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A q-Analogue for Euler's $\zeta(6) = \pi^6/945$

In honour of Prof. George Andrews on his 80th birthday

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Abstract. Recently, Sun (Two q-analogues of Euler's formula $\zeta(2) = \pi^2/6$. arXiv:1802.01473, 2018) obtained q-analogues of Euler's formula for $\zeta(2)$ and $\zeta(4)$. Sun's formulas were based on identities satisfied by triangular numbers and properties of Euler's q-Gamma function. In this paper, we obtain a q-analogue of $\zeta(6) = \pi^6/945$. Our main results are stated in Theorems 2.1 and 2.2 below.

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1. Introduction

Recently, Sun [3] obtained a very nice q-analogue of Euler's formula $\zeta(2) = \pi^2/6$.

Theorem 1.1. (Sun [3]) *For a complex q with $|q| < 1$, we have:*

$$\sum_{k=0}^{\infty} \frac{q^k(1+q^{2k+1})}{(1-q^{2k+1})^2} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^4}{(1-q^{2n-1})^4}. \tag{1.1}$$

Motivated by Theorem 1.1, the present author obtained the q-analogue of $\zeta(4) = \pi^4/90$ and noted that it was simultaneously and independently obtained by Sun in his subsequent revised paper.

Theorem 1.2. (Sun [3]) *For a complex q with $|q| < 1$, we have:*

$$\sum_{k=0}^{\infty} \frac{q^{2k}(1+4q^{2k+1}+q^{4k+2})}{(1-q^{2k+1})^4} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^8}{(1-q^{2n-1})^8}. \tag{1.2}$$

Furthermore, Sun commented that one does not know how to find q-analogues of Euler's formula for $\zeta(6)$ and beyond, similar to Theorems 1.1 and 1.2. This further motivated the author to consider the problem, and indeed, we obtained the q-analogue of $\zeta(6)$. As we shall see shortly, the q-analogue formulation of $\zeta(6)$ is more difficult as compared to $\zeta(2)$ and $\zeta(4)$ due to an extra term that shows up in the identity; however, in the limit as $q \uparrow 1$ (where $q \uparrow 1$ means q is approaching 1 from inside the unit disk), this term vanishes. We also state the q-analogue of $\zeta(4) = \pi^4/90$, since we found it independently of Sun's result; however, we skip the proof of this, since it essentially uses the same idea as Sun.

We emphasize here that the q-analogue of $\zeta(6) = \pi^6/945$ is the first non-trivial case where we notice the occurrence of an interesting extra term which essentially is the twelfth power of a well-known function of Euler (see Theorem 2.2). After obtaining this result, we obtained q-analogues of Euler's general formula for $\zeta(2k), k = 4, 5, \ldots$ (see [1]). Each of these q-analogues has an extra term that arises from the general theory of modular forms all of which approach zero in the limit $q \uparrow 1$. The case $k = 3$ or the q-analogue of $\zeta(6)$ is special, since the extra term that we obtain in this case has a beautiful product representation, and has connections to well-known identities of Euler (see below).

2. Main Theorems

Theorem 2.1. *For a complex q with $|q| < 1$, we have:*

$$\sum_{k=0}^{\infty} \frac{q^{2k}\, P_2(q^{2k+1})}{(1-q^{2k+1})^4} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^8}{(1-q^{2n-1})^8}, \tag{2.1}$$

where $P_2(x) = x^2 + 4x + 1$. In other words, (2.1) gives a q-analogue of $\zeta(4) = \pi^4/90$.

Theorem 2.2. *For a complex q with $|q| < 1$, we have:*

$$\sum_{k=0}^{\infty} \frac{q^{k}(1+q^{2k+1})\, P_4(q^{2k+1})}{(1-q^{2k+1})^6} - \phi^{12}(q) = 256q \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{12}}{(1-q^{2n-1})^{12}}, \tag{2.2}$$

where $P_4(x) = x^4 + 236x^3 + 1446x^2 + 236x + 1$ and $\phi(q) = \prod_{n=1}^{\infty}(1-q^n)$ is Euler's function. In other words, (2.2) gives a q-analogue of $\zeta(6) = \pi^6/945$.

Remark 2.3. We note that $\phi^{12}(q)$ has a beautiful product representation and is uniquely determined by:

$$\phi^{12}(q) = \sum_{k=0}^{\infty} \frac{q^{k}(1+q^{2k+1})\, P_4(q^{2k+1})}{(1-q^{2k+1})^6} - 256q \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{12}}{(1-q^{2n-1})^{12}}. \tag{2.3}$$

In the general q-analogue formulation (see [1]), we do not have very elegant representations of these functions, although we obtain expressions for them similar to (2.3).

Remark 2.4. Since the coefficients in the q-series expansion of $\phi^{12}(q)$ are related to the pentagonal numbers by Euler's pentagonal number theorem, and the coefficients of the product in the right-hand side of (2.2) are related to the triangular numbers, it will be worthwhile to understand the relationships of these coefficients via identity (2.2).

