Ann. Comb. 20 (2016) 815–829
DOI 10.1007/s00026-016-0325-x
Published online July 30, 2016

Annals of Combinatorics

Facets of the r-Stable (n, k)-Hypersimplex

Takayuki Hibi[1] and Liam Solus[2*]

[1]Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560–0043, Japan
hibi@math.sci.osaka-u.ac.jp

[2]Department of Mathematics, University of Kentucky, Lexington, KY 40506–0027, USA
liam.solus@uky.edu

Received December 25, 2014

Mathematics Subject Classification: 52B05, 52B20

Abstract. Let k, n, and r be positive integers with $k < n$ and $r \leq \lfloor \frac{n}{k} \rfloor$. We determine the facets of the r-stable n, k-hypersimplex. As a result, it turns out that the r-stable n, k-hypersimplex has exactly $2n$ facets for every $r < \lfloor \frac{n}{k} \rfloor$. We then utilize the equations of the facets to study when the r-stable hypersimplex is Gorenstein. For every $k > 0$ we identify an infinite collection of Gorenstein r-stable hypersimplices, consequently expanding the collection of r-stable hypersimplices known to have unimodal Ehrhart δ-vectors.

Keywords: r-stable hypersimplex, hypersimplex, facet, Gorenstein

1. Introduction

The (n, k)-hypersimplices are an important collection of integer polytopes arising naturally in the settings of convex optimization, matroid theory, combinatorics, and algebraic geometry. Generalizing the standard $(n-1)$-simplex, the (n, k)-hypersimplices serve as a useful collection of examples in these various contexts. While these polytopes are well studied, there remain interesting open questions about their properties in the field of Ehrhart theory, the study of integer point enumeration in dilations of rational polytopes (see, for example, [4]). The r-stable (n, k)-hypersimplices are a collection of lattice polytopes within the (n, k)-hypersimplex that were introduced in [2] for the purpose of studying unimodality of the Ehrhart δ-polynomials of the (n, k)-hypersimplices. However, they also exhibit interesting geometric similarities to the (n, k)-hypersimplices which they generalize. For example, it is shown in [2] that a regular unimodular triangulation of the (n, k)-hypersimplex, called the circuit triangulation, restricts to a triangulation of each r-stable (n, k)-hypersimplex.

* Liam Solus was supported by a 2014 National Science Foundation/Japan Society for the Promotion of Science East Asia and Pacific Summer Institute Fellowship.

In the present paper, we compute the facets of the r-stable (n,k)-hypersimplices for $1 \leq r < \lfloor \frac{n}{k} \rfloor$ and then study when they are Gorenstein. In Section 2, we compute their facet-defining inequalities (Theorem 2.1). From these computations, we see that the geometric similarities between the (n,k)-hypersimplex and the r-stable (n,k)-hypersimplices within are apparent in their minimal H-representations. Moreover, it turns out that each r-stable (n,k)-hypersimplex has exactly $2n$ facets for $1 \leq r < \lfloor \frac{n}{k} \rfloor$ (Corollary 2.2). In Section 3, we classify $1 \leq r < \lfloor \frac{n}{k} \rfloor$ for which these polytopes are Gorenstein (Theorem 3.6). We conclude that the Ehrhart δ-vector of each Gorenstein r-stable hypersimplex is unimodal (Corollary 3.7), thereby expanding the collection of r-stable hypersimplices known to have unimodal δ-polynomials.

2. The H-Representation of the r-Stable (n,k)-Hypersimplex

We first recall the definitions of the (n,k)-hypersimplices and the r-stable (n,k)-hypersimplices. For integers $0 < k < n$ let $[n] := \{1,2,\ldots,n\}$ and let $\binom{[n]}{k}$ denote the collection of all k-subsets of $[n]$. The *characteristic vector* of a subset I of $[n]$ is the $(0,1)$-vector $\varepsilon_I = (\varepsilon_1,\ldots,\varepsilon_n)$ for which $\varepsilon_i = 1$ for $i \in I$ and $\varepsilon_i = 0$ for $i \notin I$. The (n,k)*-hypersimplex* is the convex hull in $\mathbb{R}^n$ of the collection of characteristic vectors $\{\varepsilon_I : I \in \binom{[n]}{k}\}$, and it is denoted $\Delta_{n,k}$. Label the vertices of a regular n-gon embedded in $\mathbb{R}^2$ in a clockwise fashion from 1 to n. Given a third integer $1 \leq r \leq \lfloor \frac{n}{k} \rfloor$, a subset $I \subset [n]$ (and its characteristic vector) is called *r-stable* if, for each pair $i, j \in I$, the path of shortest length from i to j about the n-gon uses at least r edges. The *r-stable n,k-hypersimplex*, denoted by $\Delta_{n,k}^{\mathrm{stab}(r)}$, is the convex polytope in $\mathbb{R}^n$ which is the convex hull of the characteristic vectors of all r-stable k-subsets of $[n]$. For fixed n and k the r-stable (n,k)-hypersimplices form the nested chain of polytopes

$$\Delta_{n,k} \supset \Delta_{n,k}^{\mathrm{stab}(2)} \supset \Delta_{n,k}^{\mathrm{stab}(3)} \supset \cdots \supset \Delta_{n,k}^{\mathrm{stab}\left(\lfloor \frac{n}{k} \rfloor\right)}.$$

Notice that $\Delta_{n,k}$ is precisely the 1-stable (n,k)-hypersimplex.

The definitions of $\Delta_{n,k}$ and $\Delta_{n,k}^{\mathrm{stab}(r)}$ provided are V-representations of these polytopes. In this section, we provide the minimal H-representation of $\Delta_{n,k}^{\mathrm{stab}(r)}$, i.e., its collection of facet-defining inequalities. It is well known that the facet-defining inequalities of $\Delta_{n,k}$ are $\sum_{i=1}^n x_i = k$ together with $x_\ell \geq 0$ and $x_\ell \leq 1$ for all $\ell \in [n]$. Let H denote the hyperplane in $\mathbb{R}^n$ defined by the equation $\sum_{i=1}^n x_i = k$. For $1 \leq r \leq \lfloor \frac{n}{k} \rfloor$ and $\ell \in [n]$ consider the closed convex subsets of $\mathbb{R}^n$

$$H_\ell^{(+)} := \{(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n : x_\ell \geq 0\} \cap H, \quad \text{and}$$

$$H_{\ell,r}^{(-)} := \left\{ (x_1,x_2,\ldots,x_n) \in \mathbb{R}^n : \sum_{i=\ell}^{\ell+r-1} x_i \leq 1 \right\} \cap H.$$

In the definition of $H_{\ell,r}^{(-)}$ the indices i of the coordinates $x_1,\ldots,x_n$ are taken to be elements of $\mathbb{Z}/n\mathbb{Z}$. We also let H_ℓ and $H_{\ell,r}$ denote the $(n-2)$-flats given by strict equality in the above definitions. In the following we will say an $(n-2)$-flat is *facet-defining* (or *facet-supporting*) for $\Delta_{n,k}^{\mathrm{stab}(r)}$ if it contains a facet of $\Delta_{n,k}^{\mathrm{stab}(r)}$.

Theorem 2.1. *Let $1 < k < n-1$. For $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$ the facet-defining inequalities for $\Delta_{n,k}^{\mathrm{stab}(r)}$ are $\sum_{i=1}^{n} x_i = k$ together with $\sum_{i=\ell}^{\ell+r-1} x_i \leq 1$ and $x_\ell \geq 0$ for $\ell \in [n]$. In particular,*

$$\Delta_{n,k}^{\mathrm{stab}(r)} = \bigcap_{\ell \in [n]} H_\ell^{(+)} \cap \bigcap_{\ell \in [n]} H_{\ell,r}^{(-)}.$$

The following is an immediate corollary to these results.

