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# Exact volatility calibration based on a Dupire-type Call–Put duality for perpetual American options

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**Abstract.** This paper investigates the calibration of a model with a time-homogeneous local volatility function to the market prices of the perpetual American Call and Put options. The main step is the derivation of a Call–Put duality equality for perpetual American options similar to the equality which is equivalent to Dupire's formula (Dupire in Risk 7(1):18–20, 1994) in the European case. It turns out that in addition to the simultaneous exchanges between the spot price and the strike and between the interest and dividend rates which already appear in the European case, one has to modify the local volatility function in the American case. To show this duality equality, we exhibit non-autonomous nonlinear ODEs satisfied by the perpetual Call and Put exercise boundaries as functions of the strike variable. We obtain uniqueness for these ODEs and deduce that the mapping associating the exercise boundary with the local volatility function is one-to-one onto. Thanks to this Dupire-type duality result, we design a theoretical calibration procedure of the local volatility function from the perpetual Call and Put prices for a fixed spot price $x_0$. The knowledge of the Put (resp. Call) prices for all strikes enables to recover the local volatility function on the interval $(0, x_0)$ (resp. $(x_0, +\infty)$). We last prove that equality of the dual volatility functions only holds in the standard Black-Scholes model with constant volatility.

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## 0. Introduction

The motivation of this paper is the calibration of the local volatility function $\sigma(t, x)$ of the stock model with constant interest rate $r$ and dividend rate $\delta$

$$\begin{cases} dS_t^x = \sigma(t, S_t^x) S_t^x dW_t + (r - \delta) S_t^x dt, \quad t \geq 0 \\ S_0^x = x \end{cases} \tag{1}$$

to the market prices of the American Call and Put options written on this stock. Here $(W_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion. When the local volatility model is calibrated either to the market prices $(P(T,y),\ T,y>0)$ of the European Put options with maturity $T$ and strike $y$ or to the market prices $(C(T,y),\ T,y>0)$ of the European Call options, the local volatility function is obtained thanks to Dupire's formula [7]

$$\forall T,y>0,\ \sigma(T,y)=\sqrt{2\frac{\partial_T O(T,y)+(r-\delta)y\partial_y O(T,y)+\delta O(T,y)}{y^2\partial^2_{yy}O(T,y)}}. \tag{2}$$

where $O\in\{P,C\}$ (by the Call–Put parity equality: $C(T,y)-P(T,y)=xe^{-\delta T}-ye^{-rT}$, the right-hand-side does not depend on $O\in\{C,P\}$). Extensions to some exotic European options have been investigated in [17], but, to our knowledge, no significant progress has been made to get a similar formula when the options used for the calibration are of American type.

In the fourth part of the paper, we explain how a time-homogeneous volatility function $\sigma(x)$ can be recovered from the prices of the perpetual American Call and Put options written on an underlying evolving according to the corresponding time-homogeneous local volatility model. Although restricted to the perpetual case, our methodology is more complicated than Dupire's formula. It is based on a Call–Put duality equality for perpetual American options related to the Call–Put duality equality equivalent to Dupire's formula in the European case. Indeed, (2) implies that for each $T>0$, $c_T(t,y)=P(T-t,y)$ solves the Partial Differential Equation

$$\partial_t c_T(t,y)+\frac{\sigma^2(T-t,y)y^2}{2}\partial^2_{yy}c_T(t,y)+(\delta-r)y\partial_y c_T(t,y)-\delta c_T(t,y)=0$$

for $(t,y)\in[0,T)\times(0,+\infty)$ with terminal condition $c_T(T,y)=(y-x)^+$ for $y>0$. One recognizes the pricing PDE for the Call option with strike $x$ and maturity $T$ in the model

$$d\bar{S}^T_t=\sigma(T-t,\bar{S}^T_t)\bar{S}^T_t dW_t+(\delta-r)\bar{S}^T_t dt \tag{3}$$

with local volatility function $\sigma(T-t,y)$, interest rate $\delta$ and dividend rate $r$. Therefore, denoting by $(\bar{S}^{y,T}_t)_{t\in[0,T]}$ the solution of (3) starting from $\bar{S}^{y,T}_0=y$, one has $c_T(0,y)=\mathbb{E}[e^{-\delta T}(\bar{S}^{y,T}_T-x)^+]$. One deduces the following Dupire-type Call–Put duality equality

$$\forall T\geq 0,\ \forall x,y>0,\ \mathbb{E}\left[e^{-rT}(y-S^x_T)^+\right]=\mathbb{E}\left[e^{-\delta T}(\bar{S}^{y,T}_T-x)^+\right]. \tag{4}$$

This equality is different from the one which can be derived by the change of numéraire approach [10] with the numéraire $(e^{\delta t}S^x_t)_{t\geq 0}$, see [12,18]:

$$\forall y,x>0,\ \forall T\geq 0,\ \mathbb{E}\left[e^{-rT}(y-S^x_T)^+\right]=\mathbb{E}\left[e^{-\delta T}(\hat{S}^{y,x}_T-x)^+\right] \tag{5}$$

when uniqueness in law holds for the following SDE

$$\begin{cases} d\hat{S}_t^{y,x} = \sigma\left(t, \frac{xy}{\hat{S}_t^{y,x}}\right)\hat{S}_t^{y,x}dW_t + (\delta - r)\hat{S}_t^{y,x}dt, \quad t \in [0,T] \\ \hat{S}_0^{y,x} = y. \end{cases} \tag{6}$$

In both equalities, the spot price and the strike price (resp. the interest rate and the dividend rate) are interchanged when going from the Put option in the left-hand-side to the Call option in the right-hand-side. In the equality (4) derived from Dupire's formula, the local volatility function $\sigma(T-t,z)$ of the underlying model in the right-hand-side depends on the maturity $T$ but not on the spot and strike variables $x, y$. It is obtained by time-reversal of the primal volatility function. In contrast, in the equality (5) derived by the change of numéraire approach, the local volatility function $\sigma(t, \frac{xy}{z})$ of the underlying model in the right-hand-side depends on the spot and strike variables $x, y$ but not on the maturity $T$. It is obtained by some logarithmic spatial reversal of the primal volatility function. Therefore, even when the volatility function only depends on the time or on the spot variable, the functions $\sigma(T-t,z)$ and $\sigma(t, \frac{xy}{z})$ are a priori different and so are the duality formulas (4) and (5).

To compare the interest of these duality equalities in terms of calibration, let us denote by $(\bar{S}_t^{s,y})_{t\in[s,T]}$ the solution of (3) starting from $\bar{S}_s^{s,y} = y$. Writing at $s=0$ the pricing PDE satisfied by $\mathbb{E}[e^{-\delta s}(\bar{S}_T^{s,y} - x)^+]$ in the variables $(s,y)$, remarking that the expectation only depends on $(s,T)$ through the difference $T-s$, one recovers (2) for $O = P$ from (4). In contrast, it is not clear at all how (5) could be used to recover the local volatility function from the European Put prices. The dependence on $y$ of the local volatility function in (6) makes the derivation of a PDE in the variables $(T,y)$ from $\mathbb{E}[e^{-\delta T}(\hat{S}_T^{y,x} - x)^+]$ non trivial.

The duality equality (5) can be generalized as detailed in [6,18] to time and spot dependent interest and dividend rates, non Markovian underlying models and American options. For instance, Proposition 6 in [6] implies that for each $T \in [0,+\infty]$,

$$\sup_{\tau\le T}\mathbb{E}\left[e^{-r\tau}(y - S_\tau^x)^+\right] = \sup_{\tau\le T}\mathbb{E}\left[e^{-\delta\tau}(\hat{S}_\tau^{y,x} - x)^+\right]$$

where $\tau$ is any stopping time of the Brownian filtration. To our knowledge, (4) has only been generalized in the European case to time-dependent interest and dividend rates and to models involving a very specific form of jumps [2,9,14]. In particular, it does not seem possible to generalize this equality when interest and dividend rates depend on the spot. Moreover, the American case remains open. The main contribution of the paper is the derivation of such a Call–Put Dupire-type duality equality

$$\sup_{\tau}\mathbb{E}\left[e^{-r\tau}(y - S_\tau^x)^+\right] = \sup_{\tau}\mathbb{E}\left[e^{-\delta\tau}(\bar{S}_\tau^{y} - x)^+\right] \tag{7}$$

for perpetual ($T = +\infty$) American options when the original volatility function $\sigma(x)$ is time-homogeneous. Whereas time-homogeneous volatility functions are preserved in the European case, it turns out that in addition to the exchanges between the spot price of the underlying and the strike and between the interest and dividend rates, the volatility function is modified:

$$d\bar{S}_t^y = \eta(\bar{S}_t^y)\bar{S}_t^y dW_t + (r - \delta)\bar{S}_t^y dt.$$

Although in general different from $\sigma$, the function $\eta$ still does not depend on $x, y > 0$.

The paper is organized as follows. In the first part, we recall results concerning the pricing of the perpetual Put and Call options in such models. For a given strike $y > 0$, we introduce the exercise boundary $x^*(y)$ of the perpetual Put option such that the perpetual Put price is equal to its payoff $(y-x)^+$ if and only if the initial value $x$ of the underlying is smaller or equal to $x^*(y)$.

In the second part of the paper, we derive new results concerning the exercise boundary. Considering the exercise boundaries as functions of the strike variable, we characterize them as the unique solutions of some non-autonomous ordinary differential equations.

The third part is dedicated to our main result. We prove the perpetual American Dupire-type duality equality (7) where the dual volatility function $\eta$ has an explicit expression in terms of $\sigma$ and $x^*$. To do so, we take advantage of a very nice feature: in the continuation region, the price of the perpetual option writes as the product of a function of the underlying spot price by another function of the strike price.

The fourth part addresses calibration issues. It turns out that for a given initial value $x_0 > 0$ of the underlying one recovers the restriction of the time-homogeneous volatility function $\sigma(x)$ to $(0, x_0]$ (resp. $[x_0, +\infty)$) from the perpetual Put (resp. Call) prices for all strikes.

In the last part, we show that at least when $\delta < r$ in the class of volatility functions $\sigma$ analytic in a neighborhood of the origin, the only ones invariant by our Dupire-type duality result (i.e. such that $\eta = \sigma$) are the constants. This means that the case of the standard Black-Scholes model is very specific regarding that duality.

Last, we extend in Appendix B our results to spot-dependent dividend rates. Then, because of additional technicalities, our theory is not so nice as in the constant dividend rate case.

## 1. Perpetual American put and call pricing

### 1.1. Framework and notations

For a function $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$, we denote $\underline{f} = \inf_{x \in \mathbb{R}_+^*} f(x)$ and $\overline{f} = \sup_{x \in \mathbb{R}_+^*} f(x)$.

We consider a constant interest spot-rate $r$ that is assumed to be nonnegative and an asset $S_t$ which pays a constant dividend rate $\delta \geq 0$ and is driven by a homogeneous volatility function $\sigma : \mathbb{R}_+^* \to \mathbb{R}_+^*$ that satisfies the following hypothesis.

**Hypothesis** ($\mathcal{H}_{\mathbf{vol}}$): *$\sigma$ is continuous on $\mathbb{R}_+^*$ and such that $0 < \underline{\sigma} \leq \overline{\sigma} < +\infty$.*

In other words, $S_t$ is assumed to follow under the risk-neutral measure the SDE:

$$dS_t = S_t((r-\delta)dt + \sigma(S_t)dW_t). \tag{8}$$

With the assumption made on $\sigma$, we know that for any initial condition $x \in \mathbb{R}_+^*$, there is a unique solution denoted by $(S_t^x, t \geq 0)$ in the sense of probability law (see for example Theorem 5.15 in [15], using a log transformation). Moreover, Theorem 4.20 in [15] ensures that the strong Markov property holds for $(S_t^x, t \geq 0)$. Under that model, we denote by

$$P_\sigma(x,y) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-r\tau}(y - S_\tau^x)^+\right] \quad \text{and} \quad C_\sigma(x,y) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-r\tau}(S_\tau^x - y)^+\right]$$

respectively the prices of the American perpetual Put and Call options with strike $y > 0$ and spot $x$. Here, $\mathcal{T}_{0,\infty}$ simply denotes the set of the stopping times with respect to the natural filtration of $(S_t^x, t \geq 0)$. This setting will be called *primal world* in the sequel. The pricing of American options is an optimal stopping problem and we refer to [16] for a review of known results concerning such problems. The perpetual optimal stopping problem of regular one-dimensional diffusion processes with reward functions more general than the Call and Put payoffs has been recently considered in [5]. The authors characterize the value function as the smallest concave, in a generalized sense, majorant of the reward function. To obtain our Call–Put duality equality we rather use a characterization of the pricing function based on ODEs and take advantage of the specificity of the Call and Put payoffs.

We now introduce the *dual world*. It is mathematically identical to the primal one, but the variables have a different meaning in it. Namely, $\delta$ plays the role of the interest rate and $r$ of the dividend rate; $x$ plays the role of the strike and $y$ is the spot value of the underlying. Let $\eta : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be an homogeneous volatility function that is also assumed to satisfy ($\mathcal{H}_{\mathbf{vol}}$). We consider then $(\overline{S}_t^y, t \geq 0)$ the solution of $d\overline{S}_t = \overline{S}_t((\delta - r)dt + \eta(\overline{S}_t)dW_t)$ that starts from $y$ at time 0. Under that model, we denote respectively by

$$p_\eta(y,x) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-\delta\tau}(x - \overline{S}_\tau^y)^+\right] \quad \text{and} \quad c_\eta(y,x) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-\delta\tau}(\overline{S}_\tau^y - x)^+\right]$$

the prices of the perpetual Put and Call options with strike $x > 0$ and spot $y$.

For the primal world, we introduce the following ODE which is satisfied by $x \mapsto P_\sigma(x,y)$ and $x \mapsto C_\sigma(x,y)$ in the continuation region (see the proof of Theorem 1.2):

$$\frac{1}{2}\sigma^2(x)x^2 f''(x) + (r-\delta)xf'(x) - rf(x) = 0, \quad x > 0. \tag{9}$$

According to [4, p. 18], there are two noticeable solutions to this ODE that we denote $f_\uparrow$ and $f_\downarrow$: $f_\uparrow$ (resp. $f_\downarrow$) is the unique up to a multiplicative constant positive increasing (resp. positive decreasing (non-increasing when $r = 0$)) solution to (9). Thanks to the continuity of $\sigma$, these solutions are $\mathcal{C}^2$ on $\mathbb{R}_+^*$. In the same manner for the dual world, we introduce $g_\downarrow$ and $g_\uparrow$ as the unique decreasing (non-increasing when $\delta = 0$) and increasing positive solutions to

$$\frac{1}{2}\eta^2(x)x^2 g''(x) + (\delta - r)xg'(x) - \delta g(x) = 0. \tag{10}$$

Moreover, $g_\uparrow$ (resp. $g_\downarrow$) is the unique solution of (10) up to a multiplicative constant such that $\lim_{x\to 0} g(x) = 0$ (resp. $\limsup_{x\to+\infty}|g(x)| < \infty$).

