



On Perfect Sequence Covering Arrays

Aidan R. Gentle  and Ian M. Wanless

Abstract. A $\text{PSCA}(v, t, \lambda)$ is a multiset of permutations of the v -element alphabet $\{0, \dots, v-1\}$, such that every sequence of t distinct elements of the alphabet appears in the specified order in exactly λ of the permutations. For $v \geq t \geq 2$, we define $g(v, t)$ to be the smallest positive integer λ , such that a $\text{PSCA}(v, t, \lambda)$ exists. We show that $g(6, 3) = g(7, 3) = g(7, 4) = 2$ and $g(8, 3) = 3$. Using suitable permutation representations of groups, we make improvements to the upper bounds on $g(v, t)$ for many values of $v \leq 32$ and $3 \leq t \leq 6$. We also prove a number of restrictions on the distribution of symbols among the columns of a PSCA.

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1. Introduction

For positive integers v and t with $v \geq t$, we let $[v] = \{0, \dots, v-1\}$, \mathcal{S}_v be the set of permutations of $[v]$ and $\mathcal{S}_{v,t}$ be the set of ordered sequences of t distinct elements of $[v]$. For $\pi \in \mathcal{S}_v$ and $s = (s_0, \dots, s_{t-1}) \in \mathcal{S}_{v,t}$, we say that s is *covered* by π if $\pi^{-1}(s_i) < \pi^{-1}(s_{i+1})$ for $0 \leq i \leq t-2$. A *perfect sequence covering array* with order v , strength t , and multiplicity λ , denoted by $\text{PSCA}(v, t, \lambda)$, is a multiset X of permutations in \mathcal{S}_v such that every sequence in $\mathcal{S}_{v,t}$ is covered by exactly λ permutations in X . If we let T be a t -subset of $[v]$, then there are $t!$ orderings of the symbols of T , each of which must be covered by λ permutations in a $\text{PSCA}(v, t, \lambda)$. Furthermore, every permutation in a $\text{PSCA}(v, t, \lambda)$ covers exactly one ordering of T , so a $\text{PSCA}(v, t, \lambda)$ must consist of $t!\lambda$ permutations.

Perfect sequence covering arrays were introduced by Yuster [10] in 2020 as a variant of sequence covering arrays. *Sequence covering arrays*, denoted by $\text{SCA}(v, t)$, are sets of permutations in \mathcal{S}_v in which every sequence in $\mathcal{S}_{v,t}$ is covered by at least one permutation in the set. The study of SCAs dates

back to Spencer [9] in 1971. They are useful for constructing test suites for situations where the order of operations may be important.

For $v \geq t$ define $g(v, t)$ to be the smallest positive integer λ , such that a $\text{PSCA}(v, t, \lambda)$ exists. Observe that \mathcal{S}_v is a $\text{PSCA}(v, t, v!/t!)$, so $g(v, t)$ is well defined and $g(v, t) \leq v!/t!$. Note that if $v > t$ and we remove the symbol $v - 1$ from every permutation of a $\text{PSCA}(v, t, \lambda)$, then we obtain a $\text{PSCA}(v - 1, t, \lambda)$, and hence, $g(v, t) \geq g(v - 1, t)$. For $2 \leq t' \leq t$, a $\text{PSCA}(v, t, \lambda)$ is also a $\text{PSCA}(v, t', \lambda \binom{t}{t'})$, so $g(v, t') \leq \binom{t}{t'} g(v, t)$.

The question of when $g(v, t) = 1$ has received particular attention. Not only would a $\text{PSCA}(v, t, 1)$ be the smallest possible $\text{SCA}(v, t)$, but it is also an object of interest in coding theory. A $(v - t)$ -deletion correcting code is a set X of permutations in \mathcal{S}_v , such that every sequence in $\mathcal{S}_{v,t}$ is covered by *at most* one permutation in X . Hence, a $\text{PSCA}(v, t, 1)$ would be the largest possible $(v - t)$ -deletion correcting code. For more on deletion correcting codes, see [3, 5]. Note that \mathcal{S}_v forms a $\text{PSCA}(v, v, 1)$. At the other end of the spectrum, if $t = 2$, then we can take any permutation in \mathcal{S}_v and its reverse to form a $\text{PSCA}(v, 2, 1)$. Therefore, $g(v, v) = g(v, 2) = 1$. Levenshtein [5] proved that $g(t + 1, t) = 1$ for $t \geq 3$. Mathon and van Trung [6] proved that a $\text{PSCA}(5, 3, 1)$ does not exist (we provide a new proof of this fact in Sect. 2). As demonstrated above, $g(v, t) \geq g(v - 1, t)$ so it follows that a $\text{PSCA}(v, 3, 1)$ does not exist for $v \geq 5$. Therefore, when $t = 3$, we have $g(v, t) > 1$ for $v > t + 1$. It was initially conjectured by Levenshtein that this property would hold for any $t \geq 3$; however, that was later shown to be false for $t = 4$ by Mathon and van Trung [6], who presented a $\text{PSCA}(6, 4, 1)$. On the other hand, Mathon and van Trung computationally proved that neither a $\text{PSCA}(7, 5, 1)$ nor a $\text{PSCA}(8, 6, 1)$ exists, thus confirming Levenshtein's conjecture for $t \in \{5, 6\}$. They also found that a $\text{PSCA}(7, 4, 1)$ does not exist. A combinatorial proof of this last fact was later given by Klein [3]. Chee et al. [1] proved that $g(2t, t) > 1$ for $t \geq 3$.

Yuster [10] proved that $g(5, 3) = 2$. In Sect. 3, we show that $g(6, 3) = g(7, 3) = g(7, 4) = 2$, $g(8, 3) = 3$, and $g(8, 4) \geq 3$. We state here a definition which we expand upon further in Sect. 2.

Definition 1.1. For a multiset X of permutations of $[v]$, a symbol $w \in [v]$, and for $0 \leq i \leq v - 1$, we define

$$d_w(i) := |\{\pi \in X : \pi(i) = w\}|.$$

We refer to the vector $\mathbf{d}_w = (d_w(0), \dots, d_w(v - 1))$ as the *distribution vector* of w .

In Sect. 2, we derive restrictions on distribution vectors of symbols in PSCAs. These restrictions facilitate the computer searches that we use to exhaustively catalogue $\text{PSCA}(v, t, \lambda)$ for different sets of parameters. These searches and their results are described in Sect. 3. All computational results reported in this paper were checked by both authors using independent computations.

TABLE 1. Improvements to known upper bounds on $g(v, 3)$.
An asterisk denotes that the new bound is exact

v	New bound	Old bound
6–7	2*	8
8	3*	8
9	6	8
10–12	6	160
13–14	7	160
15–16	16	160
17–19	19	160
20–32	96	160

Although there are still few parameters (v, t) for which $g(v, t)$ has been determined exactly, there are some known results on the asymptotic behaviour of $g(v, t)$. A lower bound for $g(v, t)$ for $t > 3$ was proved by Yuster [10] using a matrix rank argument. If $t/2$ is a prime, then for $v \geq t$

$$g(v, t) \geq \frac{\binom{v}{t/2} - \binom{v}{t/2-1}}{t!}.$$

Additionally, for an absolute constant c , Kuperberg, Lovett and Peled [4] guarantee the existence of a $\text{PSCA}(v, t, \lambda)$ with $(cv)^{ct}$ permutations using probabilistic methods. Thus

$$g(v, t) \leq \frac{(cv)^{ct}}{t!}.$$

Yuster [10] also proved an upper and lower bound on $g(v, 3)$. The upper bound comes from a general construction of a PSCA with $(v, t) = (3^n, 3)$ for $n \geq 1$ that is built from an affine plane. The lower bound is proved using a similar matrix rank argument to the more general case above. Combining these bounds, we have that for an absolute constant c

$$\frac{v}{6} \leq g(v, 3) \leq cv(\log v)^{\log 7}.$$

In Sect. 4, we explore the relationship between groups and PSCAs and use this relationship to construct PSCAs with strengths 3 and 4, thereby improving upon the best known upper bounds for $g(v, 3)$ for $9 \leq v \leq 32$ and providing non-trivial upper bounds for $g(v, t)$ for $v \leq 24$ and $4 \leq t \leq 6$. Table 1 summarises the improvements to upper bounds on $g(v, 3)$, while Table 2 summarises the new results for $4 \leq t \leq 6$. The tables also incorporate the exact bounds shown in Sect. 3.

Independently, and using different methods, Na, Jedwab, and Li [8] have also considered the problem of determining $g(v, t)$. They find that $g(6, 3) = g(7, 3) = g(7, 4) = 2$, while also demonstrating that $g(8, 3) \leq 3$, $g(9, 3) \leq 4$ and $g(7, 5) \leq 4$. They also show that for $(v, t) \in \{(5, 3), (6, 3), (7, 3), (7, 4)\}$, a $\text{PSCA}(v, t, \lambda)$ exists if and only if $\lambda \geq 2$, while a $\text{PSCA}(8, 3, \lambda)$ exists for any $\lambda \geq 3$. Several of these results were originally reported in Na's Masters

TABLE 2. New upper bounds for $g(v, t)$ for $4 \leq t \leq 6$. An asterisk denotes that the new bound is exact

v	t	New bound
7	4	2*
8–12	4	18
13	4	234
14–21	4	5040
22	4	18,480
23	4	425,040
24	4	10,200,960
7–11	5	66
12	5	792
13–22	5	3696
23	5	85,008
24	5	2,040,192
8–12	6	132
13–24	6	340,032

thesis [7]; in particular, he reported that $g(7, 4) = 2$ before we computed our catalogue of $\text{PSCA}(7, 4, 2)$.

