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The large-time behavior of solutions of the Cauchy-Dirichlet problem for Hamilton-Jacobi equations

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Abstract. We investigate the large-time behavior of viscosity solutions of the Cauchy-Dirichlet problem (CD) for Hamilton-Jacobi equations on bounded domains. We establish general convergence results for viscosity solutions of (CD) by using the Aubry-Mather theory.

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1. Introduction

In this paper, we investigate the large-time behavior of solutions of the Cauchy-Dirichlet problem for the Hamilton-Jacobi equation:

$$\text{(CD)}\quad \begin{cases} u_t(x,t) + H(x, Du(x,t)) = 0 & \text{in } Q, & (1.1)\\ u(x,t) = f(x) & \text{in } \Omega \times \{0\}, & (1.2)\\ u(x,t) = g(x) & \text{on } \partial\Omega \times (0,\infty), & (1.3)\end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $Q := \Omega \times (0,\infty)$, $H = H(x,p)$ is a real-valued function on $\overline{\Omega} \times \mathbb{R}^n$ which is coercive and convex in the variable p, $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is the unknown function, $u_t := \partial u/\partial t, Du := (\partial u/\partial x_1, \dots, \partial u/\partial x_n)$, and $f : \overline{\Omega} \to \mathbb{R}$, $g : \partial\Omega \to \mathbb{R}$ are given functions. The function H will be called the Hamiltonian. We will be dealing only with viscosity solutions of Hamilton-Jacobi equations in this paper and thus we mean by "solutions", "subsolutions" and "supersolutions" viscosity solutions, viscosity subsolutions, and viscosity supersolutions, respectively.

In recent years, many researchers have investigated the large-time behavior of the solution $u(x,t)$ of (1.1) as $t \to \infty$ and established convergence results which state that under appropriate hypotheses

$$u(x,t) + ct - v(x) \to 0 \quad \text{locally uniformly for } x \in \Omega \text{ as } t \to \infty, \tag{1.4}$$

for some solution $(v, c) \in C(\Omega) \times \mathbb{R}$ of the *additive eigenvalue* problem for H:

$$H(x, Dv(x)) = c \quad \text{in } \Omega. \tag{1.5}$$

Here the additive eigenvalue problem for H is a problem of finding a pair of $v \in C(\Omega)$ and $c \in \mathbb{R}$ such that v is a solution of (1.5). If (v, c) is such a pair, we call v an *additive eigenfunction* and c an *additive eigenvalue*. A simple observation related to this is that, for any $(v, c) \in C(\Omega) \times \mathbb{R}$, the function $v(x) - ct$ is a solution of (1.1) if and only if (v, c) is a solution of the additive eigenvalue problem for H. We call such a function $v(x) - ct$ an *asymptotic solution* of (1.1).

The study of this asymptotic problem goes back to the works of Kružkov [17], Lions [18] and Barles [1], who studied the case where $\Omega = \mathbb{R}^n$ and $H = H(p)$ does not depend on the variable x. In the case where $H = H(x, p)$ depends both on x and p, the first general results were obtained by Namah-Roquejoffre [20] and Fathi [8]. One of their results assures that (1.4) holds, provided Ω is a compact manifold without boundary and $H = H(x, p)$ is smooth in (x, p) and superlinear and strictly convex in p. For this result Fathi [9] took an approach based on Aubry-Mather theory. Afterwards Roquejoffre [21] and Davini-Siconolfi [6] has refined the approach. By another approach based on the theory of partial differential equations and viscosity solutions, this type of results has been obtained by Barles-Souganidis [3]. More recently the large-time asymptotic problem of the same kind has been studied in the case where $\Omega = \mathbb{R}^n$ by Fujita-Ishii-Loreti [13], Barles-Roquejoffre [2], Ishii [15], and Ichihara-Ishii [14]. The convergence rate in (1.4) has been investigated by Fujita [11] and Fujita-Uchiyama [12].

Regarding boundary value problems, the author has recently studied the large-time asymptotic problem for Hamilton-Jacobi equations under the state constraint boundary condition (see [22, 5]) in [19]. The large-time asymptotic problem under the Dirichlet boundary condition has been treated in [1, 21]. In one of these papers, the problem

$$\begin{cases} u_t(x,t) + H(x, Du(x,t)) = 0 & \text{in} \quad Q, \\ u(x,t) = h(x) & \text{on} \quad \partial_p Q, \end{cases}$$

where $\partial_p Q = \partial\Omega \times (0, \infty) \cup \Omega \times \{0\}$ and $h \in C(\overline{\Omega})$ is a given function, is considered, where the function h is assumed to satisfy the compatibility condition

$$h(x) - h(y) \le \inf \left\{ \int_0^t L(\gamma(\lambda), \dot\gamma(\lambda))\, d\lambda \mid t > 0,\ \gamma \in \mathcal{C}(y, 0; x, t; \overline{\Omega}) \right\} \tag{1.6}$$

for any $x, y \in \overline{\Omega}$, where L is the Lagrangian of H, i.e, $L(x, \xi) := \sup_{p \in \mathbb{R}^n} \{p \cdot \xi - H(x, p)\}$ and $\mathcal{C}(y, 0; x, t; \overline{\Omega})$ denotes the spaces of those curves γ which satisfy $\gamma(s) \in \overline{\Omega}$ for all $s \in [0, t]$ as well as $\gamma(0) = y$ and $\gamma(t) = x$. In this case, solutions of (CD) satisfy the boundary condition in the pointwise sense.

