

On absolutely minimizing lipschitz extensions and PDE $\Delta_\infty(u) = 0$

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Abstract. We prove the existence of Absolutely Minimizing Lipschitz Extensions by a method which differs from those used by G. Aronsson in general metrically convex compact metric spaces and R. Jensen in Euclidean spaces. Assuming Jensen's hypotheses, our method yields numerical schemes for computing, in euclidean \mathbb{R}^n , the solution of viscosity of equation $\Delta_\infty(u) = 0$ with Dirichlet's condition.

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1 Introduction

To produce an optimal solution to Tietze's extension problem in general metrically convex compact metric space, we have introduced a class H_h of extension schemes which solve the problem [7].

In this paper we first prove in section 3 that, for any continuous Dirichlet's condition f , there exists a subsequence $(H_{h(n)}(f))_{n \in \mathbb{N}}$ which converges to an AMLE of f . Therefore, assuming Jensen's hypotheses [4], $H_h(f)$ approaches the solution of viscosity of $\Delta_\infty(u) = 0$ under Dirichlet's condition f when h tends to 0.

Unfortunately it is generally hopeless to obtain a numerical approximation of this solution on, say, a regular grid of step h by discretisation of H_h on this grid. In fact, by such a discretisation we obtain an extension which is Lipschitz-optimal not for euclidean metric but only for the geodesic metric on the grid.

To overcome this difficulty we introduce in this paper an explicit scheme of extension valid on any finite network contained in the considered metrically convex

compact metric space and prove that the extension converges to an AMLE when the network suitably densifies the metric space. As a consequence, assuming Jensen's hypotheses [4], we obtain numerical approximations of the solution of viscosity of $\Delta_\infty(u) = 0$ under Dirichlet's condition. Note here that A. Oberman [9] has obtained very similar numerical approximations in \mathbb{R}^n , based upon the same numerical scheme, proposing a proof of the convergence of the scheme based upon the Δ_∞ -approach of the problem.

In the whole paper (E, d) denotes a metrically convex compact metric space that is a compact length space with the terminology of ([2], appendix). We denote by δ the Hausdorff metric induced by d on compact non-empty subsets of E . The second part of the paper is organized as follows.

In section 4 we prove that solutions of (1.1) (see below) satisfy the maximum principle and, as a corollary, uniqueness of the solution.

In section 5 we prove the existence of the solution of (1.1) and we study the stability of this solution.

In section 6 we prove the existence of an AMLE as the limit of solutions of (1.1) for sequences $((G_n, V_n))_{n \in \mathbb{N}}$ which suitably densify E .

Definition 1.1 A network on E is a couple (G, V) where $G \subset E$ denotes a finite non-empty subset of E and V a mapping $x \in G \rightarrow V(x) \subset G$, ($V(x)$ is the neighbourhood of x) which satisfies

- (P1) for any $x \in G$, $x \in V(x)$;
- (P2) for any $x, y \in G$, $x \in V(y)$ iff $y \in V(x)$;
- (P3) for any $x, y \in G$, there exists $x_1, x_2, \dots, x_{n-1}, x_n \in G$ such that $x_1 = x$, $x_n = y$ and $x_i \in V(x_{i+1})$ for $i = 1, \dots, n-1$;
- (P4) for any $x \in G$, any $y \in G - V(x)$ there exists $z \in V(x)$ such that $d(z, y) < d(x, y)$.

To any chain such as in (P3) we associate its length $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$. We define the geodesic metric d_g on (G, V) by letting $d_g(x, y)$ be the infimum of the length of chains connecting x and y .

It follows from (P1),(P2),(P3) that d_g is a metric, that $d(x, y) \leq d_g(x, y)$ for $x, y \in G$ and that $d(x, y) = d_g(x, y)$ for $x, y \in G$, $x \in V(y)$. It follows from (P2),(P3) that if G has at least two elements (assumed from now on) then $V(x) - \{x\} \neq \emptyset$ for any $x \in G$. We shall denote $\tilde{V}(x) := V(x) - \{x\}$. Extra-condition (P4), crucial in this paper (see the end of theorem 4.1 and theorem 6.3 iii), will be used as follows:

for any $x \in G$, D non-empty subset of G , $d(x, D) > 0$, there exists $y \in V(x)$ such that $d(y, D) < d(x, D)$.

We consider the following functional equation with Dirichlet's condition :

$$\begin{cases} u(x) = \mu(u; x) & \forall x \in G - S; \\ u(s) = f(s) & \forall s \in S. \end{cases} \quad (1.1)$$

Here S denotes a non-empty subset of G , function f is the Dirichlet's condition defined on S , u is the numerical unknown function defined on G and

$$\mu(u; x) = \inf_{z \in \tilde{V}(x)} \sup_{q \in \tilde{V}(x)} M(u; z, q)(x); \quad (1.2)$$

where

$$M(u; z, q)(x) := \frac{d(x, z)u(q) + d(x, q)u(z)}{d(x, z) + d(x, q)}. \quad (1.3)$$

Remark 1.2 It can be checked that

$$\mu(u; x) = \sup_{z \in \tilde{V}(x)} \inf_{q \in \tilde{V}(x)} M(u; z, q)(x). \quad (1.4)$$

It can also be checked that

$$J(\mu(u; x)) = \inf_{\mu \in \mathbb{R}} J(\mu)$$

where

$$J(\mu) = \sup_{z \in \tilde{V}(x)} \frac{|u(z) - \mu|}{d(x, z)}.$$

Therefore $\mu(u; x)$ is the explicit solution of the problem of minimization considered by A.Oberman.

2 Basics

Let f be any function from $dom(f) \subset E$ to \mathbb{R} . We define $\kappa(f)$ by

$$\kappa(f) := \sup_{x, y \in dom(f), x \neq y} \frac{f(x) - f(y)}{d(x, y)}.$$

We call concave modulus of continuity any mapping $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which satisfies the following:

- (i) $\omega(0) = 0$ and ω is continuous at 0;
- (ii) ω is increasing: $h_1 \leq h_2 \Rightarrow \omega(h_1) \leq \omega(h_2)$;
- (iii) ω is concave.

We say that f is Ω -continuous iff there exists a concave modulus of continuity ω such that, for any $x, y \in dom(f)$,

$$|f(x) - f(y)| \leq \omega(d(x, y)). \quad (2.1)$$

For such a function f , we denote by $\omega(f)$ the lower bound of those concave moduli of continuity which satisfy (2.1).

For any $A \subset \text{dom}(f)$, we have obviously $\omega(f | A) \leq \omega(f)$ (symbol $|$ denotes restriction to).

Let us restate here results of [6] which are of constant use in this paper.

Proposition 2.1 *Let f, g be any two Ω -continuous real-valued functions of domain S and let A and B be any two compact non-empty subsets of S . Then*

$$\| \omega(f) - \omega(g) \|_{\infty, \mathbb{R}^+} \leq 2 \| f - g \|_{\infty, S} ; \quad (2.2)$$

$$\| \omega(f | A) - \omega(f | B) \|_{\infty, \mathbb{R}^+} \leq 4\omega(f; \delta(A, B)). \quad (2.3)$$

Note that (2.2) and (2.3) have been established in [6] for weak moduli of continuity. It is immediate that these inequalities hold for concave moduli of continuity with the same constants.

Remark 2.2 So, aside the obvious fact that $AM\Omega E$ (see below) are more general than AMLE, the true reason why we adopt the modulus of continuity approach rather than the Lipschitz approach in this paper is that there is no equivalent of (2.2) and (2.3) for Lipschitz functions.

Now we recall Aronsson's definition of an AMLE [1]. Let e be a Lipschitz extension of a Lipschitz function f of compact domain.

Definition 2.3 We say that e is an Absolutely Minimizing Lipschitz Extension of f if for every non empty open $D \subset E$, $D \cap \text{dom}(f) = \emptyset$ we have

$$\kappa(e | D) = \kappa(e | \partial D),$$

where ∂D denotes the boundary of D .

Characterisation below has been noticed by Aronsson [1].

Proposition 2.4 *An extension e of f is AMLE iff for any non empty open $D \subset E$, $D \cap \text{dom}(f) = \emptyset$ we have*

$$e(x) \leq \inf_{y \in \partial D} (e(y) + \kappa(e | \partial D)d(x, y)), \forall x \in D, \quad (2.4)$$

and

$$\sup_{y \in \partial D} (e(y) - \kappa(e | \partial D)d(x, y)) \leq e(x), \forall x \in D. \quad (2.5)$$

In this paper we use a slightly more general definition (see remark 2.2). Let e be a continuous extension of a Ω -continuous function f .

Definition 2.5 We say that e is an Absolutely Minimizing Ω Extension of f if for every non empty open subset D of E , $D \cap \text{dom}(f) = \emptyset$ we have

$$\omega(e \mid D) = \omega(e \mid \partial D),$$

where ∂D denotes the boundary of D .

