omega)} + \frac{1}{n^{(p-2)/2}}\left(\|f\|^2_{L^2(\Omega)} + \|f\|^{4/p}_{L^2(\Omega)}\right)\right). \end{aligned} \tag{3.5}$$

Then, from (3.4) and (3.5) we infer that

$$\begin{aligned} &(1-\delta)\int_\Omega a_n(|\nabla u_n|)^q|D^2u_n|^2 - \delta\left(\|u_n\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} + \frac{1}{n^{(r-2)/2}}\|u_n\|^2_{\mathcal{N}^{p1+2/r,r}(\Omega)}\right) \\ &\quad -\delta C_2\|u_n\|^{r-2}_{\mathcal{N}^{1+2/r,r}(\Omega)} \leq C\left(\|f\|^q_{L^q(\Omega)} + \|f\|^r_{L^q(\Omega)} + \|f\|^{2r/p}_{L^q(\Omega)}\right. \\ &\quad \left. + \frac{1}{n^{(r-2)/2}}\left(\|f\|^2_{L^2(\Omega)} + \|f\|^{4/p}_{L^2(\Omega)} + 1\right)\right), \\ &\leq C\left(\|f\|^q_{L^q(\Omega)} + \|f\|^r_{L^q(\Omega)} + \|f\|^{2r/p}_{L^q(\Omega)}\right) + \frac{C_3}{n^{(r-2)/2}}, \end{aligned}$$

where $C = C(N, p, q, \Omega)$ and $C_3 = C_3(\|f\|_{L^2(\Omega)}, N, p, q, \Omega)$.

However, inasmuch as

$$\int_\Omega a_n(|\nabla u_n|)^q|D^2u_n|^2 \geq \int_\Omega |\nabla u_n|^{q(p-2)}|D^2u_n|^2, \quad \forall n \in \mathbb{N},$$

by combining the latter estimates and Lemma 2.2, one obtains the following inequality:

$$(1-2\delta)\|u_n\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} - \frac{\delta}{n^{(r-2)/2}}\|u_n\|^2_{\mathcal{N}^{1+2/r,r}(\Omega)} - \delta C_2\|u_n\|^{r-2}_{\mathcal{N}^{1+2/r,r}(\Omega)}$$
$$\leq C\Bigg(\|f\|^q_{L^q(\Omega)} + \|f\|^r_{L^q(\Omega)} + \|f\|^{2r/p}_{L^q(\Omega)} + \frac{1}{n^{(r-2)/2}}\Big(\|f\|^2_{L^2(\Omega)} + \|f\|^{4/p}_{L^2(\Omega)} + 1\Big)\Bigg).$$
$$\leq C\Big(\|f\|^q_{L^q(\Omega)} + \|f\|^r_{L^q(\Omega)} + \|f\|^{2r/p}_{L^q(\Omega)}\Big) + \frac{C_3}{n^{(r-2)/2}}. \tag{3.6}$$

Since $r > 2$, it is clear from the last inequality that $\|u_n\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)}$ is bounded. However, it is our purpose to do some basic manipulations in order to obtain at least a subsequence of $\{u_n\}$, still denoted as $\{u_n\}$, for which the following improved version of (3.6) holds

$$\|u_n\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} \leq C\Big(\|f\|^q_{L^q(\Omega)} + \|f\|^r_{L^q(\Omega)} + \|f\|^{2r/p}_{L^q(\Omega)}\Big) + \frac{C_3}{n^{(r-2)/2}}, \tag{3.7}$$

where $C = C(N,p,q,\Omega)$, what obviously implies (1.2).

In this way, since $\|u_n\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)}$ is bounded,

$$\frac{\delta}{n^{(p-2)/2}}\|u_n\|^2_{\mathcal{N}^{1+2/r,r}(\Omega)} \to 0, \qquad \text{if } n \to +\infty,$$

and this term can be considered as a part of

$$\frac{C_3}{n^{(r-2)/2}}$$

in (3.6).