In this paper, we do not assume the compatibility condition on our initial and boundary data f, g. This gives a viewpoint which unifies the state constraint and Dirichlet boundary conditions. Another consequence is that we do not expect any more that "solutions" of (CD) satisfy the Dirichlet condition pointwise, and therefore we have to understand the boundary condition in a generalized sense. We define (viscosity) solutions of (CD) by the following. (See [4], for example.)

Definition 1.1. Let $u \in C(\overline{\Omega} \times [0,\infty))$. (i) We call u a viscosity subsolution (resp., supersolution) of (CD) if the following conditions (1)-(3) hold: (1) u is a viscosity subsolution (resp., supersolution) of (1.), (2) $u(x,0) \leq f(x)$ (resp., $u(x,0) \geq f(x)$) for all $x \in \Omega$, and (3) for any $\phi \in C^1(\overline{\Omega} \times (0,\infty))$ and any $(x_0,t_0) \in \partial\Omega \times (0,\infty)$ such that $u - \phi$ takes a local maximum (resp., minimum) at (x_0,t_0),

$$\begin{aligned} \min\{(u-g)(x_0,t_0), \phi_t(x_0,t_0) + H(x_0, D\phi(x_0,t_0))\} &\leq 0 \\ (\mathit{resp.},\ \max\{(u-g)(x_0,t_0), \phi_t(x_0,t_0) + H(x_0, D\phi(x_0,t_0))\} &\geq 0). \end{aligned}$$

(ii) We call u a viscosity solution of (CD) if it is a viscosity subsolution and a viscosity supersolution of (CD).

For any $a \in \mathbb{R}$, we consider the Dirichlet problem,

$$(\mathrm{D})_a \begin{cases} H(x, Du(x)) = a & \mathit{in}\ \Omega, & (1.7) \\ u(x) = g(x) & \mathit{on}\ \partial\Omega. & (1.8) \end{cases}$$

Proposition 3.2 below states that there exist solutions of $(\mathrm{D})_a$ if and only if $c_H \leq a$, where

$$c_H := \inf\{a \in \mathbb{R} \mid (1.7) \text{ has a solution}\}. \tag{1.9}$$

As our main theorem, Theorem 2.2, shows, the large-time behavior of solutions of (CD) changes depending on the sign of c_H. To put it more precisely, in the case where $c_H > 0$, for any solution u of (CD), there exists a solution $v \in C(\overline{\Omega})$ of the state constraint problem of the Hamilton-Jacobi equation,

$$(\mathrm{SC}) \qquad \begin{cases} H(x, Du(x)) \leq c_H & \mathit{in}\ \Omega, & (1.10) \\ H(x, Du(x)) \geq c_H & \mathit{on}\ \overline{\Omega}, & (1.11) \end{cases}$$

such that

$$u(x,t) + c_H t - v(x) \to 0 \quad \text{uniformly for } x \in \overline{\Omega} \text{ as } t \to \infty; \tag{1.12}$$

and in the case where $c_H \leq 0$, for any solution u of (CD), there exists a solution $w \in C(\overline{\Omega})$ of $(\mathrm{D})_0$ such that

$$u(x,t) - w(x) \to 0 \quad \text{uniformly for } x \in \overline{\Omega} \text{ as } t \to \infty. \tag{1.13}$$

We give representation formulas for the functions v, w in (1.12) and (1.13), respectively.

Our proof of the convergence results (1.12), (1.13) is based on the Aubry-Mather theory, which was developed recently by Fathi [7, 9] and Fathi-Siconolfi [10]. (See also Ishii-Mitake [16].)

This paper is organized as follows: in Section 2, we state our main results, Theorem 2.2, as well as the precise assumptions on H, f, g and Ω. In Section 3, we prove the existence of solutions for $(\mathrm{D})_a$ for any $a \geq c_H$ and any $g \in C(\partial\Omega)$. In Section 4, we prove the existence of a solution of (CD) and examine its basic properties. We prepare some propositions concerning Aubry sets in Section 5. In Section 6, we prove Theorem 2.2. In Section 7, we give representation formulas for asymptotic solutions.

2. Assumptions and main result

Let $A \subset \mathbb{R}^k$, $B \subset \mathbb{R}^l$, where $k, l \in \mathbb{N}$, and $r > 0$. We write $U(x, r) = \{y \in \mathbb{R}^n \mid |x - y| < r\}$. We denote by $C(A, B)$ the sets of continuous functions on A with values in B. When the set B is clear by the context, we may omit writing B in the above notation. For given $-\infty < a < b < \infty$ and $x, y \in B$, we use the symbol $\mathrm{AC}([a, b], B)$ to denote the set of absolutely continuous functions on $[a, b]$ with values in B and we set

$$\begin{aligned} \mathcal{C}(x, b; B) &:= \{\gamma \in \mathrm{AC}([a, b], B) \mid \gamma(b) = x\}, \\ \mathcal{C}(x, a; y, b; B) &:= \{\gamma \in \mathrm{AC}([a, b], B) \mid \gamma(a) = x \text{ and } \gamma(b) = y\}. \end{aligned}$$

We call a function $m : [0, \infty) \to [0, \infty)$ a *modulus* if it is continuous and nondecreasing on $[0, \infty)$ and vanishes at the origin.

We make throughout the following assumptions:

(A1) $H \in C(\overline{\Omega} \times \mathbb{R}^n)$.

(A2) The function $p \mapsto H(x, p)$ is strict convex for each $x \in \overline{\Omega}$.