The analogue of proposition 2.4 is:

Proposition 2.6 An extension e of f is $AM\Omega E$ iff for any non empty open $D \subset E$, $D \cap \text{dom}(f) = \emptyset$ we have

$$e(x) \leq \inf_{y \in \partial D} (e(y) + \omega(e \mid \partial D; d(x, y))), \forall x \in D, \quad (2.6)$$

and

$$\sup_{y \in \partial D} (e(y) - \omega(e \mid \partial D; d(x, y))) \leq e(x), \forall x \in D. \quad (2.7)$$

It follows from these definitions that, if f is Lipschitz and e is an $AM\Omega E$ of f , then e is an $AMLE$ of f .

3 Convergence of harmonious extensions to an $AM\Omega E$

Let f be a continuous function of closed domain $\text{dom}(f) \subset E$.

For any $h > 0$, for any $x \in E$ we denote by $V_h(x)$ the closed ball of center x , radius $r(x) = \inf(h, d(x, \text{dom}(f)))$.

By the [Theorem 3.3 of [7]], there exists a unique continuous extension $H_h(f)$ from E to \mathbb{R} which satisfies functional equation

$$g(x) = \frac{1}{2} \sup_{z \in V_h(x)} g(z) + \frac{1}{2} \inf_{z \in V_h(x)} g(z), \forall x \in E. \quad (3.1)$$

Moreover, as noticed in remark 3.4 of [7], the proof of [Theorem 3.3 of [7]] shows that

$$\omega(H_h(f)) = \omega(f).$$

Lemma 3.1 below shows that $H_h(f)$ is close to an $AM\Omega E$ of f . Its proof uses the arguments of Proposition 3.9 of [7].

Lemma 3.1 For any non empty open subset D of E , $D \cap \text{dom}(f) = \emptyset$, we have

$$H_h(f)(x) \leq \inf_{y \in \partial D} (H_h(f)(y) + \omega(H_h(f) \mid \partial D; d(x, y))) + 2\omega(f; h), \forall x \in D, \quad (3.2)$$

and

$$\sup_{y \in \partial D} (H_h(f)(y) - \omega(H_h(f) \mid \partial D; d(x, y))) - 2\omega(f; h) \leq H_h(f)(x), \forall x \in D. \quad (3.3)$$

Proof. Since the arguments are symmetric we prove only (3.2).

Since E is a compact metrically convex metric space, by Theorem 3.3 [7] there exists a unique extension v of $H_h(f) | \partial D$ in E such that

$$v(x) = \frac{1}{2} \sup_{z \in W_h(x)} v(z) + \frac{1}{2} \inf_{z \in W_h(x)} v(z), \forall x \in E. \quad (3.4)$$

where $W_h(x) := \{z \in E : d(x, z) \leq \inf(h, d(x, \partial D))\}$.

Moreover

$$v(x) - v(y) \leq \omega(H_h(f) | \partial D; d(x, y)), \forall x, y \in E.$$

In particular we have

$$v(x) - v(y) \leq \omega(H_h(f) | \partial D; d(x, y)), \forall y \in \partial D, \forall x \in D. \quad (3.5)$$

Now, let us bound $\sup_{x \in D} |H_h(f)(x) - v(x)|$. By symmetry we have only to bound from above: $\Delta = \sup_{x \in D} (H_h(f)(x) - v(x))$. Let

$$F = \{x \in D : H_h(f)(x) - v(x) = \Delta\}, M = \sup_{x \in F} H_h(f)(x),$$

and

$$\tilde{F} = \{x \in F : H_h(f)(x) = M\}.$$

Let $x \in \tilde{F}$ be such that

$$d(x, \partial D) = \inf_{y \in \tilde{F}} d(y, \partial D). \quad (3.6)$$

Let us first show that we cannot have $d(x, \partial D) > h$. Towards a contradiction let us assume it is the case.

Then we have $W_h(x) = \{z \in E : d(x, z) \leq h\}$.

Since $\text{dom}(f) \cap D = \emptyset$ we infer that $d(x, S) \geq d(x, \partial D) > h$, that is $V_h(x) = \{z \in E : d(x, z) \leq h\}$. Therefore $V_h(x) = W_h(x)$.

Now, using a similar argument to this of theorem 3.3-uniqueness- of [7], we infer that $V_h(x) \subset \tilde{F}$. It follows that there exists $z \in \tilde{F}$ such that $d(z, \partial D) < d(x, \partial D)$ which is a contradiction with definition (3.6) of x .

Now, since $d(x, \partial D) \leq h$, there exists $y \in \partial D$ such that $d(x, y) \leq h$ and $H_h(f)(y) = v(y)$.

By Ω -stability of both v and $H_h(f)$ we have:

$$H_h(f)(x) - v(x) = H_h(f)(x) - H_h(f)(y) - v(y) - v(x) \leq 2\omega(f; h). \text{Therefore}$$

$$\Delta \leq 2\omega(f; h). \quad (3.7)$$

Now let $x \in D$ and $y \in \partial D$. We have

$$H_h(f)(x) - H_h(f)(y) - \omega(H_h(f) | \partial D; d(x, y)) = A_1 + A_2$$

where

$$A_1 = H_h(f)(x) - v(x),$$

and

$$A_2 = v(x) - v(y) - \omega(H_h(f) | \partial D; d(x, y)).$$

From inequalities (3.5) and (3.7), we obtain $A_1 + A_2 \leq 2\omega(f; h) + 0$. \square

Now we prove the existence of $AM\Omega E$ in any metrically convex compact metric space.

Theorem 3.2 *Let $(h(n))_{n \in \mathbb{N}}$ be a sequence of positive reals which converges to 0. If sequence $(H_{h(n)}(f))_{n \in \mathbb{N}}$ converges uniformly to a continuous extension g of f then g is an $AM\Omega E$ of f .*

Proof. By symmetry we prove only (2.6).

Let D be non empty open subset of E such that $D \cap \text{dom}(f) \neq \emptyset$. For any $\epsilon > 0$, there exists $N > 0$ such that $\forall n \geq N$, $\|H_{h(n)}(f) - g\|_{\infty, E} \leq \epsilon$. For any $x \in D$, $y \in \partial D$, $n > N$ we have
 $g(x) - g(y) - \omega(g | \partial D; d(x, y)) \leq A_1 + A_2 + A_3 + A_4$, where
 $A_1 = |g(x) - H_{h(n)}(f)(x)| \leq \epsilon$;
 $A_2 = |H_{h(n)}(f)(x) - H_{h(n)}(f)(y) - \omega(H_{h(n)}(f) | \partial D; d(x, y))| \leq 2\omega(f; h(n))$;
 $A_3 = |\omega(H_{h(n)}(f) | \partial D; d(x, y)) - \omega(g | \partial D; d(x, y))| \leq 2\epsilon$;
 $A_4 = |H_{h(n)}(f)(y) - g(y)| \leq \epsilon$.

The second inequality follows from Lemma 3.1 and the third one from (2.2). We obtain $g(x) - g(y) - \omega(g | \partial D; d(x, y)) \leq 4\epsilon + 2\omega(f; h(n))$.

By letting $n \rightarrow \infty$ we have

$$g(x) - g(y) - \omega(g | \partial D; d(x, y)) \leq 4\epsilon, \forall y \in \partial D,$$

and we obtain the stated result. \square

Theorem 3.3 *For any continuous real-valued function f whose domain is a compact non-empty subset of E , there exists an $AM\Omega E$ of f .*

Proof. The set $\{H_h(f), h > 0\}$ is equicontinuous and equibounded. Therefore, by Ascoli's theorem, there exists a subsequence $(H_{h(n)}(f))_{n \in \mathbb{N}}$ which converges uniformly to a continuous extension of f which is an $AM\Omega E$ of f by theorem 3.2. \square

Remark 3.4 Moreover if, for any f , there exists a unique $AM\Omega E$ of f denoted by $H(f)$ then $\lim_{h \rightarrow 0} H_h(f) = H(f)$. In this case it follows from proposition 3.9 of [7] that

$$\|H(f | A) - H(f | B)\|_{\infty, E} \leq 4\omega(f; \delta(A, B)),$$

for any non-empty compact subsets A, B of $\text{dom}(f)$.

Remark 3.5 We can summarize the difference between Jensen's proof [4] of the existence of an AMLE and our own proof as follows. Jensen obtains the desired AMLE as a limit of local (because solutions of PDE) extensions which become more and more optimally Lipschitz. We obtain the desired AMLE as a limit of optimally Lipschitz extensions which become more and more local.

Aronsson [1] (see also [5] and [8]) proves the existence of *AMLE* by giving two explicit solutions:

$$\begin{aligned} u &= \sup\{w : w \text{ AMLE of } f \text{ from above } \}, \\ v &= \inf\{w : w \text{ AMLE of } f \text{ from below } \}. \end{aligned}$$

Our proof leads to less explicit but, assuming uniqueness, more constructive solutions than Aronsson's one.

Remark 3.6 Note the formal analogy between the process $u \rightarrow \Phi_h(u)$ of harmonious regularization defined by

$$\Phi_h(u)(x) = \frac{1}{2} \left(\sup_{y \in B_h(x)} u(y) + \inf_{y \in B_h(x)} u(y) \right)$$

which deals with *PDE* $\Delta_\infty u = 0$ and the process $u \rightarrow \Psi_h(u)$ of harmonic regularization defined by

$$\Psi_h(u)(x) = \frac{\int_{B_h(x)} u(y) dy}{\int_{B_h(x)} dy}$$

which deals with *PDE* $\Delta u = 0$.

