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Results in Mathematics

A Note on Gauged Baby Skyrmions

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Abstract. In this paper we study a gauged version of the two dimensional Skyrme model of nuclear physics; the field configurations are couples (u, A), where $u: \mathbb{R}^2 \to S^2$ is a map constant at infinity, which can be classified by its topological degree, and $A: \mathbb{R}^2 \to \mathbb{R}^2$ is the gauge field. We prove the existence of rotationally symmetric field configurations which minimize the gauged Skyrme energy on every topological sector (gauged baby skyrmions). Moreover we study the behavior of these skyrmions in the case of weak or strong coupling with the gauge field. In particular, we show that, as observed by many authors by means of numerical simulations, in the strong coupling regime the magnetic flux associated with the gauge field becomes quantized.

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1. Introduction and Statement of the Results

The baby Skyrme model is a two dimensional version of the original Skyrme model for baryons and mesons (see [16,17]); in the static case, the two dimensional Skyrme field is a map $u: \mathbb{R}^2 \to S^2$ which goes to a constant at infinity, so it can be identified with a map from S^2 to S^2 with a given topological degree $Q(u)$, where

$$Q(u) = \frac{1}{4\pi} \int_{\mathbb{R}^2} u \cdot \partial_1 u \times \partial_2 u \, dx,$$

and the problem is to minimize the energy functional

$$E(u) = \int_{\mathbb{R}^2} \Big(\frac{1}{2} |\nabla u|^2 + |\partial_1 u \times \partial_2 u|^2 + V(u) \Big) \, dx$$

over the maps with a fixed degree $Q(u) \neq 0$; these energy-minimizing fields are called baby skyrmions. The first two terms in the energy density are, respectively, the sigma model term and the two dimensional Skyrme term, quartic in derivatives. The third term is a non negative potential which is mandatory in two dimensions, otherwise $E(u)$ can be decreased by a simple scaling argument (see [6]), and the minimum is not attained.

Notice that the convergence of a minimizing sequence is not trivial because of the lack of compactness of $E(u)$. The existence of baby skyrmions is proved in [10,11] for the potential $V(u) = |u - e_3|^4$, where $e_3 = (0, 0, 1)$ is the north pole of the sphere S^2, and in [9] for more general potentials.

Gauged versions of the baby Skyrme (and related) models have been recently studied with numerical methods by many authors (see [1–3,5,8,12, 13,15] and their bibliographies), in general by using the Skyrme ansatz in order to reduce the problem to a system of ordinary differential equations.

In this paper, we consider the gauged version of $E(u)$ introduced in [8], namely the functional

$$F(u,A) = \frac{1}{2}\int_{\mathbb{R}^2} \big(|D_1u|^2 + |D_2u|^2 + |D_1u \times D_2u|^2 + 2(1 - e_3 \cdot u) \\ + \gamma(\partial_1 A_2 - \partial_2 A_1)^2\big)\, dx$$

where $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the gauge field, with $A(x) = (A_1(x), A_2(x))$, $D_i u = \partial_i u + A_i e_3 \times u$ are the covariant derivatives, and $\partial_1 A_2 - \partial_2 A_1 \equiv F_{1,2}$ is the magnetic component of the field strength. We have, for simplicity, fixed the coefficient of the Skyrme term and the potential, while $\gamma > 0$ represents the coupling strength of the gauge field.

Since $2(1 - e_3 \cdot u) \geq (1 - e_3 \cdot u)^2$, using the well known topological lower bound (see [14]):

$$\frac{1}{2}\int_{\mathbb{R}^2} \big(|D_1u|^2 + |D_2u|^2 + (1 - e_3 \cdot u)^2 + (\partial_1 A_2 - \partial_2 A_1)^2\big)\, dx \geq 4\pi|Q(u)|,$$

we get

$$F(u,A) \geq 4\pi \min(1,\gamma)|Q(u)|$$

(provided that $u(x)$ and $A(x)$ have an appropriate behavior at infinity) so that $F(u, A)$ is bounded away from zero on every non trivial topological sector. Using the Skyrme ansatz, namely functions of the form

$$u_\theta(x) = (\sin(\theta(r))\cos(k\varphi), \sin(\theta(r))\sin(k\varphi), \cos(\theta(r))),$$
$$A(x) = k\frac{a(r)}{r^2}(-x_2, x_1), \tag{1}$$