2. Distribution Vectors

Recall the definition of distribution vectors given in Definition 1.1. The distribution vector of w records the number of times a symbol w appears in each column across a multiset of permutations. In this section, we will derive several restrictions on distribution vectors for symbols in PSCAs. We begin with the following lemma which limits the number of occurrences of each symbol in a PSCA across sets of consecutive columns.

Lemma 2.1. *Let X be a $\text{PSCA}(v, t, \lambda)$ with $v \geq t \geq 2$ and $\lambda \geq 1$. Then, for $w \in [v]$ and for $0 \leq i \leq t - 1$*

$$\frac{\lambda(v-1)!}{(v-t)!} = \sum_{j=0}^{v-1} d_w(j) \binom{j}{i} \binom{v-1-j}{t-1-i}.$$

Proof. Let X be a $\text{PSCA}(v, t, \lambda)$, let $i \in \{0, \dots, t - 1\}$ and let $w \in [v]$. Let $S = \{s \in \mathcal{S}_{v,t} : s(i) = w\}$. Note that $|S| = (v - 1)!/(v - t)!$ and each sequence in S is covered by λ permutations in X . For some $j \in [v]$, let $\pi \in X$ be one of the $d_w(j)$ permutations in X , such that $\pi(j) = w$ and consider how many sequences in S are covered by π . There are j symbols that appear before w and $v - 1 - j$ symbols that appear after w in π . For every sequence in S , there are i symbols appearing before w and $t - 1 - i$ symbols appearing after w . Hence, π covers $\binom{j}{i} \binom{v-1-j}{t-1-i}$ sequences in S . The result follows. □

We can now use Lemma 2.1 to prove the following theorem.

Theorem 2.2. *Let X be a $\text{PSCA}(v, t, \lambda)$ with $v \geq t \geq 2$ and $\lambda \geq 1$. Then, for $w \in [v]$ and for $1 \leq s < t$*

$$\frac{1}{t!\lambda} \sum_{j=0}^{v-1} j^s d_w(j) = \frac{1}{v} \sum_{i=0}^{v-1} i^s. \quad (1)$$

Proof. Fix $s \in \{1, \dots, t-1\}$ and $w \in [v]$. Let $\alpha_k(w) = \sum_{j=0}^{v-1} j^k d_w(j)$ for $k \in \{1, \dots, s\}$. Now, X is a $\text{PSCA}(v, k+1, \lambda \binom{t}{k+1})$, and so, using Lemma 2.1 with $i = k$, we find that $\alpha_k(w)$ is a function of λ, v, t, k and $\alpha_1(w), \dots, \alpha_{k-1}(w)$. Therefore, proceeding by induction on k , we have that $\alpha_k(w)$ is independent of w for each $k \in \{1, \dots, s\}$. Thus

$$v\alpha_s(w) = \sum_{w \in [v]} \alpha_s(w) = \sum_{\pi \in X} \sum_{i=0}^{v-1} i^s = t!\lambda \sum_{i=0}^{v-1} i^s,$$

and hence, (1) holds. \square

We call a distribution vector that satisfies (1) for parameters (v, t, λ) a (v, t, λ) -feasible distribution. We now prove some more facts about distribution vectors when certain restrictions on v and t are imposed. The following theorem demonstrates more stringent restrictions on the distribution vector whenever t is an odd prime. Intuitively, it states that in a PSCA whose strength is an odd prime p and whose order is not divisible by p , the number of occurrences of a symbol across all columns of a given equivalence class modulo p is itself divisible by p .

Theorem 2.3. *Let X be a $\text{PSCA}(v, p, \lambda)$ with p an odd prime and with $v \not\equiv 0 \pmod{p}$. For $w \in [v]$ and $0 \leq j \leq p-1$, let $y_w(j) = \sum_{i \equiv j \pmod{p}} d_w(i)$. Then, $y_w(j) \equiv 0 \pmod{p}$.*

Proof. Let $w \in [v]$. By Theorem 2.2, if X is a $\text{PSCA}(v, p, \lambda)$, then for $1 \leq i \leq p-1$

$$\sum_{j=0}^{v-1} j^i d_w(j) = \frac{p!\lambda}{v} \sum_{j=0}^{v-1} j^i.$$

As v is not divisible by p , the right-hand side of the equation above must be divisible by p . Therefore, for $1 \leq i \leq p-1$

$$0 \equiv \sum_{j=0}^{v-1} j^i d_w(j) \equiv \sum_{j=1}^{p-1} j^i y_w(j) \pmod{p}.$$

This gives a system of $p-1$ linear equations in $p-1$ variables over the field \mathbb{F}_p . We can restate this system as

$$A \begin{pmatrix} y_w(1) \\ y_w(2) \\ \vdots \\ y_w(p-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where A is a $(p-1) \times (p-1)$ matrix over \mathbb{F}_p with $A_{i,j} = j^i$. Therefore, A is a Vandermonde matrix and thus, A is non-singular. Hence, the only solution to this system is $y_w(j) \equiv 0 \pmod p$ for all $j \in \{1, \dots, p-1\}$. As the number of permutations in X is $p!\lambda \equiv 0 \pmod p$, it also follows that $y_w(0) \equiv 0 \pmod p$. \square

When $v = t+1$ and t is even, the following lemma proves that all (v, t, λ) -feasible distribution vectors are palindromic.

Lemma 2.4. *Let t be even and d_w be a $(t+1, t, \lambda)$ -feasible distribution. Then, $d_w(t/2 - i) = d_w(t/2 + i)$ for $0 \leq i \leq t$.*

Proof. With $v = t+1$, Lemma 2.1 implies that

$$\lambda t! = (t-i)d_w(i) + (i+1)d_w(i+1)$$

for $0 \leq i \leq t-1$. Therefore, for even t

$$\begin{aligned} d_w\left(\frac{t}{2} - i\right) &= \frac{\lambda t! - \left(\frac{t}{2} - i + 1\right) d_w\left(\frac{t}{2} - i + 1\right)}{\frac{t}{2} + i} \\ d_w\left(\frac{t}{2} + i\right) &= \frac{\lambda t! - \left(\frac{t}{2} - i + 1\right) d_w\left(\frac{t}{2} + i - 1\right)}{\frac{t}{2} + i} \end{aligned}$$

for $0 \leq i \leq t/2$. Then, induction on i shows that $d_w(t/2 - i) = d_w(t/2 + i)$ for $0 \leq i \leq t/2$. \square

We continue with the case where $v = t+1$. The only possible PSCA(t, t, λ) is a multiset containing λ copies of \mathcal{S}_v . Therefore, this is exactly the PSCA we obtain by deleting any symbol from a PSCA($t+1, t, \lambda$) (throughout the paper, whenever we delete a symbol from a PSCA with ground set $[v]$, we assume the remaining symbols get relabelled to $[v-1]$ in an order preserving way). We use this fact to derive further restrictions for a PSCA($t+1, t, \lambda$). Let X be a PSCA(v, t, λ). For $w \in [v]$ and $I \subseteq [v] \setminus \{w\}$ with $|I| = i$, let $d_{I,w}$ be the number of permutations $\pi \in X$, such that $\pi(i) = w$ and $I = \{\pi(j) : 0 \leq j \leq i-1\}$.

Theorem 2.5. *Let X be a PSCA($t+1, t, 1$), let $w \in [v]$, and let $0 \leq i \leq t$. Then, for any i -subset $I \subseteq [v] \setminus \{w\}$*

$$d_{I,w} = \frac{d_w(i)}{\binom{t}{i}}.$$

Proof. Let $w \in [v]$. We proceed by induction on i . Note that the statement is trivially true for $i = 0$. Suppose the statement is true for some i with $0 \leq i \leq t-1$ and consider the statement for $i+1$. Let $I = \{u_1, \dots, u_{i+1}\} \subseteq [v] \setminus \{w\}$ and let $J = I \setminus \{u_{i+1}\}$. The array formed by removing u_{i+1} from each permutation of X is \mathcal{S}_t . The number of permutations $\tau \in \mathcal{S}_t$ for which $\tau(i) = w$ and $\{\tau(j) : 0 \leq j \leq i-1\} = J$ is $i!(t-1-i)!$. These permutations exactly correspond to the permutations $\pi \in X$, such that either $\pi(i) = w$ and $\{\pi(j) : 0 \leq j \leq i-1\} = J$ or $\pi(i+1) = w$ and $\{\pi(j) : 0 \leq j \leq i\} = I$. Thus

$$i!(t-1-i)! = d_{J,w} + d_{I,w}.$$

By the inductive hypothesis, $d_{J,w} = d_w(i)/\binom{t}{i}$. Therefore, for any two $(i+1)$ -subsets of $[v] \setminus \{w\}$, I and I' , $d_w(I) = d_w(I')$. The sum of $d_{I,w}$ as I ranges

over all $\binom{t}{i+1}$ possible $(i+1)$ -subsets of $[v] \setminus \{w\}$ must be $d_w(i+1)$. Therefore, $d_{I,w} = d_w(i+1)/\binom{t}{i+1}$, completing the induction. \square

Corollary 2.6. *Let X be a PSCA($t+1, t, 1$), and let $0 \leq i \leq t$. Then, $d_w(i)$ is divisible by $\binom{t}{i}$ for all $w \in [v]$.*

In general, if it could be shown that there are no (v, t, λ) -feasible distributions for some choice of v, t and λ , then it would imply that a PSCA(v, t, λ) does not exist. However, it is possible to find (v, t, λ) -feasible distributions for infinitely many choices of v, t , and λ . For example, if $t!\lambda$ is divisible by v , then a distribution vector with $d_w(i) = t!\lambda/v$ for $0 \leq i \leq v-1$ is (v, t, λ) -feasible.