Corollary 2.2. *All but possibly the smallest polytope in the nested chain*

$$\Delta_{n,k} \supset \Delta_{n,k}^{\mathrm{stab}(2)} \supset \Delta_{n,k}^{\mathrm{stab}(3)} \supset \cdots \supset \Delta_{n,k}^{\mathrm{stab}\left(\left\lfloor \frac{n}{k} \right\rfloor\right)}$$

has $2n$ facets.

This is an interesting geometric property since the number of vertices of these polytopes strictly decreases down the chain. To prove Theorem 2.1 we will utilize the geometry of the circuit triangulation of $\Delta_{n,k}$ as defined in [8], the construction of which we will now recall.

2.1. The Circuit Triangulation

Fix $0 < k < n$, and let $G_{n,k}$ be the labeled, directed graph with the following vertices and edges. The vertices of $G_{n,k}$ are all the vectors $\varepsilon_I \in \mathbb{R}^n$ where I is a k-subset of $[n]$. We think of the indices of a vertex of $G_{n,k}$ modulo n. Now suppose that ε and ε' are two vertices of $G_{n,k}$ such that for some $i \in [n]$ $(\varepsilon_i, \varepsilon_{i+1}) = (1,0)$ and ε' is obtained from ε by switching the order of ε_i and ε_{i+1}. Then the directed and labeled edge $\varepsilon \xrightarrow{i} \varepsilon'$ is an edge of $G_{n,k}$. Hence, an edge of $G_{n,k}$ corresponds to a move of a single 1 in a vertex ε one spot to the right, and such a move can be done if and only if the next spot is occupied by a 0.

We are interested in the circuits of minimal length in the graph $G_{n,k}$. Such a circuit is called a *minimal circuit*. Suppose that ε is a vertex in a minimal circuit of $G_{n,k}$. Then the minimal circuit can be thought of as a sequence of edges in $G_{n,k}$ that moves each 1 in ε into the position of the 1 directly to its right (modulo n). It follows that a minimal circuit in $G_{n,k}$ has length n. An example of a minimal circuit in $G_{9,3}$ is provided in Figure 1. Notice that for a fixed initial vertex of the minimal circuit the labels of the edges form a permutation $\omega = \omega_1\omega_2\cdots\omega_n \in S_n$, the symmetric group on n elements. Following the convention of [8], we associate a minimal circuit in $G_{n,k}$ with the permutation consisting of the labels of the edges of the circuit for which $\omega_n = n$. Let (ω) denote the minimal circuit in $G_{n,k}$ corresponding to the permutation $\omega \in S_n$ with $\omega_n = n$. Let $\sigma_{(\omega)}$ denote the convex hull in $\mathbb{R}^n$ of the set of vertices of (ω). Notice that $\sigma_{(\omega)}$ will always be an $(n-1)$-simplex.

Theorem 2.3. (Lam and Postnikov [8]) *The collection of simplices $\sigma_{(\omega)}$ given by the minimal circuits in $G_{n,k}$ are the maximal simplices of a triangulation of the hypersimplex $\Delta_{n,k}$. We call this triangulation the* circuit triangulation.

Denote the circuit triangulation of $\Delta_{n,k}$ by $\nabla_{n,k}$, and let $\max\nabla_{n,k}$ denote the set of maximal simplices of $\nabla_{n,k}$. To simplify notation we will write ω to denote the simplex $\sigma_{(\omega)} \in \max\nabla_{n,k}$. In [2] it is shown that the collection of simplices in $\nabla_{n,k}$ that lie completely within $\Delta_{n,k}^{\mathrm{stab}(r)}$ forms a triangulation of this polytope. We let $\nabla_{n,k}^{r}$ denote this triangulation of $\Delta_{n,k}^{\mathrm{stab}(r)}$ and let $\max\nabla_{n,k}^{r}$ denote the set of maximal simplices of $\nabla_{n,k}^{r}$. In the following, we compute the facet-defining inequalities for $\Delta_{n,k}^{\mathrm{stab}(r)}$ using the nesting of triangulations:

$$\nabla_{n,k} \supset \nabla_{n,k}^{2} \supset \nabla_{n,k}^{3} \supset \cdots \supset \nabla_{n,k}^{\lfloor \frac{n}{k} \rfloor}.$$

The method by which we will do this is outlined in the following remark.

Remark 2.4. To compute the facet-defining inequalities of $\Delta_{n,k}^{\mathrm{stab}(r)}$ we first consider the geometry of their associated facet-defining $(n-2)$-flats. Suppose that $\Delta_{n,k}^{\mathrm{stab}(r)}$ is $(n-1)$-dimensional. Since $\Delta_{n,k}^{\mathrm{stab}(r-1)} \supset \Delta_{n,k}^{\mathrm{stab}(r)}$ then a facet-defining $(n-2)$-flat of $\Delta_{n,k}^{\mathrm{stab}(r)}$ either also defines a facet of $\Delta_{n,k}^{\mathrm{stab}(r-1)}$ or it intersects $\mathrm{relint}\,\Delta_{n,k}^{\mathrm{stab}(r-1)}$, the relative interior of $\Delta_{n,k}^{\mathrm{stab}(r-1)}$. Therefore, to compute the facet-defining $(n-2)$-flats of $\Delta_{n,k}^{\mathrm{stab}(r)}$ it suffices to compute the former and latter collections of $(n-2)$-flats independently. To identify the former collection we will use an induction argument on r. To identify the latter collection we work with pairs of adjacent $(n-1)$-simplices in the set $\max\nabla_{n,k}^{r}$. Note that two simplices u, $\omega \in \max\nabla_{n,k}^{r}$ are adjacent (i.e., share a common facet) if and only if they differ by a single vertex. Therefore, their common vertices span an $(n-2)$-flat which we will denote by $H[u, \omega]$. Thus, we will identify adjacent pairs of simplices $u \in \max\nabla_{n,k}^{r-1}$ and $\omega \in \max\nabla_{n,k}^{r-1} \setminus \max\nabla_{n,k}^{r}$ for which $H[u, \omega]$ is facet-defining.

2.2. Computing Facet-Defining Inequalities via a Nesting of Triangulations

Suppose $1 < k < n-1$. In order to prove Theorem 2.1 in the fashion outlined by Remark 2.4 we require a sequence of lemmas. Notice that $\Delta_{n,k}^{\mathrm{stab}(r)}$ is contained in $H_{\ell}^{(+)}$ and $H_{\ell,r}^{(-)}$ for all $\ell \in [n]$. So in the following we simply show that H_{ℓ} and $H_{\ell,r}$ form the complete set of facet-defining $(n-2)$-flats.

Lemma 2.5. *Let* $1 \leq r < \lfloor \frac{n}{k} \rfloor$*. For all* $\ell \in [n]$*,* H_{ℓ} *is facet-defining for* $\Delta_{n,k}^{\mathrm{stab}(r)}$.

Proof. First notice that the result clearly holds for $r = 1$. So we need only show that $n-1$ affinely independent vertices of $\Delta_{n,k}^{\mathrm{stab}(r)}$ lie in H_{ℓ}. Hence, to prove the claim it suffices to identify a simplex $\omega \in \max\nabla_{n,k}^{r}$ such that H_{ℓ} supports a facet of ω. Since $r \leq \lfloor \frac{n}{k} \rfloor - 1$ it also suffices to work with $r = \lfloor \frac{n}{k} \rfloor - 1$.