where $k \in \mathbb{Z}$, $x = (x_1, x_2)$, $r = |x|$, and $\theta(r)$, $a(r)$ are functions defined for $r > 0$, the functional $F(u, A)$ becomes

$$F(\theta, a) = \pi \int_0^{+\infty} \Big(k^2(1 + a(r))^2 (1 + \theta'(r)^2) \frac{\sin^2(\theta(r))}{r^2} + \theta'(r)^2 + \gamma \frac{k^2 a'(r)^2}{r^2} + 2\big(1 - \cos(\theta(r))\big) \Big) r \, dr.$$

We aim to prove the existence of topologically non trivial minimum points (θ, a) of $F(\theta, a)$, and to study the behavior of (θ, a) as the electromagnetic coupling constant γ tends to infinity or to zero, namely in the regime of weak or strong coupling respectively. More precisely, we consider the set X of the couples (θ, a) of continuous functions $\theta, a \colon [0, +\infty[\to \mathbb{R}$ which are absolutely continuous on every compact subinterval of $]0, +\infty[$, such that $F(\theta, a) < +\infty$, and the usual boundary conditions $\theta(0) = \pi$, $\theta(\infty) = 0$, $a(0) = 0$ are satisfied. If $(\theta, a) \in X$, for the function $u_\theta(x)$ in (1) we have $Q(u_\theta) = -k$ (see Remark 1). From now on we assume $k \neq 0$. The following theorem holds.

Theorem 1.1. *There exists* $(\theta, a) \in X$ *such that* $F(\theta, a) = \inf\{F(\theta, a) \mid (\theta, a) \in X\}$, *where* $\theta \in C([0, +\infty[) \cap C^\infty(]0, +\infty[)$, $a \in C^2([0, +\infty[) \cap C^\infty(]0, +\infty[)$. *Moreover* $0 < \theta(r) < \pi$ *and* $-1 < a(r) < 0$ *for every* $r > 0$. *Finally,* $a(r)$ *is strictly decreasing on* $[0, +\infty[$ *from zero to a value* $a(\infty) \in [-1, 0[$, *and* $\theta(r)$ *is strictly decreasing for large values of* r.

In Theorem 1.1 we prove the existence of a minimizer of $F(\theta, a)$ by using analytical methods rather than numerical simulations. In the three dimensional case (without potential) an existence result has been proven in [18], always using Skyrme anzatz.

Moreover we recall that the existence of a minimizer of the *ungauged* functional $F(\theta, 0)$ is well known (see [4] or [11]).

We want now to study what happens when $\gamma \to +\infty$ or $\gamma \to 0$. To highlight the dependence of the minimizer (θ, a) on the coupling constant γ, sometimes we will denote it by $(\theta_\gamma, a_\gamma)$. Then we can state the following theorems.

Theorem 1.2. *Let* $(\gamma_n)_n$ *be a sequence of coupling constants such that* $\gamma_n \to +\infty$*; then we have* $\lim_{n\to+\infty} a_{\gamma_n}(\infty) = 0$ *and* $\lim_{n\to+\infty} a_{\gamma_n}(r) = 0$ *uniformly on* $[0, +\infty[$*; we have moreover, up subsequences,* $\lim_{n\to+\infty} \theta_{\gamma_n}(r) = \hat{\theta}(r)$ *uniformly on every compact subinterval of* $]0, +\infty[$, *where* $\hat{\theta}$ *is a minimizer of the ungauged functional* $F(\theta, 0)$.

Theorem 1.3. *Let* $(\gamma_n)_n$ *be a sequence of coupling constants such that* $\gamma_n \to 0$*; then we have* $\lim_{n\to+\infty} a_n(\infty) = -1$*; moreover, for every* $R > 0$, *we have, up subsequences,* $\lim\limits_{n\to+\infty} a_{\gamma_n}(r) = -1$ *and* $\lim\limits_{n\to+\infty} \theta_{\gamma_n}(r) = 0$ *uniformly on* $[R, +\infty[$.