On the other hand, it is possible to use (v, t, λ) -feasible distributions to disprove the existence of a PSCA(v, t, λ) even when such distributions do exist. For example, consider the $(5, 3, 1)$ -feasible distributions. By Theorem 2.3, for such a distribution, $d_w(2) \in \{0, 3, 6\}$. If $d_w(2) = 6$, then $d_w = (0, 0, 6, 0, 0)$ which violates (1) for $s = 2$. Now, suppose $d_w(2) = 3$. Again, by Theorem 2.3, $\{d_w(0) + d_w(3), d_w(1) + d_w(4)\} = \{0, 3\}$. As the reverse of a PSCA is also a PSCA, we can without loss of generality suppose $d_w(0) + d_w(3) = 0$. Then, for $s = 2$, (1) reduces to $d_w(1) + 16d_w(4) = 24$. As $d_w(1)$ and $d_w(4)$ must be nonnegative integers that sum to 3, we find that this equation has no solutions. Therefore, in any $(5, 3, 1)$ -feasible distribution, $d_w(2) = 0$. This means that if a PSCA($5, 3, 1$) exists, then it would be impossible to place any symbol in column 2. This contradiction provides an alternate proof of the non-existence of a PSCA($5, 3, 1$). See [6] for an earlier proof.

In the proof of Theorem 2.5, we were able to enforce restrictions on a PSCA($t+1, t, \lambda$) by considering the new array formed by deleting a symbol from this PSCA. We consider this kind of symbol deletion in a more general setting with the following theorem.

Theorem 2.7. *Let $\mathbf{d}_w = (d_w(0), \dots, d_w(v-1))$ be the distribution vector for a symbol w in X , a PSCA(v, t, λ). Let $\mathbf{d}'_w = (d'_w(0), \dots, d'_w(v-2))$ be the distribution vector of w in the PSCA X' obtained by deleting a symbol $w' \neq w$ from X . Then*

$$\delta_k = \sum_{i=0}^k (d'_w(i) - d_w(i))$$

satisfies $0 \leq \delta_k \leq d'_w(k)$ for $0 \leq k \leq v-2$.

Proof. Define $c_i = |\{\pi \in X : \pi^{-1}(w) = i < \pi^{-1}(w')\}|$ and $c'_i = |\{\pi \in X : \pi^{-1}(w) = i > \pi^{-1}(w')\}|$ for $0 \leq i \leq v-1$. Then, $c_i + c'_i = d_w(i)$ and $c_i + c'_{i+1} = d'_w(i)$. Now, $c'_0 = 0$ and $c'_{i+1} - c'_i = d'_w(i) - c_i - c'_i = d'_w(i) - d_w(i)$. Therefore, it follows by induction on i that $\delta_i = c'_{i+1}$ for $0 \leq i \leq v-2$. The result then follows from the fact that $c_i \geq 0$ and $c'_i \geq 0$ for each i , by definition. \square

We say that \mathbf{d}_w and \mathbf{d}'_w are *compatible* if they satisfy Theorem 2.7. This test can be used to eliminate some distributions from consideration. If \mathbf{d}_w is (v, t, λ) -feasible, it may be the case that there is no $(v-1, t, \lambda)$ -feasible

distribution \mathbf{d}'_w compatible with \mathbf{d}_w . It may even happen that there is a compatible \mathbf{d}'_w , but that all such candidates can themselves be ruled out, because they are not compatible with a $(v-2, t, \lambda)$ -feasible distribution, and so on. A concrete example is that $(2, 6, 1, 1, 6, 2)$ is a $(6, 3, 3)$ -feasible distribution. The only $(5, 3, 3)$ -feasible distribution that it is compatible with is $(3, 6, 0, 6, 3)$. However, $(3, 6, 0, 6, 3)$ is not compatible with any of the four $(4, 3, 3)$ -feasible distributions, which are $(3, 9, 0, 6)$, $(4, 6, 3, 5)$, $(5, 3, 6, 4)$, and $(6, 0, 9, 3)$. Hence, $(2, 6, 1, 1, 6, 2)$ and $(3, 6, 0, 6, 3)$ can be eliminated from consideration.

Table 3 records the number of $(v, t, 1)$ -feasible distributions for different values of v and t , as well as incorporating information about how many distributions cannot be ruled out using Theorem 2.7 in the manner just described.

3. Exhaustive Search Algorithm

We have seen in the previous section the relationship between a $\text{PSCA}(v, t, \lambda)$ and the smaller array that results from deleting a symbol from this PSCA . Specifically, we have seen that by deleting a symbol from a $\text{PSCA}(t+1, t, \lambda)$, we are left with λ copies of \mathcal{S}_t . We can extend this argument to say that by deleting $v-t$ symbols from a $\text{PSCA}(v, t, \lambda)$, we obtain λ copies of \mathcal{S}_t . In this sense, every PSCA contains λ copies of \mathcal{S}_t . This relationship between smaller and larger PSCAs with the same strength and multiplicity allows for the design of an algorithm that can exhaustively search for a $\text{PSCA}(v, t, \lambda)$ by first cataloguing all possible $\text{PSCA}(v', t, \lambda)$ for $t \leq v' < v$. Such an algorithm is further aided by the results proved in the previous section. To catalogue all possible PSCAs for a particular choice of parameters, we must first establish a definition of isomorphism for PSCAs .

Definition 3.1. Two multisets of permutations, X and Y , are *isomorphic* if Y can be obtained from X by permuting the symbols and/or reversing every permutation.

In searching for $\text{PSCA}(v, t, \lambda)$ for $v > t$, we employed two different methods. Both of these methods relied on a catalogue of isomorphism class representatives of $\text{PSCA}(v-1, t, \lambda)$. For each array in this catalogue, we tested every possible way of inserting a new symbol into each permutation of the array. In the first method, we assigned a (v, t, λ) -feasible distribution for this new symbol and found all possible PSCAs that can be formed when the new symbol obeys that distribution, before moving on to the next (v, t, λ) -feasible distribution. In the second method, we did not fix a distribution. Instead, we maintained a list of (v, t, λ) -feasible distributions that were consistent with the positions so far chosen for the new symbol. If that list ever became empty, then we knew that the current placements were unviable. Using these two search methods, we were able to independently count the number of isomorphism classes of $\text{PSCA}(v, t, \lambda)$ for different sets of parameters, as shown in Table 4. In some cases, it was not feasible to perform an exhaustive enumeration. In such cases, the number of PSCAs that we found before abandoning the search is given with a $+$ symbol, indicating that the search was incomplete. In each

TABLE 3. Number of feasible distributions. Each entry r/s indicates that there are s distributions that are (v, k, λ) -feasible, and that r of these cannot be ruled out using Theorem 2.7

t	$v=3$	4	5	6	7	8	9	10
$\lambda=1$	3	1/1	2/2	2/3	0/1	0/3	0/4	0/9
	4	–	1/1	3/3	6/6	8/13	19/30	61/119
	5	–	–	1/1	5/5	21/27	117/127	3359/3620
$\lambda=2$	3	1/1	3/3	6/8	8/12	16/28	30/55	67/165
	4	–	1/1	5/5	17/17	59/74	261/291	3940/4235
	5	–	–	1/1	9/9	79/93	900/910	106859/107947
$\lambda=3$	3	1/1	4/4	11/14	32/37	84/99	224/252	1315/1409
	4	–	1/1	7/7	35/35	195/221	1246/1296	38781/39486
	5	–	–	1/1	13/13	179/199	2933/2951	790491/793171

such case, we believe that the true number of PSCAs is much higher than the number that we quote.

In the cases when $(v, t, \lambda) \in \{(5, 3, 1), (7, 4, 1), (7, 5, 1), (8, 3, 2), (8, 4, 2)\}$, our enumeration was exhaustive, and demonstrated that no PSCA with these parameters exists. For the first three of these parameter sets, this was already known, but the last two are new results. Our computations have discovered several new values of the function g .