It is known since Gauss that any harmonic function satisfies $\Psi_h(u) = u$ for any $h > 0$. The analog of this result does not hold in general for the process of harmonious regularization : it can be seen by numerical tests that $\Phi_h(u) \neq u$ for $u(x, y) = x^{4/3} - y^{4/3}$ even in subdomains where this function is analytic.

However some functions u solutions of $\Delta_\infty u = 0$ have this property : for example linear functions, $(x_1^2 + x_2^2)^{1/2}$, $\arctan(x_1/x_2)$, in euclidean plane, $x_1^2 - x_2^2$ in the plane equipped with sup norm.

Remark 3.7 When (E, d) is $(\bar{\Omega}, \|\cdot\|_2)$ with Ω open convex non empty subset of euclidean \mathbb{R}^n , $dom(f) = \partial\Omega$, it can be shown directly (that is whitout using theorem 3.3 and the equivalence between AMLE and solution of viscosity of $\Delta_\infty u = 0$) that $H_h(f)$ converges, when h tends to 0, to the solution of viscosity of $\Delta_\infty u = 0$, $u|_{\partial\Omega} = f$. It is a consequence of Jensen's uniqueness results [4] and of a Barles-Souganidis's result [3] : see Appendix.

Remark 3.8 The results of [7] and of this section hold for spaces more general than compact metrically convex metric spaces. They hold in compact metric spaces (E, d) having the following properties:

- i) $\frac{1}{2} \sup_{q \in B_h(x)} \inf_{r \in B_h(y)} d(q, r) + \frac{1}{2} \sup_{r \in B_h(y)} \inf_{q \in B_h(x)} d(r, q) \leq d(x, y)$, for any $x, y \in E$
- ii) For any $x \in E$ and $y \notin B_h(x)$ there exist $z \in B_h(x)$ such that $d(y, z) < d(y, x)$.

It follows that the condition of convexity on Ω assumed in remark 3.7 can be removed.

Note that conditions i) and ii) can hold in metric spaces which can be very far from metrically convex metric spaces (some finite metric spaces satisfy conditions i) and ii)) : we have therefore established a theorem of existence of an AMLE under weaker hypotheses than those of Milman [8] and Juutinen [5] (however our result holds only for compact spaces).

4 Uniqueness theorem for functional equation (1.1)

As usual, we first prove a maximum principle.

Theorem 4.1 *Let f, g be any two real-valued functions both of domain S . Let u, v be two solutions of (1.1) with Dirichlet's conditions f and g respectively. Then*

$$\sup_{x \in G} (u(x) - v(x)) \leq \sup_{s \in S} (f(s) - g(s)). \quad (4.1)$$

Proof. Let us set $\Delta = \sup_{x \in G} (u(x) - v(x))$, $F = \{x \in G : u(x) - v(x) = \Delta\}$, $\Lambda = \sup_{x \in F} u(x)$ and $\tilde{F} = \{x \in F : u(x) = \Lambda\}$.

We start our proof by choosing $x \in \tilde{F}$ such that $d(x, S) = \inf_{y \in \tilde{F}} d(y, S)$.

If $d(x, S) = 0$ then equality (4.1) is true.

Else, let $z_1 \in \tilde{V}(x)$ such that

$$\sup_{q \in \tilde{V}(x)} M(v; z_1, q)(x) = \mu(v; x) .$$

We have

$$\Delta \leq \sup_{q \in \tilde{V}(x)} M(u; z_1, q)(x) - \sup_{q \in \tilde{V}(x)} M(v; z_1, q)(x) .$$

Let $q_1 \in \tilde{V}(x)$ such that

$$M(u; z_1, q_1) = \sup_{q \in \tilde{V}(x)} M(u; z_1, q) .$$

We have

$$\Delta \leq \frac{d(x, z_1)(u(q_1) - v(q_1))}{d(x, z_1) + d(x, q_1)} + \frac{d(x, q_1)(u(z_1) - v(z_1))}{d(x, z_1) + d(x, q_1)} \leq \Delta ,$$

from which it follows that $z_1, q_1 \in F$.

Since

$$u(x) = \mu(u; x) \leq \sup_{q \in \tilde{V}(x)} M(u; z_1, q)(x) = M(u; z_1, q_1) \leq u(q_1),$$

we have $q_1 \in \tilde{F}$ and $u(q_1) = u(x)$.

Since

$$u(x) \leq \frac{d(x, z_1)u(q_1) + d(x, q_1)u(z_1)}{d(x, z_1) + d(x, q_1)},$$

we have $u(z_1) = u(x)$.

Since

$$u(x) = u(q_1) = \sup_{q \in \tilde{V}(x)} M(u; z_1, q) \quad (4.2)$$

and

$$\forall q \in \tilde{V}(x), u(x) \geq M(u; z_1, q) = \frac{d(x, z_1)u(q) + d(x, q)u(z_1)}{d(x, z_1) + d(x, q)}$$

with $u(z_1) = u(x)$,

we have

$$\forall z \in \tilde{V}(x), u(z) \leq u(x). \quad (4.3)$$

We finish the proof of our assertion by proving that u is constant in $V(x)$. Towards a contradiction let $q \in \tilde{V}(x)$ such that $u(q) < u(x)$. We have

$$u(x) \leq \sup_{t \in \tilde{V}(x)} M(u; q, t) = \sup_{t \in \tilde{V}(x)} \frac{d(x, t)u(q) + d(x, q)u(t)}{d(x, t) + d(x, q)}.$$

Using (4.3), we obtain

$$u(x) \leq \sup_{t \in \tilde{V}(x)} \frac{d(x, t)u(q) + d(x, q)u(x)}{d(x, t) + d(x, q)}.$$

Let $c > 0$ such that $u(q) = u(x) - c$, we have

$$u(x) \leq \sup_{t \in \tilde{V}(x)} \frac{d(x, t)(u(x) - c) + d(x, q)u(x)}{d(x, t) + d(x, q)}.$$

On the other hand, we can write

$$u(x) \leq \sup_{t \in \tilde{V}(x)} \left(u(x) - \frac{d(x, t)c}{d(x, t) + d(x, q)} \right),$$

that is

$$\inf_{t \in \tilde{V}(x)} \frac{d(x, t)c}{d(x, t) + d(x, q)} \leq 0.$$

Since

$$\inf_{t \in \tilde{V}(x)} \frac{d(x, t)}{d(x, t) + d(x, q)} > 0, \text{ and } c > 0,$$

we obtain the desired contradiction.

Now, using inequality

$$\Delta \leq \sup_{q \in \tilde{V}(x)} M(u; z_1, q) - \sup_{q \in \tilde{V}(x)} M(v; z_1, q),$$

and $u = \text{constant}$ in $V(x)$, we have

$$\forall q \in \tilde{V}(x), \Delta \leq \frac{d(x, z_1)(u(q) - v(q))}{d(x, z_1) + d(x, q)} + \frac{d(x, q)(u(z_1) - v(z_1))}{d(x, z_1) + d(x, q)}.$$

Therefore $\Delta \leq u(q) - v(q) \leq \Delta, \forall q \in \tilde{V}(x)$, that is $\tilde{V}(x) \subset F$.

Since u is constant on $V(x)$ we have $V(x) \subset \tilde{F}$.

Since V satisfies (P4), we have

$$d(x, S) = \inf_{y \in \tilde{F}} d(y, S) \text{ and } \inf_{q \in \tilde{V}(x)} d(q, S) < d(x, S)$$

which is a contradiction with $V(x) \subset \tilde{F}$. So (4.1) is proved. \square

As immediate consequences of theorem 4.1 we obtain Theorem 4.2 and corollary 4.3:

Theorem 4.2 *Functional equation (1.1) has a unique solution.*

Corollary 4.3 *Let u a solution of (1.1) then*

$$\inf_{s \in S} f(s) \leq u(x) \leq \sup_{s \in S} f(s), \forall x \in G. \tag{4.4}$$

5 Existence and stability of solutions of (1.1)

To prove the existence of a solution of (1.1), we introduce a process of evolution $u \rightarrow \Phi(u)$ whose the stationary state $u = \Phi(u)$ is solution of (1.1). Precisely $\Phi(u) = \Psi(u; x_N) \circ \Psi(u; x_{N-1}) \circ \dots \circ \Psi(u; x_1)$ where $\{x_1, \dots, x_N\}$ is an enumeration of $G - S$ and $\Psi(u; x), x \in G - S$ is defined as follows:

$$\begin{cases} \Psi(u; x)(y) = u(y) & \text{if } y \in G - \{x\}; \\ \Psi(u; x)(x) = \mu(u; x) & \text{if } y = x \end{cases} \tag{5.1}$$

We need three lemmas useful for existence and stability.