Notice that, since

$$\int_{\mathbb{R}^2} (\partial_1 A_2 - \partial_2 A_1) \, dx = 2\pi \int_0^{+\infty} k a'_\gamma(r) \, dr = 2\pi k \, a_\gamma(\infty),$$

$a_\gamma(\infty)$ is proportional to the magnetic flux, and Theorem 1.3 says that, even if the magnetic flux can be any value in the interval $[-2\pi k, 0[$, in the strong coupling regime it is quantized, and the skyrmion profile as well as the magnetic field are localized at the origin.

The behavior of $(\theta_\gamma, a_\gamma)$ described in the theorems above was observed, by means of numerical simulations, in [8], and also, for different gauged baby Skyrme models, in [12,13,15].

2. The Functional Framework

The natural domain of the functional $F(\theta, a)$ in the Introduction is the set Y of the functions (θ, a) continuous on $]0, +\infty[$, absolutely continuous on every compact subinterval of $]0, +\infty[$, such that $F(\theta, a) < +\infty$. Clearly this implies

$$\int_0^{+\infty} \frac{a'(r)^2}{r}\, dr < +\infty,$$

so that $a'(r) = \frac{a'(r)}{\sqrt{r}}\sqrt{r} \in L^1([0, R])$ for every $R > 0$, and $a(r)$ is absolutely continuous on $[0, R]$ for every $R > 0$. In particular, $a(0) \in \mathbb{R}$. Moreover, since

$$\frac{|a(r) - a(0)|}{r} \le \frac{1}{r}\int_0^r \frac{|a'(s)|}{\sqrt{s}}\sqrt{s}\, ds \le \sqrt{\frac{1}{2}\int_0^r \frac{a'(s)^2}{s}\, ds},$$

we have $a'(0) = 0$. We have now the following simple lemmas.

Lemma 2.1. *If $(\theta, a) \in Y$ and $a(0) \neq -1$, then there exists $p \in \mathbb{Z}$ such that $\lim_{r\to 0} \theta(r) = p\pi$.*

Proof. Since $a(0) \neq -1$ and $F(\theta, a) < +\infty$, we have

$$2\int_0^\delta |\sin(\theta(r))||\theta'(r)|\, dr \le \int_0^\delta \Big(\frac{\sin^2(\theta(r))}{r} + \theta'(r)^2 r\Big)\, dr < +\infty \tag{2}$$

for some $\delta > 0$, so that $\partial_r \sin^2(\theta(r)) \in L^1([0, \delta])$, and, in particular, $\sin^2(\theta(r))$ tends to some $\ell \in [0, 1]$ for $r \to 0$; but (2) implies $\ell = 0$, and since the set $\sin^2 x < \varepsilon$ is disconnected for small ε, we get the lemma. □

Lemma 2.2. *If $(\theta, a) \in Y$, then there exists $q \in \mathbb{Z}$ such that $\lim_{r\to+\infty} \theta(r) = 2\pi q$.*

Proof. $(\theta, a) \in Y$ implies

$$\int_0^{+\infty} \big(\theta'(r)^2 r + \big(1 - \cos(\theta(r))\big) r\big)\, dr < +\infty,$$

so that $\liminf_{r\to+\infty}(1-\cos(\theta(r)))=0$. On the other hand, since for $1<R<r_1<r_2$ we have

$$\left|\big(1-\cos(\theta(r_2))\big)^2-\big(1-\cos(\theta(r_1))\big)^2\right| \leq 2\sqrt{\int_R^{+\infty}\big(1-\cos(\theta(r))\big)^2 r\,dr}\sqrt{\int_R^{+\infty}\theta'(r)^2 r\,dr},$$

and $(1-\cos x)^2\leq 2(1-\cos x)$, the function $1-\cos(\theta(r)$ verifies the Cauchy condition at infinity, so $\lim_{r\to+\infty}(1-\cos(\theta(r)))=0$, and we can conclude as in the previous lemma. □

Remark 1. If $(\theta,a)\in Y$ and $a(0)\neq -1$, the topological degree $Q(u)$ of the function $u(x)=(\sin(\theta(r))\cos(k\varphi),\sin(\theta(r))\sin(k\varphi),\cos(\theta(r)))$ is well defined and $Q(u)=0$ or $Q(u)=-k$, in fact

$$Q(u)=\frac{k}{2}\int_0^{+\infty}\sin(\theta(r))\theta'(r)\,dr=-\frac{k}{2}\big(1-\cos(\theta(0))\big)=\frac{k}{2}\big((-1)^p-1\big)$$

since $\theta(0)=p\pi$ (Lemma 2.1). Clearly $Q(u)\neq 0$ implies $k\neq 0$, p odd, and $Q(u)=-k$.