Fix $\ell \in [n]$. For $r = \lfloor \frac{n}{k} \rfloor - 1$ we construct a minimal circuit in the graph $G_{n,k}$ that corresponds to a simplex in $\max\nabla_{n,k}^{r}$ for which H_{ℓ} is facet-supporting. To this end, consider the characteristic vector of the k-subset $\{(\ell-1)-(s-1)r \colon s \in [k]\} \subset [n]$. Denote this characteristic vector by ε^{ℓ}, and think of its indices modulo n. Labeling the 1 in coordinate $(\ell-1)-(s-1)r$ of ε^{ℓ} as 1_s, we see that 1_s and 1_{s+1} are separated by $r-1$ zeros for $s \in [k-1]$. That is, the coordinate $\varepsilon_i^{\ell} = 0$ for every $(\ell-1)-sr < i < (\ell-1)-(s-1)r$ (modulo n), and there are precisely $r-1$ such coordinates.

$$\begin{tikzcd}
\epsilon^{\ell} = (0,1,0,1,0,0,0,0,1) \arrow[r, "4"] & (0,1,0,0,1,0,0,0,1) \arrow[r, "5"] & (0,1,0,0,0,1,0,0,1) \arrow[d, "2"] \\
 & & (0,0,1,0,0,1,0,0,1) \arrow[d, "9"] \\
(0,1,0,1,0,0,0,1,0) \arrow[uu, "8"] & & (1,0,1,0,0,1,0,0,0) \arrow[d, "6"] \\
(0,1,0,1,0,0,1,0,0) \arrow[u, "7"] & (1,0,0,1,0,0,1,0,0) \arrow[l, "1"'] & (1,0,1,0,0,0,1,0,0) \arrow[l, "3"']
\end{tikzcd}$$

Figure 1: The minimal circuit (ω^{ℓ}) for $n = 9, k = 3$, and $\ell = 5$ constructed in Lemma 2.5.

Moreover, since $kr = k\left(\left\lfloor \frac{n}{k} \right\rfloor - 1\right) \leq n$ then there are at least $r-1$ zeros between 1_1 and 1_k. Hence, this vertex is r-stable. From ε^{ℓ} we can now construct an r-stable circuit (ω^{ℓ}) by moving the 1's in ε^{ℓ} one coordinate to the right (modulo n), one at a time, in the following pattern:

(1) Move 1_1.
(2) Move 1_1. Then move 1_2. Then move 1_3. … Then move 1_k.
(3) Repeat step (2) $r-1$ more times.
(4) Move 1_1 until it rests in entry $\ell - 1$.

An example of (ω^{ℓ}) for $n = 9$, $k = 3$, and $\ell = 5$ is provided in Figure 1. This produces a minimal circuit in $G_{n,k}$ since each 1_s has moved precisely enough times to replace 1_{s+1}. Moreover, since $k > 1$ then $k\left(\left\lfloor \frac{n}{k} \right\rfloor - 1\right) \leq n-2$. So there are at least $r+1$ 0's between 1_1 and 1_k in ε^{ℓ}. From here, it is a straightforward exercise to check that every vertex in (ω^{ℓ}) is r-stable. Therefore, $\omega^{\ell} \in \max \nabla^{r}_{n,k}$. Finally, since $r > 1$, the simplex ω^{ℓ} has only one vertex satisfying $x_{\ell} = 1$, and this is the vertex following ε^{ℓ} in the circuit (ω^{ℓ}). Hence, all other vertices of ω^{ℓ} satisfy $x_{\ell} = 0$. So H_{ℓ} supports a facet of ω^{ℓ}. Thus, we conclude that H_{ℓ} is facet-defining for $\Delta^{\mathrm{stab}(r)}_{n,k}$ for $r < \left\lfloor \frac{n}{k} \right\rfloor$. ∎

The following theorem follows immediately from the construction of the $(n-1)$-simplex ω^{ℓ} in the proof of Lemma 2.5, and it justifies the assumption on the dimension of $\Delta^{\mathrm{stab}(r)}_{n,k}$ made in Remark 2.4.

Theorem 2.6. *The polytope $\Delta^{\mathrm{stab}(r)}_{n,k}$ is $(n-1)$-dimensional for all $r < \left\lfloor \frac{n}{k} \right\rfloor$.*

Lemma 2.7. *Suppose $r > 1$ and $\Delta^{\mathrm{stab}(r)}_{n,k}$ is $(n-1)$-dimensional. Then $H_{\ell,r-1}$ is not facet-defining for $\Delta^{\mathrm{stab}(r)}_{n,k}$.*

Proof. Suppose for the sake of contradiction that $H_{\ell,r-1}$ is facet-defining for $\Delta_{n,k}^{\mathrm{stab}(r)}$. Since $\Delta_{n,k}^{\mathrm{stab}(r)}$ is $(n-1)$-dimensional then there exists an $(n-1)$-simplex $\omega \in \max \nabla_{n,k}^{r}$ such that $H_{\ell,r-1}$ is facet-defining for ω. In other words, every vertex in (ω) satisfies $\sum_{i=\ell}^{\ell+r-2} x_i = 1$ except for exactly one vertex, say $\varepsilon^{\star}$. Since all vertices in (ω) are $(0,\,1)$-vectors, this means that all vertices other than $\varepsilon^{\star}$ have exactly one coordinate in the subvector $(\varepsilon_{\ell},\,\varepsilon_{\ell+1},\dots,\,\varepsilon_{\ell+r-2})$ being 1 and all other coordinates are 0. Similarly, this subvector is the 0-vector for $\varepsilon^{\star}$. Since (ω) is a minimal circuit this means that the move preceding the vertex $\varepsilon^{\star}$ in (ω) results in the only 1 in $(\varepsilon_{\ell},\,\varepsilon_{\ell+1},\dots,\,\varepsilon_{\ell+r-2})$ exiting the subvector to the right. Similarly, the move following the vertex $\varepsilon^{\star}$ in (ω) results in a single 1 entering the subvector on the left. Suppose that

$$\varepsilon^{\star} = (\dots,\varepsilon^{\star}_{\ell-1},\varepsilon^{\star}_{\ell},\varepsilon^{\star}_{\ell+1},\dots,\varepsilon^{\star}_{\ell+r-2},\varepsilon^{\star}_{\ell+r-1},\dots) = (\dots,1,0,0,\dots,0,1,\dots).$$

Then this situation looks like

$$\begin{tikzcd}
& \epsilon^{\star} = (\dots,1,0,0,\dots,0,1,\dots) \arrow[dr, "\ell-1"] & \\
(\dots,1,0,0,\dots,1,0,\dots) \arrow[ur, "\ell+r-2"] & & (\dots,0,1,0,\dots,0,1,\dots)
\end{tikzcd}$$

Hence, neither the vertex preceding or following the vertex $\varepsilon^{\star}$ is r-stable. For example, in the vertex following $\varepsilon^{\star}$ there is a 1 in entries ℓ and $\ell+r-1$. This contradicts the fact that $\omega \in \max \nabla_{n,k}^{r}$. ∎

To see why Lemma 2.7 will be useful, suppose that Theorem 2.1 holds for $\Delta_{n,k}^{\mathrm{stab}(r-1)}$ for some $1 \le r < \lfloor \frac{n}{k} \rfloor$. Then Lemmas 2.5 and 2.7 tell us that the collection of facet-defining $(n-2)$-flats for $\Delta_{n,k}^{\mathrm{stab}(r-1)}$ that are also facet-defining for $\Delta_{n,k}^{\mathrm{stab}(r)}$ is $\{H_{\ell} \colon \ell \in [n]\}$. This is the nature of the induction argument mentioned in Remark 2.4. To identify the facet-defining $(n-2)$-flats of $\Delta_{n,k}^{\mathrm{stab}(r)}$ that intersect $\mathrm{relint}\,\Delta_{n,k}^{\mathrm{stab}(r-1)}$ we will use the following definition.