(A3) The function H is coercive, i.e.

$$\lim_{r \to \infty} \inf\{H(x, p) \mid x \in \overline{\Omega}, p \in \mathbb{R}^n \setminus U(0, r)\} = \infty.$$

(A4) $f \in C(\overline{\Omega})$ and $g \in C(\partial\Omega)$.

(A5) $f(x) \leq g(x)$ for all $x \in \partial\Omega$.

(B) For each $z \in \partial\Omega$, there are a constant $r > 0$, a C^1-diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ and a function $b \in C(\mathbb{R}^{n-1})$ such that

$$\Phi(\Omega \cap U(z, r)) = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n > b(x')\} \cap \Phi(U(z, r)).$$

Remark 2.1. As in [14], we can replace (A2) in our main hypotheses by the following condition which is weaker than (A2): the function $p \mapsto H(x,p)$ is convex and there is a modulus ω with $\omega(r) > 0$ for all $r > 0$ such that for any $(x,p) \in \Omega \times \mathbb{R}^n$ and for any $\xi \in D_p^- H(x,p)$, $q \in \mathbb{R}^n$ if $H(x,p) = c_H$, then

$$H(x, p+q) \geq H(x,p) + \xi \cdot q + \omega(\max\{\xi \cdot q, 0\}),$$

where $D_p^- H(x,p)$ stands for the subdifferential of H with respect to the variable p.

We now state our main theorem.

Theorem 2.2. *Let $u \in C(\overline{Q})$ be the solution of* (CD). (i) *If $c_H > 0$, then there exists a solution $v \in C(\overline{\Omega})$ of* (SC) *such that*

$$u(x,t) + c_H t - v(x) \to 0 \quad \text{uniformly for } x \in \overline{\Omega} \text{ as } t \to \infty.$$

(ii) *If $c_H \leq 0$, then there exists a solution $w \in C(\overline{\Omega})$ of* $(\mathrm{D})_0$ *such that*

$$u(x,t) - w(x) \to 0 \quad \text{uniformly for } x \in \overline{\Omega} \text{ as } t \to \infty.$$

Remark 2.3. (1) Existence and uniqueness of a solution of (CD) will be established in Theorems 4.1 and 4.3. (2) We can generalize Theorem 2.2 slightly. We consider the Cauchy-Dirichlet problem,

$$(\mathrm{CD}') \qquad \begin{cases} u_t + H(x, Du(x,t)) = 0 & \text{in } Q, & (2.1) \\ u(x,t) = f(x) & \text{in } \Omega \times \{0\}, & (2.2) \\ u(x,t) = h(x,t) & \text{on } \partial\Omega \times (0,\infty), & (2.3) \end{cases}$$

where $h(x,t) = g(x) - at$ for a given constant $a \in \mathbb{R}$. Then we have: (i$'$) if $c_H > a$, then there exists a solution v of (SC) such that $u(x,t) + c_H t - v(x) \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$, and (ii$'$) if $c_H \leq a$, then there exists a solution w of $(\mathrm{D})_a$ such that $u(x,t) + at - w(x) \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$. Thus the threshold 0 in terms of c_H for the asymptotic behavior of solutions of (CD) turns into the value a for that of (CD$'$). (3) We refer to Theorem 7.1 in Section 7 for a characterization of functions v and w in Theorem 2.2.

The following example gives a simple illustration of what our main theorem asserts.

Example. Consider the one-dimensional Cauchy-Dirichlet problem

$$\begin{cases} u_t(x,t) + |Du(x,t)|^2 + a = 0 & \text{in } Q, \\ u(x,t) = 0 & \text{on } \partial_p Q, \end{cases} \qquad (2.4)$$

where $\Omega = (0,1)$ and $a \in \mathbb{R}$ is a given constant. Formula (4.1) below gives the unique solution of (CD). Accordingly, for the solution u of (2.4), we have

$$\begin{aligned} u(x,t) &= \inf \left\{ \int_{\tau}^{t} \left(\frac{1}{4} \left| \dot{\gamma}(s) \right|^2 - a \right) ds \mid \gamma \in \mathcal{C}(x,t;[0,1]), \tau \in [0,t], (\gamma(\tau),\tau) \in \partial_p Q \right\} \\ &= \min\{-at, f_a(x)\}, \end{aligned}$$

where $f_a(x) := |a|^{1/2} \min\{x, 1-x\}$ for $x \in [0,1]$. It is easy to see that $c_H = a$, where $H(x,p) = |p|^2 + a$, that any constant function on $[0,1]$ is a solution of problem (SC), with $\Omega = (0,1)$, and that if $a \leq 0$, then f_a is a solution of $(\mathrm{D})_0$, with $\Omega = (0,1)$. The above formula tells us that if $c_H > 0$, then $u(x,t) + c_H t \to 0$ uniformly for $x \in [0,1]$ as $t \to \infty$ and that if $c_H \leq 0$, then $u(x,t) \to f_a(x)$ uniformly for $x \in [0,1]$ as $t \to \infty$, which are exactly what Theorem 2.2 claims.

3. Problem $(\mathrm{D})_a$

In this section, we prove the existence of solutions of $(\mathrm{D})_a$ for any $a \geq c_H$.

Proposition 3.1. ([16, Proposition A. 1]). *For any $M > 0$ there exists a modulus ω such that if $u \in C(\Omega)$ is a solution of $|Du(x)| \leq M$ in Ω, then $|u(x) - u(y)| \leq \omega(|x-y|)$ for all $x, y \in \Omega$.*

Proposition 3.2. *Problem $(\mathrm{D})_a$ has a solution if and only if $a \geq c_H$.*

Proof. In the case where $c_H > a$, it is clear that $(\mathrm{D})_a$ has no solutions due to the definition of c_H. Thus we need only show that $(\mathrm{D})_a$ has a solution in the case where $c_H \leq a$.