Let us consider now the set X defined in Sect. 1, namely the functions $(\theta,a)\in Y$ which satisfy the boundary conditions $\theta(0)=\pi$, $\theta(\infty)=0$, $a(0)=0$. As we shall see soon, the functional $F(\theta,a)$ can be studied in the set $X_0=\{(\theta,a)\in X\mid 0\leq\theta(r)\leq\pi, -1\leq a(r)\leq 0\}$. In fact we have the following result.

Lemma 2.3. *If $(\theta,a)\in X$, then there exists $(\tilde\theta,\tilde a)\in X_0$ such that $F(\tilde\theta,\tilde a)\leq F(\theta,a)$.*

Proof. Set $h(s)=\arccos(\cos(s))$ and $\tilde\theta(r)=h(\theta(r))$. Clearly $0\leq\tilde\theta(r)\leq\pi$, $\cos(\tilde\theta(r))=\cos(\theta(r))$, $\sin^2(\tilde\theta(r))=\sin^2(\theta(r))$, and $\tilde\theta(r)$ fulfills the boundary conditions $\tilde\theta(0)=\pi$, $\tilde\theta(\infty)=0$. Since $h(s)$ is Lipschitz continuous, $\tilde\theta(r)$ is continuous on $[0,+\infty[$ and absolutely continuous on every compact subinterval of $]0,+\infty[$; moreover the chain rule holds true, namely $\tilde\theta'(r)=h'(\theta(r))\theta'(r)$ if h is derivable at $\theta(r)$, and $\tilde\theta'(r)=0$ if $\theta(r)$ is a corner point of h (see [7], Theorem 7.8). Since $h'(s)=\pm1$ for $s\neq n\pi$, we have clearly $|\tilde\theta'(r)|\leq|\theta'(r)|$ a.e. on $[0,+\infty[$, and the inequality $F(\tilde\theta,a)\leq F(\theta,a)$ follows immediately.

In a similar way, if we set now $h(s)=(|1+s|-1-||1+s|-1|)/2$ and $\tilde a(r)=h(a(r))$, it is easy to check that $\tilde a(0)=0$ and $-1\leq\tilde a(r)\leq 0$. Moreover, since $(1+h(s))^2\leq(1+s)^2$, we have $(1+\tilde a(r))^2\leq(1+a(r))^2$ for every $r\geq 0$. Finally $|\tilde a'(r)|\leq|a'(r)|$ a.e. on $[0,+\infty[$, so that $F(\tilde\theta,\tilde a)\leq F(\theta,a)$, and the lemma is proved. □

We notice the following property of $(\theta,a)\in X_0$.

Lemma 2.4. *If $(\theta, a) \in X_0$, then, for every $r \geq 1$, we have:*

$$\theta(r) \leq \sqrt{\pi^2 + 2F(\theta, a)}\frac{1}{\sqrt{r}}.$$

Proof. Since $0 \leq \theta(r) \leq \pi$, and $x^2 \leq \pi^2(1 - \cos x)/2$ on $[0, \pi]$, we have

$$\int_0^{+\infty} \big(\theta'(r)^2 + \theta(r)^2\big) r\, dr \leq F(\theta, a);$$

therefore, for every $r \geq 1$ we get:

$$\begin{aligned} r\theta(r)^2 - \theta(1)^2 &= \int_1^r \big(\theta(s)^2 + 2s\theta(s)\theta'(s)\big)\, ds \leq \int_1^r \big(\theta(s)^2 s + 2s\theta(s)\theta'(s)\big)\, ds \\ &\leq \int_1^r \theta(s)^2 s\, ds + 2\sqrt{\int_1^r \theta(s)^2 s\, ds}\sqrt{\int_1^r \theta'(s)^2 s\, ds} \\ &\leq \int_0^{+\infty} \big(\theta'(r)^2 + 2\theta(r)^2\big) r\, dr \leq 2F(\theta, a), \end{aligned}$$

so that $r\theta(r)^2 \leq \pi^2 + 2F(\theta, a)$, and we get the lemma. □

We conclude this section by proving that $F(\theta, a)$ is bounded away from zero on X.