The aim of this paper is to put in evidence a Dupire-type duality relation and interpret Put (resp. Call) prices in the primal world as Call (resp. Put) prices in the dual world for a specific volatility function $\eta = \tilde{\sigma}$ (resp. $\eta = \hat{\sigma}$). When $r = 0$ (resp. $\delta = 0$), this is trivial because we can show that $P_\sigma(x, y) = c_\eta(y, x) = y$ (resp. $C_\sigma(x, y) = p_\eta(y, x) = x$) but not really fruitful, and we take thus the following convention in the sequel.

**Convention 1.1.** *We will always assume* $r > 0$ *(resp.* $\delta > 0$*) to state properties on* $P_\sigma$ *and* $c_\eta$ *(resp.* $C_\sigma$ *and* $p_\eta$*).*

Both worlds being mathematically equivalent, *we will work with the Put price in the primal world and the Call price in the dual world* in order not to do the things twice. Following the Convention 1.1, we will consider a positive interest rate. We also denote from now on:

$$\boxed{f = f_\downarrow \quad \text{and} \;\; g = g_\uparrow.}$$

## 1.2. Pricing and free boundaries

In that section, we turn to the existence of an optimal stopping time and to the pricing issue.

**Theorem 1.2.** *For any strike* $y > 0$*, there is a unique* $x^*_\sigma(y) < y$ *such that* $\tau^P_x = \inf\{t \geq 0, S^x_t \leq x^*_\sigma(y)\}$ *(convention* $\inf \emptyset = +\infty$*) is an optimal stopping time for the Put and:*

$$\begin{aligned} &\forall x \leq x^*_\sigma(y), P_\sigma(x, y) = (y - x)^+, \\ &\forall x > x^*_\sigma(y), P_\sigma(x, y) = \frac{y - x^*_\sigma(y)}{f(x^*_\sigma(y))} f(x) > (y - x)^+. \end{aligned} \tag{11}$$

*The smooth-fit principle holds:* $\partial_x P_\sigma(x^*_\sigma(y), y) = -1$*. In addition, we have* $f'(x^*_\sigma(y)) < 0$ *and:*

$$x^*_\sigma(y) - y = \frac{f(x^*_\sigma(y))}{f'(x^*_\sigma(y))}. \tag{12}$$

*Last, there are constants* $0 < c_1 \leq c_2 < \min(1, r/\delta)$ *such that:*

$$\forall y > 0, \;\; c_1 y \leq x^*_\sigma(y) \leq c_2 y. \tag{13}$$

**Theorem 1.3.** *For any strike $x > 0$, there is a unique $y^*_\eta(x) > x$ such that $\tau^c_y = \inf\{t \geq 0, \overline{S}^y_t \geq y^*_\eta(x)\}$ is an optimal stopping time for the Call and:*

$$\forall y \geq y^*_\eta(x), c_\eta(y,x) = (y-x)^+,$$
$$\forall y < y^*_\eta(x), c_\eta(y,x) = \frac{y^*_\eta(x) - x}{g(y^*_\eta(x))} g(y) > (y-x)^+. \tag{14}$$

*The smooth-fit principle holds: $\partial_y c_\eta(y^*_\eta(x), x) = 1$. In addition, we have $g'(y^*_\eta(x)) > 0$ and:*

$$y^*_\eta(x) - x = \frac{g(y^*_\eta(x))}{g'(y^*_\eta(x))}. \tag{15}$$

*Last, there are constants $+\infty > d_2 \geq d_1 > \max(1, \delta/r)$ such that:*

$$\forall x > 0, \quad d_1 x \leq y^*_\eta(x) \leq d_2 x. \tag{16}$$

Let us define $\alpha(y) = \frac{y - x^*_\sigma(y)}{f(x^*_\sigma(y))}$ and $\beta(x) = \frac{y^*_\eta(x) - x}{g(y^*_\eta(x))}$. We get then that $\alpha$ and $\beta$ are positive functions and:

$$\forall y > 0,\ \forall x \geq x^*_\sigma(y),\ P_\sigma(x,y) = \alpha(y) f(x) \tag{17}$$
$$\forall x > 0,\ \forall y \leq y^*_\eta(x),\ c_\eta(y,x) = \beta(x) g(y). \tag{18}$$

This product form will play an important role in the derivation of the duality.

The proofs of these theorems are similar and postponed in Appendix A. To get the upper and lower bounds satisfied by the exercise boundary, we use the following convexity result derived from [8,13] and the explicit formulas obtained in the Black Scholes framework (see [11] and Section 25 in [16]).

**Proposition 1.4.** *Let us consider two volatility functions $\sigma_1$ and $\sigma_2$ (resp. $\eta_1$ and $\eta_2$) such that $\forall x > 0,\ \sigma_1(x) \leq \sigma_2(x)$ (resp. $\forall y > 0,\ \eta_1(y) \leq \eta_2(y)$) and that satisfy ($\mathcal{H}_{\mathbf{vol}}$). Then, we have*

$$\forall x, y > 0, \quad P_{\sigma_1}(x,y) \leq P_{\sigma_2}(x,y) \quad (\textit{resp. } c_{\eta_1}(y,x) \leq c_{\eta_2}(y,x)),$$

*and the functions $x \mapsto P_{\sigma_1}(x,y)$ and $y \mapsto c_{\eta_1}(y,x)$ are convex.*

**Proposition 1.5.** *For $\varsigma > 0$, let*

$$a(\varsigma) = \frac{\delta - r + \varsigma^2/2 - \sqrt{(\delta - r + \varsigma^2/2)^2 + 2r\varsigma^2}}{\varsigma^2} < 0,$$
$$b(\varsigma) = \frac{r - \delta + \varsigma^2/2 + \sqrt{(\delta - r - \varsigma^2/2)^2 + 2\delta\varsigma^2}}{\varsigma^2} = 1 - a(\varsigma) > 1.$$

*When $\sigma \equiv \varsigma$ (resp. $\eta \equiv \varsigma$), $P_\sigma$ (resp. $c_\eta$) is given by the formula (11) (resp. (14)) with $f(x) = x^{a(\varsigma)}$ (resp. $g = x^{b(\varsigma)}$) and the exercise boundary $x^*_\varsigma(y) = \frac{a(\varsigma)}{a(\varsigma)-1} y$ (resp. $y^*_\varsigma(x) = \frac{b(\varsigma)}{b(\varsigma)-1} x$).*

## 2. ODEs for the exercise boundaries

Our main result concerning the exercise boundaries considered as functions of the strike is the following theorem.

**Theorem 2.1.** *Let us assume that the volatility functions* $\sigma$ *and* $\eta$ *satisfy* $(\mathcal{H}_{\mathbf{vol}})$. *Then, the boundaries* $x^*_\sigma(y)$ *and* $y^*_\eta(x)$ *are respectively the unique increasing solutions defined on* $(0,+\infty)$ *of the ODEs*

$$(x^*_\sigma)'(y) = \frac{x^*_\sigma(y)^2\sigma(x^*_\sigma(y))^2}{2(y - x^*_\sigma(y))(ry - \delta x^*_\sigma(y))}, \tag{19}$$

$$(y^*_\eta)'(x) = \frac{\eta^2(y^*_\eta(x))y^*_\eta(x)^2}{2(y^*_\eta(x) - x)(ry^*_\eta(x) - \delta x)} \tag{20}$$

*satisfying* $\forall y > 0,\ cy \le x^*_\sigma(y) < y$ *and* $\forall x > 0,\ x < y^*_\eta(x) \le dx$ *with* $0 < c < 1 < d < +\infty$.

*Proof.* We have seen in Theorems 1.2 and 1.3 that the exercise boundaries satisfy

$$x^*_\sigma(y) - y = f(x^*_\sigma(y))/f'(x^*_\sigma(y)) \ \ (resp.\ y^*_\eta(x) - x = g(y^*_\eta(x))/g'(y^*_\eta(x))). \tag{21}$$

According to the technical Lemma A.1 stated and proved in Appendix A, $x^*_\sigma(y)$ and $y^*_\eta(x)$ are $\mathcal{C}^1$ functions on $\mathbb{R}^*_+$. Differentiating (21) with respect to $y$ (resp. $x$), we obtain that $1 = (x^*_\sigma)'(y)\frac{f(x^*_\sigma(y))f''(x^*_\sigma(y))}{f'(x^*_\sigma(y))^2}$ (resp. $1 = (y^*_\eta)'(y)\frac{g(y^*_\eta(y))g''(y^*_\eta(y))}{g'(y^*_\eta(y))^2}$). Using (21) and equation (9) written at $x = x^*_\sigma(y)$ (resp. (10) written at $x = y^*_\eta(x)$), we deduce (19) and (20).

Let us now remark that the uniqueness result for (19) is equivalent to the uniqueness result for (20). Indeed, it is easy to see that $x^*_\sigma(y)$ is solution of (19) if and only if $\hat{y}(x) := 1/(x^*_\sigma(1/x))$ is solution of (20) with the volatility function $\eta(x) = \sigma(1/x)$. This new volatility function also satisfies $(\mathcal{H}_{\mathbf{vol}})$. Moreover, $d_1 x \le \hat{y}(x) \le d_2 x$ with $d_1 > \max(1, \delta/r)$ if and only if $0 \le c_1 y \le x^*_\sigma(y) \le c_2 y$ with $0 < c_1 \le c_2 < \min(1, r/\delta)$.

Let us suppose then that $y(x)$ is an increasing function that solves (20) and satisfies $x < y(x) \le dx$ for some $d > 1$. We have:

$$\begin{aligned}
&\frac{d}{dx}y^{-1}(y^*_\eta(x))\\
&\quad= \frac{y^*_\eta(x)^2\eta(y^*_\eta(x))^2}{2(y^*_\eta(x) - x)(ry^*_\eta(x) - \delta x)} \times \frac{2(y^*_\eta(x) - y^{-1}(y^*_\eta(x)))(ry^*_\eta(x) - \delta y^{-1}(y^*_\eta(x)))}{y^*_\eta(x)^2\eta(y^*_\eta(x))^2}\\
&\quad= \frac{(y^*_\eta(x) - y^{-1}(y^*_\eta(x)))(ry^*_\eta(x) - \delta y^{-1}(y^*_\eta(x))}{(y^*_\eta(x) - x)(ry^*_\eta(x) - \delta x)}.
\end{aligned}$$

Thus, the function $\psi(x) = y^{-1}(y^*_\eta(x))/x$ solves

$$\psi'(x) = \frac{1}{x}\left[\frac{y^*_\eta(x) - \psi(x)x}{y^*_\eta(x) - x} \times \frac{ry^*_\eta(x) - \delta x\psi(x)}{ry^*_\eta(x) - \delta x} - \psi(x)\right]$$
$$= \frac{1}{x}\left[\left(1 - \frac{\psi(x) - 1}{y^*_\eta(x)/x - 1}\right)\left(1 - \frac{\psi(x) - 1}{ry^*_\eta(x)/(\delta x) - 1}\right) - \psi(x)\right]. \quad (22)$$

The estimation (16) and $x < y(x) \leq dx$ imply that:

$$\exists A > 0, \quad \forall x > 0, \quad 1/A \leq \psi(x) \leq A, \quad (23)$$

$$\forall x > 0,\ \psi(x) < \frac{y^*_\eta(x)}{x},\ \frac{y^*_\eta(x)}{x} - 1 > 0 \quad \text{and} \quad \frac{ry^*_\eta(x)}{\delta x} - 1 > 0. \quad (24)$$

Since local uniqueness holds for (22) by the Cauchy Lipschitz theorem, the only solution $\varphi$ such that $\varphi(1) = 1$ is the constant $\varphi \equiv 1$. Therefore checking that (23) does not hold for solutions $\varphi$ satisfying (24) and such that $\varphi(1) \neq 1$ is enough to conclude that $\psi \equiv 1$.

Let $\varphi$ be a solution to (22) satisfying (24). If $\varphi(1) > 1$, by local uniqueness for (22), for all $x \in \mathbb{R}^*_+$, $\varphi(x) > 1$. By (24), one deduces that for all $x \in \mathbb{R}^*_+$, $\varphi'(x) < \frac{1-\varphi(x)}{x} < 0$. Therefore, $\varphi'(x) \leq (1 - \varphi(1))/x$ for $x \in (0, 1]$, and we have

$$\varphi(x) \geq \varphi(1) + (1 - \varphi(1)) \ln(x) \underset{x\to 0}{\to} +\infty$$

which is contradictory to (23). In the same manner, if $\varphi(1) < 1$, $\varphi(x) < 1$ for $x \in \mathbb{R}^*_+$ and $\varphi$ is strictly increasing. In particular, for $x \leq 1$, $\varphi'(x) \geq (1-\varphi(1))/x$ and therefore $\varphi(1) - \varphi(x) \geq (1 - \varphi(1)) \ln(1/x) \underset{x\to 0}{\to} +\infty$ and this yields another contradiction. □

**Corollary 2.2.** *Let us denote* $\tilde{\mathcal{C}} = \{f \in \mathcal{C}^1(\mathbb{R}^*_+),\ s.t.\ f(0) = 0,\ \exists 0 < a < b,\ \forall x \geq 0, a \leq f'(x) \leq b\}$. *The application* $\sigma \mapsto x^*_\sigma$ *(resp.* $\eta \mapsto y^*_\eta$*) is one-to-one between the set* $\{\sigma \in \mathcal{C}(\mathbb{R}^*_+)$ *that satisfies* $(\mathcal{H}_{\mathbf{vol}})\}$ *and the set of function* $\tilde{\mathcal{C}}_x = \{x \in \tilde{\mathcal{C}},\ s.t.\ \exists 0 < c_1 \leq c_2 < \min(1, r/\delta), \forall y > 0,\ c_1 y \leq x(y) \leq c_2 y\}$ *(resp.* $\tilde{\mathcal{C}}_y = \{y \in \tilde{\mathcal{C}},\ s.t.\ \exists \max(1, \delta/r) < d_1 \leq d_2, \forall x > 0,\ d_1 x \leq y(x) \leq d_2 x\}$*.)*

*Proof.* If $\sigma$ is a continuous function satisfying $(\mathcal{H}_{\mathbf{vol}})$, by (19) and (13), $x^*_\sigma$ belongs to $\tilde{\mathcal{C}}_x$. The one to one property is easy to get. If $x^*_{\sigma_1} \equiv x^*_{\sigma_2}$ with $\sigma_1$ and $\sigma_2$ satisfying $(\mathcal{H}_{\mathbf{vol}})$, the ODE (19) ensures that $\sigma_1^2(x^*_{\sigma_1}(y)) = \sigma_2^2(x^*_{\sigma_2}(y))$ for $y > 0$. Therefore $\sigma_1 \equiv \sigma_2$.