We prove this by using Perron's method and it is thus sufficient to construct a supersolution ψ_1 and a subsolution ψ_2 of $(\mathrm{D})_a$ such that $\psi_2 \leq \psi_1$ on $\overline{\Omega}$.

Due to (A3), there exists a $p_0 \in \mathbb{R}^n$ such that $H(x, p_0) \geq a$ for all $x \in \overline{\Omega}$. We set $\psi_1(x) = p_0 \cdot x + C_1$, where $C_1 > 0$ is a constant chosen so that $\psi_1(x) \geq g(x)$ for all $x \in \partial\Omega$. Since $a \geq c_H$, there exists a subsolution $\psi_2 \in C(\overline{\Omega})$ of (1.7). Subtracting a sufficiently large constant from ψ_2 if necessary, we may assume that $\psi_2 \leq \psi_1$ on $\overline{\Omega}$. We define the function w on $\overline{\Omega}$ by

$$w(x) = \sup\{v \in C(\overline{\Omega}) \mid v \text{ is a subsolution of (1.7) and } \psi_2 \leq v \leq \psi_1 \text{ on } \overline{\Omega}\}.$$

In view of Proposition 3.1, we see $w \in C(\overline{\Omega})$ and conclude as a consequence of Perron's method that w is a solution of $(\mathrm{D})_a$. ∎

4. Problem (CD)

We introduce the function $u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ defined by

$$u(x,t) := \inf \left\{ \int_\tau^t L(\gamma(s), \dot\gamma(s))\, ds + h(\gamma(\tau), \tau) \mid \right.$$
$$\left. \gamma \in \mathcal{C}(x,t;\overline{\Omega}),\ 0 \le \tau \le t,\ (\gamma(\tau), \tau) \in \partial_p Q \right\}, \tag{4.1}$$

where $h : \partial_p Q \to \mathbb{R}$ denotes the function given by $h(x,0) = f(x)$ for $x \in \Omega$ and $h(x,t) = g(x)$ for $(x,t) \in \partial\Omega \times (0,\infty)$.

Theorem 4.1. *The function u is a solution of* (CD) *and it is uniformly continuous on $\overline{Q}$.*

Remark 4.2. Assumption (A5) is used to assure that the function u defined by (4.1) is continuous on $\overline{Q}$. In fact, (A5) is a necessary condition as well for u to be continuous at points in $\partial\Omega \times \{0\}$.

Theorem 4.3. *Let $T > 0$ and set $Q_T := \Omega \times (0,T)$. Let $u, v \in C(\overline{Q_T})$ be a subsolution and a supersolution of (1.1) and (1.3), respectively. Assume $u \le v$ on $\overline{\Omega} \times \{0\}$. Then $u \le v$ on $\overline{Q_T}$.*

Theorem 4.1 can be proved as in [19, Section 5] and the proof of Theorem 4.3. is similarly to that for [16, Theorem 7.3], respectively, so we do not give here the details of proofs of the above theorems.

It follows from Theorems 4.1 and 4.3 that the function given by (4.1) is a unique solution of (CD).

The following theorem gives a bound on the solution of (CD).

Theorem 4.4. *Let c_H be the constant defined by (1.9).* (i) *If $c_H > 0$, then there exists a constant $M_1 > 0$ such that $|u(x,t) + c_H t| \le M_1$ for all $(x,t) \in \overline{Q}$.* (ii) *If $c_H \le 0$, then there exists a constant $M_2 > 0$ such that $|u(x,t)| \le M_2$ for all $(x,t) \in \overline{Q}$.*

Proof. We first consider the case where $c_H > 0$. By [19, Theorem 3.4], there exists a solution $\psi \in C(\overline{\Omega})$ of (SC). We set $v^{\pm}(x,t) := \psi(x) - c_H t \pm C_1$, where $C_1 > 0$ is a constant chosen so that $v^-(x,0) \le f(x) \le v^+(x,0)$ for all $x \in \overline{\Omega}$, and observe that v^+ and v^- are a supersolution and a subsolution of (CD), respectively. We then apply Theorem 4.3, to obtain $v^- \le u \le v^+$ on $\overline{Q}$, from which we conclude that $|u(x,t) + c_H t| \le M_1$ for all $(x,t) \in \overline{Q}$ and for some constant $M_1 > 0$.

Next we consider the case where $c_H \le 0$. In view of Proposition 3.2, we may choose a solution $\phi \in C(\overline{\Omega})$ of $(\mathrm{D})_0$. We set $w^{\pm}(x,t) = \phi(x) \pm C_2$, where

$C_2 > 0$ is chosen so that $w^-(x,0) \le f(x) \le w^+(x,0)$ for all $x \in \overline{\Omega}$. We see easily that w^+ and w^- are a supersolution and a subsolution of (CD), respectively, and by Theorem 4.3, we get $w^- \le u \le w^+$ on $\overline{Q}$. Consequently, we find a constant $M_2 > 0$ such that $|u(x,t)| \le M_2$ for all $(x,t) \in \overline{Q}$. The proof is now complete. □

5. The Aubry set

We define $\mathcal{A}_{c_H}$ as the set of those $y \in \overline{\Omega}$ such that

$$\inf \left\{ \int_{-t}^{0} L_{c_H}(\gamma(s), \dot{\gamma}(s))\, ds \mid t \ge \delta, \gamma \in \mathcal{C}(y,0;y,-t;\overline{\Omega}) \right\} = 0 \text{ for all } \delta > 0, \quad (5.1)$$

where $L_{c_H}(x,\xi) = L(x,\xi) + c_H$ for all $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n$. Following [10], we call $\mathcal{A}_{c_H}$ the *Aubry set* for the Hamiltonian H. We give here a few properties of the Aubry set, for which we refer the reader to [10, 16, 19].