This yields

$$(1-2\delta)\|u_n\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} - \delta C\|u_n\|^{r-2}_{\mathcal{N}^{1+2/r,r}(\Omega)}$$
$$\leq C\Big(\|f\|^q_{L^q(\Omega)} + \|f\|^r_{L^q(\Omega)} + \|f\|^{2r/p}_{L^q(\Omega)}\Big) + \frac{C_3}{n^{(r-2)/2}}. \tag{3.8}$$

For the term $\delta C_2\|u_n\|^{r-2}_{\mathcal{N}^{1+2/r,r}(\Omega)}$, there are two possibilities. First suppose that there exists a subsequence of $\{u_n\}$, still denoted as $\{u_n\}$, such that

$$\|u_n\|^{r-2}_{\mathcal{N}^{1+2/r,r}(\Omega)} \leq 1, \quad \forall n \in \mathbb{N}.$$

Then, we have

$$(1-2\delta)\|u_n\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} - \delta C_2$$
$$\leq C\Big(\|f\|^q_{L^q(\Omega)} + \|f\|^r_{L^q(\Omega)} + \|f\|^{2r/p}_{L^q(\Omega)}\Big) + \frac{C_3}{n^{(r-2)/2}}. \tag{3.9}$$

However, since

$$0 < \delta < \Big(\|f\|^q_{L^q(\Omega)} + \|f\|^r_{L^q(\Omega)} + \|f\|^{2r/p}_{L^q(\Omega)}\Big),$$

from (3.9) we obtain (3.7).

For the other possibility, suppose that there exists $n_1 \in \mathbb{N}$ such that

$$\|u_n\|^{r-2}_{\mathcal{N}^{1+2/r,r}(\Omega)} > 1, \text{ if } n > n_1.$$

Now, it is time to fix κ . It is convenient to choose

$$\kappa = \frac{1}{4 + 2C_2}.$$

Thus by the choice of δ, we have

$$(1 - 2\delta - \delta C_2) > 1/2.$$

Hence, from (3.8), there follows that

$$\begin{aligned}&(1 - 2\delta - \delta C_2)\|u_n\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)}\\&\quad\le C\left(\|f\|^q_{L^q(\Omega)} + \|f\|^r_{L^q(\Omega)} + \|f\|^{2r/p}_{L^q(\Omega)}\right) + \frac{C_2}{n^{(r-2)/2}}.\end{aligned}$$

Well, from the analysis above we conclude that there always exists $n_2 = \max\{n_0, n_1\}$ such that if $n \ge n_2$

$$\|u_n\|^r_{\mathcal{N}^{1+2/r,r}(\Omega)} \le C\left(\|f\|^q_{L^q(\Omega)} + \|f\|^r_{L^q(\Omega)} + \|f\|^{2r/p}_{L^q(\Omega)}\right) + \frac{C_3}{n^{(r-2)/2}}.$$

We then obtain $u \in \mathcal{N}^{1+2/r,r}(\Omega)$ such that, up to subsequences,

$$u_n \rightharpoonup u \text{ in } \mathcal{N}^{1+2/r,r}(\Omega), \quad \text{if } n \to +\infty. \tag{3.10}$$

Notice that from (3.7) and (3.10), we obtain the estimate (1.2).

Finally, by the convergence (3.10), u is a weak solution of (1.1). Furthermore, since $\mathcal{N}^{1+2/r,r}(\Omega) \hookrightarrow W^{1+2/r-\epsilon,r}(\Omega)$ from (2.5) and (3.10) there holds that $\dfrac{\partial u}{\partial \nu} = 0$ on Ω, what completes the proof of Theorem 1.1. □

With Theorem 1.1 in hands, we are in position to prove the compactness result for the resolvent operator.

Proof of Corollary 1.2.
Given $f \in L^q(\Omega)$, there exists a unique $u \in W^{1,p}(\Omega)$ weak solution of (1.1). By Theorem 1.1, such u belongs to $\mathcal{N}^{1+2/r,r}(\Omega)$. However, due to the embeddings (2.7), $u \in W^{1,r}(\Omega)$, so that S is well defined. Further, (2.7) also implies that S is compact.

Now we proceed to prove that S is continuous. Indeed, let $\{f_n\} \subset L^q(\Omega)$ be such that $f_n \to f$ in $L^q(\Omega)$. Set $u_n = S(f_n)$ and $u = S(f)$. Consider a given subsequence $\{u_{n_l}\}$, which will be denoted as $\{u_l\}$. From (1.2), there exists $C = C(f) > 0$ such that

$$\|u_l\|_{\mathcal{N}^{1+2/r,r}(\Omega)} \le C, \forall l \in \mathbb{N}.$$

Then, due to (2.7), there exists another subsequence $\{u_{l_k}\}$ for which $u_{l_k} \to v$ strongly in $W^{1,r}(\Omega)$, for a certain $v \in \mathcal{N}^{1+2/r,r}(\Omega)$. In this fashion, v is a solution for (1.1) and then $v = u$.

Consequently, for every subsequence of $\{u_n\}$, there exists another subsequence converging strongly to u in $W^{1,r}(\Omega)$. Therefore

$$u_n \to u \quad \text{in } W^{1,r}(\Omega)$$

and the result follows. □

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