Let us check the onto property and consider $x^*(y) \in \tilde{\mathcal{C}}_x$. The function $\sigma$ defined by

$$\sigma(x^*(y)) = \frac{\sqrt{2(y - x^*(y))(ry - \delta x^*(y))x^{*\prime}(y)}}{x^*(y)} \quad (25)$$

is well defined thanks to the hypothesis made on $x^*$. As $x^*_\sigma$ satisfies (13) and solves the same ODE (19) as $x^*$, we have $x^* \equiv x^*_\sigma$ using Theorem 2.1.

The proof for $\eta \mapsto y^*_\eta$ is the same and gives incidentally the expression of $\eta$ in function of the exercise boundary $y^*(x)$:

$$\eta(y^*(x)) = \frac{\sqrt{2(y^*(x)-x)(ry^*(x)-\delta x)y^{*\prime}(x)}}{y^*(x)}. \tag{26}$$

□

## 3. The Call–Put Dupire-type duality

This section is devoted to the key result of the paper: for related local volatility functions $\sigma$ and $\eta$, we can interpret a Put price in the primal world as a Call price in the dual world.

### 3.1. The main result

**Theorem 3.1.** (Dupire-type duality) *The following conditions are equivalent:*

1. $$\forall x, y > 0,\ P_\sigma(x,y) = c_\eta(y,x). \tag{27}$$
2. $x^*_\sigma$ *and* $y^*_\eta$ *are reciprocal functions:* $\forall x > 0,\ x^*_\sigma(y^*_\eta(x)) = x$.
3. $\eta \equiv \tilde{\sigma}$ *where*
$$\tilde{\sigma}(y) = \frac{2(y - x^*_\sigma(y))(ry - \delta x^*_\sigma(y))}{y x^*_\sigma(y)\sigma(x^*_\sigma(y))}. \tag{28}$$
4. $\sigma \equiv \underset{\sim}{\eta}$ *where*
$$\underset{\sim}{\eta}(x) = \frac{2(y^*_\eta(x) - x)(ry^*_\eta(x) - \delta x)}{y^*_\eta(x) x \eta(y^*_\eta(x))}. \tag{29}$$

*Remark* 3.2. Thanks to relation (13) (resp. (16)), if $\sigma$ (resp. $\eta$) satisfies $(\mathcal{H}_{\mathbf{vol}})$ then the dual volatility function $\tilde{\sigma}$ defined by (28) (resp. $\underset{\sim}{\eta}$ defined by (29)) satisfies $(\mathcal{H}_{\mathbf{vol}})$.

*Proof.* 1 $\Longrightarrow$ 2: We have on the one hand $P_\sigma(x,y) = y - x$ on $\{(x,y),\ x \le x^*_\sigma(y)\}$ and $P_\sigma(x,y) > y - x$ outside, and on the other hand $c_\eta(y,x) = y - x$ on $\{(x,y),\ y \ge y^*_\eta(x)\}$ and $c_\eta(y,x) > y - x$ outside. The duality relation (27) imposes then that $\{(x,y),\ x \le x^*_\sigma(y)\} = \{(x,y),\ y \ge y^*_\eta(x)\}$ and so $y^*_\eta(x^*_\sigma(y)) = y$.

2 $\Longrightarrow$ 3, 4: Taking the derivative of the last relation, we get thanks to (19) and (20) $\frac{x^*_\sigma(y)^2\sigma(x^*_\sigma(y))^2}{2(y-x^*_\sigma(y))(ry-\delta x^*_\sigma(y))} \times \frac{\eta^2(y)y^2}{2(y-x^*_\sigma(y))(ry-\delta x^*_\sigma(y))} = 1$ and deduce (28) and (29).

3 $\Longrightarrow$ 2 (resp. 4 $\Longrightarrow$ 2): By (19) (resp. (20)) and (28) (resp. (29)), ${x^*_\sigma}^{-1}$ (resp. ${y^*_\eta}^{-1}$) satisfies (20) (resp. (19)). Since by (13) (resp. (16)) this function satisfies (16) (resp. (13)), one concludes by Theorem 2.1.

2 $\Longrightarrow$ 1: The equality (27) is clear in the exercise region since $\{(x,y), x \le x^*_\sigma(y)\} = \{(x,y), y \ge y^*_\eta(x)\}$. Let us check that it also holds in the continuation

region. Using the product form (18), and the smooth-fit principle (Theorem 1.3) we get for all $y \in \mathbb{R}_+^*$

$$\begin{cases} y - x_\sigma^*(y) = \beta(x_\sigma^*(y))g(y) \\ 1 = -\beta(x_\sigma^*(y))g'(y). \end{cases}$$

Differentiating the first equality with respect to $y$, one gets $1 - x_\sigma^*(y)' = x_\sigma^*(y)'$ $\beta'(x_\sigma^*(y))g(y) + \beta(x_\sigma^*(y))g'(y)$, which combined with the second equality gives

$$-1 = \beta'(x_\sigma^*(y))g(y).$$

Dividing by the first equality and using (21), one deduces $\frac{\beta'}{\beta}(x_\sigma^*(y)) = \frac{f'}{f}(x_\sigma^*(y))$. Since $x_\sigma^* : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a bijection, there is a constant $C \neq 0$ such that $\beta \equiv Cf$. Since $\forall y > 0, \alpha(y)f(x_\sigma^*(y)) = y - x_\sigma^*(y) = \beta(x_\sigma^*(y))g(y)$, one has $\alpha \equiv Cg$. From (17) and (18), one concludes that (27) holds. □

### 3.2. An analytic example of dual volatility functions

By (29) and (20), if $y^* \in \tilde{\mathcal{C}}_y$ (where $\tilde{\mathcal{C}}_y$ is defined in Corollary 2.2), then the reciprocal function of $y^*$ is the Put exercise boundary $x_\sigma^*$ associated to the local volatility function

$$\sigma(x) = \frac{\sqrt{2(ry^*(x) - \delta x)(y^*(x) - x)}}{x\sqrt{y^*(x)'}}.$$

Now by (26), $y^*$ is the Call exercise boundary associated with the dual volatility function:

$$\tilde{\sigma}(y) = \frac{\sqrt{2(y - x_\sigma^*(y))(ry - \delta x_\sigma^*(y))y^{*\prime}(x_\sigma^*(y))}}{y}.$$

Let us consider the family of exercise boundaries

$$y^*(x) = x\frac{x + a}{bx + c}$$

where $a, b, c$ are positive constants such that $\max(c/a, b) < \min(1, r/\delta)$ (condition ensuring $y^* \in \tilde{\mathcal{C}}_y$). Since $y^*(x)' = (bx^2 + 2cx + ac)/(bx + c)^2$, one has

$$\sigma(x) = \sqrt{2\frac{((r - \delta b)x + ra - \delta c)((1 - b)x + a - c)}{bx^2 + 2cx + ac}}, \quad x > 0.$$

Moreover, the function $x_\sigma^*(y)$ is the only positive root of the polynomial function: $X^2 + X(a - by) - cy$, that is:

$$x_\sigma^*(y) = \frac{1}{2}\left(by - a + \sqrt{(by - a)^2 + 4cy}\right)$$

and

$$\forall y > 0, \qquad \tilde{\sigma}(y) = \frac{\sqrt{2(y - x_\sigma^*(y))(ry - \delta x_\sigma^*(y))(bx_\sigma^*(y)^2 + 2cx_\sigma^*(y) + ac)}}{y(bx_\sigma^*(y) + c)}.$$

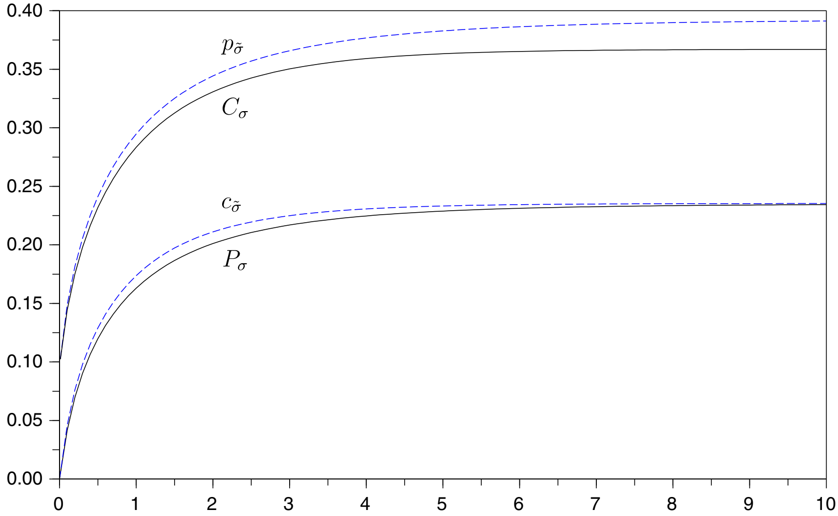


FIGURE 1. $P_\sigma(T,x,y)$ and $c_{\tilde{\sigma}}(T,y,x)$, and $C_\sigma(T,x,y)$ and $p_{\tilde{\sigma}}(T, y,x)$ as functions of the maturity $T$ for $x=0.5$, $y=0.4$, $r=0.2$, $\delta=0.1$ and the volatility parameters $(a,b,c)=(1,0.4,0.1)$

This example enables us to check numerically the duality. We have plotted in Fig. 1, the prices of an American Put $P_\sigma(T,x,y)$ in the primal world for the local volatility $\sigma(x)$ and an American Call $c_{\tilde{\sigma}}(T,y,x)$ in the dual world for the local volatility $\tilde{\sigma}(x)$ as functions of the maturity $T$. These prices have been computed using the Crank-Nicholson finite difference method. We can see at $T=10$ that the limit value is quite reached and both prices are equal. The plots are nonetheless distinct which means that the same duality does not hold for finite maturities. We have also plotted, in function of $T$, $C_\sigma(T,x,y)$ in the primal world and $p_{\tilde{\sigma}}(T,y,x)$ in the dual world to check numerically whether the volatility function $\tilde{\sigma}$ is such as $C_\sigma(x,y)=p_{\tilde{\sigma}}(y,x)$. As we can see, the curves do not seem to converge toward the same limit when $T$ is large. This means that the volatility function $\hat{\sigma}$ such that $\forall x,y>0,\ C_\sigma(x,y)=p_{\hat{\sigma}}(y,x)$ (obtained from $\sigma$ as $\underset{\sim}{\eta}$ is obtained from $\eta$ but with exchange of $r$ and $\delta$) is different from $\tilde{\sigma}$.

## 4. Exact calibration of the local volatility $\sigma(x)$

In that section, we will explain how the Dupire-type duality enables us to achieve our primary goal: the calibration of the local volatility function to the prices of perpetual American options. We suppose that we know the short interest rate $r$, the price $x_0$ of the underlying stock, its dividend rate $\delta$, and the market prices of the perpetual American Put and Call options written on this stock for any strike

$K > 0$. We name respectively $p(K)$ and $c(K)$ these prices and denote:

$$X = \sup\left\{K > 0,\ c(K) = (x_0 - K)^+\right\} \text{ and}$$
$$Y = \inf\left\{K > 0,\ p(K) = (K - x_0)^+\right\}. \tag{30}$$

Since we need to make use of the Dupire-type duality with the Call in the primal world and the Put in the dual world, we introduce:

- the exercise boundary $\Upsilon^*_\sigma(y)$ of $C_\sigma$: $[\Upsilon^*_\sigma(y), +\infty) = \{x \in \mathbb{R}^*_+, C_\sigma(x, y) = (y - x)^+\}$,
- the exercise boundary $\xi^*_\eta(x)$ of $p_\eta$: $(0, \xi^*_\eta(x)] = \{y \in \mathbb{R}^*_+, p_\eta(y, x) = (x - y)^+\}$.

We will first suppose that the Put and Call prices derive from a time-homogeneous local volatility model before relaxing this assumption.

### 4.1. The calibration procedure

Let us assume that there is a volatility function $\sigma$ satisfying $(\mathcal{H}_{\mathbf{vol}})$ such that for all $K > 0$, $p(K) = P_\sigma(x_0, K)$ and $c(K) = C_\sigma(x_0, K)$. The following proposition says that these prices characterize $\sigma$ and its proof gives a constructive way to retrieve the volatility function from the prices.

**Proposition 4.1.** *Let us consider $x_0 > 0$. The map*

$$\sigma \mapsto ((P_\sigma(x_0, K), C_\sigma(x_0, K)), K > 0)$$

*is one-to-one on the set of volatility functions satisfying $(\mathcal{H}_{\mathbf{vol}})$.*

*Proof.* We first consider the Put case. The differential equation satisfied by the Put prices in the continuation region makes only appear the values and the derivatives in $x$, $K$ being fixed. Hence, we cannot exploit directly the prices. But the Dupire-type duality relation enables to get a differential equation in the strike variable. Thanks to the duality Theorem, we have $P_\sigma(x_0, K) = c_{\tilde{\sigma}}(K, x_0)$ for some $\tilde{\sigma}$ satisfying $(\mathcal{H}_{\mathbf{vol}})$. It is then easy to calibrate $\tilde{\sigma}(.)$. Indeed, one has $\frac{K^2\tilde{\sigma}(K)^2}{2}p''(K) + K(\delta - r)p'(K) - \delta p(K) = 0$ for $K < Y = y^*_{\tilde{\sigma}}(x_0)$. Since the differential equation is valid only for $K < Y$, we only get $\tilde{\sigma}$ on $(0, Y]$ by continuity:

$$\forall K \leq Y,\ \tilde{\sigma}(K) = \frac{1}{K}\sqrt{\frac{2(\delta p(K) + K(r - \delta)p'(K))}{p''(K)}}$$

which is well defined since $p''(K) = \partial^2_K c_{\tilde{\sigma}}(K, x) > 0$ thanks to Lemma A.1 and Theorem 1.3. Then, we can calculate the exercise boundary $y^*_{\tilde{\sigma}}(x)$, for $x \in (0, x_0]$, solving (20) supplemented with the final condition $y^*_{\tilde{\sigma}}(x_0) = Y$ backward. This step only requires the knowledge of $\tilde{\sigma}$ only on the interval $(0, Y]$. Finally, we can recover the desired volatility $\sigma(x)$ for $x \leq x_0$ thanks to (28):

$$\forall x \in (0, x_0],\ \sigma(x) = \frac{2(y^*_{\tilde{\sigma}}(x) - x)(ry^*_{\tilde{\sigma}}(x) - \delta x)}{xy^*_{\tilde{\sigma}}(x)\tilde{\sigma}(y^*_{\tilde{\sigma}}(x))}. \tag{31}$$