Lemma 2.5. *For every $(\theta, a) \in X$ we have $F(\theta, a) \geq 4\pi \min(1, \gamma)|k|$.*

Proof. Clearly we can assume $(\theta, a) \in X_0$; from the decomposition (see [14]):

$$\begin{aligned} &\frac{1}{2}\Big(|D_1 u|^2 + |D_2 u|^2 + (1 - e_3 \cdot u)^2 + (\partial_1 A_2 - \partial_2 A_1)^2\Big) \\ &\quad = \pm u \cdot D_1 u \times D_2 u \pm (\partial_1 A_2 - \partial_2 A_1)(1 - e_3 \cdot u) \\ &\qquad + \frac{1}{2}|D_1 u \pm u \times D_2 u|^2 + \frac{1}{2}\Big(\partial_1 A_2 - \partial_2 A_1 \mp (1 - e_3 \cdot u)\Big)^2 \end{aligned}$$

we get

$$\begin{aligned} F(\theta, a) &\geq \pm 2\pi \min(1, \gamma) \int_0^{+\infty} \Big(\frac{k \sin(\theta(r))\theta'(r)}{r} + \frac{k\partial_r((1 - \cos(\theta(r)))a(r))}{r}\Big) r\, dr \\ &= \pm 2\pi \min(1, \gamma)\Big(-2k + k\int_0^{+\infty} \partial_r((1 - \cos(\theta(r)))a(r))\, dr\Big); \end{aligned}$$

since $a(0) = 0$, and $-1 \leq a(r) \leq 0$, the last integral is equal to zero, and the lemma follows. □

3. Existence of Gauged Baby Skyrmions

In this Section we want to show that the infimum of $F(\theta, a)$ on X is attained.

Proposition 3.1. *There exists $(\theta, a) \in X_0$ such that $F(\theta, a) = \min_X F(\theta, a)$.*

Proof. Let us consider a minimizing sequence $(\theta_n, a_n)_n$ for $F(\theta, a)$; from Lemma 2.3 we can assume $(\theta_n, a_n)_n \subset X_0$; moreover $F(\theta_n, a_n) \leq M$ for some $M > 0$ that does not depend on n. Since

$$\int_0^{+\infty} \left(\theta_n'(r)^2 + \theta_n(r)^2\right) r\, dr \leq F(\theta_n, a_n),$$

the sequence $(\theta_n)_n$ is bounded in $W^{1,2}([r_1, r_2])$ for every $[r_1, r_2] \subset]0, +\infty[$, so that, by using a standard diagonal subsequence argument (see for instance [4]), we get a function $\theta\colon]0, +\infty[\to \mathbb{R}$ such that (modulo subsequences) $\theta_n \to \theta$ weakly in $W^{1,2}([r_1, r_2])$ and strongly in $C([r_1, r_2])$ for every $[r_1, r_2] \subset]0, +\infty[$. We observe now that since $-1 \leq a_n(r) \leq 0$ and

$$\int_0^R a_n'(r)^2\, dr \leq R \int_0^R \frac{a_n'(r)^2}{r}\, dr \leq \frac{R}{\pi k^2 \gamma} F(\theta_n, a_n),$$

the sequence $(a_n)_n$ is bounded in $W^{1,2}([0, R])$ for every $R > 0$, so that, arguing as above, there exists a function $a\colon [0, +\infty[\to \mathbb{R}$ such that $a_n \to a$ weakly in $W^{1,2}([0, R])$ and strongly in $C([0, R])$ for every $R > 0$. Clearly $0 \leq \theta(r) \leq \pi$, $-1 \leq a(r) \leq 0$, and moreover $a(0) = 0$.

We have also $F(\theta, a) < +\infty$; in fact, let us denote by $F(\theta, a; [r_1, r_2])$ the integral of the energy density on the interval $[r_1, r_2] \subset]0, +\infty[$. From the weak lower semicontinuity of $F(\theta, a; [r_1, r_2])$, we get

$$F(\theta, a; [r_1, r_2]) \leq \liminf_{n \to +\infty} F(\theta_n, a_n; [r_1, r_2]) \leq \liminf_{n \to +\infty} F(\theta_n, a_n) \leq M$$

and since $[r_1, r_2]$ was arbitrary, the claim is proved.

