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Addendum to: Multiplicity of critical points in presence of a linking: application to a superlinear boundary value problem, NoDEA. Nonlinear Differential Equations Appl. 11 (2004), no. 3, 379-391, and a comment on the generalized Ambrosetti-Rabinowitz condition

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Abstract. We show the incompleteness of a usually used version of the generalized Ambrosetti–Rabinowitz condition in superlinear problems, also used in the paper cited in the title, and we propose a complete one.

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1. Introduction

Since the appearing of that milestone in partial differential equations given by the paper by Ambrosetti and Rabinowitz where the Mountain Pass was introduced (see [1]), thousands of papers have studied semilinear problems like

$$\begin{cases} -\Delta u = g(x,u) & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a domain of $\mathbb{R}^N$, $N \geq 3$, and B is a boundary operator, for example the Dirichlet or the Neumann one. Ω is allowed also to be unbounded, with obvious adaptations.

Moreover, also related quasilinear versions, for example in presence of the p-Laplacian operator $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$, $p \in (1, \infty)$,

$$\begin{cases} -\Delta_p u = g(x,u) & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

have been widely studied.

In order to study problem (1) when Ω is bounded, just to fix the ideas, the usual assumptions, introduced in [1], are:

(i) $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is (locally Hölder) *continuous*,
(ii) g is subcritical in the sense of Sobolev's Embedding Theorem at infinity,
(iii) $g(x,s) = o(|s|)$ as $s \to 0$ uniformly in Ω,
(iv) the now–called Ambrosetti–Rabinowitz condition holds: there exist $\mu > 2$ and $R \geq 0$ such that

$$0 < \mu \int_0^s g(x,t)\,dt \leq g(x,s)s \qquad \text{for any } |s| > R \text{ and } x \in \overline{\Omega}. \tag{3}$$

Since then, there have been a plenty of papers where the authors consider problem (2) with $g : \Omega \times \mathbb{R} \to \mathbb{R}$, possibly just a *Carathéodory function*, satisfying (ii) and (iii) and the following generalized Ambrosetti–Rabinowitz condition: there exist $\mu > p$ and $R \geq 0$ such that

$$0 < \mu \int_0^s g(x,t)\,dt \leq g(x,s)s \qquad \text{for any } |s| > R \quad \text{and} \quad \text{for a.e. } x \in \Omega. \tag{4}$$

At a first look the two conditions look pretty much the same, and in fact they are in the autonomous case $g(x,s) = g(s)$, but the consequences are extremely different, at least in view of the applications. Indeed, by direct integration, (3) implies that there exist $c_1 > 0$ and $c_2 \geq 0$ such that

$$G(x,s) = \int_0^s g(x,t)\,dt \geq c_1|s|^\mu - c_2 \qquad \text{for all } s \in \mathbb{R} \quad \text{and} \quad x \in \overline{\Omega}. \tag{5}$$

In a massive number of papers it is written that integrating (3) - or (4) -, we get that there exist $c_1 > 0$ and $c_2 \geq 0$ such that

$$G(x,s) \geq c_1|s|^\mu - c_2 \qquad \text{for all } s \in \mathbb{R} \quad \text{and} \quad \text{a.e. } x \in \Omega. \tag{6}$$

Not to be unfair, we only quote our [3], were such a mistake was done assuming (3) with $R = 0$ and deducing (6) with $c_2 = 0$. Luckily such a mistake was not done in [2], a natural development of [3].

However, this deduction is false. Indeed, consider the function $g : (0, \pi) \times \mathbb{R} \to \mathbb{R}$ defined as $g(x,s) = \sin x |s|^{\mu-2}s$; then it obviously verifies $\mu G(x,s) \leq g(x,s)s$ in Ω for all s, but there are no $c_1 > 0$, $c_2 \geq 0$ such that $G(x,s) \geq c_1|s|^\mu - c_2$ in Ω.

The mistake is simply in the integration and, we suppose, it is made just because the integral has not been really calculated. Indeed, (6) follows from (4) only if

$$\operatorname*{ess\,inf}_{x\in\Omega} G(x,\pm R) > 0, \tag{7}$$

a condition which is not satisfied by the example above, since (4) holds only in Ω and not in $\overline{\Omega}$.

However, it is well known that condition (6) is extremely important, for example, in order to verify mountain pass structures.

Moreover, also reversed forms of (4), like

$$\mu \int_0^s g(x,t)\,dt \geq g(x,s)s > 0 \quad \text{for any } 0 < |s| \leq R \quad \text{and} \quad \text{for a.e. } x \in \Omega, \tag{8}$$

have been extensively used, for instance in order to compute critical groups of the associated action functional, deriving too fast that in this case there exists $c_1 > 0$ such that

$$G(x,s) \geq c_1|s|^\mu \qquad \text{for all } |s| \leq R \quad \text{and} \quad \text{a.e. } x \in \Omega, \tag{9}$$

without knowing, again, that (7) holds.

In conclusion, working with functions satisfying (4), or (3) only in Ω and not in $\overline{\Omega}$, forces to *add* condition (6) to (3) or (4) and consider them as a unique hypothesis, as well as (9) should be assumed *together* with (8).

Therefore, in [3], where condition (3) was assumed with $R = 0$, one must ignore Remark 1, while condition (6) with $c_2 = 0$ must be taken as part of hypothesis (g_4).

References

[1] Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**, 349–381 (1973)

[2] Mugnai, D.: Asymptotic behaviour, nodal lines and symmetry properties for solutions of superlinear elliptic equations near an eigenvalue. ESAIM Control Optim. Calc. Var. **11**(4), 508–521 (2005)

[3] Mugnai, D.: Multiplicity of critical points in presence of a linking: application to a superlinear boundary value problem. NoDEA Nonlinear Differ. Equ. Appl. **11**(3), 379–391 (2004)

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