Now let us consider the calibration to the Call prices. This relies on the same principle, but we have to be careful because the duality Theorem is stated given to the Call interest rate $\delta$ and dividend rate $r$. So we have to interchange these variables when we apply that theorem. There is a function $\hat{\sigma}$ satisfying ($\mathcal{H}_{\mathbf{vol}}$) such that: $\forall K > 0,\ C_\sigma(x_0, K) = p_{\hat{\sigma}}(K, x_0)$. We have

$$\frac{1}{2}K^2\hat{\sigma}(x)^2c''(K) + (\delta - r)Kc'(K) - \delta c(K) = 0$$

for $K > X = \xi^*_{\hat{\sigma}}(x_0)$. Thus, we get

$$\forall K \geq X,\ \hat{\sigma}(K) = \frac{1}{K}\sqrt{\frac{2(\delta c(K) + K(r-\delta)c'(K))}{c''(K)}}$$

which is well defined for analogous reasons. We can then obtain as before the exercise boundary solving (19) forward

$$\forall y \geq x_0, \qquad \xi^*_{\hat{\sigma}}(y)' = \frac{\xi^*_{\hat{\sigma}}(y)^2\hat{\sigma}(\xi^*_{\hat{\sigma}}(y))^2}{2(y - \xi^*_{\hat{\sigma}}(y))(\delta y - r\xi^*_{\hat{\sigma}}(y))}, \quad \xi^*_{\hat{\sigma}}(x_0) = X$$

and we finally get the volatility $\sigma(y)$ for $y \geq x_0$ using the duality Theorem. More precisely, we interchange $r$ and $\delta$ in (28) to get

$$\sigma(y) = \frac{2(y - \xi^*_{\hat{\sigma}}(y))(\delta y - r\xi^*_{\hat{\sigma}}(y))}{y\xi^*_{\hat{\sigma}}(y)\hat{\sigma}(\xi^*_{\hat{\sigma}}(y))}. \tag{32}$$

□

This calibration method, although being theoretical, sheds light on a striking and interesting result: the perpetual American Put prices only give the restriction of $\sigma(x)$ to $(0, x_0]$ and the Call prices only the restriction of $\sigma(x)$ to $[x_0, +\infty)$. This has the following economical interpretation: long-term American Put prices mainly give information on the downward volatility while long-term American Call prices give information on the upward volatility. This dichotomy is remarkable. In comparison, according to Dupire's formula [7], there is no such phenomenon for European options: the knowledge of the Call prices gives the whole local volatility surface, not only one part. In other words, the European Call and Put prices give the same information on the volatility while the perpetual American Call and Put prices give complementary information.

Thus, one may think that the perpetual American Call and Put prices only depend on a part of the volatility curve. This is precised by the Proposition below that gives necessary and sufficient conditions on the volatility functions to observe the same Put prices (resp. Call prices).

**Proposition 4.2.** *Let us consider $x_0 > 0$ and $\sigma_1(.)$, $\sigma_2(.)$ two volatility functions satisfying ($\mathcal{H}_{\mathbf{vol}}$). Then, the following properties are equivalent:*

(i) $\forall y > 0,\ P_{\sigma_1}(x_0, y) = P_{\sigma_2}(x_0, y)$ *(resp.* $\forall y > 0,\ C_{\sigma_1}(x_0, y) = C_{\sigma_2}(x_0, y)$*)*

(ii) $\forall y \leq y^*_{\tilde{\sigma}_2}(x_0), \tilde{\sigma}_1(y) = \tilde{\sigma}_2(y)$. *(resp.* $\forall x \geq \xi^*_{\hat{\sigma_1}}(x_0), \hat{\sigma_1}(x) = \hat{\sigma_2}(x)$ *where* $\hat{\sigma_j}$ *denotes the local volatility function such that* $\forall x, y > 0,\ C_{\sigma_j}(x, y) = p_{\hat{\sigma_j}}(y, x)$*.)*

(iii) $\forall x \in (0, x_0],\ \sigma_1(x) = \sigma_2(x)$ *and* $y^*_{\tilde{\sigma}_1}(x_0) = y^*_{\tilde{\sigma}_2}(x_0)$. *(resp.* $\forall x \in [x_0, +\infty)$, $\sigma_1(x) = \sigma_2(x)$ *and* $\xi^*_{\hat{\sigma_1}}(x_0) = \xi^*_{\hat{\sigma_2}}(x_0)$*.)*

(iv) $\forall x \in (0, x_0], \sigma_1(x) = \sigma_2(x)$ *and* $\frac{f'_{\downarrow,\sigma_1}(x_0)}{f_{\downarrow,\sigma_1}(x_0)} = \frac{f'_{\downarrow,\sigma_2}(x_0)}{f_{\downarrow,\sigma_2}(x_0)}$. (*resp.* $\forall x \in [x_0, +\infty)$, $\sigma_1(x) = \sigma_2(x)$ *and* $\frac{f'_{\uparrow,\sigma_1}(x_0)}{f_{\uparrow,\sigma_1}(x_0)} = \frac{f'_{\uparrow,\sigma_2}(x_0)}{f_{\uparrow,\sigma_2}(x_0)}$.)

(v) $f_{\downarrow,\sigma_1}$ *and* $f_{\downarrow,\sigma_2}$ (*resp.* $f_{\uparrow,\sigma_1}$ *and* $f_{\uparrow,\sigma_2}$) *are proportional on* $(0, x_0]$ (*resp.* $[x_0, +\infty)$).

(vi) $\forall x \le x_0, \forall y > 0,\ P_{\sigma_1}(x, y) = P_{\sigma_2}(x, y)$ (*resp.* $\forall x \ge x_0, \forall y > 0,\ C_{\sigma_1}(x, y) = C_{\sigma_2}(x, y)$).

*Remark* 4.3.

- Among these many conditions, let us remark that condition (ii) on the dual volatility is much simpler than condition (iii) on the primal volatility since the latter requires the equality of the dual exercise boundaries at $x_0$.
- When $\delta = 0$, one has an explicit form for the solutions of (9):

$$f_\downarrow(x) = \frac{\varphi(x)}{\varphi(1)} \text{ where } \varphi(x) = x \int_x^{+\infty} \left( \frac{1}{v^2} \exp\left[ - \int_1^v \frac{2r}{u\sigma^2(u)} du \right] \right) dv, \quad f_\uparrow(x) = x.$$

Then, condition $(iv)$ also writes in the Put case $\forall x \in (0, x_0],\ \sigma_1(x) = \sigma_2(x)$ and

$$\int_{x_0}^{+\infty} \left( \frac{1}{v^2} \exp\left[ - \int_{x_0}^v \frac{2r}{u\sigma_1^2(u)} du \right] \right) dv = \int_{x_0}^{+\infty} \left( \frac{1}{v^2} \exp\left[ - \int_{x_0}^v \frac{2r}{u\sigma_2^2(u)} du \right] \right) dv.$$

*Proof.* We consider for example the Put case.

(i) $\Longrightarrow$ (ii): See the proof of Theorem 4.1.

(ii) $\Longrightarrow$ (iii): Let us define $\psi(x) = (y^*_{\tilde{\sigma}_1})^{-1}(y^*_{\tilde{\sigma}_2}(x))/x$. We can show as in the proof of Theorem 2.1 that $\psi(x_0) = 1$ and then $\psi \equiv 1$ on $(0, x_0]$, otherwise it would go to 0 or $+\infty$ when $x \to 0$, which is not possible thanks to (16). We get then $\forall x \in (0, x_0],\ \sigma_1(x) = \sigma_2(x)$ using (29) that express $\sigma_j$ in function of $y^*_{\tilde{\sigma}_j}$ and $\tilde{\sigma}_j$, $j \in \{1, 2\}$.

(iii) $\Longrightarrow$ (iv): Thanks to (21) and Theorem 3.1, we have $\frac{f'_{\downarrow,\sigma_1}(x_0)}{f_{\downarrow,\sigma_1}(x_0)} = \frac{-1}{y^*_{\tilde{\sigma}_1}(x_0) - x_0} = \frac{-1}{y^*_{\tilde{\sigma}_2}(x_0) - x_0} = \frac{f'_{\downarrow,\sigma_2}(x_0)}{f_{\downarrow,\sigma_2}(x_0)}$.

(iv) $\Longrightarrow$ (v):

The set of solutions to $\frac{1}{2}\sigma_1^2(x)x^2 f''(x) + (r - \delta)x f'(x) - r f(x) = 0$ on $(0, x_0]$ is a two-dimensional vector space, but thanks to the relation $\frac{f'_{\downarrow,\sigma_1}(x_0)}{f_{\downarrow,\sigma_1}(x_0)} = \frac{f'_{\downarrow,\sigma_2}(x_0)}{f_{\downarrow,\sigma_2}(x_0)}$, $f_{\downarrow,\sigma_1}$ and $f_{\downarrow,\sigma_2}$ are proportional on $(0, x_0]$:

$$\forall x \le x_0,\ f_{\downarrow,\sigma_1}(x) = \frac{f_{\downarrow,\sigma_1}(x_0)}{f_{\downarrow,\sigma_2}(x_0)} f_{\downarrow,\sigma_2}(x). \tag{33}$$

(v) $\Longrightarrow$ (vi): The proportionality implies that $\forall x \in (0, x_0],\ \frac{f_{\downarrow,\sigma_1}(x)'}{f_{\downarrow,\sigma_1}(x)} = \frac{f_{\downarrow,\sigma_2}(x)'}{f_{\downarrow,\sigma_2}(x)}$, and then $(y^*_{\tilde{\sigma}_1}(x) - x)^{-1} = (y^*_{\tilde{\sigma}_2}(x) - x)^{-1}$ using (21) and Theorem 3.1. Therefore

$$\forall x \in (0, x_0],\ y^*_{\tilde{\sigma}_1}(x) = y^*_{\tilde{\sigma}_2}(x).$$

We have $\alpha_{\sigma_1}(y^*_{\tilde{\sigma}_1}(x))f_{\downarrow,\sigma_1}(x) = \alpha_{\sigma_2}(y^*_{\tilde{\sigma}_2}(x))f_{\downarrow,\sigma_2}(x)$ using (17), and obtain from (33) that

$$\forall x \leq x_0,\ \forall y \leq y^*_{\tilde{\sigma}_1}(x_0),\ \alpha_{\sigma_1}(y) = \frac{f_{\downarrow,\sigma_2}(x_0)}{f_{\downarrow,\sigma_1}(x_0)}\alpha_{\sigma_2}(y) = \frac{f_{\downarrow,\sigma_2}(x)}{f_{\downarrow,\sigma_1}(x)}\alpha_{\sigma_2}(y). \tag{34}$$

Thus, we deduce from (17), (33) and (34) the equality of the Put prices for the low strikes

$$\forall x \leq x_0,\ \forall y \leq y^*_{\tilde{\sigma}_1}(x), P_{\sigma_1}(x,y) = P_{\sigma_2}(x,y).$$

For $y > y^*_{\tilde{\sigma}_1}(x) = y^*_{\tilde{\sigma}_2}(x)$, the equality is clear since both prices are equal to $y - x$.

(vi) $\Longrightarrow$ (i): clear. $\square$

Let us observe that the point $(ii)$ of the last proposition allows to exhibit different volatility functions with analytic expressions that give the same Put (or Call) prices. Let us consider the same family as in Sect. 3.2 coming from the Call exercise boundary $y^*_1(x) = x\frac{x+a}{bx+c}$ (assuming $a, b, c > 0$ and $\max(c/a, b) < \min(1, r/\delta)$). For $x_0 > 0$, we introduce the exercise boundary:

$$y^*_2(x) = y^*_1(x) \text{ for } x \leq x_0 \text{ and } y^*_2(x) = y^*_1(x_0) + (y^*_1)'(x_0)(x - x_0) \text{ for } x \geq x_0.$$

The condition $(y^*_2 \in \tilde{\mathcal{C}}_y)$ is satisfied provided that $(y^*_1)'(x_0) > \max(1, \delta/r)$. This is automatically ensured by the assumptions made on $a, b, c$ since $(y^*_1)'(x_0) = (bx_0^2 + 2cx_0 + ac)/(b^2x_0^2 + 2bcx_0 + c^2)$. That family is such that $\tilde{\sigma}_1(y) = \tilde{\sigma}_2(y)$ for $y \leq y^*_2(x_0)$. We can then calculate $\sigma_2$ as in Sect. 3.2 using the relation $\sigma_2(x) = \frac{\sqrt{2(ry^*_2(x)-\delta x)(y^*_2(x)-x)}}{x\sqrt{y^*_2(x)'}}$. This gives $\sigma_2(x) = \sigma(x)$ for $x \leq x_0$ and for $x \geq x_0$,

$$\sigma_2(x) = \sqrt{2\frac{[(r(y^*_1)'(x_0)-\delta)x + r(y^*_1(x_0)-x_0(y^*_1)'(x_0))][((y^*_1)'(x_0)-1)x+y^*_1(x_0)-x_0(y^*_1)'(x_0)]}{x^2(y^*_1)'(x_0)}}.$$

In Fig. 2, we have plotted the same example as in Fig. 1 ($x = 0.5$ and $y = 0.4$), adding the graph of $T \mapsto P_{\sigma_2}(T, x, y)$. The volatility function $\sigma_2$ has been calculated with the formula above with $x_0 = 0.5$. According to Proposition 4.2 and the duality, the three prices are equal when $T$ is large. In the second example ($x = 3$ and $y = 1$), we still observe that $P_\sigma(T, x, y)$ and $c_{\tilde{\sigma}}(T, y, x)$ converge toward the same value when $T$ is large. On the contrary, the limit price of $P_{\sigma_2}(T, x, y)$ is significantly different. To observe the same price, we should have taken, according to Proposition 4.2, $x_0 \geq 3$.

### 4.2. Calibration to "real" Call and Put prices

In that subsection, we address some problems that arise if one tries to apply the calibration procedure when the prices $p(K)$ and $c(K)$ do not derive from a time-homogeneous model. For arbitrage-free reasons, the function $p$ (resp. $c$) must be non-decreasing (resp. non-increasing). We assume moreover that they are smooth functions of the strike $K$, and focus for example on the calibration to Put prices.