Lemma 5.1 *For any two scalar-valued functions u, v of domain G , we have the following properties :*

$$u \leq v \implies \Psi(u; x) \leq \Psi(v; x), \forall x \in G - S; \tag{5.2}$$

$$|\Psi(u; x)(y) - \Psi(u; x)(z)| \leq \omega(u; d_g(y, z)), \forall x \in G - S, \forall y, z \in G; \tag{5.3}$$

and

$$\sup_{y \in G} (\Psi(u; x)(y) - \Psi(v; x)(y)) \leq \sup_{y \in G} (u(y) - v(y)) . \quad (5.4)$$

Proof. Let us show (5.2). Let u, v scalar-valued functions of domain G such that $u \leq v$. Let $x \in G - S$. We have

$$\Psi(u; x)(y) = u(y) \leq v(y) = \Psi(v; x)(y) \text{ for } y \in G - \{x\}.$$

Since $\forall z, q \in \tilde{V}(x)$ we have $M(u; z, q)(x) \leq M(v; z, q)(x)$, therefore $\mu(u; x) \leq \mu(v; x)$

Let us show (5.3).

It suffices to prove that $|\Psi(u; x)(x) - u(y)| \leq \omega(u; d_g(x, y))$, $\forall y \in G$.

Let $y \in G - \{x\}$ we have two case :

First case : suppose that $y \in G - \tilde{V}(x)$.

Let $z_1 \in \tilde{V}(x)$ such that $d_g(x, y) = d_g(x, z_1) + d_g(z_1, y)$.

We have

$$\Psi(u; x)(x) - u(y) \leq \sup_{q \in \tilde{V}(x)} \frac{d(x, z_1)(u(q) - u(y)) + d(x, q)(u(z_1) - u(y))}{d(x, z_1) + d(x, q)} .$$

By definition of $\omega(u)$ we have

$$\Psi(u; x)(x) - u(y) \leq \sup_{q \in \tilde{V}(x)} \left(\frac{d(x, z_1)\omega(u; d_g(q, y)) + d(x, q)\omega(u; d_g(z_1, y))}{d(x, z_1) + d(x, q)} \right).$$

By concavity of $\omega(u)$ we have

$$\Psi(u; x)(x) - u(y) \leq \sup_{q \in \tilde{V}(x)} \omega(u; \frac{d(x, z_1)d_g(q, y) + d(x, q)d_g(z_1, y)}{d(x, z_1) + d(x, q)}),$$

Since

$$d(x, z_1)d_g(q, y) + d(x, q)d_g(z_1, y) = d(x, z_1)(d_g(q, y) - d(x, q)) + d(x, q)d_g(x, y)$$

by the triangle inequality, we have

$$\frac{d(x, z_1)d_g(q, y) + d(x, q)d_g(z_1, y)}{d(x, z_1) + d(x, q)} \leq d_g(x, y) .$$

Second case. Suppose that $y \in \tilde{V}(x)$. We have

$$\Psi(u; x)(x) - u(y) \leq \sup_{q \in \tilde{V}(x)} (M(u; y, q) - u(y)) \leq \sup_{q \in \tilde{V}(x)} \frac{d(x, y)\omega(u; d_g(q, y))}{d(x, y) + d(x, q)} .$$

By concavity we have

$$\Psi(u; x)(x) - u(y) \leq \sup_{q \in \tilde{V}(x)} \omega(u; \frac{d_g(x, y)d_g(q, y)}{d_g(x, y) + d_g(x, q)}) .$$

Since

$$\frac{d_g(q, y)}{d_g(x, y) + d_g(x, q)} \leq 1 ,$$

we conclude that $\Psi(u; x)(x) - u(y) \leq \omega(u; d_g(d(x, y)))$. The arguments to prove that $u(y) - \Psi(u; x)(x) \leq \omega(u; d_g(d(x, y)))$ are symmetric using (1.4) instead of (1.2). Inequality (5.3) is therefore proved.

Inequality (5.4) holds because we have $\mu(u; x) - \mu(v; x) \leq \sup_{z \in \tilde{V}(x)} (u(z) - v(z))$. □

Lemma 5.2 *For any scalar-valued functions u, v of domain G , we have the following properties :*

$$u \leq v \implies \Phi(u) \leq \Phi(v) , \forall x \in G - S ; \tag{5.5}$$

$$| \Phi(u)(y) - \Phi(u)(z) | \leq \omega(u; d_g(y, z)) , \forall x \in G - S , \forall y, z \in G ; \tag{5.6}$$

and

$$\inf_{z \in G} u(z) \leq \Phi(u)(x) \leq \sup_{z \in G} u(z) , \forall x \in G . \tag{5.7}$$

Proof. The proof is a consequence of Lemma 5.1. □

Now let U_0 be defined by

$$U_0(x) = \inf_{s \in S} (f(s) + \omega(f; d_g(x, s))) , \forall x \in G .$$

Function U_0 looks like classical M° Shane maximal Lipschitz-optimal extension of f on G . But here U_0 is defined with both d (in $\omega(f)$) and d_g (in $d_g(x, s)$). Therefore we have to check that U_0 is an extension of f .

Lemma 5.3 *We have*

$$U_0(s) = f(s), \text{ for } s \in S ; \tag{5.8}$$

$$| U_0(x) - U_0(y) | \leq \omega(f; d_g(x, y)), \text{ for } x, y \in G ; \tag{5.9}$$

$$\inf_{s \in S} f(s) \leq U_0(x) \leq \sup_{s \in S} f(s), \text{ for } x \in G . \tag{5.10}$$

Proof. Let $\tilde{s} \in S$ we have

$$U_0(\tilde{s}) - f(\tilde{s}) \leq f(\tilde{s}) - \omega(f; d_g(\tilde{s}, \tilde{s})) - f(\tilde{s}) \leq 0 ;$$

and

$$f(\tilde{s}) - U_0(\tilde{s}) = \sup_{s \in S} (f(\tilde{s}) - f(s) - \omega(f; d_g(\tilde{s}, s))) ,$$

therefore

$$f(\tilde{s}) - U_0(\tilde{s}) \leq \sup_{s \in S} (\omega(f; d(\tilde{s}, s)) - \omega(f; d_g(\tilde{s}, s))) .$$

Since $d \leq d_g$ we have $f(\tilde{s}) - U_0(\tilde{s}) \leq 0$ and $f(\tilde{s}) = U_0(\tilde{s})$.

Let $x, y \in G$ we have

$$U_0(x) - U_0(y) \leq \sup_{s \in S} (f(s) + \omega(f; d_g(x, s)) - f(s) - \omega(f; d_g(y, s))) ,$$

Therefore

$$U_0(x) - U_0(y) \leq \sup_{s \in S} (\omega(f; d_g(x, s)) - \omega(f; d_g(y, s))) \leq \omega(f; d_g(x, y)) .$$

Let $x \in G$ we have

$$U_0(x) \leq \inf_{s \in S} f(s) + \sup_{s \in S} \omega(f; d_g(x, s)) .$$

since $\omega(f; d_g(x, s)) \leq \sup_{s_1, s_2 \in S} (f(s_1) - f(s_2))$, we have

$$U_0(x) \leq \inf_{s \in S} f(s) + \sup_{s_1, s_2 \in S} (f(s_1) - f(s_2)) \leq \sup_{s \in S} f(s) .$$

Since $\omega(f; d_g(x, s)) \geq 0$, we have $U_0(x) \geq \inf_{s \in S} f(s)$. □

Now we are ready to prove the existence of a solution of (1.1).

Theorem 5.4 *Let $(U_n)_{n \in \mathbb{N}}$ the sequence defined inductively by $U_{n+1} = \Phi(U_n)$, $\forall n \in \mathbb{N}$. This sequence converges to a solution of (1.1) denoted by $K(f)$:*

$$K(f)(s) = f(s) , \forall s \in S \tag{5.11}$$

$$K(f)(x) = \mu(K(f); x) , \forall x \in G - S. \tag{5.12}$$

Moreover we have

$$|K(f)(x) - K(f)(y)| \leq \omega(f; d_g(x, y)) , \forall x, y \in G . \tag{5.13}$$

Proof. Let us show that $(U_n)_{n \in \mathbb{N}}$ is decreasing. By Lemma 5.2, it is sufficient to prove that $U_1 \leq U_0$. Given an arbitrary $k \in \{1, \dots, N\}$, we have

$$\Psi(U_0; x_k)(x_k) - U_0(x_k) = \mu(U_0; x_k) - U_0(x_k) .$$

Let $s_k \in S$ such that

$$f(s_k) + \omega(f; d_g(x_k, s_k)) = \inf_{s \in S} (f(s) + \omega(f; d_g(x_k, s))) .$$

By lemma 5.1, we have

$$\begin{aligned} & \Psi(U_0; x_k)(x_k) - f(s_k) \\ & \leq \inf_{z \in \tilde{V}(x_k)} \sup_{q \in \tilde{V}(x_k)} \left(\frac{d_g(x_k, z)\omega(f; d_g(q, s_k)) + d_g(x_k, q)\omega(f; d_g(z, s_k))}{d_g(x_k, z) + d_g(x_k, q)} \right). \end{aligned}$$

By concavity we have

$$\begin{aligned} & \Psi(U_0; x_k)(x_k) - f(s_k) \\ & \leq \inf_{z \in \tilde{V}(x_k)} \sup_{q \in \tilde{V}(x_k)} \omega\left(f; \frac{d_g(x_k, z)d_g(q, s_k) + d_g(x_k, q)d_g(z, s_k)}{d_g(x_k, z) + d_g(x_k, q)}\right). \end{aligned}$$

First case. If $s_k \notin \tilde{V}(x_k)$, then $\exists z' \in \tilde{V}(x_k)$ such that $d_g(x_k, s_k) = d_g(x_k, z') + d_g(z', s_k)$.

