

Concavity maximum principle for viscosity solutions of singular equations

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Abstract. We prove a concavity maximum principle for the viscosity solutions of certain fully nonlinear and singular elliptic and parabolic partial differential equations. Our results parallel and extend those obtained by Korevaar and Kennington for classical solutions of quasilinear equations. Applications are given in the case of the singular infinity Laplace operator.

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1. Introduction

It is often natural to expect that a solution to a partial differential equation, equipped with suitable boundary conditions, reflects the geometric properties of the domain in which it is defined. Perhaps the simplest example of such a phenomenon is a radially symmetric solution in a ball.

A reasonable generalization of the rotationally symmetric case is that of a convex domain. That is, one would like to show that, under suitable assumptions on the equation and boundary values, a solution defined in a convex domain has convex level sets. The rich history and the various techniques used to study this question have been extensively discussed in [10], and we recall only the things that are relevant to our current work. One way to establish the convexity of the level sets of a solution v is to show that $u = f(v)$ is convex for some (monotone) real function f . This idea was used already by Brascamp and Lieb [2], but from our point of view the papers [13] and [14] of Korevaar are the most influential ones. Korevaar proved, roughly speaking, that the “concavity function”

$$\mathcal{C}(x, y, \lambda) = u(\lambda x + (1 - \lambda)y) - \lambda u(x) - (1 - \lambda)u(y)$$

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obeys maximum principle, and thus showed that u is convex in Ω if it is convex “near $\partial\Omega$ ”. Korevaar’s results were extended and improved by Caffarelli and Spruck [3], Kennington [12] and Kawohl [11]. All of these papers were dealing with classical solutions and quasilinear equations of the form

$$-\operatorname{trace}(A(Du)D^2u) + B(x, u, Du) = 0,$$

the principal assumptions being the joint concavity (or some weaker version of it) of B with respect to (x, u) and the monotonicity of B with respect to u .

The main goal of this paper is to prove versions of Korevaar’s concavity maximum principle for viscosity solutions of certain fully nonlinear equations. Our first result, Theorem 3.1, states that such a maximum principle holds for the solutions of the degenerate elliptic equation $F(x, u, Du, D^2u) = 0$ provided that F is jointly concave with respect to (x, u, D^2u) and increasing with respect to u ; here F can be singular at the points where the derivative Du vanishes so as to allow operators such as the singular infinity Laplacian. For quasilinear equations we are able to relax the concavity assumption in (x, u) to harmonic concavity. This result, Theorem 3.4, parallels those obtained by Kennington [12] and Kawohl [11]. Finally, we observe that both of the aforementioned theorems have natural parabolic counterparts. It should be noted that in the case of viscosity solutions, which are a priori only continuous, convexity is also a strong regularity statement as it implies local Lipschitz continuity and almost everywhere twice differentiability.

Our original motivation to study the convexity properties of viscosity solutions arose from eigenvalue problems that involve the infinity Laplace operator

$$\Delta_\infty u := \operatorname{trace} \left(\frac{Du}{|Du|} \otimes \frac{Du}{|Du|} D^2u \right).$$

These include the equation $-\Delta_\infty u = \lambda u$, considered in [8], and the so-called infinity eigenvalue problem,

$$\min\{|Du| - \Lambda_1 u, -\Delta_\infty u\} = 0,$$

which was introduced in [9]. This latter one is the limit, as $p \rightarrow \infty$, of the eigenvalue problems for the p -Laplace operator $\operatorname{div}(|Du|^{p-2}Du)$, and hence it follows from the result of Sakaguchi [16] that it has at least one log-concave solution in a convex domain, see [15]. However, as the simplicity of the eigenvalue Λ_1 is still an open problem, it is not known if all eigenfunctions have this property. Furthermore, it would be desirable to have a direct proof for the log-concavity. Although we have not yet succeeded in proving this result, we believe that the maximum principles obtained in this paper will serve as valuable tools in the coming efforts to settle the matter. We discuss the topic briefly in Sect. 4.2 below.

During the preparation of this manuscript it was pointed out to us by Bernd Kawohl that the convexity properties of viscosity solutions (with state constraints boundary conditions) have been studied by Alvarez et al. in [1]. These authors do not prove a concavity maximum principle. Instead, their way of showing the convexity of a solution is based on the comparison principle

and the fact that if u is a viscosity supersolution to $F(x, u, Du, D^2u) = 0$ in $\overline{\Omega}$, then the convex envelope of u (i.e., the largest convex function below u) is also a supersolution in $\overline{\Omega}$ (see [1, Prop. 3]). Observe that it is required that u is a supersolution in the closure $\overline{\Omega}$; this is a much stronger condition than u being a supersolution in Ω . The assumptions on F in [1] differ slightly from ours. For example, they do not allow for singular equations and, instead of the (strict) monotonicity of F with respect to u , assume that the comparison principle holds; see e.g. [6] for more on the relation of these two conditions. Moreover, their assumption on the dependence of F on the Hessian D^2u is weaker than ours, see Remark 3.6 below. All in all, these two methods are clearly closely related and seem to complement each other quite nicely.

2. Preliminaries

In this paper, we consider equations of the form

$$F(x, u, Du, D^2u) = 0, \quad (2.1)$$

where $F: \Omega \times \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \times S^{n \times n} \rightarrow \mathbb{R}$ is assumed to be continuous¹ and degenerate elliptic, that is,

$$F(x, r, \xi, X) \leq F(x, r, \xi, Y) \quad \text{for } (x, r, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \text{ and } X \geq Y. \quad (2.2)$$

Moreover, we assume that

$$-\infty < F_i(x, r, 0, 0) = F_s(x, r, 0, 0) < \infty \quad (2.3)$$

for all $x \in \Omega$ and $r \in \mathbb{R}$, where

$$F_i(x, r, \xi, M) := \liminf_{\varepsilon \rightarrow 0} \{F(y, s, \zeta, N) : \zeta \neq 0, |(x, r, \xi, M) - (y, s, \zeta, N)| < \varepsilon\}$$

and

$$F_s(x, r, \xi, M) := \limsup_{\varepsilon \rightarrow 0} \{F(y, s, \zeta, N) : \zeta \neq 0, |(x, r, \xi, M) - (y, s, \zeta, N)| < \varepsilon\}$$

are the semicontinuous envelopes of F , defined in $\Omega \times \mathbb{R} \times \mathbb{R}^n \times S^{n \times n}$. The degenerate ellipticity (2.2) is a standard assumption in the theory of viscosity solutions, and it ensures that classical solutions and smooth viscosity solutions coincide. Condition (2.3) in turn is instrumental in many papers dealing with singular equations (cf. [7] and references therein).