3. Some Useful Lemmas

Let $q = e^{2\pi i\tau}$, $\tau \in \mathcal{H}$ where $\mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. Then, the Dedekind η-function defined by:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \tag{3.1}$$

is a modular form of weight 1/2. Also, let us denote by $\psi(q)$ the following sum:

$$\psi(q) = \sum_{n=0}^{\infty} q^{T_n}, \tag{3.2}$$

where $T_n = \dfrac{n(n+1)}{2}$ (for $n = 0, 1, 2, \ldots$) are triangular numbers. Then, we have the following well-known result due to Gauss:

Lemma 3.1.

$$\psi(q) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})}{(1 - q^{2n-1})}. \tag{3.3}$$

Thus, we have from Lemma 3.1 that:

$$\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{12}}{(1 - q^{2n-1})^{12}} = \psi^{12}(q) = \sum_{n=1}^{\infty} t_{12}(n) q^n, \tag{3.4}$$

where $t_{12}(n)$ is the number of ways of representing a positive integer n as a sum of 12 triangular numbers. Next, we have the following well-known result of Ono, Robins and Wahl [2].

Theorem 3.2. *Let $\eta^{12}(2\tau) = \sum_{k=0}^{\infty} a(2k+1) q^{2k+1}$. Then, for a positive integer n, we have:*

$$t_{12}(n) = \frac{\sigma_5(2n+3) - a(2n+3)}{256}, \tag{3.5}$$

where

$$\sigma_5(n) = \sum_{d|n} d^5. \tag{3.6}$$

4. Proof of Theorem 2.2

Since $\zeta(6)=\dfrac{\pi^6}{945}$ has the following equivalent form:

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^6}=\frac{63}{64}\zeta(6)=\frac{\pi^6}{960}, \tag{4.1}$$

it will be sufficient to get the q-analogue of (4.1). Now, from q-analogue of Euler's Gamma function, we know that:

$$\lim_{q\uparrow 1}(1-q)\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2}{(1-q^{2n-1})^2}=\frac{\pi}{2}, \tag{4.2}$$

so that from (4.2), we have:

$$\lim_{q\uparrow 1}(1-q)^6\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{12}}{(1-q^{2n-1})^{12}}=\frac{\pi^6}{64}. \tag{4.3}$$

Next, we consider the following infinite series

$$S_6(q):=\sum_{k=0}^{\infty}\frac{q^k(1+q^{2k+1})\,P_4(q^{2k+1})}{(1-q^{2k+1})^6}, \tag{4.4}$$

where $P_4(x)=x^4+236x^3+1446x^2+236x+1$.

By partial fractions, we have:

$$\begin{aligned} S_6(q)=\sum_{k=0}^{\infty}q^k\Bigg\{&\frac{3840}{(1-q^{2k+1})^6}-\frac{9600}{(1-q^{2k+1})^5}+\frac{8160}{(1-q^{2k+1})^4}\\ &-\frac{2640}{(1-q^{2k+1})^3}+\frac{242}{(1-q^{2k+1})^2}-\frac{1}{(1-q^{2k+1})}\Bigg\}. \end{aligned} \tag{4.5}$$

Lemma 4.1. *With $S_6(q)$ represented by (4.5), we have:*

$$S_6(q)=256q\sum_{n=0}^{\infty}t_{12}(n)q^n+\phi^{12}(q). \tag{4.6}$$

Proof. From (4.5), we have:

$$\begin{aligned} S_6(q)&=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}q^k\Bigg\{3840\binom{-6}{j}-9600\binom{-5}{j}+8160\binom{-4}{j}\\ &\quad-2640\binom{-3}{j}+242\binom{-2}{j}-\binom{-1}{j}\Bigg\}(-q)^{j(2k+1)}\\ &=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\{32(j+1)(j+2)(j+3)(j+4)(j+5)\\ &\quad-400(j+1)(j+2)(j+3)(j+4)+1360(j+1)(j+2)(j+3)\\ &\quad-1320(j+1)(j+2)+242(j+1)-1\}\,q^{k+j(2k+1)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(2j+1)^5 q^{\frac{(2j+1)(2k+1)-1}{2}} \\
&= \sum_{n=0}^{\infty} \sigma_5(2n+1)q^n \\
&= 1 + \sum_{n=1}^{\infty} \sigma_5(2n+1)q^n \\
&= 1 + q\sum_{n=0}^{\infty} \sigma_5(2n+3)q^n.
\end{aligned}
$$

Also from (3.1), we have:

$$
\begin{aligned}
\phi^{12}(q) &= \frac{\eta^{12}(\tau)}{q^{\frac{1}{2}}} \\
&= \sum_{n=0}^{\infty} a(2n+1)q^n \\
&= 1 + \sum_{n=1}^{\infty} a(2n+1)q^n \\
&= 1 + q\sum_{n=0}^{\infty} a(2n+3)q^n.
\end{aligned}
$$

Thus, from above, we have:

$$
\begin{aligned}
S_6(q) - \phi^{12}(q) &= q\sum_{n=0}^{\infty} \left\{\sigma_5(2n+3) - a(2n+3)\right\} q^n \\
&= 256\, q\sum_{n=0}^{\infty} t_{12}(n)q^n,
\end{aligned}
$$

where the last step follows from Theorem 3.2. This completes the proof of Theorem 2.2. □

We also note that

$$
\begin{aligned}
\lim_{q\uparrow 1}\,(1-q)^6(S_6(q) - \phi^{12}(q)) &= \lim_{q\uparrow 1}\,(1-q)^6 S_6(q) - \lim_{q\uparrow 1}\,(1-q)^6\phi^{12}(q) \\
&= \sum_{k=0}^{\infty} \frac{3840}{(2k+1)^6},
\end{aligned}
\tag{4.7}
$$

where $\lim_{q\uparrow 1}\,(1-q)^6\phi^{12}(q) = 0$ and $q \uparrow 1$ indicates $q \to 1$ from within the unit disk. Hence, combining Eqs. (4.1), (4.3), (4.7), and Lemma 4.1, Theorem 2.2 follows.

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