We have

$$\Psi(U_0; x_k)(x_k) - f(s_k) \leq \sup_{q \in \tilde{V}(x_k)} \omega\left(f; \frac{d_g(x_k, z')d_g(q, s_k) + d_g(x_k, q)d_g(z', s_k)}{d_g(x_k, z') + d_g(x_k, q)}\right).$$

Since

$$\begin{aligned} & d_g(x_k, z')d_g(q, s_k) + d_g(x_k, q)d_g(z', s_k) \\ & = d_g(x_k, z')(d_g(q, s_k) - d_g(x_k, q)) + d_g(x_k, q)d_g(x_k, s_k) \end{aligned}$$

by the triangle inequality we have

$$\begin{aligned} & \Psi(U_0; x_k)(x_k) - f(s_k) \\ & \leq \omega\left(f; \frac{d_g(x_k, z')d_g(s_k, x_k) + d_g(x_k, q)d_g(x_k, s_k)}{d_g(x_k, z') + d_g(x_k, q)}\right) = \omega(f; d_g(s_k, x_k)). \end{aligned}$$

This last inequality clearly implies $\Psi(U_0; x_k)(x_k) \leq U_0(x_k)$.

Second case. if $s_k \in \tilde{V}(x_k)$, then

$$\Psi(U_0; x_k)(x_k) - f(s_k) \leq \sup_{q \in \tilde{V}(x_k)} \omega\left(f; \frac{d_g(x_k, s_k)d_g(q, s_k)}{d_g(x_k, s_k) + d_g(x_k, q)}\right).$$

Since

$$\frac{d_g(q, s_k)}{d_g(x_k, s_k) + d_g(x_k, q)} \leq 1$$

we have also $\Psi(U_0; x_k)(x_k) \leq U_0(x_k)$. We conclude that

$$\forall k = 1, \dots, N, \Psi(U_0; x_k) \leq U_0. \quad (5.14)$$

By Lemma 5.1 and this last inequality, we prove inductively that $\forall k = 1, \dots, N - 1, \Psi(U_0; x_{k+1}), \dots, \circ \Psi(U_0; x_1) \leq U_0$.

Therefore we have $U_1 \leq U_0$ and we deduce from Lemma (5.2) that sequence $(U_n)_{n \in \mathbb{N}}$ is decreasing.

By Lemmas 5.1,5.2,5.3 we prove inductively that

$$U_n \geq \inf_{s \in S} U_0(s).$$

The sequence $(U_n)_{n \in \mathbb{N}}$ is lower bounded and decreasing and therefore converges to a function denoted by $K(f)$. It remains to check that (5.11),(5.12) and (5.13) hold.

For any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$0 \leq \sup_{y \in G-S} (U_n(y) - K(f)(y)) \leq \epsilon, \forall n \geq N.$$

For $x_k \in G - S$ and $n > N$ we have

$$U_{n+1}(x_k) = \mu(\tilde{\Psi}; x_k)$$

with

$$\tilde{\Psi} = \Psi(U_n; x_{k-1}) \circ \Psi(U_n; x_{k-2}) \circ \dots \circ \Psi(U_n; x_1).$$

Since $\tilde{\Psi}(y) \leq U_n(y)$, $\forall y \in G - S$ we have

$$0 \leq \tilde{\Psi}(y) - K(f)(y) \leq \epsilon.$$

We can write

$$| \mu(K(f); x_k) - K(f)(x_k) | \leq | \mu(K(f); x_k) - \mu(\tilde{\Psi}; x_k) | + | U_{n+1}(x_k) - K(f)(x_k) |.$$

Since

$$| \mu(K(f); x_k) - \mu(\tilde{\Psi}; x_k) | \leq \sup_{y \in V(x_k)} | \tilde{\Psi}(y) - K(f)(y) | \leq \epsilon,$$

we obtain

$$| \mu(K(f); x_k) - K(f)(x_k) | \leq 2\epsilon.$$

By lemma 5.3, we have $U_0(s) = f(s)$, $\forall s \in S$ and $U_n(s) = f(s)$, $\forall s \in S$, and $\forall n \in \mathbb{N}$. Therefore $K(f)$ is an extension of f and we obtain the stated result. \square

Now, combining theorems 5.4 and 4.2, functional equation (1.1) has $K(f)$ as unique solution.

As a consequence of lemmas 5.1, 5.2, 5.3 and of theorems 4.1, 4.2 and 5.4 we have the following properties of stability of the extension scheme K :

Theorem 5.5 *Let f, g any two real-valued functions both of domain S . Then*

$$| K(f)(x) - K(f)(y) | \leq \omega(f; d_g(x, y)), \forall x, y \in G; \quad (5.15)$$

$$\sup_{x \in G} (K(f)(x) - K(g)(x)) \leq \sup_{s \in S} (f(s) - g(s)); \quad (5.16)$$

for any non-empty subsets A, B of S , we have

$$\sup_{x \in G} (K(f | A)(x) - K(f | B)(x)) \leq 4\omega(f; \delta_g(A, B)) , \tag{5.17}$$

where $f | A$ and $f | B$ denote the restrictions of f to A and B and δ_g Hausdorff metric constructed on geodesic metric d_g .

6 Approximation of an AMLE

Let f denote any Ω -continuous real-valued function whose domain is a compact non-empty subset S of E .

In this section we shall consider sequences $(G_n, V_n)_{n \in \mathbb{N}}$ of networks having the following properties:

(Q1) $\lim_{n \rightarrow \infty} r_n = 0$

where $r_n := \sup(\delta(G_n, E), \delta(S_n, S))$ and $S_n := S \cap G_n$;

(Q2) $\lim_{n \rightarrow \infty} \rho_n = 0$ where

$$\rho_n = \sup_{x \in G_n} \sup_{y \in V_n(x)} d(x, y);$$

(Q3) $\lim_{n \rightarrow \infty} \|d_n - d\| = 0$

where d_n denotes geodesic metric on (G_n, V_n) and

$$\|d_n - d\| := \sup_{x, y \in G_n} |d_n(x, y) - d(x, y)| .$$

We note $b_n(x)$ the open ball of center $x \in E$, radius r_n , and $B_n(x)$ the closed ball of center $x \in E$, radius ρ_n .

Lemma 6.1 shows that such sequences $(G_n, V_n)_{n \in \mathbb{N}}$ exist in any metrically convex metric space.

Lemma 6.1 *Sequences $(G_n, V_n)_{n \in \mathbb{N}}$ exist which satisfy properties (Q1), (Q2), (Q3).*

Proof. Let $(r_n)_{n \in \mathbb{N}}$ and $(\rho_n)_{n \in \mathbb{N}}$ be any two sequences of positive reals such that:

$$r_n \leq \rho_n, n \in \mathbb{N}; \tag{6.1}$$

$$\lim_{n \rightarrow \infty} \rho_n = 0; \tag{6.2}$$

$$\lim_{n \rightarrow \infty} \frac{r_n}{\rho_n} = 0. \tag{6.3}$$

For any $x \in E$, $n \in \mathbb{N}$, let us set $b_n(x) := \{y \in E : d(x, y) < r_n\}$, $B_n(x) := \{y \in E : d(x, y) \leq \rho_n\}$. Define (G_n, V_n) as follows. Since S is a compact subset of E , S is covered by balls $b_n(x)$, $x \in S$. Therefore there exists $x_1, \dots, x_k \in S$ such that $b_n(x_i), i = 1, \dots, k$ cover S . Now $E - \cup_{i=1}^k b_n(x_i)$ is a compact subset of E .

Let $x_{k+1}, \dots, x_m \in E - \cup_{i=1}^k b_n(x_i)$ such that $\cup_{i=k+1}^m b_n(x_i)$ cover $E - \cup_{i=1}^k b_n(x_i)$. Set $G_n := \{x_1, \dots, x_m\}$ and, for $x \in G_n$, set $V_n(x) := G_n \cap B_n(x)$. Note that, by construction, we have $\delta(S_n, S) \leq r_n$ and $\delta(G_n, E) \leq r_n$. Therefore properties (Q1), (Q2) are obviously satisfied.

Now let us show that for $n \in \mathbb{N}$ sufficiently large we have

- i) (G_n, V_n) is a network;
- ii) $\lim_{n \rightarrow \infty} \|d_n - d\| = 0$.

Properties (P1) and (P2) are immediate. To prove (P4) let $x, y \in G_n$, $y \notin V_n(x)$. By metrical convexity of E there exists $t \in E$ such that $d(x, t) = \rho_n - r_n$ and $d(x, y) = d(y, t) + d(t, x)$. Let $z \in G_n$ such that $d(z, t) \leq r_n$. One has $d(x, z) \leq d(x, t) + d(t, z) \leq \rho_n - r_n + r_n = \rho_n$. Therefore $z \in V_n(x)$. Moreover $d(y, z) \leq d(y, t) + d(t, z) \leq r_n + d(x, y) - \rho_n + r_n$. Since for n sufficiently large we have $2r_n - \rho_n < 0$, we infer that $d(x, z) < d(x, y)$.