Example 2.1. Let $A: \mathbb{R}^n \setminus \{0\} \rightarrow S^{n \times n}$ be continuous, $A(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and assume that there is $C > 0$ such that

$$\|A(\xi)\| \leq C \quad \text{for all } 0 < |\xi| \leq 1,$$

that is, $A(\cdot)$ is bounded near the origin. Then the quasilinear equation

$$-\operatorname{trace}(A(Du)D^2u) + B(x, u, Du) = 0,$$

¹Furthermore, we require that the mapping $r \mapsto F(x, r, \xi, M)$ is continuous uniformly with respect to the other variables.

where $B \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, satisfies (2.2) and (2.3). In particular, our results apply to the singular infinity Laplace operator,

$$-\Delta_\infty u := -D^2 u \frac{Du}{|Du|} \cdot \frac{Du}{|Du|},$$

which is obtained by choosing $A(\xi) = \frac{1}{|\xi|^2} \xi \otimes \xi$.

For completeness, we recall the definition of viscosity solutions for singular equations satisfying (2.3). For further information on this topic, see e.g. [7].

Definition 2.2. An upper semicontinuous function $u : \Omega \rightarrow \mathbb{R}$ is a *viscosity subsolution* to (2.1) if, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that

$$0 = u(x_0) - \phi(x_0) > u(x) - \phi(x) \quad \text{for all } x \neq x_0,$$

then

$$F_i(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0.$$

A lower semicontinuous function $v : \Omega \rightarrow \mathbb{R}$ is a *viscosity supersolution* to (2.1) if, whenever $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$ are such that

$$0 = v(x_0) - \psi(x_0) < v(x) - \psi(x) \quad \text{for all } x \neq x_0,$$

then

$$F_s(x_0, \psi(x_0), D\psi(x_0), D^2\psi(x_0)) \geq 0.$$

Finally, $u \in C(\Omega)$ is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution.

3. Concavity maximum principle

3.1. Fully non-linear equations

Let $u \in C(\bar{\Omega})$, $\Omega \subset \mathbb{R}^n$ a bounded convex domain, and define the *concavity function* $C : \bar{\Omega} \times \bar{\Omega} \rightarrow \mathbb{R}$ by setting

$$C(x, y) := 2u\left(\frac{x+y}{2}\right) - u(x) - u(y).$$

Observe that if $C(x, y) \leq 0$ for all $x, y \in \Omega$, then u is convex in Ω .

Theorem 3.1. Let $u \in C(\bar{\Omega})$ be a viscosity solution to (2.1) in Ω , and suppose that

$$(x, r, M) \mapsto F(x, r, \xi, M) \quad \text{is concave} \tag{3.1}$$

for all $\xi \neq 0$ and

$$r \mapsto F(x, r, \xi, M) \quad \text{is increasing,} \tag{3.2}$$

uniformly for all $(x, \xi, M) \in \Omega \times \mathbb{R}^n \setminus \{0\} \times S^{n \times n}$. Then

$$\sup_{(x,y) \in \Omega \times \Omega} C(x, y) = \sup_{(x,y) \in \partial(\Omega \times \Omega)} C(x, y),$$

provided that C is positive at some point.

Remark 3.2. The assumption (3.2) means that the quantity

$$\rho(s, r) := \inf_{\substack{x \in \Omega \\ \xi \in \mathbb{R}^n \setminus \{0\} \\ M \in S^{n \times n}}} (F(x, s, \xi, M) - F(x, r, \xi, M))$$

is positive for any $s, r \in \mathbb{R}$ such that $s > r$. Note that since $r \mapsto F(x, r, \xi, M)$ is continuous, uniformly with respect to the other variables, ρ is continuous.

Proof. We begin by defining a sequence of functions that approximate the concavity function C . For $j = 1, 2, \dots$, let

$$\Phi_j(x, y, z) = 2u(z) - u(x) - u(y) - \psi_j(x, y, z),$$

where

$$\psi_j(x, y, z) = \frac{j}{4} \left| \frac{x+y}{2} - z \right|^4,$$

and let $(x_j, y_j, z_j) \in \overline{\Omega}^3$ be such that

$$\Phi_j(x_j, y_j, z_j) = \sup_{(x, y, z) \in \overline{\Omega}^3} \Phi_j(x, y, z).$$

Then

$$j \left| \frac{x_j + y_j}{2} - z_j \right|^4 \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

and, up to a subsequence,

$$x_j \rightarrow \hat{x}, \quad y_j \rightarrow \hat{y}, \quad z_j \rightarrow \frac{\hat{x} + \hat{y}}{2}$$

as $j \rightarrow \infty$ and

$$C(\hat{x}, \hat{y}) = \sup_{(x, y) \in \overline{\Omega} \times \overline{\Omega}} C(x, y) =: \gamma, \quad (3.3)$$

where $\gamma > 0$ by assumption. We refer to [6, Proposition 3.7] for the proof of these facts. Moreover, by continuity, we have

$$\frac{u(x_j) + u(y_j)}{2} < u(z_j) - \frac{\gamma}{4} \quad (3.4)$$

for j large.