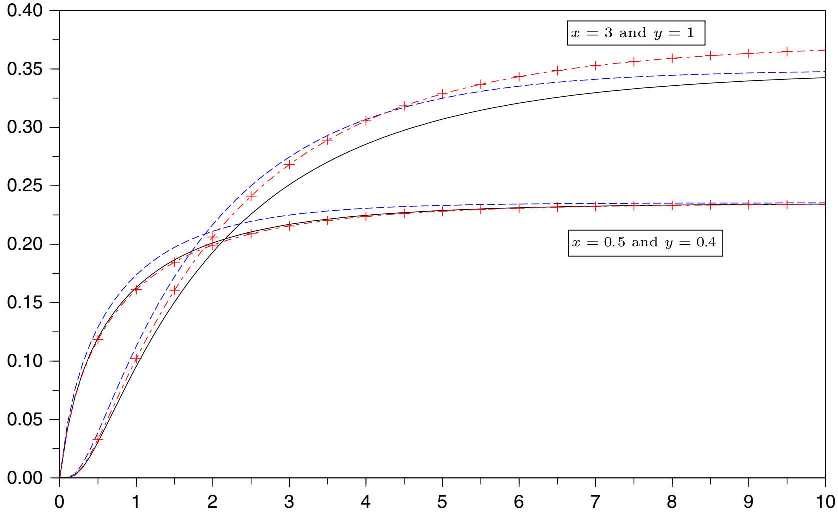


FIGURE 2. $P_\sigma(T,x,y)$ (*solid line*), $P_{\sigma_2}(T,x,y)$ (*dashed line with crosses*) and $c_{\tilde{\sigma}}(T,y,x)$ (*dashed line*) in function of the time $T$ for $a=1$, $b=0.4$, $c=0.1$, $x_0=0.5$, $r=0.2$ and $\delta=0.1$

Firstly, let us observe that the arbitrage-free theory allows to define a dual volatility as previously by $(0,Y]$:

$$\forall K < Y, \qquad \eta_p(K) = \frac{1}{K}\sqrt{\frac{2(\delta p(K) + K(r-\delta)p'(K))}{p''(K)}}. \tag{35}$$

Indeed, the payoff convexity in $K$ ensures the positivity of $p''(K)$ and the arbitrage-free assumption ensures that $\delta p(K) + K(r-\delta)p'(K)$ is nonnegative, so that the square-root is well defined. Let us prove the last point and suppose the contrary (i.e. $\exists y>0$ such that $\frac{d}{dy}e^{\delta y}p(e^{(r-\delta)y}) < 0$) to exhibit an arbitrage opportunity. In that case, there is $z>y$ such that $e^{\delta y}p(e^{(r-\delta)y}) > e^{\delta z}p(e^{(r-\delta)z})$. We then sell one Put with strike $e^{(r-\delta)y}$ and buy $e^{\delta(z-y)}$ Puts with strike $e^{(r-\delta)z}$. This initial transaction generates a positive flow. The hedging works as follows: naming $\tau$ the time at which the Put sold is exercised, we have to pay $e^{(r-\delta)y} - S_\tau$. In other words, we receive one share and borrow $e^{(r-\delta)y}$ in cash. We keep this position until time $\tau + z - y$. At this time, we have exactly $e^{\delta(z-y)}$ shares and Puts with strike $e^{(r-\delta)z}$. Thus, we obtain at least $e^{\delta(z-y)}e^{(r-\delta)z} = e^{(r-\delta)y}e^{r(z-y)}$ and we cancel the debt.

The next proposition gives sufficient conditions that allow to construct an homogeneous volatility which is consistent to the observed prices.

**Proposition 4.4.** *Let us assume that* $K \in \mathbb{R}_+^* \mapsto p(K)$ *is a* $\mathcal{C}^1$ *function,* $\mathcal{C}^2$ *on* $\mathbb{R}_+^* \backslash \{Y\}$ *with* $Y = \inf\{K>0 : p(K) = K - x_0\} < +\infty$*. Let us also assume that* $\eta_p$ *defined by* (35) *is bounded from below and above by two positive constants and* *admits a left-hand limit in $Y$. Then, if we extend $\eta_p$ in any continuous function on $(0, +\infty)$ satisfying $(\mathcal{H}_{\mathbf{vol}})$ still denoted by $\eta_p$, we have*

$$\forall K > 0, P_{\underset{\sim}{\eta_p}}(x_0, K) = p(K).$$

Notice that once we choose the extended function $\eta_p$, we obtain $\underset{\sim}{\eta_p}$ by first solving (20) on $\mathbb{R}_+^*$ starting from $x_0$ with the condition $y^*_{\eta_p}(x_0) = Y$ and then using (29).

*Proof.* The functions $K \mapsto p(K)$ and $K \mapsto c_{\eta_p}(K, x_0)$ solve (10). Since we have $0 \leq p(K) \leq K$ for arbitrage-free reasons, both functions go to 0 when $K \to 0$. As recalled in Sect. 1.1, $g_\uparrow$ is the unique solution to (10) up to a multiplicative constant that satisfies $\lim_{x \to 0} g(x) = 0$. Therefore, there is $\lambda > 0$ such that:

$$\forall K \leq Y, \; p(K) = \lambda c_{\eta_p}(K, x_0).$$

The $\mathcal{C}^1$ assumption made on $p$ ensures $p(Y) = Y - x_0$ and $p'(Y) = 1$. This gives $g_\uparrow(Y)/g'_\uparrow(Y) = Y - x_0$ and therefore $Y = y^*_{\eta_p}(x_0)$ using Lemma A.1. Thus, $c_{\eta_p}(Y, x_0) = Y - x_0 = p(Y)$ and $\lambda = 1$. One concludes with Theorem 3.1. □

In the same manner, we obtain the following result from the Call prices.

**Proposition 4.5.** *Let us assume that $K \in \mathbb{R}_+^* \mapsto c(K)$ is a $\mathcal{C}^1$ function, $\mathcal{C}^2$ on $\mathbb{R}_+^* \backslash \{X\}$ with $X = \sup\{K > 0, \; c(K) = x_0 - K\} > 0$. Let us also assume that $\eta_c$ defined by*

$$\forall K > X, \; \eta_c(K) = \frac{1}{K}\sqrt{\frac{2(\delta c(K) + K(r - \delta)c'(K))}{c''(K)}}$$

*is bounded from below and above by two positive constants and admits a right-hand limit in $X$. Then, if we extend $\eta_c$ in any continuous function on $(0, +\infty)$ satisfying $(\mathcal{H}_{\mathbf{vol}})$ still denoted by $\eta_c$, we have*

$$\forall K > 0, C_{\underset{\smile}{\eta_c}}(x_0, K) = c(K)$$

*where $\underset{\smile}{\eta_c}$ is obtained from $\eta_c$ like $\sigma$ from $\hat{\sigma}$ in the end of the proof of Proposition 4.1.*

Therefore, we are able to find volatility functions that give exactly the Put prices and others that give exactly the Call prices. Now, the natural question is whether one can find a volatility function $\sigma$ that is consistent to both the Put and Call prices. According to Proposition 4.2, all the volatility functions $\underset{\sim}{\eta_p}$ (resp. $\underset{\smile}{\eta_c}$) giving the Put (resp. Call) prices coincide on $(0, x_0)$ (resp. $(x_0, +\infty)$). The only volatility function possibly giving both the Put and Call prices is

$$\sigma(x) = \begin{cases} \underset{\sim}{\eta_p}(x) & \text{if } x < x_0 \\ \underset{\smile}{\eta_c}(x) & \text{if } x > x_0 \end{cases}.$$

We deduce from Proposition 4.2:

**Proposition 4.6.** *Assume that* $\eta_{\underset{\sim}{p}}(x_0^-) = \eta_{\check{c}}(x_0^+)$. *Then,*

$$\forall K > 0, p(K) = P_\sigma(x_0, K) \quad \textit{and} \quad c(K) = C_\sigma(x_0, K)$$
$$\textit{iff } x^*_\sigma(Y) = x_0 \quad \textit{and} \quad \Upsilon^*_\sigma(X) = x_0.$$

## 5. The Black-Scholes model: the unique model invariant through this Dupire-type duality

The purpose of that section is to put in evidence the particular role played by the Black-Scholes model for the perpetual American Call–Put Dupire-type duality. We have recalled in the introduction that in the European case, the Call–Put Dupire-type duality holds for all maturities without any change of the volatility function when it is time-homogeneous. Here, on the contrary, we are going to prove that if the duality holds for the perpetual American options with the same volatility:

$$\forall x, y > 0,\ P_\sigma(x, y) = c_\sigma(y, x) \tag{36}$$

then, under some technical assumptions, necessarily $\sigma(.)$ is a constant function.

**Proposition 5.1.** *Let us consider a positive interest rate* $r$ *and a nonnegative dividend rate* $\delta < r$. *We suppose that the volatility function* $\sigma$ *satisfies* $(\mathcal{H}_{\mathbf{vol}})$, *and is analytic in a neighborhood of* $0$, *i.e.*

$$\exists \rho > 0, \forall x \in [0, \rho), \quad \sigma(x) = \sum_{k=0}^{\infty} \sigma_k x^k. \tag{37}$$

*Then,* (36) *holds if and only if* $\forall x \geq 0,\ \sigma(x) = \sigma_0$.

We have already shown in the introduction that (36) holds in the Black-Scholes case. So we only have to prove the necessary condition. We decompose the proof into the three following lemmas.

**Lemma 5.2.** *Let us consider a volatility function that satisfies* $(\mathcal{H}_{\mathbf{vol}})$. *If the dual volatility function* $\tilde{\sigma}$ *is analytic in a neighborhood of* $0$, *then the boundaries* $x^*_\sigma$ *and* $y^*_{\tilde{\sigma}}$ *are also analytic in a neighborhood of* $0$.

**Lemma 5.3.** *Let us suppose that* $\sigma$ *satisfies* $(\mathcal{H}_{\mathbf{vol}})$ *and is analytic in a neighborhood of* $0$. *Let us assume moreover that* $r > \delta$. *If the equality* (36) *holds,* $\sigma$ *is constant in a neighborhood of 0:*

$$\exists \rho > 0, \forall y \in [0, \rho], \quad \sigma(y) = \sigma_0.$$

**Lemma 5.4.** *Let us suppose that* $\sigma$ *is a constant function on* $[0, \rho]$ *for* $\rho > 0$ *satisfying* $(\mathcal{H}_{\mathbf{vol}})$ *and* (36). *Then,* $\sigma$ *is constant on* $\mathbb{R}_+$ *(and* $x^*_\sigma$ *and* $y^*_{\tilde{\sigma}}$ *are linear functions).*

*Proof of Lemma 5.2.* Let us first show that $x^*_\sigma$ is analytic in 0. Thanks to the relation (21), we have $\frac{g(y^*_{\tilde\sigma}(x))}{g'(y^*_{\tilde\sigma}(x))} = y^*_{\tilde\sigma}(x) - x$, and therefore $\frac{g(y)}{g'(y)} = y - x^*_\sigma(y)$. Thus, $x^*_\sigma(y)$ is analytic in 0 iff $\phi(y) = \frac{g(y)}{g'(y)}$ is analytic in 0. Using (10) and $\phi' = 1 - \frac{g''}{g'}\phi$, we get that $\phi$ is solution of

$$\phi'(y) = 1 + \frac{2}{\tilde\sigma^2(y)}\left((\delta - r)\phi(y)/y - \delta(\phi(y)/y)^2\right). \tag{38}$$

Notice that $\phi(y) = y - x^*_\sigma(y)$ and (13) imply that if $\phi$ is analytic in 0 then the coefficient of order 0 in its expansion vanishes and the coefficient of order 1 belongs to $(0,1)$.

To complete the proof we are first going to check that if $\psi(y) = \sum_{k=1}^\infty \phi_k y^k$ with $\phi_1 \in (0,1)$ solves (38) in a neighborhood of 0 then $\phi \equiv \psi$ in this neighborhood. Then we will prove existence of such an analytic solution $\psi$. We have $\psi(0) = 0$, and the function $\psi$ being analytic with $\phi_1 \neq 0$, its zeros are isolated points. There is therefore a neighborhood of 0, $(0, 2\epsilon)$ where $\psi$ does not vanish. Let us consider $\gamma$ a solution of $\gamma' - \frac{1}{\psi}\gamma = 0$ starting from $\gamma(\epsilon) \neq 0$ in $\epsilon$: $\gamma(x) = \gamma(\epsilon)\exp(\int_\epsilon^x \frac{1}{\psi(u)}du)$. Since $\psi$ solves (38), it is not hard to check that $\gamma$ is solution of (10) with $\eta = \tilde\sigma$. The limit condition $\gamma(x) \underset{x\to 0}{\to} 0$ is satisfied since we have $\frac{1}{\psi(u)} \underset{u\to 0}{\sim} \frac{1}{\phi_1 u}$ and so $\int_\epsilon^x \frac{1}{\psi(u)}du \underset{x\to 0}{\to} -\infty$. As $g$ is the unique solution to (10) up to a multiplicative constant that satisfies $\lim_{x\to 0} g(x) = 0$ (see Sect. 1.1), there is $c \neq 0$ such that $\gamma(y) = cg(y)$, and thus $\psi(y) = g(y)/g'(y) = \phi(y) = y - x^*_\sigma(y)$. We can then write $x^*_\sigma(y) = (1-\phi_1)y - \sum_{k=2}^\infty \phi_k y^k$ in the neighborhood of 0 with $1 - \phi_1 > 0$. It is well-known that in that case, the reciprocal function $y^*_{\tilde\sigma}$ is also analytic in 0.

Let us turn to the existence of $\psi$. Since $\sigma_0 \geq \underline{\sigma} > 0$, $y \to \frac{2}{\tilde\sigma^2(y)}$ is an analytic function in the neighborhood of 0. Thus, there is $\rho_0 > 0$ and $a_0 > 0$ such that

$$\forall y \in [0, \rho_0], \quad \frac{2}{\tilde\sigma^2(y)} = \sum_{k=0}^\infty a_k y^k \quad \text{and} \quad \sum_{k=0}^\infty |a_k|\rho_0^k < \infty.$$

The analytic function $\sum_{k\geq 1}\phi_k y^k$ solves (38) if and only if

$$\sum_{k=0}^\infty (k+1)\phi_{k+1}y^k = 1 + (\delta - r)\sum_{k=0}^\infty \left(\sum_{i+j=k} a_i\phi_{j+1}\right) y^k - \delta\sum_{k=0}^\infty \left(\sum_{i+j+l=k} a_i\phi_{j+1}\phi_{l+1}\right) y^k.$$

Identifying the terms of order 0, we get that $\phi_1$ solves $P(\phi_1) = 0$ where $P(x) = \delta a_0 x^2 + (1 - (\delta - r)a_0)x - 1$. Since $P(0) = -1 < 0$ and $P(1) = ra_0 > 0$, the polynomial $P$ admits a unique root on $(0,1)$ and we choose $\phi_1$ equal to this root.

Then, by identification of the terms with order $k$, we define the sequence $(\phi_k)_{k\geq 1}$ inductively by

$$\phi_{k+1} = \frac{(\delta - r)\sum_{i+j=k, j\neq k} a_i\phi_{j+1} - \delta\sum_{i+j+l=k, j\neq k, l\neq k} a_i\phi_{j+1}\phi_{l+1}}{k+1+(r-\delta)a_0 + 2\delta a_0\phi_1}.$$

This ratio is well defined since $(r-\delta)a_0 + 2\delta a_0\phi_1 = \delta a_0\phi_1 + 1/\phi_1 - 1 > 0$.