Now let us prove both (P3) and ii). Let $N \in \mathbb{N}$, $N \geq 1$, and $x, y \in G_n$. By metrical convexity there exists elements of E $y_0 = x, y_1, \dots, y_N = y$ such that $d(y_i, y_{i+1}) = d(x, y)/N$ and $d(x, y) = \sum_{i=0}^{N-1} d(y_i, y_{i+1})$.

For each $i = 1, \dots, N-1$, choose $z_i \in G_n$ such that $d(z_i, y_i) \leq r_n$. We have

$$d(z_i, z_{i+1}) \leq d(z_i, y_i) + d(y_i, y_{i+1}) + d(y_{i+1}, z_{i+1}) \leq d(x, y)/N + 2r_n.$$

Now, choosing N such that

$$\frac{d(x, y)}{N} + 2r_n \leq \rho_n, \quad (6.4)$$

we have $z_i \in V_n(z_{i+1})$. Property (P3) is therefore proved. Moreover $d_g(x, y) \leq \sum_{i=0}^{N-1} d(z_i, z_{i+1})$. It follows that

$$d_g(x, y) - d(x, y) \leq \sum_{i=0}^{N-1} (d(z_i, z_{i+1}) - d(y_i, y_{i+1})) \leq \sum_{i=0}^{N-1} 2r_n = 2Nr_n.$$

Now, for n sufficiently large, one has $\rho_n > 2r_n$. Therefore (6.4) is satisfied by taking $N =$ the smaller integer larger than $d(x, y)/(\rho_n - 2r_n)$. It follows that

$$2Nr_n \leq 2d(x, y)/((\rho_n/r_n) - 2) + 2r_n.$$

Therefore $\|d_n - d\| \leq 2\Delta/((\rho_n/r_n) - 2) + 2r_n$ where Δ denotes the diameter of (E, d) . Since $\lim_{n \rightarrow \infty} r_n = 0$ and $\lim_{n \rightarrow \infty} \rho_n/r_n = 0$, lemma 6.1 is proved. \square

For each $n \in \mathbb{N}$, let us define:

- 1) the real-valued function f_n of domain S_n by

$$f_n(s) = f(s), \quad x \in S_n; \quad (6.5)$$

2) the real-valued function W_n of domain E by

$$W_n(x) = \inf_{s \in G_n} (K_n(f_n)(s) + \omega(f; d(s, x))), \quad x \in E; \quad (6.6)$$

where $K_n(f_n)$ (K_n for short) denotes the solution of (1.1) for network (G_n, V_n) under Dirichlet's condition f_n .

Lemma 6.2 *We have*

$$|W_n(x) - W_n(y)| \leq \omega(f; d(x, y)), \quad \forall x, y \in E; \quad (6.7)$$

$$\forall x \in E, \quad \inf_{s \in \text{dom}(f)} f(s) \leq W_n(x) \leq \sup_{s \in \text{dom}(f)} f(s) + \omega(f; r_n); \quad (6.8)$$

and

$$K_n(s) - \sup_{t \in G_n} \omega(f; d_n(t, s) - d(t, s)) \leq W_n(s) \leq K_n(f_n)(s), \quad \forall s \in G_n; \quad (6.9)$$

from which we infer

$$K_n(s) - \omega(f; \|d_n - d\|) \leq W_n(s) \leq K_n(s), \quad \forall s \in G_n. \quad (6.10)$$

Proof. For any $x, y \in E$, we have

$$W_n(x) - W_n(y) \leq \sup_{s \in G_n} (\omega(f; d(s, x)) - \omega(f; d(s, y))).$$

By triangular inequality we have $d(s, x) \leq d(s, y) + d(y, x)$.

By growth and subadditivity of $\omega(f)$ we have:

$$\omega(f; d(s, x)) \leq \omega(f; d(s, y) + d(y, x)) \leq \omega(f; d(s, y)) + \omega(f; d(y, x)).$$

Therefore $W_n(x) - W_n(y) \leq \omega(f; d(y, x))$.

For any $x \in E$ we have

$$W_n(x) \leq \inf_{s \in G_n} K_n(f_n)(s) + \sup_{s \in G_n} \omega(f; d(x, s)).$$

By property of moduli of continuity we have

$$W_n(x) \leq \inf_{s \in G_n} K_n(f_n)(s) + \sup_{s \in S} f(s) - \inf_{s \in S} f(s).$$

Using (5.16) we have $\inf_{s \in G_n} K_n(f_n)(s) = \inf_{s \in S_n} f(s)$.

Therefore

$$W_n(x) \leq \sup_{s \in S} f(s) + \inf_{s \in S_n} f(s) - \inf_{s \in S} f(s)$$

$$\text{and } W_n(x) \leq \sup_{s \in S} f(s) + \omega(f; r_n).$$

Moreover

$$W_n(x) \geq \inf_{s \in G_n} K_n(f_n)(s).$$

Using (5.16) again we have $\inf_{s \in G_n} K_n(f_n)(s) = \inf_{s \in S_n} f(s)$.

Therefore

$$W_n(x) \geq \inf_{s \in S} f(s). \text{ So (6.8) is proved.}$$

Let any $s_0 \in G_n$ we have

$$W_n(s_0) - K_n(f_n)(s_0) \leq K_n(f_n)(s_0) + \omega(f; d(s_0, s_0)) - K_n(f_n)(s_0) \leq 0,$$

and

$K_n(f_n)(s_0) - W_n(s_0) \leq \sup_{s \in G_n} (K_n(f_n)(s_0) - K_n(f_n)(s) - \omega(f; d(s_0, s)))$.
 By Ω -stability of $K_n(f_n)$ we have $K_n(f_n)(s) - K_n(f_n)(s_0) \leq \omega(f; d_n(s_0, s))$
 Therefore $K_n(f_n)(s_0) - W_n(s_0) \leq \sup_{s \in G_n} (\omega(f; d_n(s_0, s)) - \omega(f; d(s_0, s)))$
 and

$$K_n(f_n)(s_0) - W_n(s_0) \leq \sup_{s \in G_n} \omega(f; d_n(s_0, s) - d(s_0, s)) \leq \omega(f; \|d_n - d\|).$$

So (6.9) and (6.10) are proved. \square

Indeed, from Lemma 6.2, sequence $(W_n)_{n \in \mathbb{N}}$ is equicontinuous and equibounded. Therefore, by Ascoli's theorem, there exists a subsequence $(W_{\alpha(n)})_{n \in \mathbb{N}}$ which converges to a continuous function denoted by u .

Theorem 6.3 *The function u is an AM Ω E of f .*

Proof. We must prove that :

(i) u is an extension of f

$$u(s) = f(s), \forall s \in \text{dom}(f); \quad (6.11)$$

(ii) u is Ω -optimally continuous

$$|u(x) - u(y)| \leq \omega(f; d(x, y)), \forall x, y \in E; \quad (6.12)$$

(iii) for any open $D \subset E$, such that $D \cap \text{dom}(f) = \emptyset$, for any $x \in D$, we have

$$\sup_{y \in \delta D} (u(y) - \omega(u|_{\partial D}; d(x, y))) \leq u(x) \leq \inf_{y \in \delta D} (u(y) + \omega(u|_{\partial D}; d(x, y))). \quad (6.13)$$

For typographical convenience let us assume in the proof that subsequence $(W_{\alpha(n)})_{n \in \mathbb{N}}$ is sequence $(W_n)_{n \in \mathbb{N}}$ itself (the true proof can easily be restated: replace n by $\alpha(n)$ almost everywhere).

Let us show (i).

For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N, \|W_n - u\|_{\infty, E} \leq \epsilon$.

For any $n \geq N$ and $s \in \text{dom}(f)$ there exists by (Q1) $s_n \in S_n$ such that $d(s, s_n) \leq r_n$. We have

$$|u(s) - f(s)| \leq A_1 + A_2 + A_3,$$

where

$$A_1 = |u(s) - W_n(s)|, \quad A_2 = |W_n(s) - K_n(s)|,$$

$$A_3 = |K_n(s) - K_n(s_n)| + |f(s_n) - f(s)|.$$

We have $A_1 \leq \epsilon$. Using (6.10) we have $A_2 \leq \omega(f; \|d_n - d\|)$. Using (5.13) we have

$$|K_n(s) - K_n(s_n)| \leq \omega(f; d_n(s_n, s)) \leq \omega(f; r_n) + \omega(f; \|d_n - d\|).$$

In definitive we have

$$|u(s) - f(s)| \leq \epsilon + 2\omega(f; r_n) + 2\omega(f; \|d_n - d\|).$$

Since this inequality is true $\forall n \geq N$ and $\forall \epsilon > 0$ then, using (Q1),(Q3) and letting n tend to ∞ , we conclude that $u(s) = f(s)$ so we have proved (i).