Definition 2.8. *Suppose u and ω are a pair of simplices in* $\max \nabla_{n,k}$ *satisfying*

- $u \in \max \nabla_{n,k}^{r}$,
- $\omega \in \max \nabla_{n,k}^{r-1} \setminus \max \nabla_{n,k}^{r}$, *and*
- *ω uses exactly one vertex that is not r-stable, called the* key vertex, *and this is the only vertex by which u and ω differ.*

We say that the ordered pair of simplices $(u,\,\omega)$ is an r-supporting pair *of $H[u,\,\omega]$, where $H[u,\,\omega]$ is the flat spanned by the common vertices of u and ω.*

Lemma 2.9. *Suppose $1 < r < \lfloor \frac{n}{k} \rfloor$. Suppose also that H_F is a $(n-2)$-flat defining a facet F of $\Delta_{n,k}^{\mathrm{stab}(r)}$ such that $H_F \cap \mathrm{relint}\,\Delta_{n,k}^{\mathrm{stab}(r-1)} \neq \emptyset$. Then $H_F = H[u,\,\omega]$ for some r-supporting pair of simplices $(u,\,\omega)$.*

Proof. Since $H_F \cap \operatorname{relint}\Delta_{n,k}^{\mathrm{stab}(r-1)} \neq \emptyset$ and $\Delta_{n,k}^{\mathrm{stab}(r)}$ is contained in $\Delta_{n,k}^{\mathrm{stab}(r-1)}$ then $F \cap \operatorname{relint}\Delta_{n,k}^{\mathrm{stab}(r-1)} \neq \emptyset$. That is, there exists some $\alpha \in F$ such that $\alpha \in \operatorname{relint}\Delta_{n,k}^{\mathrm{stab}(r-1)}$. Recall that $\nabla_{n,k}^{r-1}$ is a triangulation of $\Delta_{n,k}^{\mathrm{stab}(r-1)}$ that restricts to a triangulation $\nabla_{n,k}^{r}$ of $\Delta_{n,k}^{\mathrm{stab}(r)}$. It follows that $\nabla_{n,k}^{r}$ and $\nabla_{n,k}^{r-1} \backslash \nabla_{n,k}^{r}$ give identical triangulations of $\partial\Delta_{n,k}^{\mathrm{stab}(r)} \cap \operatorname{relint}\Delta_{n,k}^{\mathrm{stab}(r-1)}$. Since $\Delta_{n,k}^{\mathrm{stab}(r)}$ is $(n-1)$-dimensional we may assume, without loss of generality, that α lies in the relative interior of an $(n-2)$-dimensional simplex in the triangulation of $\partial\Delta_{n,k}^{\mathrm{stab}(r)} \cap \operatorname{relint}\Delta_{n,k}^{\mathrm{stab}(r-1)}$ induced by $\nabla_{n,k}^{r}$ and $\nabla_{n,k}^{r-1} \backslash \nabla_{n,k}^{r}$. Therefore, there exists some $u \in \max\nabla_{n,k}^{r}$ such that H_F is facet-defining for u and $\alpha \in u \cap H_F$, and there exists some $\omega \in \max\nabla_{n,k}^{r-1} \backslash \max\nabla_{n,k}^{r}$ such that $\alpha \in \omega \cap H_F$. Since $\nabla_{n,k}^{r-1}$ is a triangulation of $\Delta_{n,k}^{\mathrm{stab}(r-1)}$ it follows that $u \cap H_F = \omega \cap H_F$. Hence, u and ω are adjacent simplices that share the facet-defining $(n-2)$-flat H_F, and they form an r-supporting pair (u, ω) with $H[u, \omega] = H_F$. ∎

It will be helpful to understand the key vertex of an r-supporting pair (u, ω). To do so, we will use the following definition.

Definition 2.10. *Let $\varepsilon \in \mathbb{R}^n$ be a vertex of $\Delta_{n,k}$. A* pair *of* 1*'s in ε is an ordered pair of two coordinates of ε, (i, j), such that $\varepsilon_i = \varepsilon_j = 1$, and $\varepsilon_t = 0$ for all $i < t < j$ (modulo n). A pair of* 1*'s is called an* r-stable pair *if there are at least $r-1$ 0's separating the two 1's.*

Lemma 2.11. *Suppose (u, ω) is an r-supporting pair, and let ε be the key vertex of this pair. Then ε contains precisely one $(r-1)$-stable but not r-stable pair, $(\ell, \ell+r-1)$. Moreover, $H[u, \omega] = H_{\ell,r}$.*

Proof. We first show that ε has precisely one $(r-1)$-stable but not r-stable pair, $(\ell, \ell+r-1)$. To see this, consider the minimal circuit (ω) in the graph $G_{n,k}$ associated with the simplex ω. Think of the key vertex ε as the initial vertex of this circuit, and recall that each edge of the circuit corresponds to a move of exactly one 1 to the right by exactly one entry. Hence, in the circuit (ω) the vertex following ε differs from ε by a single right move of a single 1. Since ε is the only vertex in (ω) that is $(r-1)$-stable but not r-stable, then the move of this single 1 to the right by one entry must eliminate all pairs that are $(r-1)$-stable but not r-stable. Moreover, this move cannot introduce any new $(r-1)$-stable but not r-stable pairs. Since a single 1 can be in at most two pairs, and this 1 must move exactly one entry to the right, then this 1 must be in entry j in the pairs (i, j) and (j, t) where (i, j) is $(r-1)$-stable but not r-stable, and (j, t) is $(r+1)$-stable. Moreover, since the move of the 1 in entry j can only change the stability of the pairs (i, j) and (j, t), then it must be that all other pairs are r-stable.

Finally, since ω has the unique $(r-1)$-stable but not r-stable vertex ε, and since ε has the unique $(r-1)$-stable but not r-stable pair $(\ell, \ell+r-1)$ then all other vertices in ω satisfy $\sum_{i=\ell}^{\ell+r-1} x_i = 1$. Hence, $H[u, \omega] = H_{\ell,r}$. ∎

Lemma 2.12. *Suppose $1 < r < \left\lfloor \frac{n}{k} \right\rfloor$. Suppose also that H_F is an $(n-2)$-flat defining a facet F of $\Delta_{n,k}^{\mathrm{stab}(r)}$ and $H_F \cap \operatorname{relint}\Delta_{n,k}^{\mathrm{stab}(r-1)} \neq \emptyset$. Then $H_F = H_{\ell,r}$ for some $\ell \in [n]$.*

Proof. By Lemma 2.9 $H_F = H[u, \omega]$ for some r-supporting pair (u, ω). By Lemma 2.11 ω has a unique vertex that is $(r-1)$-stable but not r-stable with a unique $(r-1)$-stable but not r-stable pair $(\ell, \ell+r-1)$ for some $\ell \in [n]$. Thus, $H_F = H[u, \omega] = H_{\ell,r}$. ∎

We now show that $H_{\ell,r}$ is indeed facet-defining for $\Delta_{n,k}^{\mathrm{stab}(r)}$ for all $\ell \in [n]$.