Theorem 3.2. $g(6, 3) = g(7, 3) = g(7, 4) = 2$ and $g(8, 3) = 3$. In addition, $g(8, 4) > 2$.

Proof. Given the non-existence results just mentioned, it suffices to display a $\text{PSCA}(7, 3, 2)$, a $\text{PSCA}(8, 3, 3)$, and a $\text{PSCA}(7, 4, 2)$:

PSCA(7, 3, 2)		PSCA(8, 3, 3)		PSCA(7, 4, 2)			
0123465	0642315	04712563	05672341	0123465	0254163	0351264	0432165
1540362	1634052	06432157	07351462	0621435	0634125	0651432	0652341
2405163	2610543	16547203	17453026	1045263	1254063	1432560	1530264
3054261	3625401	17630245	25476301	1632045	1635402	1640253	1652043
4312560	4651230	26751043	27410365	2045361	2103564	2341560	2530164
5231064	5603124	31526074	34675102	2601534	2635104	2643015	2645103
		37206154	42351067	3015462	3214065	3402561	3520461
		46051327	50213476	3604521	3610254	3614520	3625401
		53764201	61234075	4015362	4123065	4351062	4520163
				4610352	4620351	4621530	4653012
				5103462	5214360	5341260	5402361
				5603214	5604123	5612340	5643210

□

There are 260,664 isomorphism classes of $\text{PSCA}(5, 3, 3)$. We took the one which has the largest automorphism group and extended it in all possible ways. Doing so produced 3072, 481 765 and 51 448 isomorphism classes of PSCAs with parameters $(6, 3, 3)$, $(7, 3, 3)$ and $(8, 3, 3)$, respectively. However, none of these extended to a $\text{PSCA}(9, 3, 3)$. We also performed a search for all $\text{PSCA}(6, 3, 3)$ in which every symbol has distribution vector $(3, 3, 3, 3, 3, 3)$. Using Theorem 2.7, we were able to find all $(5, 3, 3)$ -feasible distributions that are compatible with this uniform distribution and thus could determine the $\text{PSCA}(5, 3, 3)$ that could potentially extend to such a $\text{PSCA}(6, 3, 3)$. From them, we found 1 053 700 $\text{PSCA}(6, 3, 3)$ up to isomorphism. These arrays extend to 35 872 460 $\text{PSCA}(7, 3, 3)$ and 1 992 709 $\text{PSCA}(8, 3, 3)$ up to isomorphism. Again, none of these arrays extend to a $\text{PSCA}(9, 3, 3)$. We also built some other $\text{PSCA}(8, 3, 3)$ via several other routes, but were unable to find a $\text{PSCA}(9, 3, 3)$.

The last column of Table 4 lists the number of isomorphism classes in our catalogue which contain a PSCA for which the corresponding set (ignoring multiplicity of repeated permutations) of permutations forms a group. To test

TABLE 4. Number of PSCAs generated by adding one symbol at a time

t	λ	v	PSCAs	Groups
3	1	3	1	1
3	1	4	1	0
3	1	5	0	0
3	2	3	1	1
3	2	4	12	1
3	2	5	314	0
3	2	6	1957	5
3	2	7	146	0
3	2	8	0	0
3	3	3	1	1
3	3	4	37	0
3	3	5	260,664	0
3	3	6	29,100,897+	0+
3	3	7	14,943,804+	0+
3	3	8	2,111,540+	0+
4	1	4	1	1
4	1	5	4	0
4	1	6	2	1
4	1	7	0	0
4	2	4	1	1
4	2	5	12,351	0
4	2	6	32,507	2
4	2	7	1826	0
4	2	8	0	0
5	1	5	1	1
5	1	6	3461	0
5	1	7	0	0

if a PSCA is isomorphic to a group it suffices to permute the symbols to ensure that one permutation (it does not matter which) is the identity, and then check that the resulting set of permutations is closed under composition. PSCAs that form groups will be studied further in the next section, which will provide details of all of the groups included in Table 4 (except the trivial cases when $v = t$).

In Table 3, we showed how many distributions might be achieved by symbols in PSCAs. In the “realised distributions” column of Table 5, we show how many of these distributions are actually realised within some PSCA. For comparison, the column headed “compatible distributions” repeats the smaller of the two bounds we had computed in Table 3. Table 5 covers all cases where we computed (non-empty) exhaustive catalogues. It also covers the case $(v, t, \lambda) = (6, 3, 3)$, where we were able to rule out 6 distributions with

TABLE 5. Number of realised distributions for different parameter sets

v	t	λ	Realised distributions	Compatible distributions
3	3	1	1	1
4	3	1	2	2
3	3	2	1	1
4	3	2	3	3
5	3	2	6	6
6	3	2	4	8
7	3	2	2	16
3	3	3	1	1
4	3	3	4	4
5	3	3	11	11
6	3	3	26	32
4	4	1	1	1
5	4	1	3	3
6	4	1	1	6
4	4	2	1	1
5	4	2	5	5
6	4	2	10	17
7	4	2	16	59
5	5	1	1	1
6	5	1	5	5

targeted searches, assisted by Theorem 2.7. The 6 unrealised distributions were $(0,9,1,3,0,5)$, $(2,6,0,4,3,3)$, $(3,1,8,0,2,4)$, and their reverses. The other 26 distributions from Table 3 appeared in our partial catalogue.

4. PSCAs from Permutation Groups

In this section, we consider PSCAs which can be constructed from permutation groups. For permutations $f, g \in \mathcal{S}_v$, the composition $f \circ g$ is the permutation $(f \circ g)(x) = f(g(x))$. For a subgroup $H \leq G$ and for $g \in G$, the right coset Hg is the set $\{hg : h \in H\}$, whereas the left coset gH is the set $\{gh : h \in H\}$. If H is a subgroup of \mathcal{S}_v , then the right coset Hg permutes the columns of H according to g , while the left coset gH permutes the symbols of H according to g . Throughout this section, for $s \in \mathcal{S}_{v,t}$, we use the notation $s = (s_0, \dots, s_{t-1})$. Moreover, G will always denote a group, such that if G has order v , then the elements of G are $\{0, \dots, v-1\}$, and ψ will denote an injective homomorphism $\psi : G \rightarrow \mathcal{S}_v, g \mapsto \psi_g$. We can then consider the action of G on $\mathcal{S}_{v,t}$ where, if $s = (s_0, \dots, s_{t-1}) \in \mathcal{S}_{v,t}$ and $g \in G$, then $gs = (\psi_g(s_0), \dots, \psi_g(s_{t-1}))$.

Mathon and van Trung [6] found that there are exactly two non-isomorphic $\text{PSCA}(6, 4, 1)$; one forms a group isomorphic to S_4 , the other forms three cosets of a group isomorphic to D_8 . While noting the connection between their PSCAs and groups, their search methods did not focus on building PSCAs from groups (the same is true of our work in Sect. 3). However, several connections between PSCAs and groups have been formalised by Na, Jedwab, and Li [8] and they found a number of examples of PSCAs based on groups. Note that the permutation composition convention used in [8] differs from the convention used here.

Lemma 4.1. *Let G be a group, $\psi : G \rightarrow \mathcal{S}_v$ be an injective homomorphism, T be the image of ψ , and let Th be a right coset of T . If x and y are sequences belonging to the same orbit under the action of G on $\mathcal{S}_{v,t}$, then x and y are covered by the same number of permutations in Th .*

Proof. Let x and y be elements of $\mathcal{S}_{v,t}$ that belong to the same orbit under the action of G . Then, $gx = y$ for some $g \in G$. Let $0 \leq c_0 < \dots < c_{t-1} \leq v-1$ and let $f \in Th$ such that $f(c_i) = x_i$ for $0 \leq i \leq t-1$. Then, f covers x . Now, consider $\psi_g \circ f$. As $f(c_i) = x_i$, $(\psi_g \circ f)(c_i) = \psi_g(x_i)$ for $0 \leq i \leq t-1$. Therefore, $\psi_g \circ f$ covers y . Therefore, for every permutation in Th that covers x , we can find a corresponding permutation that covers y . Therefore, the number of permutations in Th that cover y is at least the number of permutations in Th that cover x . By reversing the argument, and noting $x = g^{-1}y$, we find that the number of permutations in Th that cover x is at least the number of permutations in Th that cover y . Thus, x and y are covered by the same number of permutations in Th . \square

A consequence of Lemma 4.1 is that in a right coset of a permutation group $\psi(G)$, we can determine the number of permutations covering each sequence in the orbit of a sequence x under the action of G on $\mathcal{S}_{v,t}$ by simply finding the number of permutations in the coset that cover x . We will develop this point further in the context of transitive permutation groups in Lemma 4.2. Recall that a set X of permutations in \mathcal{S}_v is *transitive* if for each $i, j \in [v]$, there is a permutation $\pi \in X$, such that $\pi(i) = j$. If for each i and j , the permutation π is unique, then X is *sharply transitive*.