By the maximum principle for semicontinuous functions, see [6, Theorem 3.2] or [5], there exist symmetric $n \times n$ matrices X_j, Y_j and Z_j such that

$$\begin{aligned} (D_x \psi_j, X_j) &\in \overline{J}_{\overline{\Omega}}^{2,+}(-u)(x_j), \\ (D_y \psi_j, Y_j) &\in \overline{J}_{\overline{\Omega}}^{2,+}(-u)(y_j), \\ (D_z \psi_j, Z_j) &\in \overline{J}_{\overline{\Omega}}^{2,+}(2u)(z_j), \end{aligned} \quad (3.5)$$

and

$$\begin{pmatrix} X_j & 0 & 0 \\ 0 & Y_j & 0 \\ 0 & 0 & Z_j \end{pmatrix} \leq D^2 \psi_j(x_j, y_j, z_j) + \frac{1}{j} [D^2 \psi_j(x_j, y_j, z_j)]^2. \quad (3.6)$$

Upon denoting $\zeta_j = z_j - \frac{x_j+y_j}{2}$, $\eta_j = \frac{j}{2}|\zeta_j|^2\zeta_j$ and

$$M = \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix},$$

(3.5) and (3.6) read as

$$\begin{aligned} (\eta_j, -X_j) &\in \overline{J}_{\overline{\Omega}}^{2,-} u(x_j), \\ (\eta_j, -Y_j) &\in \overline{J}_{\overline{\Omega}}^{2,-} u(y_j), \\ \left(\eta_j, \frac{Z_j}{2}\right) &\in \overline{J}_{\overline{\Omega}}^{2,+} u(z_j), \end{aligned} \quad (3.7)$$

and

$$\begin{pmatrix} X_j & 0 & 0 \\ 0 & Y_j & 0 \\ 0 & 0 & Z_j \end{pmatrix} \leq \left(\frac{j}{4}|\zeta_j|^2 I + \frac{j}{2}\zeta_j \otimes \zeta_j + \frac{6j}{16}|\zeta_j|^4 I + 3j|\zeta_j|^2\zeta_j \otimes \zeta_j \right) M, \quad (3.8)$$

where the notation means that each $n \times n$ block of M is multiplied by the matrix in parentheses. Notice that the semijets $J^{2,+}$ and $J^{2,-}$ are taken with respect to the closure $\overline{\Omega}$ since x_j, y_j, z_j might be boundary points of Ω .

In order to prove the claim, we argue by contradiction and suppose that

$$\gamma = \sup_{(x,y) \in \Omega \times \Omega} C(x,y) > \sup_{(x,y) \in \partial(\Omega \times \Omega)} C(x,y).$$

Then $\hat{x}, \hat{y} \in \Omega$ and also $x_j, y_j, z_j \in \Omega$ for all j large enough. Moreover, (3.7) holds with $J_{\overline{\Omega}}^{2,+}$ and $J_{\overline{\Omega}}^{2,-}$ replaced by $J_{\Omega}^{2,+}$ and $J_{\Omega}^{2,-}$, respectively.

Let us first suppose that $z_j \neq \frac{x_j+y_j}{2}$ for infinitely many j 's, and let us consider only those indexes in what follows. Since M annihilates vectors of the form $(\alpha\xi, \beta\xi, \frac{\alpha+\beta}{2}\xi) \in \mathbb{R}^{3n}$ for any $\alpha, \beta \in \mathbb{R}$, (3.8) implies that

$$\alpha^2 X_j + \beta^2 Y_j + \left(\frac{\alpha+\beta}{2}\right)^2 Z_j \leq 0. \quad (3.9)$$

In particular, by choosing $\alpha = \beta = 1$, we have $X_j + Y_j + Z_j \leq 0$.

Since u is a solution (and hence both a subsolution and a supersolution) and $\eta_j \neq 0$, (3.7) gives

$$\begin{aligned} F(x_j, u(x_j), \eta_j, -X_j) &\geq 0, \\ F(y_j, u(y_j), \eta_j, -Y_j) &\geq 0, \\ F(z_j, u(z_j), \eta_j, \frac{Z_j}{2}) &\leq 0. \end{aligned} \quad (3.10)$$

Recalling (2.2), (3.1) and $Z_j \leq -X_j - Y_j$, we have

$$\begin{aligned} F\left(\frac{x_j+y_j}{2}, \frac{u(x_j)+u(y_j)}{2}, \eta_j, \frac{Z_j}{2}\right) &\geq F\left(\frac{x_j+y_j}{2}, \frac{u(x_j)+u(y_j)}{2}, \eta_j, \frac{-X_j-Y_j}{2}\right) \\ &\geq \frac{F(x_j, u(x_j), \eta_j, -X_j) + F(y_j, u(y_j), \eta_j, -Y_j)}{2}, \end{aligned}$$

whereas (3.10) yields

$$\frac{F(x_j, u(x_j), \eta_j, -X_j) + F(y_j, u(x_j), \eta_j, -Y_j)}{2} \geq 0 \geq F\left(z_j, u(z_j), \eta_j, \frac{Z_j}{2}\right).$$

In particular, we have

$$F\left(\frac{x_j + y_j}{2}, \frac{u(x_j) + u(y_j)}{2}, \eta_j, \frac{Z_j}{2}\right) \geq F\left(z_j, u(z_j), \eta_j, \frac{Z_j}{2}\right).$$

Since $|z_j - \frac{x_j + y_j}{2}| \rightarrow 0$ as $j \rightarrow \infty$,

$$\frac{u(x_j) + u(y_j)}{2} < u(z_j) - \frac{\gamma}{4}$$

for j large enough and $r \mapsto F(x, r, \xi, M)$ is (strictly) increasing, we have a contradiction.

Next we consider the second alternative, that $z_j = \frac{x_j + y_j}{2}$ for all j large. Then $\eta_j = \zeta_j = 0$. Since

$$\begin{aligned} & 2u(z_j) - u(x_j) - u(y_j) - \psi_j(x, y_j, z_j) \\ & \leq 2u(z_j) - u(x_j) - u(y_j) - \psi_j(x_j, y_j, z_j) \end{aligned}$$

for all $x \in \Omega$, the function

$$\varphi(x) = u(x_j) + \psi_j(x_j, y_j, z_j) - \psi_j(x, y_j, z_j)$$

touches u from below at x_j , and hence

$$(0, 0) = (D\varphi(x_j), D^2\varphi(x_j)) \in J^{2,-}u(x_j);$$

this is the reason why we have the distance to power 4 (in fact any power greater than 2 would do) in the definition of ψ_j . In a similar way we see that

$$(0, 0) \in J^{2,-}u(y_j), \quad (0, 0) \in J^{2,+}u(z_j).$$

Since u is a solution, these imply

$$F_i(z_j, u(z_j), 0, 0) \leq 0 \leq \frac{F_s(x_j, u(x_j), 0, 0) + F_s(y_j, u(y_j), 0, 0)}{2}.$$

On the other hand, owing to (2.3) and Lemma 5.1 in Appendix,

$$\begin{aligned} \frac{F_s(x_j, u(x_j), 0, 0) + F_s(y_j, u(y_j), 0, 0)}{2} &= \frac{F_i(x_j, u(x_j), 0, 0) + F_i(y_j, u(y_j), 0, 0)}{2} \\ &\leq F_i\left(z_j, \frac{u(x_j) + u(y_j)}{2}, 0, 0\right). \end{aligned}$$

By combining these two inequalities we have

$$F_i(z_j, u(z_j), 0, 0) \leq F_i\left(z_j, \frac{u(x_j) + u(y_j)}{2}, 0, 0\right),$$

which, upon recalling Lemma 5.1 and the fact that $r \mapsto F(x, r, \xi, M)$ is increasing, contradicts (3.4). \square

In the applications, it is often the case that, instead of convexity, one would like to prove the concavity of a solution. Thus it is convenient to state a “convexity maximum principle” as well; the proof is entirely analogous to that of Theorem 3.1.