From $F(\theta, a) < +\infty$ and Lemma 2.2 we get $\theta(+\infty) = 0$. To conclude the proof, it remains to show the crucial fact that $\theta(0) = \pi$, and therefore the minimum is not topologically trivial.

In fact, since $a(0) = 0$ and $a_n \to a$ uniformly on every interval $[0, R]$, the sequence $(1 + a_n(r))_n$ is bounded away from zero on $[0, \delta]$ for some $\delta > 0$, so that the bound $F(\theta_n, a_n) \leq M$ implies

$$\int_0^\delta \frac{\sin^2(\theta_n(r))}{r} \theta_n'(r)^2\, dr \leq c,$$

where $c > 0$ does not depend on n. Then, since $\theta_n(0) = \pi$, for every $r \in]0, \delta[$ we have

$$\begin{aligned}\cos(\theta_n(r)) + 1 &\leq \int_0^r \frac{|\sin(\theta_n(s))||\theta_n'(s)|}{\sqrt{s}} \sqrt{s}\, ds \\ &\leq \sqrt{\int_0^r \frac{\sin^2(\theta_n(s))}{s} \theta_n'(s)^2\, ds}\, \frac{r}{\sqrt{2}} \leq \frac{\sqrt{c}}{\sqrt{2}} r.\end{aligned}$$

For $n \to +\infty$ we get $\cos(\theta(r)) + 1 \leq \frac{\sqrt{c}}{\sqrt{2}} r$, an so $\theta(0) = \pi$. □

Let $(\theta, a) \in X_0$ be a minimum point for the functional $F(\theta, a)$; by using variations of the form $(\theta + \varepsilon_1\varphi_1, a + \varepsilon_2\varphi_2)$, where φ_i, $i = 1, 2$ are smooth functions with compact support in $]0, +\infty[$, we obtain the Euler–Lagrange equations for $F(\theta, a)$; namely (see also [8]):

$$\Big(1 + k^2\frac{(1 + a(r))^2}{r^2}\sin^2(\theta(r))\Big)\theta''(r) = -k^2(1 + a(r))^2\frac{\sin(2\theta(r))}{2r^2}\theta'(r)^2$$
$$-\frac{1}{r}\Big(1 - k^2(1 + a(r))(1 + a(r) - 2ra'(r)\Big)\frac{\sin^2(\theta(r))}{r^2}\theta'(r)$$
$$+k^2(1 + a(r))^2\frac{\sin(2\theta(r))}{2r^2} + \sin(\theta(r)) \tag{3}$$

and

$$\partial_r\frac{a'(r)}{r} = \frac{1}{\gamma}(1 + a(r))(1 + \theta'(r)^2)\frac{\sin^2(\theta(r))}{r}. \tag{4}$$

To shorten notations we will write in the following the first Euler–Lagrange equation as

$$A(r)\theta''(r) = B(r)\theta'(r)^2 + C(r)\theta'(r) + D(r). \tag{5}$$

From (3) we get $\theta' \in W^{1,1}([r_1, r_2])$ for every $[r_1, r_2] \subset]0, +\infty[$, so that $\theta \in C^1(]0, +\infty[)$, and, from (4), $a \in C^2(]0, +\infty[)$. We observe now that the right hand side of (4) is summable on every interval $[0, R]$; in fact, since $1 + a(r)$ is bounded away from zero on some $[0, \delta]$, from $F(\theta, a) < +\infty$ we get the summability on $[0, \delta]$; moreover

$$\int_\delta^R \frac{1}{\gamma}(1 + a(r))(1 + \theta'(r)^2)\frac{\sin^2(\theta(r))}{r}\,dr \le \int_\delta^R \frac{1}{\gamma}(1 + \theta'(r)^2)\frac{1}{r}\,dr$$
$$\le \frac{1}{\delta^2}\int_\delta^R \frac{1}{\gamma}(1 + \theta'(r)^2)r\,dr < +\infty$$

because of $\int_0^{+\infty}\theta'(r)^2 r\,dr < +\infty$. Then $r \to a'(r)/r$ is uniformly continuous on $]0, R]$ for every $R > 0$; in particular, $\lim_{r\to 0} a'(r)/r = a''(0) \in \mathbb{R}$, and $a \in C^2([0, +\infty[)$. Proceeding in the same way, we get the regularity stated in Theorem 1.1.