Lemma 2.13. *Suppose* $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$ *or* $n = kr+1$. *Then* $H_{\ell,r}$ *is facet-defining for* $\Delta_{n,k}^{\mathrm{stab}(r)}$ *for all* $\ell \in [n]$.

Proof. First we note that the result is clearly true for $r = 1$. So in the following we assume $r > 1$. To prove the claim we show that $H_{\ell,r}$ supports an $(n-1)$-simplex $\omega \in \max \nabla_{n,k}^{r}$.

To this end, consider the characteristic vector of the k-subset $\{(\ell-1)+(s-1)r \colon s \in [k]\} \subset [n]$. Denote this characteristic vector by ε^{ℓ}, and think of its indices modulo n. Labeling the 1 in coordinate $(\ell-1)+(s-1)r$ of ε^{ℓ} as 1_s, it is quick to see that 1_s and 1_{s+1} are separated by $r-1$ zeros for every $s \in [k]$. That is, $\varepsilon_i^{\ell} = 0$ for every $(\ell-1)+(s-1)r < i < (\ell-1)+sr$ (modulo n), and there are precisely $r-1$ such coordinates. Moreover, since $r < \left\lfloor \frac{n}{k} \right\rfloor$ or $n = kr+1$ then $n \geq kr+1$. So there are at least r zeros between 1_1 and 1_k. Hence, this vertex is r-stable. From ε^{ℓ} we can now construct an r-stable circuit (ω^{ℓ}) by moving the 1's in ε^{ℓ} one coordinate to the right (modulo n), one at a time, in the following pattern:

(1) Move 1_k. Then move 1_{k-1}. Then move 1_{k-2}. $\dots$ Then move 1_1.
(2) Repeat step (1) $r-1$ more times.
(3) Move 1_k to entry ℓ.

Each move in this pattern produces a new r-stable vertex since there are always at least $r-1$ zeros between each pair of 1's. So $\omega^{\ell} \in \max \nabla_{n,k}^{r}$ and $H_{\ell,r}$ supports ω^{ℓ} since every vertex of (ω^{ℓ}) lies in $H_{\ell,r}$ except for the vertex preceding the first move of 1_1 in the circuit (ω^{ℓ}). ∎

Remark 2.14. When $n = kr+1$ then $\omega^{\ell} = \Delta_{n,k}^{\mathrm{stab}(r)}$ for all $\ell \in [n]$. So the facet-defining inequalities for $\omega^{\ell} = \Delta_{n,k}^{\mathrm{stab}(r)}$ are precisely $H_{\ell,r}^{(-)}$ for $\ell \in [n]$.

From Lemmas 2.12 and 2.13 we see that when $1 < r < \left\lfloor \frac{n}{k} \right\rfloor$ the facet-defining $(n-2)$-flats for $\Delta_{n,k}^{\mathrm{stab}(r)}$ that intersect $\operatorname{relint} \Delta_{n,k}^{\mathrm{stab}(r-1)}$ are precisely $H_{\ell,r}$ for $\ell \in [n]$. We are now ready to prove Theorem 2.1.

2.2.1. Proof of Theorem 2.1

First recall that Theorem 2.1 is known to be true for $r = 1$. Now let $1 < r < \left\lfloor \frac{n}{k} \right\rfloor$. By Theorem 2.6 we know that $\Delta_{n,k}^{\mathrm{stab}(r)}$ is $(n-1)$-dimensional. First let $r = 2$. By Lemma 2.5 we know that H_{ℓ} is facet-defining for $\Delta_{n,k}^{\mathrm{stab}(2)}$ for all $\ell \in [n]$. By Lemma 2.7 we know that for every $\ell \in [n]$ $H_{\ell,1}$ is not facet-defining for $\Delta_{n,k}^{\mathrm{stab}(2)}$. Thus, the collection of facet-defining $(n-2)$-flats for $\Delta_{n,k}$ that are also facet-defining for $\Delta_{n,k}^{\mathrm{stab}(2)}$ are

$\{H_\ell\colon \ell \in [n]\}$, and all other facet-defining $(n-2)$-flats for $\Delta_{n,k}^{\text{stab}(2)}$ must intersect the relative interior of $\Delta_{n,k}$. Therefore, by Lemmas 2.12 and 2.13 the remaining facet-defining $(n-2)$-flats for $\Delta_{n,k}^{\text{stab}(2)}$ are $H_{\ell,2}$ for $\ell \in [n]$. Since $\Delta_{n,k}^{\text{stab}(r)}$ is contained in $H_\ell^{(+)}$ and $H_{\ell,r}^{(-)}$, this proves the result for $r = 2$. Theorem 2.1 then follows by iterating this argument for $2 < r < \lfloor \frac{n}{k} \rfloor$.

3. Gorenstein r-Stable Hypersimplices

In [2], the authors note that the r-stable hypersimplices appear to have unimodal Ehrhart δ-vectors, and they verify this observation for a collection of these polytopes in the $k = 2$ case. In [3], it is shown that a Gorenstein integer polytope with a regular unimodular triangulation has a unimodal δ-vector. In [2], it is shown that $\Delta_{n,k}^{\text{stab}(r)}$ has a regular unimodular triangulation. One application for the equations of the facets of a rational convex polytope is to determine whether or not the polytope is Gorenstein [6]. We now utilize Theorem 2.1 to identify $1 \leq r < \lfloor \frac{n}{k} \rfloor$ for which $\Delta_{n,k}^{\text{stab}(r)}$ is Gorenstein. We identify a collection of such polytopes for every $k \geq 2$, thereby expanding the collection of r-stable hypersimplices known to have unimodal δ-vectors. In this section we let $1 < k < n-1$. This is because $\Delta_{n,1}$ and $\Delta_{n,n-1}$ are simply copies of the standard $(n-1)$-simplex, which are well known to be Gorenstein [1, p. 29].

First we recall the definition of a Gorenstein polytope. Let $P \subset \mathbb{R}^N$ be a rational convex polytope of dimension d, and for an integer $q \geq 1$ let $qP := \{q\alpha\colon \alpha \in P\}$. Let $x_1, x_2, \ldots, x_N$, and z be indeterminates over some field K. Given an integer $q \geq 1$, let $A(P)_q$ denote the vector space over K spanned by the monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N} z^q$ for $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in qP \cap \mathbb{Z}^N$. Since P is convex we have that $A(P)_p A(P)_q \subset A(P)_{p+q}$ for all p and q. It then follows that the graded algebra

$$A(P) := \bigoplus_{q=0}^{\infty} A(P)_q$$

is finitely generated over $K = A(P)_0$. We call $A(P)$ the *Ehrhart Ring* of P, and we say that P is *Gorenstein* if $A(P)$ is Gorenstein.