Lemma 4.2. *Let G be a group, let $\psi : G \rightarrow \mathcal{S}_v$ be an injective homomorphism, such that the image, T , of ψ is a transitive permutation group and let X be an array constructed from right cosets of T . Furthermore, let $w \in [v]$, $0 \leq i \leq t-1$ and let $S = \{s \in \mathcal{S}_{v,t} : s_i = w\}$. If every sequence in S is covered by λ permutations in X , then X is a $\text{PSCA}(v, t, \lambda)$.*

Proof. Let $s \in \mathcal{S}_{v,t}$. Then, as T is transitive, there is a $g \in G$, such that $\psi_g(s_i) = w$. Therefore, the orbit of s contains a sequence in S . As every orbit of the action of G on $\mathcal{S}_{v,t}$ contains a representative from S , then by Lemma 4.1, if every sequence in S is covered by λ permutations in X , then every sequence in $\mathcal{S}_{v,t}$ is also covered by λ permutations in X . \square

0	1	2	3	0	2	1	3	0	3	1	2
1	0	3	2	1	3	0	2	1	2	0	3
2	3	0	1	2	0	3	1	2	1	3	0
3	2	1	0	3	1	2	0	3	0	2	1
Type A				Type B				Type C			
0	1	3	2	0	2	3	1	0	3	2	1
1	0	2	3	1	3	2	0	1	2	3	0
2	3	1	0	2	0	1	3	2	1	0	3
3	2	0	1	3	1	0	2	3	0	1	2

FIGURE 1. Three types of cosets of E_4

TABLE 6. Triples uncovered by cosets of Type A, B and C

Type A	Type B	Type C
021	012	013
031	032	023
120	103	102
130	123	132
203	210	201
213	230	231
302	301	310
312	321	320

Elementary Abelian 2-groups

Throughout this subsection, we use E_v to denote an elementary abelian 2-group on the set $[v]$ with identity 0 and operation \oplus . Then, for a group E_v , we fix $\psi : E_v \rightarrow \mathcal{S}_v$ to be the homomorphism that maps $g \mapsto \psi_g$ where $\psi_g(x) = g \oplus x$. We then let T be the image of ψ . Under this homomorphism, $gs = (g \oplus s_0, \dots, g \oplus s_{t-1})$ for $g \in E_v$ and $s \in \mathcal{S}_{v,t}$. By construction, T is a sharply transitive set of permutations, a fact that will be critical in what follows. We begin our analysis of elementary abelian 2-groups with an overview of PSCAs built from E_4 . Within \mathcal{S}_4 , there are several subgroups isomorphic to E_4 . However, the only one of these subgroups that is sharply transitive (and hence may be represented within T) is the following:

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

The cosets of this group within \mathcal{S}_4 are shown in Fig. 1. We refer to the cosets on the left as having Type A coverage, the cosets in the middle as having Type B coverage, and the cosets on the right as having Type C coverage. Cosets of the same type cover the same set of triples. Each coset covers 16 triples of $\mathcal{S}_{4,3}$ exactly once, leaving 8 triples uncovered. These uncovered triples are recorded in Table 6.

Observe that the sets of triples uncovered by Type A, Type B and Type C cosets partition $\mathcal{S}_{4,3}$. Suppose X is a $\text{PSCA}(4, 3, \lambda)$ which is built from a combination of cosets of our E_4 permutation group. As the number of permutations in X is 6λ , the total number of cosets that make up X is $3\lambda/2$. Consider the triple 012. This triple is covered by Type A and Type C cosets, but is not covered by Type B cosets. Given that the number of permutations that cover 012 is λ , there must be $\lambda/2$ Type B cosets. Similar arguments involving other triples (e.g., 021 and 013) demonstrate that X must be built from $\lambda/2$ of each type of coset. Furthermore, because of the coverage properties of each coset type, any combination of $\lambda/2$ Type A cosets, $\lambda/2$ Type B cosets, and $\lambda/2$ Type C cosets will form a $\text{PSCA}(4, 3, \lambda)$. Therefore, an array built from a combination of cosets of E_4 will form a $\text{PSCA}(4, 3, \lambda)$ if and only if the array contains an equal number of each type of coset.

We use this characterisation to aid us in our search for PSCAs from cosets of permutation representations of the elementary abelian 2-group of order v with $v > 4$. Obviously, these larger groups contain many subgroups isomorphic to E_4 . As in the general case above, we isolate a subset of triples of $\mathcal{S}_{v,3}$, such that balanced coverage on these triples implies balanced coverage for every triple in $\mathcal{S}_{v,3}$.

Lemma 4.3. *Let \mathcal{H} be the set of order 4 subgroups of E_v and let S be the set of triples defined by*

$$S = \{(s_0, s_1, s_2) \in \mathcal{S}_{v,3} : \{s_0, s_1, s_2\} \subset H \text{ for some } H \in \mathcal{H}\}.$$

Let X be an array constructed from right cosets of T in \mathcal{S}_v . If every triple in S is covered by λ permutations in X , then X is a $\text{PSCA}(v, 3, \lambda)$.

Proof. First, we observe that if $\{x, y, z\}$ is a 3-subset of an elementary abelian 2-group, then $\{x, y, z, x \oplus y \oplus z\}$ is a coset of the order 4 subgroup $\{0, x \oplus y, x \oplus z, y \oplus z\}$. Furthermore, $x \oplus y \oplus z$ is the only element we can include with $\{x, y, z\}$ to form an order 4 coset.

Let $(x, y, z) \in \mathcal{S}_{v,3}$. If ψ_x acts on (x, y, z) , we obtain the triple $(0, x \oplus y, x \oplus z)$. As per the previous paragraph, $\{0, x \oplus y, x \oplus z\}$ forms a subset of an order 4 subgroup, so $(0, x \oplus y, x \oplus z) \in S$. Hence, each orbit of $\mathcal{S}_{v,3}$ under the action of E_v contains a triple from S . Therefore, by Lemma 4.1, if every triple in S is covered by λ permutations in X , then X is a $\text{PSCA}(v, 3, \lambda)$. \square

Definition 4.4. Let $X \subseteq \mathcal{S}_v$ be a multiset of permutations. For $W \subseteq [v]$, the *reduced array of X on W* , denoted by $X[W]$, is the array we obtain by removing every symbol of $[v] \setminus W$ from X .

Let Y be a right coset of T in \mathcal{S}_v , let H be an order 4 subgroup of E_v , and consider the reduced array $Y[H]$. If we partition the rows of $Y[H]$ according to the cosets of H , then each part will form a coset of the sharply transitive E_4 permutation group. By taking X to be a collection of right cosets of T , we can determine whether $X[H]$ forms a PSCA by analysing the coverage type of each coset of E_4 that appears in $X[H]$. As a result of Lemma 4.3, if the reduced array $X[H]$ is a PSCA of strength 3 for each $H \in \mathcal{H}$, then X will be a PSCA of strength 3.

Lemma 4.5. *Let f be an order n automorphism of E_v and let X be the array*

$$X = \bigcup_{i=0}^{n-1} T f^i.$$

Let H be an order 4 subgroup of E_v . If the reduced array $X[H]$ is a $\text{PSCA}(4, 3, \lambda)$, then $X[f^i(H)]$ will also be a $\text{PSCA}(4, 3, \lambda)$ for $1 \leq i \leq n-1$.

Proof. First, we show that $T = f^{-1}Tf$. Let $g \in E_v$. Then, we can consider $\psi_g \in T$ and the composition $f^{-1}\psi_g f$. Let $x \in E_v$. Then, $f^{-1}\psi_g f(x) = f^{-1}(g \oplus f(x))$. As f is an automorphism of E_v , so too is f^{-1} . Hence, $f^{-1}(g \oplus f(x)) = f^{-1}(g) \oplus x$. Therefore, $f^{-1}\psi_g f = \psi_{f^{-1}(g)}$ and hence, $f^{-1}Tf \subseteq T$. Now, $\psi_g = \psi_{f^{-1}(f(g))} = f^{-1}\psi_{f(g)}f$ by the above argument. So, $T \subseteq f^{-1}Tf$, and thus, $T = f^{-1}Tf$. Therefore, $fT = Tf$, and so, we can consider Tf as being an array in which the symbols of T have been permuted according to f . As a result, the reduced array $T[H]$ is isomorphic to $Tf[f(H)]$. More generally, the reduced array $Tf^i[H]$ is isomorphic to $Tf^{i+1}[f(H)]$ for $0 \leq i \leq n-1$. Moreover, the isomorphism in each case is the restriction of f to H . Therefore, $X[H]$ is isomorphic to $X[f(H)]$. Applying this argument to the subgroups $f^i(H)$ and $f^{i+1}(H)$ for $0 \leq i \leq n-1$, we find that $X[H]$ is isomorphic to $X[f^i(H)]$ for $1 \leq i \leq n-1$. Therefore, if $X[H]$ is a $\text{PSCA}(4, 3, \lambda)$, then so is $X[f^i(H)]$ for $1 \leq i \leq n-1$. \square

Essentially, Lemma 4.3 demonstrates that in a collection of right cosets of T , it suffices to check the coverage of triples whose elements form a subset of an order 4 subgroup E_v to determine whether the cosets form a PSCA. When these cosets are related by an automorphism of E_v , we are able to further restrict what triples need to be checked by allowing us to consider only certain subgroups, depending upon the automorphism f . In each case, the reduced array on any order 4 subgroup H will form a collection of cosets of E_4 and so we can use the characterisation at the start of this section to determine whether these reduced arrays form PSCAs. Using these methods, we have been able to find PSCAs of orders 4, 8, 16 and 32 with strength 3.