The proof of (ii) is immediate by letting n tend to ∞ in inequality (6.7).

Let us show the right inequality of (iii). Let D an open subset of E such that $D \cap \text{dom}(f) = \emptyset$. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$ $\|W_n - u\|_{\infty, E} \leq \epsilon$.

Let $n \geq N$, $x \in D$ and $y \in \partial D$.

Using (2.2) we have

$$\omega(W_n|_{\partial D}; d(x, y)) - \omega(u|_{\partial D}; d(x, y)) \leq 2 \|W_n - u\|_{\infty, E}.$$

Now, setting

$$A := u(x) - u(y) - \omega(u|_{\partial D}; d(x, y)),$$

we have

$$A \leq 4\epsilon + A_1,$$

where

$$A_1 := W_n(x) - W_n(y) - \omega(W_n|_{\partial D}; d(x, y)).$$

Let $D_n := \{z \in G_n : b_n(z) \cap D \neq \emptyset\}$ and $\partial D_n = \{z \in G_n : b_n(z) \cap \partial D \neq \emptyset\}$.

Since $D \subset \cup_{z \in D_n} b_n(z)$ and $\partial D \subset \cup_{z \in \partial D_n} b_n(z)$, we have $\delta(\partial D, \partial D_n) \leq r_n$ and there exists $\tilde{x} \in D_n, \tilde{y} \in \partial D_n$ such that

$$d(\tilde{x}, x) \leq \delta(G_n, E) \leq r_n \text{ and } d(\tilde{y}, y) \leq \delta(G_n, E) \leq r_n.$$

By lemma 6.2, we have

$$W_n(x) - K_n(\tilde{x}) \leq \omega(f; r_n) + \omega(f; \|d_n - d\|),$$

and

$$W_n(y) - K_n(\tilde{y}) \leq \omega(f; r_n) + \omega(f; \|d_n - d\|).$$

By (2.3), we have

$$\omega(W_n|_{\partial D}; d(x, y)) - \omega(W_n|_{\partial D_n}; d(x, y)) \leq 4\omega(W_n; \delta(\partial D, \partial D_n)),$$

Since from Lemma 6.2, we have $\omega(W_n) \leq \omega(f)$, then

$$\omega(W_n|_{\partial D}; d(x, y)) - \omega(W_n|_{\partial D_n}; d(x, y)) \leq 4\omega(f; \delta(\partial D, \partial D_n)) \leq 4\omega(f; r_n).$$

Therefore

$$A_1 \leq A_2 + 6\omega(f; r_n) + 2\omega(f; \|d_n - d\|).$$

where

$$A_2 := K_n(\tilde{x}) - K_n(\tilde{y}) - \omega(W_n|_{\partial D_n}; d(x, y)).$$

Now let us bound A_2 from above. We write $A_2 := A_3 + A_4$ where

$$A_3 := \omega(K_n|_{\partial D_n}; d(\tilde{x}, \tilde{y})) - \omega(W_n|_{\partial D_n}; d(x, y)) \text{ and}$$

$$A_4 := K_n(\tilde{x}) - K_n(\tilde{y}) - \omega(K_n|_{\partial D_n}; d(\tilde{x}, \tilde{y})).$$

We have $d(\tilde{x}, \tilde{y}) - d(x, y) \leq d(\tilde{x}, x) + d(\tilde{y}, y) \leq 2r_n$.

Using the subadditivity of $\omega(W_n |_{\partial D_n})$, we infer that

$$\omega(W_n |_{\partial D_n}; d(\tilde{x}, \tilde{y})) - \omega(W_n |_{\partial D_n}; d(x, y)) \leq 2\omega(W_n |_{\partial D_n}; r_n) \leq 2\omega(f; r_n).$$

Furthermore, using (2.2) we have

$$\| \omega(K_n |_{\partial D_n}) - \omega(W_n |_{\partial D_n}) \|_{\infty, \mathbb{R}^+} \leq 2 \| K_n - W_n \|_{\infty, \partial D_n} \leq 2 \| K_n - W_n \|_{\infty, G_n}.$$

Now, using (6.10) we have

$$\| K_n - W_n \|_{\infty, G_n} \leq \omega(f; \| d_n - d \|). \text{ Therefore}$$

$$A_3 \leq A_4 + 2\omega(f; r_n) + \omega(f; \| d_n - d \|).$$

Now, we bound A_4 from above. By theorems 5.4 and 4.2, there exists a unique extension v of $K_n |_{\partial D_n}$ in G_n such that

$$\begin{cases} v(z) = \mu(v; z) & \forall z \in G_n - \partial D_n; \\ v(z) = K_n(z) & \forall z \in \partial D_n. \end{cases} \quad (6.14)$$

Moreover

$$v(z) - v(q) \leq \omega(K_n |_{\partial D_n}; d_n(z, q)), \quad \forall z, q \in G_n.$$

In particular we have

$$v(z) - v(q) - \omega(K_n |_{\partial D_n}; d_n(z, q)) \leq 0, \quad \forall q \in \partial D_n, \forall z \in D_n. \quad (6.15)$$

Now, we bound $\sup_{z \in D_n} |K_n(z) - v(z)|$. By symmetry we have only to bound $\Delta = \sup_{z \in D_n} (K_n(z) - v(z))$ from above. Let

$$F = \{z \in D_n : K_n(z) - v(z) = \Delta\}, \quad M = \sup_{z \in F} K_n(z),$$

and

$$\tilde{F} = \{z \in F : K_n(z) = M\}.$$

Let $z_0 \in \tilde{F}$ be such that

$$d(z_0, (S_n \cup \partial D_n)) = \inf_{z \in \tilde{F}} d(z, (S_n \cup \partial D_n)). \quad (6.16)$$

Let us first show that we cannot have $d(z_0, (S_n \cup \partial D_n)) > 0$. Towards a contradiction let us assume it is the case. We have

$v(z_0) = \mu(v; z_0)$ and $K_n(z_0) = \mu(K_n; z_0)$. Using a similar argument to this of theorem 4.1, we infer that $V_n(z_0) \subset \tilde{F}$. Using property (P4) there exists $y \in V_n(z_0)$ such that

$$d(y, (S_n \cup \partial D_n)) < d(z_0, (S_n \cup \partial D_n))$$

which is a contradiction with definition (6.16) of z_0 .

Now, $d(z_0, (S_n \cup \partial D_n)) = 0$. If $z_0 \in \partial D_n$, since $K_n(z_0) = v(z_0)$ we have $\Delta = 0$.

If $z_0 \in S_n$ we remark that $z_0 \in D_n$. Since $z_0 \in S_n \cap D_n$, $S_n \cap D = \emptyset$ and $D_n := \{z \in G_n : b_n(z) \cap D \neq \emptyset\}$ there exists $y_0 \in D$ such that $d(z_0, y_0) \leq r_n$.

Moreover since $z_0 \notin D$, and by metrical convexity of E there exists $p_0 \in \partial D$ such that $d(z_0, y_0) = d(z_0, p_0) + d(p_0, y_0) \leq r_n$.

By definition of ∂D_n there exists $q_0 \in \partial D_n$ such that $d(p_0, q_0) \leq r_n$. Therefore $d(z_0, q_0) \leq d(z_0, p_0) + d(p_0, q_0) \leq 2r_n$.

We conclude that

$$\Delta = K_n(z_0) - K_n(q_0) + v(q_0) - v(z_0) \leq 4\omega(f; r_n). \quad (6.17)$$

From inequalities (6.15) and (6.17), we obtain $A_4 \leq 4\omega(f; r_n)$.

Finally we obtain

$$A \leq 4\epsilon + 12\omega(f; r_n) + 3\omega(f; \|d_n - d\|).$$

Since this inequality is true $\forall n \geq N$ then, using (Q1),(Q2),(Q3) and letting n tend to ∞ , we conclude that

$$A \leq 4\epsilon,$$

which proves the right inequality of (iii). The proof of the left inequality of (iii) is similar but not symmetric because of choice of W_n . However it leads to similar bounds. \square

7 Numerical tests

The tests of this section are done for the following network: G_n is the set of points $(i.h, j.h)$ $i, j = 0, \dots, n$, $h = 1/n$ which densifies $\Omega = [0, 1] \times [0, 1]$ (eventually zoomed and shifted), $b_n(x)$ is the ball of center $x \in G_n$ radius h , $V_n(x)$ the ball of center $x \in G_n$ radius $k.h$. Since norms on \mathbb{R}^2 are equivalent, balls $b_n(x)$ and $V_n(x)$ can (and will for convenience of implementation), be chosen to be those corresponding to $\|\cdot\|_\infty$ or $\|\cdot\|_1$. Note that, for fixed n , geodesic metric on (G_n, V_n) will approach euclidean metric on Ω better and better when k increases. The errors in the following tables are

$$e_{n,k} = \sup_{x,y \in G_n} |u_{n,k}(x,y) - u(x,y)|$$

where $u_{n,k}$ is the solution of 1.1 of section 1 for $S = S_n = \partial\Omega \cap G_n$ and $f = u|_{S_n}$. We first test the algorithm in situations where the solution of the continuous problem is unique and known. $u(x,y) = r, \theta, r^{1/2}e^{\theta/2}$ in polar coordinates, $x^{4/3} - y^{4/3}$, in euclidean plane. It is seen that, for a fixed k , error becomes stationary when n increases.