We now recall the combinatorial criterion given in [5] for an integral convex polytope P to be Gorenstein. Let ∂P denote the boundary of P and let $\text{relint}(P) = P - \partial P$. We say that P is of *standard type* if $d = N$ and the origin in $\mathbb{R}^d$ is contained in $\text{relint}(P)$. When $P \subset \mathbb{R}^d$ is of standard type we define its polar set

$$P^\star = \left\{ (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d\colon \sum_{i=1}^{d} \alpha_i \beta_i \leq 1 \text{ for every } (\beta_1, \beta_2, \ldots, \beta_d) \in P \right\}.$$

The polar set $P^\star$ is again a convex polytope of standard type, and $(P^\star)^\star = P$. We call $P^\star$ the *dual polytope* of P. Suppose $(\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d$, and K is the hyperplane in $\mathbb{R}^d$ defined by the equation $\sum_{i=1}^{d} \alpha_i x_i = 1$. A well-known fact is that $(\alpha_1, \alpha_2, \ldots, \alpha_d)$ is a vertex of $P^\star$ if and only if $K \cap P$ is a facet of P. It follows that the dual polytope of a rational polytope is always rational. However, it need not be that the dual of an integral polytope is always integral. If P is an integral polytope with integral dual we say that P is *reflexive*. This idea plays a key role in the following combinatorial characterization of Gorenstein polytopes.

Theorem 3.1. (De Negri and Hibi [5]) *Let $P \subset \mathbb{R}^d$ be an integral polytope of dimension d, and let q denote the smallest positive integer for which*

$$q(\mathrm{relint}(P)) \cap \mathbb{Z}^d \neq \emptyset.$$

Fix an integer point $\alpha \in q(\mathrm{relint}(P)) \cap \mathbb{Z}^d$, and let Q denote the integral polytope $qP - \alpha \subset \mathbb{R}^d$. Then the polytope P is Gorenstein if and only if the polytope Q is reflexive.

Since Theorem 3.1 requires that the polytope be full-dimensional we consider $\varphi^{-1}\left(\Delta_{n,k}^{\mathrm{stab}(r)}\right)$, where $\varphi\colon \mathbb{R}^{n-1} \longrightarrow H$ is the affine isomorphism

$$\varphi\colon (\alpha_1, \alpha_2, \dots, \alpha_{n-1}) \longmapsto \left(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, k - \left(\sum_{i=1}^{n-1} \alpha_i\right)\right).$$

Notice that φ is also a lattice isomorphism. Hence, we have the isomorphism of Ehrhart Rings as graded algebras

$$A\left(\varphi^{-1}\left(\Delta_{n,k}^{\mathrm{stab}(r)}\right)\right) \cong A\left(\Delta_{n,k}^{\mathrm{stab}(r)}\right).$$

Let $P_{n,k}^{\mathrm{stab}(r)} := \varphi^{-1}\left(\Delta_{n,k}^{\mathrm{stab}(r)}\right)$, and recall from Theorem 2.1 that

$$\Delta_{n,k}^{\mathrm{stab}(r)} = \left(\bigcap_{\ell=1}^{n} H_\ell^{(+)}\right) \cap \left(\bigcap_{\ell=1}^{n} H_{\ell,r}^{(-)}\right).$$

3.1. The H-Representation for $P_{n,k}^{\mathrm{stab}(r)}$

We now give a description of the facet-defining inequalities for $P_{n,k}^{\mathrm{stab}(r)}$ in terms of those defining $\Delta_{n,k}^{\mathrm{stab}(r)}$. In the following, it will be convenient to let $T(\ell) = \{\ell, \ell+1, \ell+2, \dots, \ell+r-1\}$ for $\ell \in [n]$. We also let $T(\ell)^c$ denote the complement of $T(\ell)$ in $[n]$. Notice that for a fixed $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$ and $\ell \in [n]$, the set $T(\ell)$ is precisely the set of summands in the defining equation of the $(n-2)$-flat $H_{\ell,r}$. The defining inequalities of $P_{n,k}^{\mathrm{stab}(r)}$ corresponding to the $(n-2)$-flats $H_{\ell,r}$ come in two types, dependent on whether $n \notin T(\ell)$ or $n \in T(\ell)$. If $n \notin T(\ell)$ then

$$K_{\ell,r}^{(-)} := \varphi^{-1}\left(H_{\ell,r}^{(-)}\right) = \left\{(x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}\colon \sum_{i \in T(\ell)} x_i \leq 1\right\}.$$

If $n \in T(\ell)$ then

$$\widetilde{K}_{\ell,r}^{(+)} := \varphi^{-1}\left(H_{\ell,r}^{(-)}\right) = \left\{(x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}\colon \sum_{i \in T(\ell)^c} x_i \geq k-1\right\}.$$

Similarly, if $\ell \neq n$ then

$$K_\ell^{(+)} := \varphi^{-1}\left(H_\ell^{(+)}\right) = \left\{(x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : x_\ell \geq 0\right\}.$$

Finally, if $\ell = n$ then

$$K_n^{(-)} := \varphi^{-1}\left(H_n^{(+)}\right) = \left\{(x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} x_i \leq k\right\}.$$

Thus, we may write $P_{n,k}^{\mathrm{stab}(r)}$ as the intersection of closed halfspaces in $\mathbb{R}^{n-1}$

$$P_{n,k}^{\mathrm{stab}(r)} = \left(\bigcap_{n \notin T(\ell)} K_{\ell,r}^{(-)}\right) \cap \left(\bigcap_{n \in T(\ell)} \widetilde{K}_{\ell,r}^{(+)}\right) \cap \left(\bigcap_{i=1}^{n-1} K_\ell^{(+)}\right) \cap K_n^{(-)}.$$

To denote the supporting hyperplanes corresponding to these halfspaces we simply drop the superscripts $(+)$ and $(-)$.

3.2. The Codegree of $P_{n,k}^{\mathrm{stab}(r)}$

Given the above description of $P_{n,k}^{\mathrm{stab}(r)}$, we would now like to determine the smallest positive integer q for which $qP_{n,k}^{\mathrm{stab}(r)}$ contains a lattice point in its relative interior. To do so, recall that for a lattice polytope P of dimension d we can define the (*Ehrhart*) *δ-polynomial* of P. If we write this polynomial as

$$\delta_P(z) = \delta_0 + \delta_1 z + \delta_2 z^2 + \cdots + \delta_d z^d,$$

then we call the coefficient vector $\delta(P) = (\delta_0, \delta_1, \delta_2, \ldots, \delta_d)$ the *δ-vector* of P. We let s denote the degree of $\delta_P(z)$, and we call $q = (d+1) - s$ the *codegree* of P. It is a consequence of Ehrhart Reciprocity that q is the smallest positive integer such that qP contains a lattice point in its relative interior [1]. Hence, we would like to compute the codegree of $P_{n,k}^{\mathrm{stab}(r)}$. To do so requires that we first prove two lemmas. In the following let $q = \left\lceil \frac{n}{k} \right\rceil$. Our first goal is to show that there is at least one integer point in $\mathrm{relint}\left(qP_{n,k}^{\mathrm{stab}(r)}\right)$ for $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$. We then show that q is the smallest positive integer for which this is true. Recall that $q = \frac{n+\alpha}{k}$ for some $\alpha \in \{0, 1, \ldots, k-1\}$. Also recall that for fixed n and k we have the nesting of polytopes

$$P_{n,k} \supset P_{n,k}^{\mathrm{stab}(2)} \supset P_{n,k}^{\mathrm{stab}(3)} \supset \cdots \supset P_{n,k}^{\mathrm{stab}\left(\left\lfloor \frac{n}{k} \right\rfloor - 1\right)} \supset P_{n,k}^{\mathrm{stab}\left(\left\lfloor \frac{n}{k} \right\rfloor\right)}.$$

Hence, if we identify an integer point inside $\mathrm{relint}\left(qP_{n,k}^{\mathrm{stab}\left(\left\lfloor \frac{n}{k} \right\rfloor - 1\right)}\right)$ then this same integer point lives inside $\mathrm{relint}\left(qP_{n,k}^{\mathrm{stab}(r)}\right)$ for every $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$. Given these facts, we now prove two lemmas.