The $\text{PSCA}(4, 3, 2)$ forms a permutation group isomorphic to the alternating group A_4 . The $\text{PSCA}(8, 3, 4)$ forms a permutation group isomorphic to $A_4 \times C_2$. We also have the following PSCAs of orders 16 and 32.

Theorem 4.6. $g(v, 3) \leq 16$ for $v \leq 16$ and $g(v, 3) \leq 96$ for $v \leq 32$.

Proof. To prove the first part of the theorem, we need only present a PSCA $(16, 3, 16)$. We let G be the group isomorphic to E_{16} generated by the permutations

The following are examples of a PSCA(4, 3, 2) and a PSCA(8, 3, 4) (note that Na, Jedwab, and Li [8] also found a PSCA(8, 3, 4)).

PSCA(4, 3, 2)	PSCA(8, 3, 4)	
0123	01234567	42671053
1032	10543276	53106742
2301	25076143	60435217
3210	34701652	71342506
0231	43610725	07245316
1320	52167034	16532407
2013	67452301	23061754
3102	76325410	32716045
0312	06253471	45607132
1203	17524360	54170623
2130	24017635	61423570
3021	35760124	70354261

$$\begin{aligned}
 &(0\ 1)(2\ 3)(4\ 5)(6\ 7)(8\ 9)(10\ 11)(12\ 13)(14\ 15), \\
 &(0\ 2)(1\ 3)(4\ 14)(5\ 15)(6\ 12)(7\ 13)(8\ 10)(9\ 11), \\
 &(0\ 4)(1\ 5)(2\ 14)(3\ 15)(6\ 10)(7\ 11)(8\ 12)(9\ 13), \\
 &(0\ 8)(1\ 9)(2\ 10)(3\ 11)(4\ 12)(5\ 13)(6\ 14)(7\ 15).
 \end{aligned}$$

We then let $f = (1\ 8\ 9)(2\ 4\ 15\ 11\ 5\ 7)(3\ 12\ 6\ 10\ 13\ 14)$. Then

$$X = \bigcup_{i=0}^5 Gf^i$$

forms a PSCA(16, 3, 16). The 96 permutations of this PSCA also form a group which can be generated by

$$\begin{aligned}
 &(1\ 8\ 9)(2\ 4\ 15\ 11\ 5\ 7)(3\ 12\ 6\ 10\ 13\ 14), \\
 &(0\ 4\ 7)(1\ 13\ 15)(2\ 3\ 10)(5\ 14\ 8)(6\ 9\ 12).
 \end{aligned}$$

As a result of Lemma 4.3, to check whether X forms a PSCA, we need only check that the reduced arrays of X corresponding to the 35 order 4 subgroups of G form PSCA(4, 3, 16). As the cosets of G from which X is constructed are related by an automorphism, we can use Lemma 4.5 to further limit the number of reduced arrays of X that we need to check to verify that X is a PSCA. The orbits of the 35 order 4 subgroups of G under f are as follows:

$$\begin{aligned}
 &\{\{0, 1, 2, 3\}, \{0, 4, 8, 12\}, \{0, 6, 9, 15\}, \{0, 1, 10, 11\}, \{0, 5, 8, 13\}, \{0, 7, 9, 14\}\} \\
 &\{\{0, 1, 6, 7\}, \{0, 2, 8, 10\}, \{0, 4, 9, 13\}, \{0, 1, 14, 15\}, \{0, 3, 8, 11\}, \{0, 5, 9, 12\}\} \\
 &\{\{0, 2, 4, 14\}, \{0, 3, 4, 15\}, \{0, 11, 12, 15\}, \{0, 5, 6, 11\}, \{0, 5, 7, 10\}, \{0, 2, 7, 13\}\} \\
 &\{\{0, 2, 6, 12\}, \{0, 4, 6, 10\}, \{0, 10, 13, 15\}, \{0, 11, 13, 14\}, \{0, 3, 5, 14\}, \{0, 3, 7, 12\}\}
 \end{aligned}$$

$$\begin{aligned}
& \{\{0, 1, 4, 5\}, \{0, 7, 8, 15\}, \{0, 2, 9, 11\}\} \\
& \{\{0, 1, 12, 13\}, \{0, 6, 8, 14\}, \{0, 3, 9, 10\}\} \\
& \{\{0, 2, 5, 15\}, \{0, 4, 7, 11\}\} \\
& \{\{0, 3, 6, 13\}, \{0, 10, 12, 14\}\} \\
& \{\{0, 1, 8, 9\}\}.
\end{aligned}$$

Hence, by Lemma 4.5, we need only check the reduced array of one subgroup from each of these 9 orbits to verify that X is a PSCA.

For the second part of the theorem, we present a PSCA(32, 3, 96). We let G_{32} be the group isomorphic to E_{32} generated by the permutations

$$\begin{aligned}
& (0\ 1)\ (2\ 3)\ (4\ 5)\ (6\ 7)\ (8\ 9)\ (10\ 11)\ (12\ 13)\ (14\ 15)\ (16\ 17)\ (18\ 19) \\
& \quad (20\ 21)\ (22\ 23)\ (24\ 25)\ (26\ 27)\ (28\ 29)\ (30\ 31), \\
& (0\ 2)\ (1\ 3)\ (4\ 28)\ (5\ 29)\ (6\ 30)\ (7\ 31)\ (8\ 10)\ (9\ 11) \\
& \quad (12\ 20)\ (13\ 21)\ (14\ 22)\ (15\ 23)\ (16\ 18)\ (17\ 19)\ (24\ 26)\ (25\ 27), \\
& (0\ 4)\ (1\ 5)\ (2\ 28)\ (3\ 29)\ (6\ 26)\ (7\ 27)\ (8\ 14)\ (9\ 15)\ (10\ 22) \\
& \quad (11\ 23)\ (12\ 18)\ (13\ 19)\ (16\ 20)\ (17\ 21)\ (24\ 30)\ (25\ 31), \\
& (0\ 8)\ (1\ 9)\ (2\ 10)\ (3\ 11)\ (4\ 14)\ (5\ 15)\ (6\ 12)\ (7\ 13)\ (16\ 24) \\
& \quad (17\ 25)\ (18\ 26)\ (19\ 27)\ (20\ 30)\ (21\ 31)\ (22\ 28)\ (23\ 29), \\
& (0\ 16)\ (1\ 17)\ (2\ 18)\ (3\ 19)\ (4\ 20)\ (5\ 21)\ (6\ 22)\ (7\ 23)\ (8\ 24) \\
& \quad (9\ 25)\ (10\ 26)\ (11\ 27)\ (12\ 28)\ (13\ 29)\ (14\ 30)\ (15\ 31).
\end{aligned}$$

We then let f_1 be the following order 2 automorphism of G_{32} :

$$\begin{aligned}
& (2\ 8)\ (3\ 9)\ (4\ 6)\ (5\ 7)\ (12\ 28)\ (13\ 29)\ (14\ 30)\ (15\ 31)\ (18\ 24) \\
& \quad (19\ 25)\ (20\ 22)\ (21\ 23).
\end{aligned}$$

We then let $G_{64} = G_{32} \cup G_{32}f_1$. Observe that G_{64} also forms a group. Then, we let f_2 be the following order 3 automorphism of G_{32} :

$$(2\ 12\ 24)\ (3\ 13\ 25)\ (4\ 6\ 10)\ (5\ 7\ 11)(8\ 18\ 28)\ (9\ 19\ 29)(20\ 22\ 26)(21\ 23\ 27).$$

We then let $G_{192} = G_{64} \cup G_{64}f_2 \cup G_{64}f_2^2$. Again, G_{192} forms a group. Finally, we let f_3 be the following order 3 automorphism of G_{32} :

$$(1\ 16\ 17)(3\ 18\ 19)(5\ 20\ 21)(6\ 7\ 23)(9\ 24\ 25)(11\ 26\ 27)(12\ 13\ 29)(15\ 30\ 31).$$

Then, $G_{192} \cup f_3 G_{192} \cup f_3^2 G_{192}$ is a PSCA(32, 3, 96). Although this construction is not of the form described in Lemma 4.5, it is a collection of right cosets of G_{32} . Therefore, we can use Lemma 4.3 to check that this array is indeed a PSCA. \square

We remark that even though G_{32} , G_{64} , and G_{192} are groups, the PSCA (32, 3, 96) described in Theorem 4.6 is not a group. We also note that while f_3 is an automorphism of G_{32} , it is not an automorphism of G_{192} . As such, the shift to left cosets in the final step of the construction is significant as taking right cosets would not form a PSCA.