We still have to check that the series $\sum_{k\geq 1}\phi_k y^k$ is defined in a neighborhood of 0. To do so, we are going to show that there is $\rho > 0$ such that the sequence $(|\phi_k|\rho^k)_{k\geq 1}$ is bounded. We have for $1 \leq k \leq n$:

$$|\phi_{k+1}|\rho^k \leq \frac{|\delta - r|\sum_{j=0}^{k-1}|a_{k-j}|\rho^{k-j}|\phi_{j+1}|\rho^j + \delta\sum_{i=0}^{k}\left(\sum_{\substack{j+l=k-i\\ j\neq k, l\neq k}}|\phi_{j+1}|\rho^j|\phi_{l+1}|\rho^l\right)|a_i|\rho^i}{k+1}.$$

Let us suppose that for $1 \leq j < k$, $|\phi_{j+1}|\rho^j \leq 1/(j+1)$. Then,

$$|\phi_{k+1}|\rho^k \leq \frac{|\delta - r|\rho\sum_{j=1}^{k}|a_j|\rho^{j-1} + \delta\sum_{i=0}^{k}\left(\sum_{j+l=k-i}\frac{1}{j+1}\frac{1}{l+1}\right)|a_i|\rho^i}{k+1}.$$

We remark that $\sum_{j+l=k-i}\frac{1}{j+1}\frac{1}{l+1} = \frac{1}{k-i+2}\sum_{j+l=k-i}\frac{1}{j+1}+\frac{1}{l+1} \leq 2\frac{\ln(k-i+1)+1}{k-i+2}$, and we finally get:

$$|\phi_{k+1}|\rho^k \leq \frac{2\delta|a_0|\frac{\ln(k+1)+1}{k+2} + \rho(|\delta - r| + 2\delta)\sum_{j=1}^{k}|a_j|\rho^{j-1}}{k+1} \tag{39}$$

since $\frac{\ln(k-i+1)+1}{k-i+2} \leq 1$. Let us now consider $k_0$ such that $\forall k \geq k_0, 2\delta|a_0|\frac{\ln(k+1)+1}{k+2} < 1/2$. Now, we chose $\rho \in (0, \rho_0)$ small enough such that $\forall k \leq k_0, |\phi_{k+1}|\rho^k \leq 1/(k+1)$ and $\rho(|\delta - r| + 2\delta)\sum_{j=1}^{\infty}|a_j|\rho^{j-1} < 1/2$. Then we get by induction from (39) that $\forall k \geq k_0, |\phi_{k+1}|\rho^k \leq 1/(k+1)$. □

*Proof of Lemma 5.3.* On the one hand, thanks to the assumption, $\sigma = \tilde\sigma$ is analytic in 0, and therefore $x^*_\sigma$ is analytic in 0 thanks to Lemma 5.2:

$$\exists\rho > 0, \quad \forall y \in [0,\rho), x^*_\sigma(y) = \sum_{i=1}^{\infty} x_i y^i \quad \text{and} \quad \sigma(y) = \sum_{i=0}^{\infty}\sigma_i y^i.$$

On the other hand, it is not hard then to deduce from (28), $\sigma = \tilde\sigma$ and the differential equation (19) that

$$x^*_\sigma(y)' = \frac{2(y - x^*_\sigma(y))(ry - \delta x^*_\sigma(y))}{y^2\sigma(y)^2}. \tag{40}$$

From Corollary 2.2 and (25), we get

$$x^*_\sigma(y)' = \frac{(y - x^*_\sigma(y))(ry - \delta x^*_\sigma(y))((x^*_\sigma)^{-1})'(y)}{((x^*_\sigma)^{-1}(y) - y)(r(x^*_\sigma)^{-1}(y) - \delta y)}. \tag{41}$$

Now, we consider $n = \inf\{i \geq 2, x_i \neq 0\}$ and suppose it finite. We can get easily that:

$$x_\sigma^*(y) = x_1 y + x_n y^n + \dots \qquad x_\sigma^*(y)' = x_1 + n x_n y^{n-1} + \dots$$
$$(x_\sigma^*)^{-1}(y) = \tfrac{1}{x_1} y - \tfrac{x_n}{x_1^{n+1}} y^n + \dots \quad ((x_\sigma^*)^{-1})'(y) = \tfrac{1}{x_1}(1 - \tfrac{n x_n}{x_1^n} y^{n-1}) + \dots$$

and then

$$(1-x_\sigma^*(y)/y)(r-\delta x_\sigma^*(y)/y) = (1-x_1)(r-\delta x_1) + x_n(2\delta x_1 - (r+\delta))y^{n-1} + \cdots$$
$$\left(\frac{(x_\sigma^*)^{-1}(y)}{y} - 1\right)\left(r\frac{(x_\sigma^*)^{-1}(y)}{y} - \delta\right) = \frac{1}{x_1^2}\Big\{(1-x_1)(r-\delta x_1) + \frac{x_n}{x_1^n}((r+\delta)x_1 - 2r)y^{n-1}\Big\} + \cdots$$

The right hand side of (41) has then the following expansion:

$$x_1\left\{1 + \frac{x_n}{(1-x_1)(r-\delta x_1)}\left[2\delta x_1 - (r+\delta) + \frac{2r}{x_1^n} - \frac{r+\delta}{x_1^{n-1}}\right]y^{n-1} - \frac{n x_n}{x_1^n} y^{n-1}\right\} + \cdots$$

The equality of the terms of order $n-1$ in (41) then leads to:

$$n x_n x_1^{n-1} = \frac{x_n}{(1-x_1)(r-\delta x_1)}\left[2\delta x_1^{n+1} - (r+\delta)x_1^n - (r+\delta)x_1 + 2r\right] - n x_n.$$

Since $x_n \neq 0$ and with a simplification we get

$$n(1 + x_1^{n-1}) = \frac{1}{r-\delta x_1}\left[-2\delta x_1^n + (r-\delta)\sum_{k=1}^{n-1} x_1^k + 2r\right]. \tag{42}$$

In the case $\delta = 0$ this gives $n(1+x_1^{n-1}) = x_1^{n-1} + \cdots + x_1 + 2$ which is not possible because $x_1 \in (0,1)$. When $0 < \delta < r$, we denote $\alpha = r/\delta > 1$ and rewrite (42):

$$\begin{aligned} n(1 + x_1^{n-1})(\alpha - x_1) &= -2x_1^n + (\alpha-1)x_1^{n-1} + \cdots + (\alpha-1)x_1 + 2\alpha \\ &= \alpha - x_1^n + (\alpha - x_1)\frac{1-x_1^n}{1-x_1}. \end{aligned}$$

Therefore, $n(1 + x_1^{n-1}) = \frac{\alpha - x_1^n}{\alpha - x_1} + \frac{1-x_1^n}{1-x_1} < 2\frac{1-x_1^n}{1-x_1}$ because $\beta \mapsto \frac{\beta - x_1^n}{\beta - x_1}$ is decreasing on $(1,\alpha)$ $(x_1^n < x_1)$. To show that this is impossible, we consider $P_n(x) = n(1+x^{n-1}) - 2\sum_{k=0}^{n-1} x^k$. We have $P_n(1) = 0$ and for $x < 1$, $P_n'(x) = n(n-1) x^{n-2} - 2\sum_{k=1}^{n-1} k x^{k-1} = 2\sum_{k=1}^{n-1} k(x^{n-2} - x^{k-1}) < 0$. Thus $P_n$ is positive on $[0,1)$ and $P_n(x_1) > 0$ which is a contradiction. □

*Proof of Lemma 5.4.* It is easy to get from (19) and $\sigma = \tilde{\sigma}$ that

$$x_\sigma^*(y)' = \frac{x_\sigma^*(y)\sigma(x_\sigma^*(y))}{y\sigma(y)}. \tag{43}$$

We have $\sigma(x) = \sigma_0$ for $x \in [0,\rho]$. Since $x_\sigma^*(y)$ solves (43) and $x_\sigma^*(y) \leq y$, $x_\sigma^*(y)' = x_\sigma^*(y)/y$ on $[0,\rho]$. Therefore, $x_\sigma^*(y) = x_1 y$ for $y \in [0,\rho]$. Thanks to (19), $x_1$ is the unique root in $(0, \min(1, r/\delta))$ of

$$x_1\sigma_0^2 = 2(1-x_1)(r-\delta x_1).$$

Now let us observe that (19) gives for $y \in (0, y^*_{\tilde{\sigma}}(\rho)]$, $x^*_\sigma(y)' = \frac{x^*_\sigma(y)^2\sigma_0^2}{2(y-x^*_\sigma(y))(ry-\delta x^*_\sigma(y))}$ with $x^*_\sigma(\rho) = x_1\rho$. Since $y \to x_1 y$ solves this ODE, for which local uniqueness holds thanks to the Cauchy Lipschitz theorem, we then have $x^*_\sigma(y) = x_1 y$ on $[\rho, y^*_{\tilde{\sigma}}(\rho)]$ and so $y^*_{\tilde{\sigma}}(\rho) = (x^*_\sigma)^{-1}(\rho) = \rho/x_1$. Then, (43) gives $\sigma_0/\sigma(y) = 1$ on $[\rho, \rho/x_1]$. Thus, we prove by induction on $n$ that $x^*_\sigma(y) = x_1 y$ and $\sigma(y) = \sigma_0$ for $y \in [0, \rho/(x_1)^n]$. This shows the desired result. □

## 6. Conclusion

In this paper, we have derived a Dupire-type Put Call duality equality for perpetual American options. Like Dupire's formula for European options, this equality permits the calibration of the time-homogeneous local volatility function in the underlying model from the prices of perpetual Put and Call options. For a given value $x_0 > 0$ of the underlying, one respectively recovers the restrictions of the volatility function to $(0, x_0]$ and to $[x_0, +\infty)$ from the prices of the perpetual Put options and the prices of the perpetual Call options.

Addressing Dupire like Call–Put duality for American options with finite maturity in models with time-dependent local volatility functions like (1) would be of great interest for calibration purposes. If $P(T, x, y)$ denotes the initial price of the American Put option with maturity $T$ and strike $y$ in the model (1) and $x^*(T, y)$ stands for the corresponding exercise boundary such that $P(T, x, y) = (y - x)^+$ if and only if $x \leq x^*(T, y)$, then the smooth-fit principle writes

$$\begin{cases} P(T, x^*(T, y), y) = y - x^*(T, y) \\ \partial_x P(T, x^*(T, y), y) = -1 \end{cases}.$$

Differentiating the former equality with respect to $y$ yields

$$\partial_x P(T, x^*(T, y), y)\partial_y x^*(T, y) + \partial_y P(T, x^*(T, y), y) = 1 - \partial_y x^*(T, y).$$

With the second equality, one deduces that $\partial_y P(T, x^*(T, y), y) = 1$. Therefore the smooth-fit principle automatically holds for the dual Call option if there exists any. In spite of this encouraging remark, we have not been able so far to treat the finite maturity case. According to our numerical experiments (see Fig. 2), American Put and Call prices computed in infinite maturity dual models may differ for finite maturities. This means that in the case of a time-homogeneous primal local volatility function $\varsigma(t, x) = \sigma(x)$, if there exists a dual local volatility function for some finite maturity $T$, then this volatility function is either time-dependent or depends on the maturity $T$.

We have not been more successful in the a priori simpler case of an exponentially distributed random maturity independent from $(W_t)_{t \geq 0}$. Then, unlike in the perpetual case, the price of the option no longer writes as the product of a function of the underlying spot price by another function of the strike price in the continuation region. As a consequence, we did not succeed in generalizing our first key result for the perpetual case: the derivation of an ODE for the exercise boundary considered as a function of the strike variable.

So the only two generalizations of the theory presented in this paper that we have been able to work out are limited to the perpetual maturity case. First, in [1], we extend our Dupire-type duality by replacing the Put-Call payoff function $(y-x)^+$ by a nonnegative and continuous payoff function $\phi(x,y)$ such that $\Phi = \{(x,y) : \phi(x,y) > 0\} \neq \emptyset$, $\phi$ is $\mathcal{C}^2$ on $\Phi$ and

$$\forall (x,y) \in \Phi,\ \partial_x \phi(x,y) < 0,\ \partial_y \phi(x,y) > 0,\ \partial^2_{xx}\phi(x,y) \leq 0 \quad \text{and} \quad \partial^2_{yy}\phi(x,y) \leq 0.$$

Unfortunately, for such general payoff functions, it may happen that no dual local volatility function $\eta$ can be associated with $\sigma$. Secondly, in the Appendix B of the present paper, we briefly explain how to deal with a dividend rate $\delta(x)$ depending on the underlying spot price under reinforced regularity assumptions on the local volatility functions. Essentially, we are able to generalize our theory when the function $\delta$ is nonincreasing and such that $x \mapsto x\delta(x)$ is increasing.

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## Appendix A: Proofs of technical results

*Proof of Theorem 1.2.* We give here a direct proof that generalizes the approach developed by Beibel and Lerche [3] in the Black-Scholes case. Let us define:

$$\forall z > 0,\ h(z) = \frac{(y-z)^+}{f(z)} \quad \text{and} \quad h^* = \sup_{z>0} h(z).$$

The function $h$ is continuous such that $h(z) = 0$ for $z \geq y$ and that $h(0+) = 0$ since we know from [4] that $\lim_{0+} f = +\infty$. Therefore, $x^*_\sigma(y) \stackrel{\text{def}}{=} \sup\{z > 0,\ h(z) = h^*\}$ belongs to $(0,y)$ and is such that $h(x^*_\sigma(y)) = h^*$. Since the function $h$ is $\mathcal{C}^2$ on $(0,y)$, we have $h'(x^*_\sigma(y)) = 0$ and $h''(x^*_\sigma(y)) \leq 0$. These conditions give easily

$$f(x^*_\sigma(y)) + (y - x^*_\sigma(y))f'(x^*_\sigma(y)) = 0 \quad \text{and} \quad f''(x^*_\sigma(y)) \geq 0.$$

Since $f$ is positive and $x^*_\sigma(y) < y$, we have $f'(x^*_\sigma(y)) < 0$ and deduce (12). The second order condition and Eq. (9) then give $x^*_\sigma(y)(r-\delta)f'(x^*_\sigma(y)) - rf(x^*_\sigma(y)) \leq 0$ and so

$$ry - \delta x^*_\sigma(y) \geq 0. \tag{44}$$

Now let us check the optimality of $\tau_x^P$ and consider $\tau \in \mathcal{T}_{0,\infty}$. By Fatou's lemma and Doob's optional sampling theorem, we have

$$\begin{aligned}\mathbb{E}[e^{-r\tau}(y - S_\tau^x)^+] &\le \liminf_{t\to+\infty} \mathbb{E}\left[e^{-r\tau\wedge t}(y - S_{\tau\wedge t}^x)^+\right] \\ &= \liminf_{t\to+\infty} \mathbb{E}\left[e^{-r\tau\wedge t} f(S_{\tau\wedge t}^x) h(S_{\tau\wedge t}^x)\right] \\ &\le h(x_\sigma^*(y)) \liminf_{t\to+\infty} \mathbb{E}\left[e^{-r\tau\wedge t} f(S_{\tau\wedge t}^x)\right] \le h(x_\sigma^*(y)) f(x)\end{aligned}$$

since $e^{-rt} f(S_t^x) = f(x) + \int_0^t e^{-ru}\sigma(S_u^x) S_u^x f'(S_u^x) dW_u$ is a nonnegative local martingale and therefore a supermartingale. If $x \ge x_\sigma^*(y)$, we have

$$\mathbb{E}[e^{-r\tau_x^P}(y - S_{\tau_x^P}^x)^+] = (y - x_\sigma^*(y))\mathbb{E}[e^{-r\tau_x^P}] = h(x_\sigma^*(y)) f(x).$$

Indeed, the latter expectation is equal to $f(x)/f(x_\sigma^*(y))$ by Doob's optional sampling theorem applied to the non-negative martingale $(e^{-rt\wedge\tau_x^P} f(S_{t\wedge\tau_x^P}^x))_{t\ge 0}$ which is bounded by $f(x_\sigma^*(y))$ since $f$ is non-increasing. Thus, $\tau_x^P$ is optimal for $x \ge x_\sigma^*(y)$. Since $x_\sigma^*(y) = \sup\{z > 0,\ h(z) = h^*\}$, we have $(y - x)^+ = h(x) f(x) < f(x) h(x_\sigma^*(y))$ for $x > x_\sigma^*(y)$, and finally deduces (11) for $x \ge x_\sigma^*(y)$.