Lemma 3.2. *Suppose that* $q = \left\lceil \frac{n}{k} \right\rceil = \frac{n+\alpha}{k}$ *where* $\alpha \in \{0, 1\}$. *Then the integer point* $(1, 1, \ldots, 1) \in \mathbb{R}^{n-1}$ *lies inside* $\mathrm{relint}\left(qP_{n,k}^{\mathrm{stab}(r)}\right)$ *for every* $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$.

Proof. It suffices to show that $(x_1, x_2, \ldots, x_{n-1}) = (1, 1, \ldots, 1)$ satisfies the set of inequalities

(i) $x_i > 0$, for $i \in [n-1]$,
(ii) $\sum_{i=1}^{n-1} x_i < kq$,
(iii) $\sum_{i \in T(\ell)} x_i < q$, for $n \notin T(\ell)$, and
(iv) $\sum_{i \in T(\ell)^c} x_i > (k-1)q$, for $n \in T(\ell)$.

We do this in two cases. First suppose that $\alpha = 0$. Then k divides n and $q = \frac{n}{k}$. Clearly, (i) is satisfied. To see that (ii) is also satisfied simply notice that $n-1 < kq$. To see that (iii) is satisfied recall that $\#T(\ell) = r$ and $r < \left\lfloor \frac{n}{k} \right\rfloor = q$. Finally, to see that (iv) is satisfied notice that $\#T(\ell)^c = n - r$. So we would like that $n - r > (k-1)q$. However, this follows quickly from the fact that $r < \frac{n}{k}$.

Now consider the case where $\alpha = 1$. Recall that it suffices to consider the case when $r = \left\lfloor \frac{n}{k} \right\rfloor - 1$. Inequalities (i), (ii), and (iii) are all satisfied in the same fashion as the case when $\alpha = 0$. So we need only check that (iv) is also satisfied. Again we would like that $n - r > (k-1)q$. Notice since $\alpha = 1$ then k does not divide n, and so $\left\lceil \frac{n}{k} \right\rceil = \left\lfloor \frac{n}{k} \right\rfloor + 1$. Hence, $q = r + 2$. The desired inequality then follows from $n + 2 > n + \alpha$. Thus, whenever $\alpha \in \{0, 1\}$, the lattice point $(1, 1, \ldots, 1) \in \operatorname{relint}\left(qP_{n,k}^{\operatorname{stab}(r)}\right)$ for every $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$. ∎

Next we would like to identify an integer point in the relative interior of $qP_{n,k}^{\operatorname{stab}(r)}$ for $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$ when $\alpha \geq 2$. In this case, the point $(1, 1, \ldots, 1)$ does not always work, so we must identify another point. Recall that it suffices to identify such a point for $r = \left\lfloor \frac{n}{k} \right\rfloor - 1$. To do so, we construct the desired point using the notions of r-stability. Fix n and k such that $q = \frac{n+\alpha}{k}$ for $\alpha \geq 2$, and let $r = \left\lfloor \frac{n}{k} \right\rfloor - 1$. This also fixes the value $\alpha \in \{2, 3, \ldots, k-1\}$. Since $r = \left\lfloor \frac{n}{k} \right\rfloor - 1$ we may construct an r-stable vertex in $\mathbb{R}^n$ as the characteristic vector of the set

$$\{n - r, n - 2r, n - 3r, \ldots, n - (k-1)r\} \subset [n].$$

Notice that there are at least r 0's between the n^{th} coordinate of the vertex and the $n - (k-1)r^{th}$ coordinate (read from right-to-left modulo n). In particular, this implies that the n^{th} coordinate (and the 1^{st} coordinate) is occupied by a 0. To construct the desired vertex replace the 1's in coordinates

$$n - (\alpha + 1)r, n - (\alpha + 2)r, \ldots, n - (k-1)r$$

with 0's. Now add 1 to each coordinate of this lattice point. If the resulting point is $(x_1, x_2, \ldots, x_n)$ then replace $x_n = 1$ with the value $kq - \left(\sum_{i=1}^{n-1} x_i\right)$. Call the resulting vertex ε^α, and consider the isomorphism $\widetilde{\varphi}\colon \mathbb{R}^{n-1} \longrightarrow H_q$, defined analogously to φ, where H_q is the hyperplane in $\mathbb{R}^n$ defined by the equation $\sum_{i=1}^n x_i = kq$. Notice that by our construction of ε^α, the point $\widetilde{\varphi}^{-1}(\varepsilon^\alpha)$ is simply ε^α with the last coordinate projected off.

Lemma 3.3. *Suppose that* $q = \left\lceil \frac{n}{k} \right\rceil = \frac{n+\alpha}{k}$ *for* $\alpha \in \{2, 3, \ldots, k-1\}$. *Then the lattice point* $\widetilde{\varphi}^{-1}(\varepsilon^\alpha)$ *lies inside* $\operatorname{relint}\left(qP_{n,k}^{\operatorname{stab}(r)}\right)$ *for every* $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$.

Proof. It suffices to show that when $r = \lfloor \frac{n}{k} \rfloor - 1$ the lattice point $(x_1, x_2, \dots, x_{n-1}) = \widetilde{\varphi}^{-1}(\varepsilon^\alpha)$ satisfies inequalities (i), (ii), (iii), and (iv) from the proof of Lemma 3.2. It is clear that (i) is satisfied. To see that (ii) is satisfied notice that $\sum_{i=1}^{n-1} x_i = n - 1 + \alpha$. This is because α coordinates of $\widetilde{\varphi}^{-1}$ are occupied by 2's and all other coordinates are occupied by 1's. Thus, inequality (ii) is satisfied since $n - 1 + \alpha < kq$. To see that (iii) is satisfied first notice that for $T(\ell)$ with $n \notin T(\ell)$

$$\sum_{i \in T(\ell)} x_i = \begin{cases} r, & \text{if } T(\ell) \text{ contains no entry with value 2,} \\ r+1, & \text{otherwise.} \end{cases}$$

This is because we have chosen the 2's to be separated by at least $r - 1$ 0's. Thus, since k does not divide n we have that $\sum_{i \in T(\ell)} x_i \leq r + 1 = \lfloor \frac{n}{k} \rfloor < q$. Finally, to see that (iv) is satisfied first notice that for $T(\ell)$ with $n \in T(\ell)$

$$\sum_{i \in T(\ell)^c} x_i = \begin{cases} n - r + \alpha - 1, & \text{if } T(\ell) \text{ contains an entry with value 2,} \\ n - r + \alpha, & \text{otherwise.} \end{cases}$$

Hence, we must show that $n - r + \alpha - 1 > (k-1)q$. However, since $\lceil \frac{n}{k} \rceil = \lfloor \frac{n}{k} \rfloor + 1$ then $r = q - 2$, and so the desired inequality follows from $n + \alpha + 1 > n + \alpha$. Therefore, $\widetilde{\varphi}^{-1}(\varepsilon^\alpha) \in \operatorname{relint}\left(qP_{n,k}^{\operatorname{stab}(r)}\right)$ for every $1 \leq r < \lfloor \frac{n}{k} \rfloor$. ∎

Using Lemmas 3.2 and 3.3 we now show that $q = \lceil \frac{n}{k} \rceil$ is indeed the codegree of these polytopes.