TABLE 7. Strength 3 PSCAs that are permutation groups

(v, λ)	Group	Generators
$(4, 2)$	A_4	$\langle (1, 2, 3), (0, 1, 2) \rangle$
$(6, 2)$	A_4	$\langle (0, 5, 4)(1, 2, 3), (0, 5, 1)(2, 3, 4) \rangle$
	D_{12}	$\langle (0, 5, 4, 2, 1, 3), (0, 5)(1, 2)(3, 4) \rangle$
		$\langle (0, 4, 5, 2, 1, 3), (0, 4)(1, 2)(3, 5) \rangle$
		$\langle (0, 3, 1, 5, 4, 2), (0, 5)(1, 3)(2, 4) \rangle$
		$\langle (0, 3, 1, 4, 5, 2), (0, 4)(1, 3)(2, 5) \rangle$
		$\langle (0, 5, 1, 2, 3, 4), (0, 5, 4)(1, 2, 3) \rangle$
		$\langle (0, 2)(1, 3)(4, 5), (0, 4, 5)(1, 3, 2) \rangle$
		$\langle (0, 2)(1, 3)(4, 5), (0, 4, 5)(1, 2, 3) \rangle$
		$\langle (0, 5)(1, 3)(2, 4), (0, 2, 4)(1, 3, 5) \rangle$
		$\langle (1, 3)(4, 5), (0, 1, 5)(2, 4, 3) \rangle$
		$\langle (1, 3)(2, 4), (0, 1, 4)(2, 3, 5) \rangle$
		$\langle (0, 7, 4, 2)(1, 5, 3, 6), (0, 7, 1)(2, 3, 4) \rangle$
		$\langle (0, 7, 6, 4)(1, 3, 2, 5), (0, 7, 3)(4, 5, 6) \rangle$
		$\langle (0, 5, 4, 2)(1, 7, 3, 6), (0, 7, 4)(1, 3, 2) \rangle$
		$\langle (0, 5, 4, 2)(1, 6, 3, 7), (0, 7, 4)(1, 3, 5) \rangle$
		$\langle (0, 5, 6, 3)(1, 4, 7, 2), (0, 7, 6)(2, 4, 5) \rangle$
		$\langle (0, 5, 7, 3)(1, 4, 6, 2), (0, 7, 6)(2, 5, 4) \rangle$
		$\langle (0, 5, 7, 4)(1, 3, 6, 2), (0, 7, 6)(2, 5, 3) \rangle$
		$\langle (0, 5, 6, 4)(1, 3, 7, 2), (0, 7, 6)(2, 3, 5) \rangle$
		$\langle (0, 6, 4, 3)(1, 7, 5, 2), (0, 5, 4)(2, 7, 6) \rangle$
		$\langle (0, 7, 3, 6)(1, 2, 4, 5), (0, 5, 3)(1, 4, 7) \rangle$
		$\langle (0, 7, 3, 6)(1, 5, 4, 2), (0, 5, 3)(1, 4, 6) \rangle$
		$\langle (0, 7, 4, 3)(1, 6, 5, 2), (0, 5, 4)(2, 6, 7) \rangle$
$(6, 4)$	$C_2 \times A_4$	
	\mathcal{S}_4	
$(8, 4)$	$SL(2, 3)$	
	\mathcal{S}_4	

TABLE 7. continued

(v, λ)	Group	Generators
(12,6)	$C_2 \times A_4$	$\langle (0, 7, 4, 6)(1, 3, 5, 2), (0, 5, 4)(2, 3, 7) \rangle$
		$\langle (0, 7, 5, 2)(1, 6, 4, 3), (0, 5, 4)(2, 3, 6) \rangle$
		$\langle (1, 4, 7)(2, 5, 3), (0, 1)(2, 7)(3, 6)(4, 5) \rangle$
		$\langle (1, 4, 6)(2, 5, 3), (0, 1)(2, 6)(3, 7)(4, 5) \rangle$
		$\langle (1, 2, 5)(4, 7, 6), (0, 1)(2, 7)(3, 6)(4, 5) \rangle$
		$\langle (1, 2, 5)(4, 6, 7), (0, 1)(2, 6)(3, 7)(4, 5) \rangle$
		$\langle (1, 6, 7)(2, 5, 4), (0, 2)(1, 5)(3, 6)(4, 7) \rangle$
		$\langle (1, 6, 7)(2, 4, 5), (0, 2)(1, 5)(3, 7)(4, 6) \rangle$
		$\langle (1, 6, 7)(2, 5, 3), (0, 2)(1, 5)(3, 7)(4, 6) \rangle$
		$\langle (1, 6, 7)(2, 3, 5), (0, 2)(1, 5)(3, 6)(4, 7) \rangle$
		$\langle (0, 11, 9, 10, 1, 4)(2, 7, 6, 3, 8, 5), (0, 8, 9, 2, 1, 6)(3, 4, 5, 11, 7, 10) \rangle$
		$\langle (0, 11, 9, 10, 1, 4)(2, 5, 8, 3, 6, 7), (0, 8, 9, 2, 1, 6)(3, 4, 7, 10, 5, 11) \rangle$
(14,7)	$C_6 \times \mathcal{S}_3$ $\mathcal{S}_3 \times \mathcal{S}_3$ $C_3 \times A_4$	$\langle (2, 5, 9)(3, 6, 8)(4, 10, 11), (0, 2, 4)(1, 10, 8)(3, 9, 7)(5, 11, 6) \rangle$
		$\langle (2, 5, 9)(3, 6, 8)(4, 11, 10), (0, 2, 4)(1, 11, 8)(3, 9, 7)(5, 10, 6) \rangle$
		$\langle (2, 5, 9)(3, 6, 8)(4, 7, 11), (0, 2, 4)(1, 7, 8)(3, 9, 10)(5, 11, 6) \rangle$
		$\langle (2, 5, 9)(3, 6, 8)(4, 7, 10), (0, 2, 4)(1, 7, 8)(3, 9, 11)(5, 10, 6) \rangle$
		$\langle (2, 5, 9)(3, 6, 8)(4, 11, 7), (0, 2, 4)(1, 11, 8)(3, 9, 10)(5, 7, 6) \rangle$
		$\langle (2, 5, 9)(3, 6, 8)(4, 10, 7), (0, 2, 4)(1, 10, 8)(3, 9, 11)(5, 7, 6) \rangle$
		$\langle (1, 7, 9)(2, 6, 5)(4, 10, 8), (0, 1, 8)(2, 7, 10)(3, 5, 9)(4, 6, 11) \rangle$
		$\langle (1, 4, 7)(2, 11, 5)(3, 9, 13)(6, 8, 12), (0, 2)(1, 11)(3, 10)(4, 8)(5, 13)(6, 7)(9, 12) \rangle$
		$\langle (1, 4, 7)(2, 11, 5)(3, 9, 12)(6, 8, 13), (0, 2)(1, 11)(3, 10)(4, 8)(5, 12)(6, 7)(9, 13) \rangle$
		$\langle (1, 8, 9)(2, 4, 15, 11, 5, 7)(3, 12, 6, 10, 13, 14),$
		$\langle (0, 4, 7)(1, 13, 15)(2, 3, 10)(5, 14, 8)(6, 9, 12) \rangle$
		$\langle (1, 11, 5, 18, 15, 9)(2, 7, 12, 8, 3, 13)(4, 6, 10, 14, 16, 17),$
		$\langle (0, 1, 2, 3, 4, 13, 16, 5, 11, 10, 17, 15, 9, 6, 12, 14, 7, 8, 18) \rangle$
(16,16)	$(E_{16} \rtimes C_2) \rtimes C_3$	
(19,19)	$C_{19} \rtimes C_6$	

TABLE 8. Strength 4 PSCAs that are permutation groups

(v, λ)	Group	Classes	Example
(6, 1)	S_4	1	$\langle (1, 3)(4, 5), (0, 1, 4)(2, 5, 3) \rangle$
(6, 2)	$C_2 \times S_4$	1	$\langle (0, 5, 2, 1), (0, 1, 3, 2, 5, 4) \rangle$
(7, 7)	$PSL(3, 2)$	9	$\langle (0, 2, 3, 4, 6, 5, 1), (0, 5, 4)(2, 6, 3) \rangle$
(8, 56)	$E_8 \rtimes PSL(3, 2)$	22	$\langle (0, 7, 4, 2, 3, 1, 5), (0, 1, 3, 4)(2, 6, 7, 5) \rangle$
(9, 18)	$((E_9 \rtimes Q_8) \rtimes C_3) \rtimes C_2$	38	$\langle (0, 8)(1, 3)(4, 5), (0, 3, 8)(1, 6, 4)(2, 7, 5) \rangle$
(10, 30)	S_6	102	$\langle (0, 7)(2, 9)(3, 4), (0, 7, 5, 9, 1)(2, 3, 6, 8, 4) \rangle$
	$A_6.C_2$	51	$\langle (0, 7, 3)(1, 2, 6)(4, 8, 5), (0, 5, 1, 6, 2, 4, 7, 3)(8, 9) \rangle$
(12, 18)	$((E_9 \rtimes Q_8) \rtimes C_3) \rtimes C_2$	24	$\langle (2, 8)(3, 11)(6, 9)(7, 10), (0, 1, 9)(2, 4, 11)(3, 7, 5)(6, 10, 8) \rangle$
(13, 234)	$PSL(3, 3)$	77040	$\langle (3, 9)(5, 7)(8, 10)(11, 12), (0, 1, 2, 3)(4, 11, 9, 8)(5, 12)(6, 10) \rangle$
(21, 5040)	$PSL(3, 4) \rtimes S_3$?	$\langle (0, 16, 8, 9)(1, 4, 3, 20, 5, 13, 18, 19)(2, 6, 10, 7, 14, 17, 11, 12) \\ (0, 17, 7, 11, 10, 5, 4, 19)(1, 2, 16, 14, 15, 9, 12, 3)(8, 13, 18, 20) \rangle$