Proposition 5.1. ([19, Proposition 6.4]). *$\mathcal{A}_{c_H}$ is a nonempty compact set.*

Theorem 5.2. ([16, Theorem 7.3]). *Let $u, v \in C(\overline{\Omega})$ be solutions of* (SC)*. If $u \le v$ on $\mathcal{A}_{c_H}$, then $u \le v$ on $\overline{\Omega}$.*

Theorem 5.3. *Let $a \ge c_H$ and let $u, v \in C(\overline{\Omega})$ be a subsolution and a supersolution of* $(\mathrm{D})_a$*, respectively. If $a > c_H$, then $u \le v$ on $\overline{\Omega}$. Also, if $a = c_H$ and $u \le v$ on $\mathcal{A}_{c_H}$, then $u \le v$ on $\overline{\Omega}$.*

The proof of Theorem 5.3 is similar to [16, Theorem 7.3], so we omit it here.

6. Convergence

This section will be devoted to the proof of Theorem 2.2. In order to prove Theorem 2.2, we follow the generalized dynamical approach as in [6].

Let u be the solution of (CD). We first consider the case where $c_H = 0$. In view of Theorems 4.1 and 4.4 (ii), we may define the functions $w^\pm \in C(\overline{\Omega})$ by $w^+(x) = \limsup_{t\to\infty} u(x,t)$ and $w^-(x) = \liminf_{t\to\infty} u(x,t)$, respectively. Noting that

$$w^+(x) = \lim_{t\to\infty} \sup\{u(y,s) \mid y \in \overline{\Omega}, |x-y| \le 1/t, s > t\},$$
$$w^-(x) = \lim_{t\to\infty} \inf\{u(y,s) \mid y \in \overline{\Omega}, |x-y| \le 1/t, s > t\},$$

we see that w^+ and w^- are a subsolution and a supersolution of $(\mathrm{D})_0$, respectively. Moreover, since the functions $H(x,\cdot)$ are convex, we see that w^- is a solution of $(\mathrm{D})_0$. As is now standard, in order to prove Theorem 2.2 in the case where $c_H = 0$, it is enough to show that $w^+(x) \le w^-(x)$ for all $x \in \overline{\Omega}$.

Lemma 6.1. ([19, Proposition 5.2]). *Let $\phi \in C(\overline{\Omega})$ be a solution of (1.10) and $a, b \in \mathbb{R}$ with $a < b$. Then, for any $\gamma \in \mathrm{AC}([a, b], \overline{\Omega})$,*

$$\phi(\gamma(b)) - \phi(\gamma(a)) \le \int_a^b L_{c_H}(\gamma(t), \dot\gamma(t))dt.$$

Proposition 6.2. *Let $\phi \in C(\overline{\Omega})$ be a solution of* (1.) *and $y \in \mathcal{A}_{c_H}$. Then there exists a curve $\gamma \in C((-\infty, 0], \overline{\Omega})$ such that $\gamma(0) = y$ and for any $[a, b] \subset (-\infty, 0]$,*

$$\gamma \in \mathrm{AC}([a, b], \overline{\Omega}) \quad \textit{and} \quad \int_a^b L_{c_H}(\gamma(s), \dot\gamma(s))\, ds = \phi(\gamma(b)) - \phi(\gamma(a)). \tag{6.1}$$

Following [21, 15], we call curves satisfying (6.1) *extremal curves* for ϕ and hereinafter we write $\mathcal{E}(\phi)$ to denote the set of all extremal curves for ϕ and we also write $\mathcal{E}(\phi, y)$ to denote the set of all $\gamma \in \mathcal{E}(\phi)$ such that $\gamma(0) = y$.

Proof. By the definition of $\mathcal{A}_{c_H}$, for each $k \in \mathbb{N}$, we may choose $t_k \ge k$ and $\eta_k \in \mathcal{C}(y, 0; y, -t_k; \overline{\Omega})$ such that

$$\int_{-t_k}^0 L_{c_H}(\eta_k(s), \dot\eta_k(s))\, ds \le 2^{-k}. \tag{6.2}$$

Let $a, b \in (-\infty, 0]$ satisfy $a < b$. If k is sufficiently large, we have $-t_k \le a$ and by Lemma 6.1,

$$\phi(\eta_k(0)) - \phi(\eta_k(b)) \;\le\; \int_b^0 L_{c_H}(\eta_k(s), \dot\eta_k(s))\, ds, \tag{6.3}$$

$$\phi(\eta_k(a)) - \phi(\eta_k(-t_k)) \;\le\; \int_{-t_k}^a L_{c_H}(\eta_k(s), \dot\eta_k(s))\, ds. \tag{6.4}$$

Adding (6.3) and (6.4) and noting (6.2) and $\eta_k(0) = \eta_k(-t_k) = y$, we obtain

$$\phi(\eta_k(a)) - \phi(\eta_k(b)) + \int_a^b L_{c_H}(\eta_k(s), \dot\eta_k(s))\, ds \le \int_{-t_k}^0 L_{c_H}(\eta_k(s), \dot\eta_k(s))\, ds \le 2^{-k}.$$