Theorem 3.3. *Let $u \in C(\bar{\Omega})$ be a viscosity solution to (2.1) in Ω , and suppose that*

$$(x, r, M) \mapsto F(x, r, \xi, M) \text{ is convex}$$

for all $\xi \neq 0$ and

$$r \mapsto F(x, r, \xi, M) \text{ is increasing,}$$

uniformly for all $(x, \xi, M) \in \Omega \times \mathbb{R}^n \setminus \{0\} \times S^{n \times n}$. Then

$$\inf_{(x,y) \in \Omega \times \Omega} C(x, y) = \inf_{(x,y) \in \partial(\Omega \times \Omega)} C(x, y),$$

provided that C is negative at some point.

3.2. Quasilinear equations

In the proof of Theorem 3.1, we used the inequality (3.8) only to conclude that $Z_j \leq -X_j - Y_j$. In order to prove viscosity versions of the results of Kennington [12] and Kawohl [11], we need to extract more information out of (3.8).

A continuous function $f: \mathcal{O} \rightarrow \mathbb{R}$, defined on a convex domain $\mathcal{O} \subset \mathbb{R}^m$, is said to be *harmonic concave* when

$$\frac{f(x) + f(y)}{2} f\left(\frac{x+y}{2}\right) \geq f(x)f(y) \quad \text{if } f(x) + f(y) > 0, \quad (3.11)$$

and

$$f\left(\frac{x+y}{2}\right) \geq 0 \quad \text{if } f(x) = f(y) = 0. \quad (3.12)$$

It is easy to check that if f is positive, then it is harmonic concave if and only if $\frac{1}{f}$ is convex. Moreover, a concave function is harmonic concave. On the other hand, *any* (continuous) negative function is harmonic concave.

Theorem 3.4. *Let $u \in C(\bar{\Omega})$ be a viscosity solution to*

$$-\operatorname{trace}(A(Du)D^2u) + B(x, u, Du) = 0 \quad (3.13)$$

in Ω , where $A(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and there is $C \geq 0$ such that

$$\|A(\xi)\| \leq C \quad \text{for } 0 < |\xi| \leq 1. \quad (3.14)$$

Moreover, we assume that $B \geq 0$, $r \mapsto B(x, r, \xi)$ is increasing, uniformly for all $(x, \xi) \in \Omega \times \mathbb{R}^n \setminus \{0\}$, and for each $\xi \neq 0$, $(x, r) \mapsto B(x, r, \xi)$ is harmonic concave in $\Omega \times \mathbb{R}$. Then

$$\sup_{(x,y) \in \Omega \times \Omega} C(x, y) = \sup_{(x,y) \in \partial(\Omega \times \Omega)} C(x, y),$$

provided that C is positive at some point.

Proof. By arguing as in the proof of Theorem 3.1, we find points $x_j, y_j, z_j \in \Omega$ and matrices $X_j, Y_j, Z_j \in S^{n \times n}$ such that (3.3), (3.4), (3.7) and (3.8) hold.

As before, we suppose first that $z_j \neq \frac{x_j+y_j}{2}$ for infinitely many j 's, and recall that (3.8) implies

$$\alpha^2 X_j + \beta^2 Y_j + \left(\frac{\alpha+\beta}{2} \right)^2 Z_j \leq 0$$

for all $\alpha, \beta \in \mathbb{R}$. Owing to (3.7),

$$\begin{aligned} \text{trace}(A(\eta_j)X_j) + B(x_j, u(x_j), \eta_j) &\geq 0, \\ \text{trace}(A(\eta_j)Y_j) + B(y_j, u(y_j), \eta_j) &\geq 0, \\ -\text{trace}(A(\eta_j)\frac{Z_j}{2}) + B(z_j, u(z_j), \eta_j) &\leq 0. \end{aligned}$$

Hence, as $A(\eta_j) \geq 0$, we have (see [12, Lemma A1])

$$\alpha^2 B(x_j, u(x_j), \eta_j) + \beta^2 B(y_j, u(y_j), \eta_j) \geq \frac{(\alpha+\beta)^2}{2} B(z_j, u(z_j), \eta_j). \quad (3.15)$$

With the choices $\alpha = B(y_j) := B(y_j, u(y_j), \eta_j)$ and $\beta = B(x_j) := B(x_j, u(x_j), \eta_j)$, this yields

$$B(y_j)B(x_j)(B(y_j) + B(x_j)) \geq \frac{(B(x_j) + B(y_j))^2}{2} B(z_j, u(z_j), \eta_j). \quad (3.16)$$

If $B(x_j) + B(y_j) > 0$, (3.11) implies

$$\frac{B(x_j) + B(y_j)}{2} B\left(\frac{x_j + y_j}{2}, \frac{u(x_j) + u(y_j)}{2}, \eta_j\right) \geq B(x_j)B(y_j). \quad (3.17)$$

Putting (3.16) and (3.17) together gives

$$\begin{aligned} &\frac{(B(x_j) + B(y_j))^2}{2} B\left(\frac{x_j + y_j}{2}, \frac{u(x_j) + u(y_j)}{2}, \eta_j\right) \\ &\geq \frac{(B(x_j) + B(y_j))^2}{2} B(z_j, u(z_j), \eta_j), \end{aligned}$$

and hence

$$B\left(\frac{x_j + y_j}{2}, \frac{u(x_j) + u(y_j)}{2}, \eta_j\right) \geq B(z_j, u(z_j), \eta_j),$$

which contradicts (3.4) for j large enough. On the other hand, if $B(x_j) = B(y_j) = 0$, then by (3.15) (choose, say, $\alpha = \beta = 1$) and (3.12) we have

$$B\left(\frac{x_j + y_j}{2}, \frac{u(x_j) + u(y_j)}{2}, \eta_j\right) \geq 0 \geq B(z_j, u(z_j), \eta_j),$$

and thereby obtain a contradiction just as above.