Motivated by those PSCAs that we had earlier found which turned out to be permutation representations of groups, we decided to search for such objects directly. Fix v, t and λ . We sought a representation in \mathcal{S}_v of some group of order $n = t!\lambda$. We began by deciding on positive integers g_1, g_2 and possibly g_3 . We then chose permutations of orders g_1, g_2 (and possibly g_3) and checked whether they generate a group of order n . For each group that we discovered in this way, we then tried to find a conjugate that was a PSCA. This was done by building up the PSCA one column at a time, backtracking whenever some t -sequence would be covered too many times. As the conjugate $h^{-1}Gh$ of a group G is isomorphic in terms of sequence coverage to Gh , searching over all column permutations of G for a PSCA is equivalent to searching over all conjugates of G . Since we checked all conjugates of each group that we found, we were free to insist that the generator of order g_1 that we chose was lexicographically maximal amongst all of its conjugates. In particular, this meant we only had to consider one choice for each possible cycle structure of that generator. Note that this method did not prejudge which group it was going to build. Many non-isomorphic groups of order n may have generators of the specified orders. For example, there are 15 groups of order 24, but they all have a generating set with $(g_1, g_2) \in \{(12, 4), (12, 2), (8, 3), (6, 4), (3, 2)\}$ or $(g_1, g_2, g_3) = (6, 6, 2)$. Similarly, the 5 groups of order 18 all have a generating set with $(g_1, g_2) \in \{(9, 2), (6, 6)\}$ or $(g_1, g_2, g_3) = (3, 3, 2)$. Of course, groups will typically have many different generating sets with suitable orders, and hence will be built multiple times. However, we could be confident that every group of order n that has some representation in \mathcal{S}_v would be built, and thus, that our catalogue of PSCAs that are groups is exhaustive for $v \leq 14$ and $n \leq 42$.

In an alternative computation, we used GAP [2] to generate representatives of conjugacy classes of subgroups of \mathcal{S}_v and used the backtracking process described above to search over each conjugacy class. We have also performed ad hoc computations on some doubly transitive permutation groups. Some of those groups had too many conjugates to search exhaustively, so we randomly sampled conjugates instead. Our results are recorded in two tables. The first, Table 7, records permutation groups that are strength 3 PSCAs but not strength 4 PSCAs. The second table, Table 8, records permutation groups that are strength 4 PSCAs but not strength 5 PSCAs. In Table 7, a representative of each PSCA-isomorphism class of each group is presented. As a crosscheck, we note that these results agree with those presented in Table 4, which were found by a completely separate method. For reasons of space, in Table 8, we do not list representatives of each PSCA-isomorphism class. Rather, we just give the number of such classes (or a ? when random sampling of conjugates was used instead of an exhaustive search).

We know of few permutation groups that are PSCAs of strength 5, other than symmetric and alternating groups. These necessarily include the 5-transitive Mathieu groups M_{12} and M_{24} . Perhaps, more interestingly, we also found that

$$\langle (1, 7)(2, 8)(3, 4)(6, 9), (0, 2, 10, 6)(3, 7, 5, 8) \rangle \quad (2)$$

is one of 108 presentations of the (4-transitive) Mathieu group M_{11} in \mathcal{S}_{11} that form PSCAs of strength 5. No subgroup of \mathcal{S}_{11} forms a PSCA of strength 4, other than those isomorphic to M_{11} , A_{11} or \mathcal{S}_{11} . Similarly

$$\langle (2, 11, 8, 6)(3, 10, 4, 5), (0, 1, 2, 3, 4, 5, 11, 6, 7, 10, 8), (0, 9)(1, 8)(2, 5)(3, 6)(4, 7)(10, 11) \rangle \quad (3)$$

is one of 161 presentations of the (5-transitive) Mathieu group M_{12} in \mathcal{S}_{12} that form PSCAs of strength 6. The presentations of M_{11} are conjugates of each other, and similarly for M_{12} . If we let $r \in \mathcal{S}_v$ be the reverse permutation, i.e., $r(i) = (v - 1 - i)$, then for a permutation group $G \leq \mathcal{S}_v$, we will find that G and rGr are isomorphic in terms of sequence coverage. Hence, it is plausible that we may find presentations of the same group that are isomorphic as PSCAs. Indeed, this is the case for M_{11} where the 108 presentations that form PSCAs of strength 5 can be reduced to 54 isomorphism classes. Meanwhile, the presentation of M_{12} given in (3) is the only one of the 161 strength 6 PSCAs for which conjugation by r leaves the underlying set of permutations unchanged. Thus, these 161 presentations that form PSCAs of strength 6 reduce to 81 isomorphism classes.

For the larger Mathieu groups, we were unable to do exhaustive computations and again relied on random sampling. We found that

$$\begin{aligned} &\langle (0, 1, 20, 4, 2)(3, 8, 9, 12, 13)(5, 16, 10, 11, 18)(6, 7, 15, 19, 14), \\ &\quad (0, 13, 16, 5, 10)(1, 14, 19, 4, 2)(3, 18, 7, 12, 15)(9, 21, 11, 20, 17) \rangle \end{aligned} \quad (4)$$

is a presentation of the (3-transitive) Mathieu group M_{22} in \mathcal{S}_{22} that forms a PSCA of strength 5. Also

$$\begin{aligned} &\langle (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22), \\ &\quad (0, 23)(1, 22)(2, 11)(3, 15)(4, 17)(5, 9)(6, 19)(7, 13)(8, 20)(10, 16)(12, 21)(14, 18), \\ &\quad (2, 16, 9, 6, 8)(3, 12, 13, 18, 4)(7, 17, 10, 11, 22)(14, 19, 21, 20, 15) \rangle \end{aligned} \quad (5)$$

is a presentation of the (5-transitive) Mathieu group M_{24} in \mathcal{S}_{24} that forms a PSCA of strength 6. Its point stabilisers provide PSCAs of strength 5 in \mathcal{S}_{23} that are presentations of M_{23} .

Table 7 also includes the PSCA(16, 3, 16) found earlier in the section, and a PSCA(19, 3, 19). Exhaustive searches were not undertaken for either of these parameter sets. However, a partial search found 17116 and 232 isomorphism classes, respectively, of PSCA(16, 3, 16) and PSCA(19, 3, 19) that are conjugate to the examples given in the table. Note that since isomorphism includes the option to freely permute symbols, the only material effect of conjugation in this context is to permute the columns of a PSCA.

A striking feature of results summarised in Tables 7 and 8 is that there are a number of cases of non-isomorphic PSCAs being produced by similar sets of generators. For example, starting from the PSCA(6, 4, 1), if we conjugate the generating set by the transposition (2, 5), we reach a PSCA(6, 3, 4). A similar thing happens if we use the transposition (4, 5). Conjugating the generating set by a transposition has the effect of interchanging two columns of the PSCA

(and then exchanging two symbols to once again achieve the property of having one row equal to the identity permutation).

Summarising our bounds on $g(v, t)$ derived from group presentations, we have the following:

Theorem 4.7.

- For $v \leq 11$, we have $g(v, 5) \leq 66$.
- For $v \leq 12$, we have $g(v, 4) \leq 18$, $g(v, 5) \leq 792$ and $g(v, 6) \leq 132$.
- For $v \leq 13$, we have $g(v, 4) \leq 234$.
- For $v \leq 21$, we have $g(v, 4) \leq 5040$, and
- For $v \leq 22$, we have $g(v, 5) \leq 3696$, and hence, $g(v, 4) \leq 18\,480$.
- For $v \leq 23$, we have $g(v, 5) \leq 85\,008$, and hence, $g(v, 4) \leq 425\,040$.
- For $v \leq 24$, we have $g(v, 6) \leq 340\,032$, and hence, $g(v, 5) \leq 2\,040\,192$ and $g(v, 4) \leq 10\,200\,960$.

Proof. Examples of a PSCA(12, 4, 18), a PSCA(13, 4, 234) and a PSCA(21, 4, 5040) are given in Table 8. Also, we gave a PSCA(11, 5, 66) in (2), a PSCA(12, 6, 132) in (3), a PSCA(22, 5, 3696) in (4), and a PSCA(24, 6, 340 032) in (5), from which we derived a PSCA(23, 5, 85 008). \square

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Aidan R. Gentle and Ian M. Wanless
School of Mathematics
Monash University
Clayton
VIC
3800
Australia
e-mail: aidan.gentle@monash.edu

Ian M. Wanless
e-mail: ian.wanless@monash.edu

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