As in the proof of [15, Lemma 6.3], we may assume by passing to a subsequence if necessary that the sequence $\{\eta_k\}$ converges to a function $\gamma \in C((-\infty, 0], \overline{\Omega})$

in the topology of uniform convergence on bounded sets. Moreover we have $\gamma \in \mathrm{AC}([-T, 0])$ for any $T > 0$ and

$$\int_a^b L_{c_H}(\gamma(s), \dot{\gamma}(s))\, ds \le \liminf_{k\to\infty} \int_a^b L_{c_H}(\eta_k(s), \dot{\eta_k}(s))\, ds.$$

From this we get

$$\phi(\gamma(b)) - \phi(\gamma(a)) \ge \int_a^b L_{c_H}(\gamma(s), \dot{\gamma}(s))\, ds. \tag{6.5}$$

The opposite inequality is obtained directly by Lemma 6.1. Noting that $\gamma(0) = y$, we complete the proof. □

Proposition 6.3. *There exist a constant $\delta \in (0, 1)$ and a modulus ω for which if ϕ is a solution of (1.10), any $x \in \mathcal{A}_{c_H}$, $\gamma \in \mathcal{E}(\phi, x)$, $0 \le s \le t$ and $s/(t - s) \le \delta$, then*

$$u(x, t) - u(\gamma(-t), s) + c_H(t - s) \le \phi(x) - \phi(\gamma(-t)) + \frac{ts}{t-s}\omega\big(\frac{s}{t-s}\big). \tag{6.6}$$

Proof. We first observe by the dynamic programming principle that for any $x \in \overline{\Omega}$ and $t, s \ge 0$,

$$\begin{aligned} u(x, t+s) &= \inf\Bigg\{ \int_\tau^t L(\gamma(\lambda), \dot{\gamma}(\lambda))\, d\lambda + u(\gamma(\tau), s+\tau) \mid \\ &\qquad \gamma \in \mathcal{C}(x, t; \overline{\Omega}),\ 0 \le \tau \le t,\ (\gamma(\tau), \tau) \in \partial_p Q \Bigg\} \\ &\le \inf\Bigg\{ \int_0^t L(\gamma(\lambda), \dot{\gamma}(\lambda))\, d\lambda + u(\gamma(0), s) \mid \gamma \in \mathcal{C}(x, t; \overline{\Omega}) \Bigg\}. \end{aligned}$$

Then, using this inequality, we follow the proof of [19, Proposition 6.2], to obtain (6.6). □

Now, we fix any $x \in \mathcal{A}_{c_H}$. Choose an extremal curve $\gamma \in \mathcal{E}(w^-, x)$ and a divergent sequence $\{t_j\}_{j\in\mathbb{N}}$ such that $u(x, t_j) \to w^+(x)$. Since $\overline{\Omega}$ is a compact set, by replacing $\{t_j\}_{j\in\mathbb{N}}$ by its subsequence if necessary, we may assume that $\gamma(-t_j) \to y$ as $j \to \infty$ for some $y \in \overline{\Omega}$. Fix any $\epsilon > 0$ and choose $s > 0$ such that $w^-(y) + \epsilon > u(y, s)$. Let $\delta \in (0, 1)$ and ω be those from Proposition 6.3. Let $j \in \mathbb{N}$ be so large that $s(t_j - s)^{-1} \le \delta$. By Proposition 6.3, we get

$$u(x, t_j) \le u(\gamma(-t_j), s) + w^-(x) - w^-(\gamma(-t_j)) + \frac{st_j}{t_j - s}\omega\left(\frac{s}{t_j - s}\right).$$

Sending $j \to \infty$ yields

$$w^+(x) \le u(y,s) + w^-(x) - w^-(y) < w^-(y) + \epsilon + w^-(x) - w^-(y) = w^-(x) + \epsilon,$$

from which we get $w^+ \le w^-$ on $\mathcal{A}_{c_H}$. By Theorem 5.3, we see that $w^+ \le w^-$ on $\overline{\Omega}$, which completes the proof of Theorem 2.2 in the case where $c_H = 0$.

Next we turn to the case where $c_H > 0$. The following proposition reduces (CD) to the state constraint problem.

Proposition 6.4. *Assume that $c_H > 0$. Let $v(x,t) = u(x,t) + c_H t$ and M_1 be the constant given in Theorem 4.4* (i). *For any $T \ge (1/c_H)(M_1 + 1 + \max_{\partial\Omega} |g|)$,*

$$v(x,t+T) = \inf \left\{ \int_0^t L_{c_H}(\gamma(s), \dot\gamma(s))\, ds + v(\gamma(0),T) \mid \gamma \in \mathcal{C}(x,t;\overline{\Omega}) \right\}. \tag{6.7}$$

Proof. It follows that v satisfies

$$\begin{cases} v_t + H(x, Dv(x,t)) = c_H & \text{in } Q, \\ v(x,t) = f(x) & \text{in } \Omega \times \{0\}, \\ v(x,t) = g(x) + c_H t & \text{on } \partial\Omega \times (0,\infty) \end{cases}$$

in the viscosity sense. The dynamic programming principle yields

$$\begin{aligned} v(x,t+T) = \inf \Bigg\{ & \int_\tau^t L_{c_H}(\gamma(\lambda), \dot\gamma(\lambda))\, d\lambda + h_T(\gamma(\tau),\tau) \mid \\ & \gamma \in \mathcal{C}(x,t;\overline{\Omega}),\ \tau \in [0,t],\ (\gamma(\tau),\tau) \in \partial_p Q \Bigg\}, \end{aligned} \tag{6.8}$$

where $h_T(x,t) := g(x) + c_H(T+t)$ for $(x,t) \in \partial\Omega \times (0,\infty)$ and $h_T(x,0) := v(x,T)$ for $x \in \overline{\Omega}$. Since $T \ge (1/c_H)(M_1 + 1 + \max_{\partial\Omega} |g|)$, for any $\gamma \in \mathcal{C}(x,t;\overline{\Omega})$ and any $\tau \in (0,t]$, if $(\gamma(\tau),\tau) \in \partial_p Q$, then

$$\begin{aligned} h_T(\gamma(\tau),\tau) &\ge c_H T - \max_{\partial\Omega} |g| \ge M_1 + 1 \ge v(\gamma(0),T) + 1 \\ &> v(\gamma(0),T) = h_T(\gamma(0),0). \end{aligned} \tag{6.9}$$