Theorem 3.4. *Let $1 \leq r < \lfloor \frac{n}{k} \rfloor$. The codegree of $P_{n,k}^{\operatorname{stab}(r)}$ is $q = \lceil \frac{n}{k} \rceil$.*

Proof. First recall that $P_{n,k}^{\operatorname{stab}(r)}$ is a subpolytope of $\Delta_{n,k}$. By [9, Theorem 3.3], it then follows that $\delta\left(P_{n,k}^{\operatorname{stab}(r)}\right) \leq \delta(\Delta_{n,k})$. Therefore, the codegree of $P_{n,k}^{\operatorname{stab}(r)}$ is no smaller than the codegree of $\Delta_{n,k}$. In [7, Corollary 2.6], Katzman determines that the codegree of $\Delta_{n,k}$ is $q = \lceil \frac{n}{k} \rceil$. Since Lemmas 3.2 and 3.3 imply that $qP_{n,k}^{\operatorname{stab}(r)}$ contains a lattice point inside its relative interior we conclude that the codegree of $P_{n,k}^{\operatorname{stab}(r)}$ is $q = \lceil \frac{n}{k} \rceil$. ∎

Recall that if an integral polytope P of dimension d with codegree q is Gorenstein then

$$\#\left(\operatorname{relint}(qP) \cap \mathbb{Z}^d\right) = 1.$$

With this fact in hand, we have the following corollary.

Corollary 3.5. *Suppose that $q = \lceil \frac{n}{k} \rceil = \frac{n+\alpha}{k}$, where $\alpha \in \{2, 3, \dots, k-1\}$. Then the polytope $\Delta_{n,k}^{\operatorname{stab}(r)}$ is not Gorenstein for every $1 \leq r < \lfloor \frac{n}{k} \rfloor$.*

Proof. Recall the vertex $(x_1, x_2, \dots, x_n)$ from which we produce ε^α. Since $x_1 = 1$ then cyclically shifting the entries of this vertex one entry to the left, and then applying the construction for ε^α results in a second vertex, say ζ^α, such that $\widetilde{\varphi}(\zeta^\alpha)^{-1}$ also lies in the relative interior of $qP_{n,k}^{\operatorname{stab}(r)}$. Thus, $\#\left(\operatorname{relint}\left(qP_{n,k}^{\operatorname{stab}(r)}\right) \cap \mathbb{Z}^d\right) > 1$, and we conclude that $\Delta_{n,k}^{\operatorname{stab}(r)}$ is not Gorenstein. ∎

3.3. Gorenstein r-Stable Hypersimplices and Unimodal δ-Vectors

Notice that by Corollary 3.5 we need only consider those r-stable hypersimplices satisfying the conditions of Lemma 3.2. For these polytopes we now consider the translated integral polytope

$$Q := qP_{n,k}^{\mathrm{stab}(r)} - (1,1,\ldots,1).$$

From our H-representation of $P_{n,k}^{\mathrm{stab}(r)}$ we see that the facets of Q are supported by the hyperplanes

(a) $x_i = -1$, for $i \in [n-1]$,
(b) $\sum_{i=1}^{n-1} x_i = kq-(n-1)$,
(c) $\sum_{i\in T(\ell)} x_i = q-r$, for $n \notin T(\ell)$, and
(d) $\sum_{i\in T(\ell)^c} x_i = (k-1)q-(n-r)$, for $n \in T(\ell)$.

Given this collection of hyperplanes we may now prove the following theorem.

Theorem 3.6. *Let* $1 \le r < \lfloor \frac{n}{k} \rfloor$. *Then* $\Delta_{n,k}^{\mathrm{stab}(r)}$ *is Gorenstein if and only if* $n = kr+k$.

Proof. By Theorem 3.1 we must determine when all the vertices of $Q^\star$ are integral. We do so by means of the inclusion-reversing bijection between the faces of Q and the faces of $Q^\star$. It is immediate that the vertices of $Q^\star$ corresponding to hyperplanes given in (a) are integral. So consider the hyperplane given in (b). Recall that $q = \lceil \frac{n}{k} \rceil = \frac{n+\alpha}{k}$ for some $\alpha \in \{0,1\}$. Hence, this hyperplane is equivalently expressed as

$$\sum_{i=1}^{n-1} \frac{1}{\alpha+1} x_i = 1.$$

Therefore, the corresponding vertex in $Q^\star$ is integral only if $\alpha = 0$. Notice next that the hyperplanes given in (c) will have corresponding vertex of $Q^\star$ integral only if $q-r=1$. Since $\alpha=0$ we have that $q=\frac{n}{k}$ where k divides n, and so it must be that $n=kr+k$. Finally, when $n=kr+k$ the hyperplanes given in (d) reduce to

$$\sum_{i\in T(\ell)^c} x_i = -1.$$

Hence, the corresponding vertex of $Q^\star$ is integral, and we conclude that, for $1 \le r < \lfloor \frac{n}{k} \rfloor$, the polytope $\Delta_{n,k}^{\mathrm{stab}(r)}$ is Gorenstein if and only if $n=kr+k$. ∎

Theorem 3.6 demonstrates that the Gorenstein property is quite rare amongst the r-stable hypersimplices. It also enables us to expand the collection of r-stable hypersimplices known to have unimodal δ-vectors. Previously, this collection was limited to the case when $k=2$ or when $\Delta_{n,k}^{\mathrm{stab}(r)}$ is a simplex [2]. Theorem 3.6 provides a collection of r-stable hypersimplices with unimodal δ-vectors for every $k \ge 1$.

Corollary 3.7. *Let* $k \ge 1$. *The r-stable n,k-hypersimplices* $\Delta_{n,k}^{\mathrm{stab}(r)}$ *for* $r \ge 1$ *and* $n = kr+k$ *have unimodal δ-vectors.*

Proof. By [2, Corollary 2.6], there exists a regular unimodular triangulation of $\Delta_{n,k}^{\mathrm{stab}(r)}$. By Theorem 3.6, the polytope $\Delta_{n,k}^{\mathrm{stab}(r)}$ is Gorenstein for $n = kr + k$ when $k > 1$. By [3, Theorem 1] we conclude that the δ-vector of $\Delta_{n,k}^{\mathrm{stab}(r)}$ is unimodal. Finally, notice that when $k = 1$ these polytopes are just the standard $(n-1)$-simplices. ∎

References

1. Beck, M., Robins, S.: Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra. Springer-Verlag, New York (2007)
2. Braun, B., Solus, L.: Shellability, Ehrhart theory, and r-stable hypersimplices. Preprint. available at: `http://arxiv.org/pdf/1408.4713v3.pdf` (2016)
3. Bruns, W., Römer, T.: h-vectors of Gorenstein polytopes. J. Combin. Theory Ser. A 114(1), 65–76 (2007)
4. De Loera, J., Haws, D., Köppe, M.: Ehrhart polynomials of matroid polytopes and polymatroids. Discrete Comput. Geom. 42(4), 670–702 (2009)
5. De Negri, E., Hibi, T.: Gorenstein algebras of Veronese type. J. Algebra 193(2), 629–639 (1997)
6. Hibi, T.: Dual polytopes of rational convex polytopes. Combinatorica 12(2), 237–240 (1992)
7. Katzman, M.: The Hilbert series of algebras of the Veronese type. Comm. Algebra 33(4), 1141–1146 (2005)
8. Lam, T., Postnikov, A.: Alcoved polytopes, I. Discrete Comput. Geom. 38(3), 453–478 (2007)
9. Stanley, R.: A monotonicity property of h-vectors and h^*-vectors. European J. Combin. 14(3), 251–258 (1993)

Acknowledgments Open access funding provided by [Austrian university].