We consider now the complementary case $x \in (0, x_\sigma^*(y))$, and set $\tau \in \mathcal{T}_{0,\infty}$. Using the strong Markov property and the optimality result when the initial spot is $x_\sigma^*(y)$, we get

$$\mathbb{E}[e^{-r\tau}(y - S_\tau^x)^+] \le \mathbb{E}[e^{-r\tau\wedge\tilde\tau}(y - S_{\tau\wedge\tilde\tau}^x)^+], \text{ where } \tilde\tau = \inf\{t \ge 0, S_t^x = x_\sigma^*(y)\}.$$

On $\{t < \tilde\tau\}$, we have $S_t^x < x_\sigma^*(y)$, $de^{-rt}(y - S_t^x) = e^{-rt}(\delta S_t^x - ry)dt - e^{-rt}\sigma(S_t^x) S_t^x\, dW_t$, where the first term in the right-hand-side is non positive by (44). Thus, $\mathbb{E}[e^{-r\tau\wedge\tilde\tau}(y - S_{\tau\wedge\tilde\tau}^x)^+] \le \liminf_{t\to+\infty} \mathbb{E}[e^{-r\tau\wedge\tilde\tau\wedge t}(y - S_{\tau\wedge\tilde\tau\wedge t}^x)^+] \le (y - x)$.

It remains to show (13). By $(\mathcal{H}_{\mathbf{vol}})$, $0 < \underline{\sigma} \le \sigma(x) \le \overline{\sigma} < +\infty$. Thanks to Proposition 1.4, we have:

$$\forall x > 0, \quad (y - x)^+ \le P_{\underline{\sigma}}(x, y) \le P_\sigma(x, y) \le P_{\overline{\sigma}}(x, y). \tag{45}$$

Thanks to Proposition 1.5, $P_{\overline{\sigma}}(x, y) = (y - x)$ for $x \le \frac{a(\overline{\sigma})}{a(\overline{\sigma})-1} y$ and $P_{\underline{\sigma}}(x, y) > (y - x)^+$ for $x > \frac{a(\underline{\sigma})}{a(\underline{\sigma})-1} y$. Since $x_\sigma^*(y) = \sup\{x > 0, P_\sigma(x, y) = (y - x)^+\}$, we get from (45) that $x_\sigma^*(y) \in [\frac{a(\overline{\sigma})}{a(\overline{\sigma})-1} y, \frac{a(\underline{\sigma})}{a(\underline{\sigma})-1} y]$. Let us observe now that $\frac{a(\underline{\sigma})}{a(\underline{\sigma})-1}$ is a root of $Q(z) = \delta z^2 - (r + \delta + \underline{\sigma}^2/2)z + r$. As $Q(z) = 0 \iff \underline{\sigma}^2 z = 2(1 - z)(r - \delta z)$ and since $a(\underline{\sigma})/(a(\underline{\sigma}) - 1) \in (0, 1)$, we then deduce that $a(\underline{\sigma})/(a(\underline{\sigma}) - 1) < r/\delta$. □

*Proof of Theorem 1.3.* The proof works like for the Put, and we just hint the differences. We define

$$\forall z > 0, h(z) = \frac{(z - x)^+}{g(z)} \quad \text{and} \quad h^* = \sup_{z>0} h(z).$$

Let us admit for a while that $\lim_{z\to+\infty} h(z) = 0$. Then, since $h(z) = 0$ for $z \le x$ and $h$ is continuous, $h$ reaches its maximum in $y_\eta^*(x) \stackrel{\text{def}}{=} \inf\{z > 0,\ h(z) = h^*\}$, and

$y^*_\eta(x) \in (x,\infty)$. Then $h'(y^*_\eta(x)) = 0$ and $h''(y^*_\eta(x)) \le 0$ give (15) and $ry^*_\eta(x) - \delta x \ge 0$. Optimality of $\tau^c_y$ is obtained like in the Put case.

Now, let us check that $\lim_{z\to+\infty} h(z) = 0$. It's Formula gives

$$de^{-\delta t}(\bar S^1_t)^{1+a} = e^{-\delta t}(\bar S^1_t)^{1+a}\left\{(a+1)\eta(\bar S^1_t)dW_t + [a(\delta + (a+1)\eta^2(\bar S^1_t)/2) \right. \\ \left. -(a+1)r]dt\right\}.$$

When $a > 0$, the drift term is bounded from above by $a(\delta + (a+1)\overline{\eta}^2/2) - (a+1)r$ and we can find $a > 0$ such that this bound is negative since $a(\delta + (a+1)\overline{\eta}^2/2) - (a+1)r \underset{a\to 0}{\to} -r < 0$. Let $y \ge 1$. Defining $\bar\tau_y = \inf\{t \ge 0,\ \bar S^1_t \ge y\}$, we have $\mathbb{E}[e^{-\delta\bar\tau_y\wedge t}(\bar S^1_{\bar\tau_y\wedge t})^{1+a}] \le 1$ thanks to Doob's optional sampling theorem. The Fatou lemma then gives $\mathbb{E}[e^{-\delta\bar\tau_y}(\bar S^1_{\bar\tau_y})^{1+a}] \le 1$. Therefore, $\mathbb{E}[e^{-\delta\bar\tau_y}] \le 1/y^{1+a}$ where by convention $\mathbb{E}[e^{-\delta\bar\tau_y}] = \mathbb{P}(\bar\tau_y < \infty)$ if $\delta = 0$. This expectation is equal to $g(1)/g(y)$ by Doob's optional sampling theorem applied to the non-negative martingale $(e^{-\delta t\wedge\bar\tau_y} g(\bar S^1_{t\wedge\bar\tau_y}))_{t\ge 0}$ which is bounded by $g(y)$ since $g$ is non-decreasing. This shows $\lim_{z\to+\infty} h(z) = 0$. □

**Lemma A.1.** *The function $f'$ (resp. $g'$) is negative (resp. positive) and $f''$ (resp. $g''$) is positive on $(0,+\infty)$. Moreover, the boundaries $x^*_\sigma(y)$ and $y^*_\eta(x)$ are respectively the unique solution to $y-x+f(x)/f'(x) = 0$ and $y-x-g(y)/g'(y) = 0$. Last, $x^*_\sigma(y)$, $\alpha(y)$, $y^*_\eta(x)$ and $\beta(x)$ are $\mathcal{C}^1$ functions on $\mathbb{R}^*_+$.*

*Proof.* We only give the proof in the Put case, the argument being similar for the Call (When $\delta = 0$, $g'$ is positive since this function is equal to $\exp(\int_1^x \frac{2r}{y\eta^2(y)}dy)$ up to a positive constant factor). By (9), for $x > 0$, $f''(x)$ has the same sign as $h(x) = rf(x) + (\delta - r)xf'(x)$. If for some $x > 0$, $f'(x) = 0$, then since $f$ is positive, $f''(x) > 0$. Therefore $x$ is a local minimum point of $f$ which contradicts the decreasing property of this function. Hence $f'$ is a negative function.

When $\delta \le r$, $h$ and therefore $f''$ are positive functions. When $\delta > r$, we remark that if $f''(x) = 0$ then $h'(x) = \delta f'(x) < 0$. Since the continuous function $f''$ and $h$ have the same sign, this implies that

$$\forall x > \inf\{z > 0 : f''(z) \le 0\}, \quad f''(x) < 0. \tag{46}$$

Now for $y > 0$, by (9) then (21), we have

$$\frac{x^*_\sigma(y)^2\sigma(x^*_\sigma(y))^2}{2}\frac{f''(x^*_\sigma(y))}{f'(x^*_\sigma(y))} = r\frac{f(x^*_\sigma(y))}{f'(x^*_\sigma(y))} - (r-\delta)x^*_\sigma(y) = \delta x^*_\sigma(y) - ry. \tag{47}$$

By (13), the right-hand-side is negative and moreover $\lim_{y\to+\infty} x^*_\sigma(y) = +\infty$. Hence $\sup\{z > 0 : f''(z) > 0\} = +\infty$ and with (46), we conclude that $f'' > 0$.

According to (21), $F(x^*_\sigma(y), y) = 0$ where

$$F(x,y) = y - x + f(x)/f'(x).$$

The function $F$ is $\mathcal{C}^1$ on $(0,+\infty)\times(0,+\infty)$ and such that

$$\forall x,y > 0,\ \partial_x F(x,y) = -f(x)f''(x)/f'(x)^2 < 0.$$

Therefore for fixed $y > 0$, $x^*(y)$ is the unique solution to $F(x,y) = 0$. Moreover, $y \to x^*(y)$ is $C^1$ by the implicit function theorem. Last, one deduces from (21) that $\alpha(y)$ is a $C^1$ function. □

## Appendix B: Extension to spot-dependent dividend rates

In this section, we extend the Dupire-type duality result for perpetual American Options when the stock follows the following dynamics with a spot-dependent dividend rate:

$$dS_t^x = S_t^x((r - \delta(S_t^x))dt + \sigma(S_t^x)dW_t), \quad S_0^x = x > 0. \tag{48}$$

Let $P_\sigma(x,y) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}[e^{-r\tau}(y - S_\tau^x)^+]$ denote the perpetual Put price. Our purpose is to find a dual volatility function $\eta$ such that

$$\forall x, y > 0, \quad P_\sigma(x,y) = c_\eta(y,x)$$

where $c_\eta(y,x) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}[e^{-\int_0^\tau \delta(\overline{S}_t^y)dt}(\overline{S}_\tau^y - x)^+]$ denotes the perpetual Call price in the dual model: $d\overline{S}_t^y = \overline{S}_t^y((\delta(\overline{S}_t^y) - r)dt + \eta(\overline{S}_t^y)dW_t)$, $\overline{S}_0^y = y$.

We also define

$$x_\sigma^*(y) = \sup\left\{x > 0, P_\sigma(x,y) = (y-x)^+\right\} \le y, \tag{49}$$

$$y_\eta^*(x) = \inf\left\{y > 0, c_\eta(y,x) = (y-x)^+\right\} \ge x. \tag{50}$$

At this stage, it is not clear that $P_\sigma(x,y)$ (resp. $c_\eta(y,x)$) is equal to $(y-x)^+$ for all $x \in (0, x_\sigma^*(y)]$ (resp. $y \in [y_\eta^*(x), +\infty)$).

We make the following assumptions:

$$\sigma, \eta \in \left\{f : \mathbb{R}_+^* \to \mathbb{R}_+^*, \mathcal{C}^1, \overline{f} < \infty\right\} \tag{51}$$

$$\delta : \mathbb{R}_+^* \to \mathbb{R}_+, \mathcal{C}^1, \forall x > 0, x\delta'(x) + \delta(x) > 0, \tag{52}$$

$$\overline{\delta} - \underline{\delta} < r, \tag{53}$$

but no longer suppose $\underline{\sigma} > 0$.

Let $\mathbf{P}(x,y)$ (resp. $\mathbf{c}(y,x)$) be the price of a perpetual American Put (resp. Call) option with spot $x$ (resp. $y$), strike $y$ (resp. $x$), volatility function $\sigma$ (resp. $\eta$), interest rate $r$ (resp. $\underline{\delta}$) and dividend rate $\overline{\delta}$ (resp. $r - (\overline{\delta} - \underline{\delta})$). Let also $\mathbf{x}^*(y)$ (resp. $\mathbf{y}^*(x)$) denote the corresponding exercise boundary. We have the following comparison result.

**Proposition B.1.** *Under assumptions (51)–(53), one has*

$$\forall x, y > 0, \quad P_\sigma(x,y) \le \mathbf{P}(x,y) \quad \text{and} \quad c_\eta(y,x) \le \mathbf{c}(y,x), \tag{54}$$

$$\exists c \in (0,1), \forall y > 0, \ x_\sigma^*(y) \ge \mathbf{x}^*(y) \ge cy \quad \text{and} \quad \forall x > 0, \ y_\eta^*(x) \le \mathbf{y}^*(x) \le x/c. \tag{55}$$

*Proof.* Let $(\mathbf{S}_t^x, t \geq 0)$ (resp. $(\overline{\mathbf{S}}_t^y, t \geq 0)$) denote the solution of $dS_t = S_t((r - \overline{\delta})dt + \sigma(S_t)dW_t)$ starting from $x$ (resp. $d\overline{S}_t = \overline{S}_t((\overline{\delta} - r)dt + \eta(\overline{S}_t)dW_t)$ starting from $y$). Thanks to (51) and (52), by an easy adaptation of Proposition 2.18 [15, p. 293] to the case of locally Lipschitz continuous coefficients, we have a.s. $\forall t \geq 0, S_t^x \geq \mathbf{S}_t^x$ (resp. $\overline{S}_t^y \leq \overline{\mathbf{S}}_t^y$). Therefore, for any stopping time $\tau$:

$$e^{-r\tau}(y - S_\tau^x)^+ \leq e^{-r\tau}(y - \mathbf{S}_\tau^x)^+, a.s.$$
$$(resp.\ e^{-\int_0^\tau \delta(\overline{S}_t^y)dt}(\overline{S}_\tau^y - x)^+ \leq e^{-\underline{\delta}\tau}(\overline{\mathbf{S}}_\tau^y - x)^+,\ a.s.).$$

Thanks to (53), we deduce immediately (54) and thus $\forall y > 0, x_\sigma^*(y) \geq \mathbf{x}^*(y)$ (resp. $\forall x > 0, y_\eta^*(x) \leq \mathbf{y}^*(x)$). We conclude using the inequality that is derived from the upper bound of the volatility function in (13) (resp. (16)). □

Thanks to this result, the exercise regions are non empty. Following [4], we introduce $f$ (resp. $g$) the unique, up to a multiplicative constant, positive decreasing (resp. increasing) solution to $\frac{1}{2}\sigma^2(x)x^2 f''(x) + (r - \delta(x))xf'(x) - rf(x) = 0$, $x > 0$ (resp. $\frac{1}{2}\eta^2(x)x^2 g''(x) + (\delta(x) - r)xg'(x) - \delta(x)g(x) = 0, x > 0$). Following the second part of the proof of Theorem 1.2, we obtain the two first statements in the following proposition.