Formula (6.8) and (6.9) together yield (6.7). □

By Proposition 6.4, we see that the function $v(x,t) := u(x,t) + c_H t$ is a solution of the Cauchy problem with state constraints, with the initial time $T > 0$ sufficiently large. Thus Theorem 2.2 in the case where $c_H > 0$ follow from [19, Theorem 2.1].

Finally, we consider the case where $c_H < 0$. Due to Theorem 5.3, the uniqueness of solutions of $(\mathrm{D})_0$ now holds, which makes the argument below easier than the previous cases.

Here let $\{t_n\}_{n\in\mathbb{N}}$ be any diverging sequence. Set $u_n(x) := u(x,t_n)$ for all $x \in \overline{\Omega}, n \in \mathbb{N}$. Then $\{u_n\}_{n\in\mathbb{N}}$ is equicontinuous and uniformly bounded in view of Theorems 4.1 and 4.4 (ii), and therefore there exists a subsequence $\{u_{n'}\}_{n'\in\mathbb{N}} \subset \{u_n\}_{n\in\mathbb{N}}$ such that $\{u_{n'}\}_{n'\in\mathbb{N}}$ converges uniformly on $\overline{\Omega}$. Let w be the unique solution of $(\mathrm{D})_0$. It is now easy to see by the stability of viscosity property that $\{u_{n'}\}$ converges to w uniformly on $\overline{\Omega}$. Moreover, we may conclude from this that $u(\cdot,t)$ converges to w uniformly on $\overline{\Omega}$ as $t \to \infty$. The proof of Theorem 2.2 is now complete.

7. Representation formulas for asymptotic solutions

In this section, we give a representation formula for asymptotic solutions which appear in Theorem 2.2.

For $a \geq c_H$, we introduce the function $d_a : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ by

$$d_a(x,y) = \sup\{v(x) - v(y) \mid v \in C(\overline{\Omega}) \text{ is a subsolution of } (1.7)\}. \tag{7.1}$$

It is easily seen that $d_a(x,x) = 0$ for all $x \in \overline{\Omega}$, that if $v \in C(\overline{\Omega})$ is a subsolution of (1.7), then $v(x) - v(y) \leq d_a(x,y)$ for all $x, y \in \overline{\Omega}$ and that the functions $d_a(\cdot,y)$, with $y \in \overline{\Omega}$, are subsolutions of (1.7). It is known (see [10, 15, 16]) that for any $y \in \overline{\Omega}$, the function $d_{c_H}(\cdot,y)$ is a solution of (1.11) in $\overline{\Omega} \setminus \{y\}$ and that for any $y \in \overline{\Omega}$, the inclusion $y \in \mathcal{A}_{c_H}$ holds if and only if the function $d_{c_H}(\cdot,y)$ is a solution of (SC).

We next define the functions v_i, w_i, $w_b \in C(\overline{\Omega})$ by

$$\begin{aligned} v_i(x) &= \min\{d_{c_H}(x,z) + f(z) \mid z \in \overline{\Omega}\}, \\ w_i(x) &= \min\{d_{c_H}(x,y) + d_{c_H}(y,z) + f(z) \mid y \in \mathcal{A}_{c_H},\ z \in \overline{\Omega}\}, \\ w_b(x) &= \min\{d_0(x,y) + g(y) \mid y \in \partial\Omega\}. \end{aligned}$$

It is easily seen that v_i is a solution of (1.10), that $w_i(x) = \min\{v_i(y) + d_{c_H}(x,y) \mid y \in \mathcal{A}_{c_H}\}$ for all $x \in \overline{\Omega}$ and hence w_i is a solution of (SC), and that w_b is a solution of $(\mathrm{D})_0$.

Let $u : \overline{\Omega} \times [0,\infty)$ be the solution of (CD) and $v, w : \overline{\Omega} \to \mathbb{R}$ be the functions given in Theorem 2.2, i.e. $v(x) = \lim_{t\to\infty}(u(x,t) + c_H t)$ if $c_H > 0$ and $w(x) = \lim_{t\to\infty} u(x,t)$ if $c_H \leq 0$. The following theorem gives a characterization of the functions v and w.

Theorem 7.1. (i) *If $c_H > 0$, then $v = w_i$.* (ii) *If $c_H < 0$, then $w = w_b$.* (iii) *If $c_H = 0$, then $w = \min\{w_i, w_b\}$, where $\min\{w_i, w_b\}$ denotes the function $w \in C(\overline{\Omega})$ given by $w(x) = \min\{w_i(x), w_b(x)\}$.*

We need the next proposition in the following discussion.