The second alternative is that $z_j = \frac{x_j+y_j}{2}$ for all j large enough. Then $\eta_j = 0$, and by arguing as in the proof of Theorem 3.1, we see that $(0, 0) \in$

$J^{2,-}u(x_j)$, $(0, 0) \in J^{2,-}u(y_j)$ and $(0, 0) \in J^{2,+}u(z_j)$. Since

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \{\text{trace}(A(\zeta)N) : \zeta \neq 0, |(\zeta, N)| < \varepsilon\} \\ &= \liminf_{\varepsilon \rightarrow 0} \{\text{trace}(A(\zeta)N) : \zeta \neq 0, |(\zeta, N)| < \varepsilon\} \\ &= 0, \end{aligned}$$

by (3.14), the inequality (3.15) holds again for all $\alpha, \beta \in \mathbb{R}$. The rest of the proof runs now exactly as above. \square

The corresponding concavity maximum principle is stated again without a proof.

Theorem 3.5. *Let $u \in C(\bar{\Omega})$ be a viscosity solution to*

$$-\text{trace}(A(Du)D^2u) + B(x, u, Du) = 0$$

in Ω , where $A(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and there is $C \geq 0$ such that

$$\|A(\xi)\| \leq C \quad \text{for } 0 < |\xi| \leq 1.$$

Moreover, we assume that $B \geq 0$, $r \mapsto B(x, r, \xi)$ is increasing, uniformly for all $(x, \xi) \in \Omega \times \mathbb{R}^n \setminus \{0\}$, and for each $\xi \neq 0$, $(x, r) \mapsto -B(x, r, \xi)$ is harmonic concave in $\Omega \times \mathbb{R}$. Then

$$\inf_{(x,y) \in \Omega \times \Omega} C(x, y) = \inf_{(x,y) \in \partial(\Omega \times \Omega)} C(x, y),$$

provided that C is negative at some point.

Remark 3.6. In [1], the convexity of solutions with state constraints boundary conditions is proved under the assumption that

$$(x, r, P) \mapsto F(x, r, \xi, P^{-1}) \quad \text{is concave;} \tag{3.18}$$

in view of the convexity of $P \mapsto P^{-1}$ (see [1]), this condition is weaker than the concavity of $(x, r, P) \mapsto F(x, r, \xi, P)$. It is natural to ask whether our method could be extended to cover this case as well.

An inspection of the proof of Theorem 3.1 reveals that a key inequality is

$$\begin{aligned} & F\left(\frac{x_j + y_j}{2}, \frac{u(x_j) + u(y_j)}{2}, \eta_j, \frac{Z_j}{2}\right) \\ & \geq \frac{F(x_j, u(x_j), \eta_j, -X_j) + F(y_j, u(y_j), \eta_j, -Y_j)}{2}. \end{aligned}$$

This is derived by using the ellipticity (2.2), the inequality $Z_j \leq -X_j - Y_j$, and the concavity of $(x, r, P) \mapsto F(x, r, \xi, P)$. If we weaken our concavity assumption to (3.18), then the matrix inequality we need is

$$\frac{Z_j}{2} \leq \left(\frac{-X_j^{-1} - Y_j^{-1}}{2}\right)^{-1}.$$

Although we do not know how to obtain this, except when $n = 1$, we believe that the following stronger version of (3.9) plays a crucial role in its proof:

$$\left(\frac{P+Q}{2} \right)^T Z_j \left(\frac{P+Q}{2} \right) + P^T X_j P + Q^T Y_j Q \leq 0. \quad (3.19)$$

This inequality is proved by noting that the matrix M in the proof of Theorem 3.1 annihilates all vectors of the form $(P\xi, Q\xi, \frac{P+Q}{2}\xi)$, where P and Q are any $n \times n$ matrices. Note that by choosing $P = -Q = I$, we obtain $X_j + Y_j \leq 0$.

If $n = 1$, we may assume that both X_j and Y_j are non-zero and, say, $X_j \leq 0$. Upon choosing $P = X_j^{-1} = -Q$, (3.19) yields

$$X_j^{-1}(I + Y_j X_j^{-1}) \leq 0,$$

which implies that $I + Y_j X_j^{-1} \geq 0$, that is, $X_j^{-1} + Y_j^{-1} \geq 0$. Then, by choosing $P = X_j^{-1}$ and $Q = Y_j^{-1}$ in (3.19), we have

$$(X_j^{-1} + Y_j^{-1}) \left(\frac{Z_j}{4}(X_j^{-1} + Y_j^{-1}) + I \right) \leq 0.$$

Since $X_j^{-1} + Y_j^{-1} \geq 0$, this yields $\frac{Z_j}{4}(X_j^{-1} + Y_j^{-1}) \leq -I$, that is,

$$\frac{Z_j}{2} \leq \left(\frac{-X_j^{-1} - Y_j^{-1}}{2} \right)^{-1},$$

as desired.

3.3. The parabolic case

As in [14], we can easily modify the proof of elliptic concavity maximum principle and obtain a corresponding result for parabolic equations. Notice that below we assume only that $r \mapsto F(x, t, r, \xi, M)$ is non-decreasing, whereas in the elliptic case it was required that this function is increasing.