We can prove now Theorem 1.1.

Proof of Theorem 1.1. The existence of a minimizer (θ, a) and the regularity of $\theta(r)$ and $a(r)$ have been proved above. Moreover we know that $0 \le \theta(r) \le \pi$ and $-1 \le a(r) \le 0$; if $\theta(r_0) = \pi$ for some $r_0 > 0$ then $\theta'(r_0) = 0$, and, from the Euler–Lagrange equations we get $\theta(r) = \pi$ for every $r > 0$, which is impossible, so that $\theta(r) < \pi$. Arguing in the same way we have $0 < \theta(r) < \pi$ and $-1 < a(r) < 0$ for every $r > 0$.

Moreover the function $a'(r)/r$ is strictly increasing because of (4), so that $a'(r) < 0$ (for if not, we would have $a'(r_0)/r_0 > 0$ for some $r_0 > 0$, and then $a'(r) > a'(r_0)r/r_0$ for $r > r_0$, so that $a(+\infty) = +\infty$); then $a(r)$ is strictly decreasing on $[0, +\infty[$.

It remain to prove that $\theta(r)$ is strictly decreasing for large value of r; more precisely, since $\theta(+\infty) = 0$, there exists $R > 0$ such that $\theta(r) < \pi/2$ for $r > R$. We claim that $\theta(r)$ is strictly decreasing on $[R, +\infty[$. In fact, let us suppose that $\theta'(r_1) > 0$ for some $r_1 \geq R$; then there exists $r_2 > r_1$ such that $\theta(r_1) = \theta(r_2) \equiv t$ and $\theta(r) > t$ on $]r_1, r_2[$; let us consider now the function $\tilde{\theta}(r) = t$ for $r \in [r_1, r_2]$, and $\tilde{\theta}(r) = \theta(r)$ for $r \notin [r_1, r_2]$; since $\sin^2 x$ and $2(1 - \cos x)$ are increasing on $[0, \pi/2]$, we get $F[\tilde{\theta}, a] < F(\theta, a)$, which is a contradiction, so $\theta'(r) \leq 0$ on $[R, +\infty[$. On the other hand, from Eqs. (3), (4) we see that $\theta(r)$ can not be constant on a subinterval of $]0, +\infty[$, and the claim is proved. □

Remark 2. Let (θ, a) be a minimizer of $F(\theta, a)$ on X as in Theorem 1.1; for future references we point out that

$$\lim_{r \to +\infty} r a'(r) = 0. \tag{6}$$

and

$$-\frac{2}{r} \leq a'(r) < 0 \tag{7}$$

for every $r > 0$.

In fact, since $\sin^2 x \leq \pi^2(1 - \cos x)/2$ for $x \in [0, \pi]$ and $F(\theta, a) < +\infty$, we have $\sin^2(\theta(r))r \in L^1(]0, +\infty[)$. Then, from (4) we get

$$\begin{aligned} 0 < r^2 \partial_r \frac{a'(r)}{r} &< \frac{1}{\gamma}(1 + \theta'(r)^2) \sin^2(\theta(r)) r \\ &\leq \frac{1}{\gamma}(\sin^2(\theta(r)) r + \theta'(r)^2 r) \in L^1(]0, \infty[); \end{aligned}$$

but $\partial_r(ra'(r)) = 2a'(r) + r^2 \partial_r(a'(r)/r)$, so that $\partial_r(ra'(r)) \in L^1(]0, +\infty[)$, and the limit (6) exists and it is ≤ 0. Clearly $\lim_{r \to +\infty} ra'(r) < 0$ implies $ra'(r) < K < 0$ on some interval $[R, +\infty[$, and this gives $a(+\infty) = -\infty$, whereas $a(+\infty) \geq -1$, so that (6) is proved.

Moreover, for every s, $r > 0$, with $s \leq r$, we have $a'(s)/s \leq a'(r)/r$, so that $a'(s) \leq (a'(r)/r)s$; integrating over $[0, r]$ we get (7).

4. The Weak Coupling Regime

In this section we have to prove Theorem 1.2. We start by proving that $(F(\theta_\gamma, a_\gamma))_\gamma$ is bounded from above.