Proposition 7.2. *Let $x, y \in \overline{\Omega}$. Then*

$$d_a(x,y) = \inf \left\{ \int_0^t L_a(\gamma(\lambda), \dot\gamma(\lambda))\, d\lambda \mid t > 0,\ \gamma \in \mathcal{C}(x,t;y,0;\overline{\Omega}) \right\},$$

where $L_a(x,\xi) = L(x,\xi) + a$ for all $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n$.

For a proof of the above proposition we refer to [15, Proposition 8.2].

Proof of Theorem 7.1. We first prove assertion (ii). Assume that $c_H < 0$. According to Theorem 5.3., the uniqueness of solutions of $(\mathrm{D})_0$ holds. Since w_b and w are solutions of $(\mathrm{D})_0$, we see that $w = w_b$.

Next we turn to (i) and assume that $c_H > 0$. The functions $u(x,t) + c_H t$ and $v_i(x)$ are a solution and a subsolution of

$$\begin{cases} v_t + H(x, Dv(x,t)) = c_H & \text{in } Q, \\ v(x,t) = f(x) & \text{in } \Omega \times \{0\}, \\ v(x,t) = g(x) + c_H t & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

respectively. Hence, by Theorem 4.3, we get $v_i(x) \le u(x,t) + c_H t$ for all $(x,t) \in \overline{Q}$, and therefore, $v_i(x) \le v(x)$ for all $x \in \overline{\Omega}$. Since $w_i(x) = v_i(x)$ for all $x \in \mathcal{A}_{c_H}$, we have $w_i(x) \le v(x)$ for all $x \in \mathcal{A}_{c_H}$. Moreover, by Theorem 5.2, we see that $w_i \le v$ on $\overline{\Omega}$.

Now fix any $x \in \overline{\Omega}$ and $\epsilon > 0$. Noting the definition of w_i and Proposition 7.2, we may choose $y \in \mathcal{A}_{c_H}$, $z \in \overline{\Omega}$, $\gamma \in \mathcal{C}(x,\tau;y,0;\overline{\Omega})$ and $\eta \in \mathcal{C}(y,\sigma;z,0;\overline{\Omega})$, where $\tau > 0$ and $\sigma > 0$, so that

$$w_i(x) + \epsilon > \int_0^\tau L_{c_H}(\gamma(s), \dot\gamma(s))\, ds + \int_0^\sigma L_{c_H}(\eta(s), \dot\eta(s))\, ds + f(z).$$

Since $y \in \mathcal{A}_{c_H}$, for each $k \in \mathbb{N}$ we may choose a $\xi_k \in \mathcal{C}(y,t_k;y,0;\overline{\Omega})$, with $t_k > k$, so that

$$\int_0^{t_k} L_{c_H}(\xi_k(s), \dot\xi_k(s))\, ds < \epsilon.$$

We define $\zeta_k \in \mathcal{C}(x,r_k;z,0;\overline{\Omega})$, with $r_k := \tau + \sigma + t_k$, by

$$\zeta_k(s) = \begin{cases} \eta(s) & \text{for } s \in [0,\sigma), \\ \xi_k(s-\sigma) & \text{for } s \in [\sigma, \sigma + t_k), \\ \gamma(s - \sigma - t_k) & \text{for } s \in [\sigma + t_k, r_k], \end{cases}$$

and observe that

$$\begin{aligned} w_i(x) + 2\epsilon &> \int_0^{r_k} L_{c_H}(\zeta_k(s), \dot{\zeta}_k(s))\, ds + f(\zeta_k(0)) \\ &= \int_0^{r_k} L(\zeta_k(s), \dot{\zeta}_k(s))\, ds + f(\zeta_k(0)) + c_H r_k \geq u(x, r_k) + c_H r_k. \end{aligned}$$

Sending $k \to \infty$ yields $w_i(x) \geq v(x)$ and we conclude that $w_i = v$ on $\overline{\Omega}$.

Now we deal with the case where $c_H = 0$. The function w_b is a solution of $(\mathrm{D})_0$ and v_i is a subsolution of (SC). Therefore, noting that $v_i \leq f$ in Ω, we see that the function $\min\{v_i,\, w_b\}$ is a subsolution of (CD). By Theorem 4.3, we get $\min\{v_i,\, w_b\}(x) \leq u(x,t)$ for all $(x,t) \in \overline{Q}$ and consequently, $\min\{v_i,\, w_b\} \leq w$ on $\overline{\Omega}$. Moreover, we get $\min\{w_i,\, w_b\}(x) \leq w(x)$ for all $x \in \mathcal{A}_{c_H}$. Noting that $\min\{w_i,\, w_b\} \leq g$ on $\partial\Omega$, w_i and w_b satisfy a supersolution property of $(\mathrm{D})_0$ and $H(x,\cdot)$ is convex, we have that $\min\{w_i,\, w_b\}$ is a solution of $(\mathrm{D})_0$. In view of Theorem 5.3, we see that $\min\{w_i,\, w_b\} \leq w$ on $\overline{\Omega}$.

Arguing as in the previous case, we see that $w(x) \leq w_i(x)$. Also, since the function w is a solution of $(\mathrm{D})_0$, we have $w(x) - w(y) \leq d_0(x,y)$ for all $x, y \in \overline{\Omega}$. Moreover, noting that $u(x,t) \leq g(x)$ for all $x \in \partial\Omega$, we have $w(x) \leq d_0(x,y) + g(y)$ for all $x \in \overline{\Omega}$ and $y \in \partial\Omega$, from which we obtain $w(x) \leq w_b(x)$. Consequently we have $w \leq \min\{w_i,\, w_b\}$ on $\overline{\Omega}$, and conclude that $w = \min\{w_i,\, w_b\}$ on $\overline{\Omega}$.

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