We assume, as before, that $\Omega \subset \mathbb{R}^n$ is a bounded convex domain and $0 < T < \infty$. Let $C(x, y, t) := 2u(\frac{x+y}{2}, t) - u(x, t) - u(y, t)$, and by $\partial_p(\Omega \times \Omega \times (0, T))$ denote the parabolic boundary of $\Omega \times \Omega \times (0, T)$, that is,

$$\partial_p(\Omega \times \Omega \times (0, T)) = (\partial(\Omega \times \Omega) \times (0, T)) \cup (\Omega \times \Omega \times \{0\}).$$

Theorem 3.7. *Let $u \in C(\bar{\Omega} \times [0, T])$ be a viscosity solution to (2.1) in $\Omega \times (0, T)$, and suppose that*

$$(x, r, M) \mapsto F(x, t, r, \xi, M) \text{ is concave} \quad (3.20)$$

for all $\xi \neq 0$, $t \in (0, T)$, and

$$r \mapsto F(x, t, r, \xi, M) \text{ is non-decreasing.} \quad (3.21)$$

Then

$$\sup_{(x,y,t) \in \Omega \times \Omega \times (0, T)} C(x, y, t) = \sup_{(x,y,t) \in \partial_p(\Omega \times \Omega \times (0, T))} C(x, y, t),$$

provided that C is positive at some point.

Proof. The proof is rather similar to that of Theorem 3.1, and we only discuss the main points. Let us first assume that $r \mapsto F(x, t, r, \xi, M)$ is increasing, and let

$$\Phi_j(x, y, z, t) = 2u(z, t) - u(x, t) - u(y, t) - \psi_j(x, y, z, t),$$

where

$$\psi_j(x, y, z, t) = \frac{j}{4} \left| \frac{x+y}{2} - z \right|^4 + \frac{1}{j(T-t)},$$

and let $(x_j, y_j, z_j, t_j) \in \overline{\Omega}^3 \times [0, T]$ be such that

$$\Phi_j(x_j, y_j, z_j, t_j) = \sup_{(x, y, z, t) \in \overline{\Omega}^3 \times [0, T]} \Phi_j(x, y, z, t).$$

Then, up to a subsequence,

$$t_j \rightarrow \hat{t}, \quad x_j \rightarrow \hat{x}, \quad y_j \rightarrow \hat{y}, \quad z_j \rightarrow \frac{\hat{x} + \hat{y}}{2}$$

as $j \rightarrow \infty$ and

$$C(\hat{x}, \hat{y}, \hat{t}) = \sup_{(x, y, t) \in \overline{\Omega}^2 \times [0, T]} C(x, y, t) =: \gamma > 0. \quad (3.22)$$

Moreover, we have

$$\frac{u(x_j, t_j) + u(y_j, t_j)}{2} < u(z_j, t_j) - \frac{\gamma}{4} \quad (3.23)$$

for j large.

By the parabolic version of the maximum principle for semicontinuous functions, see e.g. [6, Theorem 8.3], there exist symmetric $n \times n$ matrices X_j, Y_j and Z_j such that

$$\begin{aligned} (-\mu_j, \eta_j, -X_j) &\in \overline{P}_{\overline{\Omega}}^{2,-} u(x_j, t_j), \\ (-\nu_j, \eta_j, -Y_j) &\in \overline{P}_{\overline{\Omega}}^{2,-} u(y_j, t_j), \\ \left(\frac{\lambda_j}{2}, \eta_j, \frac{Z_j}{2} \right) &\in \overline{P}_{\overline{\Omega}}^{2,+} u(z_j, t_j), \end{aligned} \quad (3.24)$$

with $\mu_j + \nu_j + \lambda_j = D_t \psi_j = -\frac{1}{j(T-t_j)^2}$ and $X_j + Y_j + Z_j \leq 0$; here η_j is defined as in the proof of Theorem 3.1.

Again we argue by contradiction and suppose that

$$\gamma = \sup_{(x, y, t) \in \Omega^2 \times (0, T)} C(x, y, t) > \sup_{(x, y) \in \partial_p(\Omega^2 \times (0, T))} C(x, y, t).$$

Then $t_j > 0$ and $x_j, y_j, z_j \in \Omega$ for j large enough, and we conclude using (2.2), (3.20) and (3.24) that

$$\begin{aligned} &F\left(\frac{x_j + y_j}{2}, t_j, \frac{u(x_j) + u(y_j)}{2}, \eta_j, \frac{Z_j}{2}\right) \\ &\geq F\left(\frac{x_j + y_j}{2}, t_j, \frac{u(x_j) + u(y_j)}{2}, \eta_j, \frac{-X_j - Y_j}{2}\right) \\ &\geq \frac{F(x_j, t_j, u(x_j), \eta_j, -X_j) + F(y_j, t_j, u(y_j), \eta_j, -Y_j)}{2} \end{aligned}$$

$$\begin{aligned}
&\geq \frac{-\mu_j - \nu_j}{2} = \frac{1}{2j(T-t_j)^2} + \frac{\lambda_j}{2} \\
&\geq \frac{1}{2j(T-t_j)^2} + F\left(z_j, t_j, u(z_j), \eta_j, \frac{Z_j}{2}\right) \\
&\geq F\left(z_j, t_j, u(z_j), \eta_j, \frac{Z_j}{2}\right),
\end{aligned}$$

provided that $\eta_j \neq 0$. A contradiction with (3.21) is now reached as before. If $\eta_j = 0$ for all large j 's, we note that [6, Theorem 8.3] implies $(-\mu_j, 0, 0) \in P^{2,-}u(x_j, t_j)$, $(-\nu_j, 0, 0) \in P^{2,-}u(y_j, t_j)$, and $(\frac{\lambda_j}{2}, 0, 0) \in P^{2,+}u(z_j, t_j)$, and then proceed as in the proof of Theorem 3.1.

If $r \mapsto F(x, t, r, Du, D^2u)$ is only non-decreasing, we let

$$u^\varepsilon(x, t) = e^{\varepsilon t}u(x, t), \quad \varepsilon > 0,$$

and notice that u^ε satisfies $u_t^\varepsilon = \tilde{F}(x, t, u^\varepsilon, Du^\varepsilon, D^2u^\varepsilon)$, where

$$\tilde{F}(x, t, r, \xi, M) = e^{\varepsilon t}F(x, t, e^{-\varepsilon t}r, e^{-\varepsilon t}\xi, e^{-\varepsilon t}M) + \varepsilon r$$

is increasing in r . As $u^\varepsilon \rightarrow u$ uniformly when $\varepsilon \rightarrow 0$ and our claim holds for u^ε , it also holds for u . \square

It is clear that one can also prove a parabolic version of Theorem 3.4 by following the reasoning above. Since we do not need such a result in the applications, a detailed formulation and a proof are left to the reader.