Table 7.1: $u(x, y) = r$

k/n	8	16	32	64	128	256
1	0.023	0.023	0.023	0.023	0.023	0.02
2	0.0063	0.0063	0.0066	0.0069	0.007	0.0067
3	0.0062	0.0031	0.0031	0.0031	0.0032	0.0032
4	0.007	0.0037	0.00205	0.0018	0.0018	0.0018
5	0.0074	0.0037	0.0021	0.001143	0.00118	0.0018
6	0.0074	0.004	0.0022	0.001135	0.000822	0.00082
7	0.0079	0.004	0.0023	0.001178	0.000602	0.000571

(7.1)
Table 7.2: $u(x, y) = \theta$

k/n	8	16	32	64	128	256
1	0.0251	0.0141	0.0139	0.0138	0.0138	0.0138
2	0.165	0.125	0.0347	0.00867	0.00556	0.00421
3	0.236	0.154	0.0814	0.0203	0.0054	0.00236
4	0.244	0.191	0.0958	0.0347	0.0088	0.0012

(7.2)
Table 7.3: $u(x, y) = r^{1/2}e^{\theta/2}$

k/n	8	16	32	64	128
1	0.156	0.112	0.0792	0.0557	0.0402
2	0.22	0.159	0.1123	0.0812	0.0557
3	0.22	0.195	0.1377	0.0971	0.068

(7.3)

Note that we obtain better approximations if we give “thickness kh ” to the boundary that is if we approach the solution of PDE $\Delta_\infty u = 0$ under Dirichlet’s condition $u|_{\partial_{kh}\Omega} = u_0$ where $\partial_\epsilon\Omega = [0, 1] \times [0, 1] -]\epsilon, 1 - \epsilon[\times]\epsilon, 1 - \epsilon[$.

Table 7.4: $u(x, y) = r$

k/n	8	16	32	64	128
2	0.0037	0.0040	0.0047	0.0057	0.0057
3	0.0014	0.0015	0.0015	0.0018	0.0023
4	0.0000	0.0008	0.0008	0.0008	0.00088
5	0.0000	0.0004	0.0005	0.0005	0.0005

(7.4)

Next we test the algorithm in situations where uniqueness of the solution of the continuous problem is not known: $u_1(x, y) = x^2 - y^2$ for $\|\cdot\|_\infty$, $u_2(x, y) = |x| - |y|$ for $\|\cdot\|_1$.

We note that, in these cases, geodesic metric on G_n coincides, for any k , with metric on $[0, 1] \times [0, 1]$. So, in these cases, we can take $k = 1$. Numerical

tests 7.5 show that the algorithm computes exactly u_2 and that error is linear in h for u_1 (see Table 7.5).

Table 7.5: $u_1(x, y) = x^2 - y^2, u_2(x, y) = |x| - |y|$.

n	8	16	32	64	128	256
e_1	0.06	0.03	0.015	0.0076	0.0038	0.0019
e_2	0.	0.	0.	0.	0.	0.

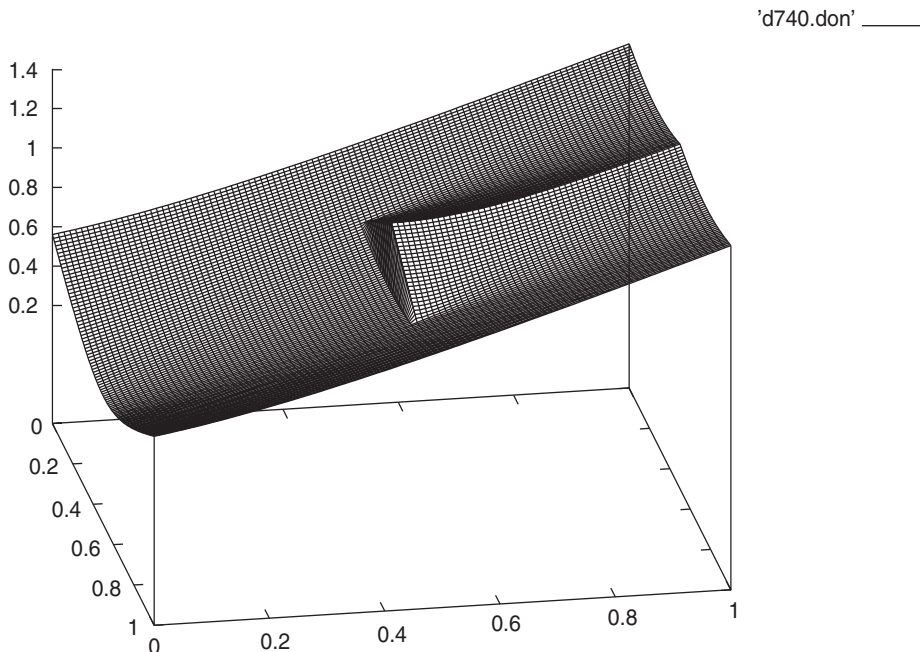
(7.5)


Figure 7.1: $u_0(x, y), n = 100$

To finish we consider the following two examples. In these examples the metrically convex metric space (E, d) is $E = [0, 1] \times [0, 1] -]1/4 - \epsilon, 3/4 + \epsilon[\times]1/2 - \epsilon, 1/2 + \epsilon[$ for $\epsilon > 0$ small and metric d on E is the geodesic metric constructed from local euclidean metric. We set $u_0(x, y) = d((1/2, 0), (x, y))$. Figure 7.1 shows function u_0 in restriction to G_n . In the first example we compute the unique solution of $\Delta_\infty u = 0, u | \Gamma = u_0$ where Γ is the union of the boundaries of internal and external rectangles. Numerical tests show that the solution in E is different from u_0 : this observation corroborates the fact that geodesic cones are not AMLE in general metrically convex metric space (see [2], appendix).

In the second example we compute $\Delta_\infty u = 0, u | \Gamma = u_0$ where Γ is now the boundary of the external rectangle alone. We obtain a solution which is different

from u_0 and from the solution of the first example. Note the difference between the two examples. In the first one we really compute the solution of PDE $\Delta_\infty u = 0$ because $E - \Gamma$ is locally euclidean. It is not the case in the second example because the space (E, d) is not locally euclidean at points of the “free internal boundary”. In fact, in this second example it is likely (we are not insured of the convergence of sequence $(W_n)_{n \in \mathbb{N}}$ in Theorem 6.3) that we compute a AMLE of u_0 .

8 Appendix

As announced in Remark 3.7 we prove that

$$\left(\frac{1}{2} \sup_{y \in B_h(x)} u(y) + \frac{1}{2} \inf_{y \in B_h(x)} u(y) - u(x)\right)/h^2 = \frac{1}{2} \Delta_\infty u(x) + o(h)$$

when u is smooth and $Du(x) \neq 0$.

Here $\Delta_\infty u = D^2u(D^*u, D^*u)$ and $D^*u = Du / |Du|$.

Since $Du(x) \neq 0$ then, for h sufficiently small, we have $Du(y) \neq 0$ for any $y \in B_h(x)$. Therefore u attains its maximum u^+ and its minimum u^- on the boundary of $B_h(x)$. Let us denote x^+ and x^- any points of this boundary such that $u(x^+) = u^+$, $u(x^-) = u^-$. Since x^+ is a maximum of $u(y)$ under the constraint $|y - x| = h$, vectors $x^+ - x$ and $Du(x^+)$ have the same direction that is $x^+ - x = hD^*u(x^+)$. For the same reason vectors $x^- - x$ and $Du(x^-)$ have opposite direction that is $x^- - x = -hD^*u(x^-)$.

Now, using these expressions of $x^+ - x$ and $x^- - x$ and Taylor formula at x^+ and x^- we obtain

$$2u(x) = u(x^+) + u(x^-) + A + B + h^2o(h)$$

where $A = h(-|Du|(x^+) + |Du|(x^-))$ and $B = \frac{1}{2}h^2(D^2u(x^+; D^*(x^+), D^*(x^+)) + D^2u(x^-; D^*(x^-), D^*(x^-)))$.

Now, using Taylor formula at x for $|Du|(x^+)$ we have

$$|Du|(x^+) = |Du|(x) + hD(|Du|)(x; D^*u(x^+)) + ho(h).$$

By continuity of $y \rightarrow D^*u(y)$, it follows that

$$|Du|(x^+) = |Du|(x) + hD(|Du|)(x; D^*u(x)) + ho(h).$$

Similarly,

$$|Du|(x^-) = |Du|(x) - hD(|Du|)(x; D^*u(x)) + ho(h).$$

Since a straightforward computation shows that $D(|Du|)(x; D^*u(x)) = \Delta_\infty u(x)$, and since maps $y \rightarrow D^*u(y)$ and $y \rightarrow D^2u(y)$ are continuous we obtain in definitive

$$2u(x) = u(x^+) + u(x^-) - 2h^2\Delta_\infty u(x) + h^2\Delta_\infty u(x) + h^2o(h),$$

which is the announced formula.

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