**Proposition B.2.** *The pricing formula* (11) *in Theorem* 1.2 (*resp.* (14) *in Theorem* 1.3) *holds for* $x \geq x_\sigma^*(y)$ (*resp.* $y \leq y_\eta^*(x)$). *Moreover,* (12) *and* (15) *hold. Last,* $f''(x_\sigma^*(y)) \geq 0,\ g''(y_\eta^*(x)) \geq 0$.

*Proof.* Let us check that $f''(x_\sigma^*(y)) \geq 0$. Setting $h(x) = \frac{y-x}{f(x)}$, we have $h(x_\sigma^*(y)) \geq h(x)$ for $x \geq x_\sigma^*(y)$ because $P_\sigma(x, y) \geq y - x$. An easy computation using (12) gives $h'(x_\sigma^*(y)) = 0$. We deduce $h''(x_\sigma^*(y)) \leq 0$, which combined with (12) leads to $f''(x_\sigma^*(y)) \geq 0$. □

From now on, we make the additional assumption

$$\forall x > 0,\ rg'(x) + \delta'(x)(g(x) - xg'(x)) > 0, \tag{56}$$

which is automatically satisfied when $\delta'(x) \leq 0$ thanks to the next result.

**Corollary B.3.** *We have* $\forall y > 0, g(y) - yg'(y) < 0$.

*Proof.* Let us introduce the function $z(y) = g(y)/y$. We have $z'(y) = (yg'(y) - g(y))/y^2$, and it is sufficient to show that $z' > 0$. Thanks to the ODE satisfied by $g$, when $z'(y) = 0$ then $\frac{1}{2}y^3\eta(y)^2 z''(y) = ryz(y) > 0$. Therefore, $z'$ can vanish at most one time on $(0, +\infty)$ and $z'$ is positive on $(\inf\{y > 0 : z'(y) > 0\}, +\infty)$. By (15), $z'(y_\eta^*(x)) = xg'(y_\eta^*(x))/(y_\eta^*(x))^2 > 0$. With (55), one concludes that $\inf\{y : z'(y) > 0\} = 0$. □

**Lemma B.4.** *The functions* $-f'$, $g'$, $f''$ *and* $g''$ *are positive on* $(0, +\infty)$. *Moreover, the boundaries* $x_\sigma^*(y)$ *and* $y_\eta^*(x)$ *are respectively the unique solution to* $y - x + f(x)/f'(x) = 0$ *and* $y - x - g(y)/g'(y) = 0$. *Last,* $x_\sigma^*(y)$, $y_\eta^*(x)$ *are* $\mathcal{C}^1$ *functions on* $\mathbb{R}_+^*$ *and satisfy:*

$$\forall y > 0,\ ry > \delta(x_\sigma^*(y))x_\sigma^*(y),\ \forall x > 0,\ ry_\eta^*(x) > \delta(y_\eta^*(x))x. \tag{57}$$

*Proof.* The positivity of $g'$ is a consequence of Corollary B.3 and the negativity of $f'$ is obtained like in the first step of the proof of Lemma A.1. The function $f''$ (resp. $g''$) has the same sign as $h(x) = rf(x) + (\delta(x) - r)xf'(x)$ (resp. $k(x) = \delta(x)g(x) + (r - \delta(x))xg'(x)$). The proof is similar to the one of Lemma A.1 (the nonnegativity of $f''(x^*_\sigma(y))$ and $g''(y^*_\eta(x))$ being given by the Proposition B.2), and the Assumption (52) (resp. (56)) just ensures that when $f''(x) = 0$ (resp. $g''(x) = 0$), $h'(x) = f'(x)(\delta(x) + x\delta'(x)) < 0$ (resp. $k'(x) = rg'(x) + \delta'(x)g(x) - \delta'(x)xg'(x) > 0$). Last, (57) comes immediately from the positivity of $f''$ and $g''$, writing respectively the ODEs satisfied by $f$ and $g$ at the points $x^*_\sigma(y)$ and $y^*_\eta(x)$. □

**Corollary B.5.** *For $x \le x^*_\sigma(y)$ (resp. $y \ge y^*_\eta(x)$), we have $P_\sigma(x, y) = y - x$ (resp. $c_\eta(y, x) = y - x$). In particular $x \mapsto P_\sigma(x, y)$ (resp. $y \mapsto c_\eta(y, x)$) is a convex function.*

*Proof.* For any $y > 0$, $x^*_\sigma(y)$ is the unique solution to $y - x + f(x)/f'(x) = 0$, which excludes any other point where the smooth-fit holds. Therefore $\{x > 0, P_\sigma(x, y) = (y - x)^+\}$ is either equal to $(0, x^*_\sigma(y)]$ or $\{x^*_\sigma(y)\}$, but the latter case is not possible according to (54). The proof is similar for the Call. □

**Theorem B.6.** *The boundaries $x^*_\sigma(y)$ and $y^*_\eta(x)$ are respectively the unique increasing solutions defined on $(0, +\infty)$ of the ODEs*

$$(x^*_\sigma)'(y) = \frac{x^*_\sigma(y)^2\sigma(x^*_\sigma(y))^2}{2(y - x^*_\sigma(y))(ry - \delta(x^*_\sigma(y))x^*_\sigma(y))},$$

$$(y^*_\eta)'(x) = \frac{\eta^2(y^*_\eta(x))y^*_\eta(x)^2}{2(y^*_\eta(x) - x)(ry^*_\eta(x) - \delta(y^*_\eta(x))x)} \tag{58}$$

*satisfying $\forall y > 0,\ cy \le x^*_\sigma(y) < y$ and $\forall x > 0,\ x < y^*_\eta(x) \le dx$ with $0 < c < 1 < d < +\infty$.*

*Proof.* The proof is similar to the one of Theorem 2.1, but the uniqueness of $x^*_\sigma$ cannot be deduced as easily as before from the uniqueness of $y^*_\eta$. Let us consider $x(y)$ another increasing solution of the ODE satisfied by $x^*_\sigma(y)$ that satisfies the required bounds. The function $\phi(y) = x^{-1}(x^*_\sigma(y))/y$ solves:

$$\phi'(y) = \frac{1}{y}\left[\frac{\phi(y) - x^*_\sigma(y)/y}{1 - x^*_\sigma(y)/y} \times \frac{r\phi(y)y - \delta(x^*_\sigma(y))x^*_\sigma(y)}{ry - \delta(x^*_\sigma(y))x^*_\sigma(y)} - \phi(y)\right].$$

Thanks to the Cauchy-Lipschitz theorem, three cases are possible: $\phi \equiv 1$, $\phi > 1$ and $\phi < 1$. If $\phi > 1$, according to (57), the second fraction is greater than 1 and we have: $\phi'(y) > \frac{1}{y}[\frac{\phi(y) - x^*_\sigma(y)/y}{1 - x^*_\sigma(y)/y} - \phi(y)] = \frac{\phi(y) - 1}{y}\frac{x^*_\sigma(y)/y}{1 - x^*_\sigma(y)/y} \ge c\frac{\phi(y) - 1}{y} > 0$, using (55). Therefore, $\phi$ is increasing and we have for $y \ge 1$, $\phi(y) - \phi(1) \ge c(\phi(1) - 1)\ln(y) \to +\infty$ when $y \to +\infty$. This is in contradiction with the bounds on $x$ and $x^*_\sigma$. We get a similar contradiction when $\phi < 1$, and deduce that $\phi \equiv 1$. □

We are now able to state a Dupire-type duality result similar to Theorem 3.1.

**Theorem B.7.** *Under assumptions (51)–(53) and (56), the following conditions are equivalent:*

1.
$$\forall x, y > 0,\ P_\sigma(x,y) = c_\eta(y,x). \tag{59}$$

2. $x^*_\sigma$ *and* $y^*_\eta$ *are reciprocal functions:* $\forall x > 0,\ x^*_\sigma(y^*_\eta(x)) = x$.
3. $\forall y > 0, x^*_\sigma(y) < ry/\delta(y)$ *and*
$$\eta(y) = \frac{2(y - x^*_\sigma(y))\sqrt{(ry - \delta(x^*_\sigma(y))x^*_\sigma(y))(ry - \delta(y)x^*_\sigma(y))}}{y x^*_\sigma(y)\sigma(x^*_\sigma(y))}. \tag{60}$$
4. $\forall x > 0, y^*_\eta(x) > \delta(x)x/r$ *and*
$$\sigma(x) = \frac{2(y^*_\eta(x) - x)\sqrt{(ry^*_\eta(x) - \delta(x)x)(ry^*_\eta(x) - \delta(y^*_\eta(x))x)}}{x y^*_\eta(x)\eta(y^*_\eta(x))}. \tag{61}$$

*Remark* B.8. In the proportional dividend rate case, according to Remark 3.2, existence of a dual volatility function $\eta$ satisfying $(\mathcal{H}_{\mathbf{vol}})$ is guaranteed as soon as $\sigma$ satisfies $(\mathcal{H}_{\mathbf{vol}})$. Here, in the general case, either the existence of the right-hand-side of (60) may fail (if for some $y > 0$, $x^*_\sigma(y) > ry/\delta(y)$) or this right-hand-side may not satisfy (51). Nonetheless, if $\delta$ is a nonincreasing function, then (57) implies $x^*_\sigma(y) < ry/\delta(y)$ for all $y > 0$. If moreover $\underline{\sigma} > 0$, then the function defined by the right-hand-side of (60) satisfies (51). Therefore under both assumptions, the duality holds and it is possible to calibrate the local volatility function $\sigma$ under the spot level from the perpetual Put prices like in the proportional dividend rate case.

*Remark* B.9. In contrast with the proportional dividend case, the duality between $C_\sigma(x,y) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}[e^{-r\tau}(S^x_\tau - y)^+]$ and $p_\eta(y,x) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}[e^{-\int_0^\tau \delta(\overline{S}^y_t)dt}(x - \overline{S}^y_\tau)^+]$ is not mathematically the same as the one between $P_\sigma(x,y)$ and $c_\eta(y,x)$. Let $g_\downarrow$ denote the positive nonincreasing solution to $\frac{1}{2}\eta^2(x)x^2 g''(x) + (\delta(x) - r)xg'(x) - \delta(x)g(x) = 0, x > 0$ ([4]). Let us assume (51), (52) and

$$\underline{\delta} > 0,\ \forall x > 0, rg'_\downarrow(x) + \delta'(x)(g_\downarrow(x) - xg'_\downarrow(x)) < 0. \tag{62}$$

Then, we can show that $C_\sigma(x,y) \le \mathbf{C}(x,y)$ (resp. $p_\eta(y,x) \le \mathbf{p}(y,x)$) where $\mathbf{C}(x,y)$ (resp. $\mathbf{p}(y,x)$) is the perpetual American Call (resp. Put) price with interest rate $r$ (resp. $\underline{\delta}$), dividend rate $\underline{\delta}$ (resp. $r$), spot $x$ (resp. $y$), strike $y$ (resp $x$) and volatility function $\sigma$ (resp. $\eta$). Similarly, the exercise boundary $\Upsilon^*_\sigma(y)$ (resp. $\xi^*_\eta(x)$) exists for $C_\sigma(x,y)$ (resp. $p_\eta(y,x)$) and solves the ODE:

$$\Upsilon^*_\sigma(y)' = \frac{\Upsilon^*_\sigma(y)^2\sigma(\Upsilon^*_\sigma(y))^2}{2(\Upsilon^*_\sigma(y) - y)(\delta(\Upsilon^*_\sigma(y))\Upsilon^*_\sigma(y) - ry)}$$
$$\left(\textit{resp. } \xi^*_\eta(x)' = \frac{\xi^*_\eta(x)^2\eta(\xi^*_\eta(x))^2}{2(x - \xi^*_\eta(x))(\delta(\xi^*_\eta(x))x - r\xi^*_\eta(x))}\right).$$

Last, we obtain equivalence between:

1. 
$$\forall x, y > 0,\ C_\sigma(x, y) = p_\eta(y, x). \tag{63}$$

2. $\Upsilon^*_\sigma$ and $\xi^*_\eta$ are reciprocal functions: $\forall x > 0,\ \Upsilon^*_\sigma(\xi^*_\eta(x)) = x$.
3. $\forall y > 0, \Upsilon^*_\sigma(y) > ry/\delta(y)$ and
$$\eta(y) = \frac{2(\Upsilon^*_\sigma(y) - y)\sqrt{(\delta(y)\Upsilon^*_\sigma(y) - ry)(\delta(\Upsilon^*_\sigma(y))\Upsilon^*_\sigma(y) - ry)}}{y\Upsilon^*_\sigma(y)\sigma(\Upsilon^*_\sigma(y))}. \tag{64}$$
4. $\forall x > 0, \xi^*_\eta(x) < x\delta(x)/r$ and
$$\sigma(x) = \frac{2(x - \xi^*_\eta(x))\sqrt{(\delta(\xi^*_\eta(x))x - r\xi^*_\eta(x))(\delta(x)x - r\xi^*_\eta(x))}}{x\xi^*_\eta(x)\eta(\xi^*_\eta(x))}. \tag{65}$$

When $\delta$ in nonincreasing and $\underline{\sigma} > 0$, then the right-hand-side of (64) exists and satisfies (52). As a consequence, the duality holds, and it is possible to calibrate the local volatility function $\sigma$ above the spot level from the perpetual Call prices.

*Example* B.10. Let $\delta(x) = \delta_0 + \alpha/(x + 1)$ with $0 < \delta_0$, $0 \leq \alpha < r$. The function $\delta$ is decreasing, satisfies conditions (51)–(53), (56) and (62). When $\underline{\sigma} > 0$, the Dupire-type duality holds for $P_\sigma$ and $C_\sigma$ and it is possible to calibrate exactly the whole volatility function $\sigma$ from the perpetual Put and Call prices.

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