4. Applications

In order to prove the convexity of a solution by using the concavity maximum principle, one must establish the convexity of a solution “near $\partial\Omega$ ”, that is, show that $C(x, y) \leq 0$ on $\partial(\Omega \times \Omega)$. In the framework of classical solutions, this was done by Korevaar [14]. However, his argument relies on the continuity of the first and second derivatives of the solution, and thus cannot be applied directly in our situation.

Rather than trying to formulate a generic result on the convexity of a solution near $\partial\Omega$, we study an example² involving the singular infinity Laplace operator. After that, we briefly discuss the eigenvalue problems that originally motivated this work.

4.1. Asymptotic profiles and nonlinear torsion problem

Consider the equation

$$-\Delta_\infty v = v^q, \tag{4.1}$$

where

$$\Delta_\infty v := \frac{1}{|Dv|^2}(D^2vDv) \cdot Dv$$

²This example is actually a version of Examples 3 and 4 in [11].

is the singular infinity Laplacian and $0 \leq q < 1$. For $q = 0$ we have the nonlinear torsion problem $-\Delta_\infty v = 1$, whereas for $0 < q < 1$ equation (4.1) (formally) describes the asymptotic profiles of the parabolic problem $\frac{\partial(w^m)}{\partial t} = \Delta_\infty w$.

Suppose that Ω is a bounded, convex domain with smooth boundary, and $v \in C(\bar{\Omega})$ is a positive viscosity solution to (4.1) with $v = 0$ on $\partial\Omega$. We claim that $v^{\frac{1-q}{2}}$ is concave.

To see this, let $u = -v^{\frac{1-q}{2}}$ and notice that v satisfies (in the viscosity sense)

$$\begin{cases} -\Delta_\infty u - \frac{1}{u} \left(\frac{1-q}{2} + \frac{1+q}{1-q} |Du|^2 \right) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This equation can be written as $-\operatorname{trace}(A(Du)D^2u) + B(x, u, Du) = 0$ with the choices

$$A(\xi) = \frac{1}{|\xi|^2} \xi \otimes \xi, \quad B(x, r, \xi) = -\frac{1}{r} \left(\frac{1-q}{2} + \frac{1+q}{1-q} |\xi|^2 \right),$$

and B satisfies the assumptions of Theorem 3.4 for $r < 0$.

Let $C(x, y) = 2u(\frac{x+y}{2}) - u(x) - u(y)$ and suppose that $\sup_{\bar{\Omega} \times \bar{\Omega}} C(x, y) > 0$. We define Φ_j and ψ_j as in the proof of Theorem 3.1, and conclude that there exist $x_j, y_j, z_j \in \bar{\Omega}$ and $X_j, Y_j, Z_j \in S^{n \times n}$ such that (x_j, y_j, z_j) is a maximum point of Φ_j in $\bar{\Omega}^3$ and (3.7) holds. Then, up to a subsequence,

$$x_j \rightarrow \hat{x}, \quad y_j \rightarrow \hat{y}, \quad z_j \rightarrow \frac{\hat{x} + \hat{y}}{2}$$

as $j \rightarrow \infty$ and

$$C(\hat{x}, \hat{y}) = \sup_{(x, y) \in \bar{\Omega} \times \bar{\Omega}} C(x, y).$$

By Theorem 3.1, $0 < \sup_{\bar{\Omega} \times \bar{\Omega}} C(x, y) = \sup_{\partial(\Omega \times \Omega)} C(x, y)$, and thus we may assume that $\hat{x} \in \partial\Omega$ and $\hat{y} \in \Omega$. In fact, we may assume that $y_j, z_j \in \Omega$ and $x_j \in \partial\Omega$ for all j large, for if $x_j \in \Omega$ the argument used to prove Theorem 3.1 would yield a contradiction. Hence, in conclusion, we have $(\eta_j, -X_j) \in \overline{J}_{\bar{\Omega}}^{2,-} u(x_j)$ and $x_j \in \partial\Omega$, which contradicts Lemma 4.1 below.

Lemma 4.1. *Let v, u and Ω be as above, and let $z \in \partial\Omega$. Then $J_{\bar{\Omega}}^{2,-} u(z) = \emptyset$.*

Proof. Let $B_r(y) \subset \Omega$ be such that $\partial B_r(y) \cap \partial\Omega = \{z\}$. Then, since $-\Delta_\infty v \geq 0$ and $v > 0$ in Ω ,

$$v(x) \geq v(y) \left(1 - \frac{|x-y|}{r} \right) \quad \text{for all } x \in B_r(y),$$

(see e.g. [4]) and thus

$$u(x) \leq u(y) \left(1 - \frac{|x-y|}{r} \right)^{\frac{1-q}{2}} \quad \text{in } B_r(y).$$

Denoting $e = \frac{y-z}{|y-z|}$, this implies that

$$\lim_{\lambda \rightarrow 0} \frac{u(z + \lambda e) - u(z)}{\lambda} \leq \lim_{\lambda \rightarrow 0} \frac{u(y) \left(\frac{\lambda}{r}\right)^{\frac{1-a}{2}}}{\lambda} = -\infty,$$

which yields $J_{\overline{\Omega}}^{2,-} u(z) = \emptyset$. \square

4.2. Eigenvalue problems

In eigenvalue problems one typically expects that the eigenfunctions are log-concave if Ω is convex. The first difficulty in proving this by using a concavity maximum principle arises from the fact that the original boundary condition $v = 0$ on $\partial\Omega$ translates to $u = \log v = -\infty$ on $\partial\Omega$, which is much more difficult to handle. Roughly speaking, in order to apply the results in this paper, one should prove that the concavity function C (of u) is asymptotically non-negative at $\partial\Omega$:

$$\liminf C(x, y) \geq 0 \quad \text{as } \text{dist}((x, y), \partial(\Omega \times \Omega)) \rightarrow 0.$$

However, this is often not the only problem. Namely, the equation one obtains for $u = \log v$ is for a large class of operators independent of the u -variable, and thus not increasing in u like our theorems require. To illustrate this, let $u \in C(\overline{\Omega})$ be a positive solution to the eigenvalue problem

$$\begin{cases} -\Delta_\infty v = \lambda v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega; \end{cases} \quad (4.2)$$

this is the problem studied in [8]. It is easy to check that $u = \log v$ satisfies

$$-\Delta_\infty u - |Du|^2 = \lambda \quad \text{in } \Omega.$$

So, in order to apply Theorem 3.5, we would like to find a family of functions g_ε such that $u_\varepsilon := g_\varepsilon(u) \rightarrow u$ locally uniformly as $\varepsilon \rightarrow 0$ and u_ε satisfies an equation for which all the assumptions of Theorem 3.5 hold; bear in mind that we are dealing with problems where the simplicity of the eigenvalue is not known.

To simplify notation, we let $w = g(u)$ and $h = g^{-1}$, and compute the equation w satisfies. To this end, we formally have

$$\lambda = -\Delta_\infty u - |Du|^2 = -h'(w)\Delta_\infty w - h''(w)|Dw|^2 - (h'(w))^2|Dw|^2,$$

whence

$$-\Delta_\infty w - \left(\frac{h''(w)}{h'(w)} + h'(w) \right) |Dw|^2 - \frac{\lambda}{h'(w)} = 0. \quad (4.3)$$

Now the question is, can we choose h in such a way that this equation satisfies the assumptions of Theorem 3.5?

An obvious candidate for the change of dependent variable appears in [9]. For $\alpha > 1$ and $A > 1$, let

$$g(t) = \frac{1}{\alpha} \log(1 + A(e^{\alpha t} - 1))$$

and

$$h(s) = g^{-1}(s) = \frac{1}{\alpha} \log \left(1 + \frac{1}{A} (e^{\alpha t} - 1) \right).$$

Then, since,

$$\frac{h''(w)}{h'(w)} + h'(w) = \alpha + (1 - \alpha)h'(w),$$

(4.3) reads

$$-\Delta_\infty w - (\alpha + (1 - \alpha)h'(w))|Dw|^2 - \frac{\lambda}{h'(w)} = 0.$$

To check the validity of the assumptions of Theorem 3.5, let us denote

$$B(x, r, \xi) = - \left((\alpha + (1 - \alpha)h'(r))|\xi|^2 + \frac{\lambda}{h'(r)} \right).$$

We have

$$\begin{aligned} \frac{\partial B}{\partial r} &= (\alpha - 1)|\xi|^2 h''(r) + \lambda \frac{h''(r)}{h'(r)^2} \\ &= \alpha(\alpha - 1)|\xi|^2 (h'(r) - h'(r)^2) + \lambda\alpha \left(\frac{1}{h'(r)} - 1 \right). \end{aligned}$$

Since $h'(r) < 1$, it follows that $r \mapsto B(x, r, \xi)$ is indeed increasing. However, it can be checked that $-B$ is not harmonic concave, and thus this change of dependent variable is not the one we are after.

Appendix

We show here that the semicontinuous extensions F_i and F_s inherit enough concavity and monotonicity from F for the proof of Theorem 3.1 to work. A key assumption in this section is condition (2.3).

Lemma 5.1. *If F satisfies (2.3) and $(x, r, M) \mapsto F(x, r, \xi, M)$ is concave for all $\xi \neq 0$, then*

$$(x, r) \mapsto F_i(x, r, 0, 0) = F_s(x, r, 0, 0)$$

is concave. Similarly, if $r \mapsto F(x, r, \xi, M)$ is increasing, uniformly for all $(x, \xi, M) \in \Omega \times \mathbb{R}^n \setminus \{0\} \times S^{n \times n}$, then

$$r \mapsto F_i(x, r, 0, 0) = F_s(x, r, 0, 0)$$

is increasing.

Proof. Let $x, y \in \Omega$ and $r, s \in \mathbb{R}$, and fix $(x_k, r_k, \xi_k, M_k), (y_k, s_k, \eta_k, N_k)$ such that $\xi_k, \eta_k \neq 0$ and

$$F_s(x, r, 0, 0) = \lim_{k \rightarrow \infty} F(x_k, r_k, \xi_k, M_k), \quad F_s(y, s, 0, 0) = \lim_{k \rightarrow \infty} F(y_k, s_k, \eta_k, N_k).$$

Then, owing to the concavity of F and (2.3), we have

$$\begin{aligned}
 & F_s(x, r, 0, 0) + F_s(y, s, 0, 0) \\
 &= \lim_{k \rightarrow \infty} (F(x_k, r_k, \xi_k, M_k) + F(y_k, s_k, \eta_k, N_k)) \\
 &\leq \limsup_{k \rightarrow \infty} F\left(\frac{x_k + y_k}{2}, \frac{r_k + s_k}{2}, \xi_k, \frac{M_k + N_k}{2}\right) \\
 &\quad + \limsup_{k \rightarrow \infty} (F(y_k, s_k, \eta_k, N_k) - F(y_k, s_k, \xi_k, N_k)) \\
 &\leq F_s\left(\frac{x+y}{2}, \frac{r+s}{2}, 0, 0\right) + F_s(y, s, 0, 0) - F_i(y, s, 0, 0) \\
 &= F_s\left(\frac{x+y}{2}, \frac{r+s}{2}, 0, 0\right).
 \end{aligned}$$

The strict monotonicity of $r \mapsto F_i(x, r, 0, 0) = F_s(x, r, 0, 0)$ is proved in a similar way. Let $x \in \Omega$ and $r, s \in \mathbb{R}$, and fix $(x_k, r_k, \xi_k, M_k), (y_k, s_k, \eta_k, N_k)$ such that $F_s(x, r, 0, 0) = \lim_{k \rightarrow \infty} F(x_k, r_k, \xi_k, M_k)$, $F_s(x, s, 0, 0) = \lim_{k \rightarrow \infty} F(y_k, s_k, \eta_k, N_k)$. Then

$$\begin{aligned}
 F_s(x, s, 0, 0) - F_s(x, r, 0, 0) &= \lim_{k \rightarrow \infty} (F(y_k, s_k, \eta_k, N_k) - F(x_k, r_k, \xi_k, M_k)) \\
 &\geq \lim_{k \rightarrow \infty} (\rho(s_k, r_k) + F(y_k, r_k, \eta_k, N_k) \\
 &\quad - F(x_k, r_k, \xi_k, M_k)) \\
 &\geq \rho(s, r) + F_i(x, r, 0, 0) - F_s(x, r, 0, 0) = \rho(s, r);
 \end{aligned}$$

see Remark 3.2 for the definition of ρ . □

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