Lemma 4.1. *There exists $C > 0$ such that, for every $\gamma > 0$, we have $F(\theta_\gamma, a_\gamma) \leq C$.*

Proof. Let $\theta(r)$ be such that

$$\pi \int_0^{+\infty} \left(k^2(1 + \theta'(r)^2) \frac{\sin^2(\theta(r))}{r^2} + \theta'(r)^2 + 2\big(1 - \cos(\theta(r))\big) \right) r\, dr \equiv C_1 < +\infty,$$

and let

$$\tilde{a}(r) = \begin{cases} -\frac{1}{\gamma} r^2 & \text{if } 0 \le r \le \sqrt{\gamma} \\ -1 & \text{if } r > \sqrt{\gamma} \end{cases}$$

Then

$$F(\theta, \tilde{a}) \le C_1 + \pi \int_0^{+\infty} \gamma \frac{k^2 \tilde{a}'(r)^2}{r} \, dr = C_1 + 2k^2 \pi,$$

and since $F(\theta_\gamma, a_\gamma) \le F(\theta, \tilde{a})$, the lemma is proved. □

The following lemma show that $F(\theta, a)$ can be written in a simpler form at a minimizer $(\theta_\gamma, a_\gamma)$.

Lemma 4.2. *For every $\gamma > 0$ we have:*

$$F(\theta_\gamma, a_\gamma) = \pi \int_0^{+\infty} \Big(k^2 (1 + a_\gamma(r))(1 + \theta_\gamma'(r)^2) \frac{\sin^2(\theta_\gamma(r))}{r^2} + \theta_\gamma'(r)^2 + 2\big(1 - \cos(\theta_\gamma(r))\big)\Big) r \, dr.$$

Proof. By multiplying the Eq. (4) by $a_\gamma(r)$ and integrating, we have

$$\int_0^{+\infty} \frac{a_\gamma'(r)^2}{r} \, dr = -\int_0^{+\infty} \frac{1}{\gamma}(1 + a_\gamma(r)) a_\gamma(r)(1 + \theta_\gamma'(r)^2) \frac{\sin^2(\theta_\gamma(r))}{r} \, dr,$$

so, inserting the right hand side of this equation in the expression of $F(\theta_\gamma, a_\gamma)$ we get the lemma. □

Proof of Theorem 1.2. Let $(\gamma_n)_n$ be a sequence of coupling constants such that $\gamma_n \to +\infty$, and set for brevity, $(\theta_{\gamma_n}, a_{\gamma_n}) = (\theta_n, a_n)$. By multiplying the Eq. (4) by r^2 and integrating we have (see also (6)):

$$-a_n(\infty) = \int_0^{+\infty} \frac{1}{2\gamma_n}(1 + a_n(r))(1 + \theta_n'(r)^2) \sin^2(\theta_n(r)) r \, dr. \tag{8}$$

But, since $\sin^2 x \le \pi^2(1 - \cos x)/2$ on $[0, \pi]$, and by using the Lemma 4.1:

$$\int_0^{+\infty} (1 + a_n(r))(1 + \theta_n'(r)^2) \sin^2(\theta_n(r)) r \, dr \le \int_0^{+\infty} \Big(\frac{\pi^2}{2}\big(1 - \cos(\theta_n(r))\big) + \theta_n'(r)^2 \Big) r \, dr \le F(\theta_n, a_n) \le C,$$

so that $-a_n(\infty) \le C/2\gamma_n$; passing to the limit for $n \to +\infty$, we get the first claim of Theorem 1.2.

To complete the proof, let us denote by $\bar{X}$ the set of continuous functions $\theta\colon [0, +\infty[\to \mathbb{R}$ which are absolutely continuous on every compact subinterval of $]0, +\infty[$, satisfies the boundary conditions $\theta(0) = \pi$, $\theta(\infty) = 0$ and, moreover, $F(\theta, 0) < +\infty$. Clearly $(\theta_n)_n \subset \bar{X}$; we claim that $(\theta_n)_n$ is a minimizing sequence for the ungauged functional $F(\theta, 0)$. In fact, we have

$$F(\theta_n, a_n) \le \inf_{\theta \in \bar{X}} F(\theta, 0) \le F(\theta_n, 0);$$

moreover, since $a_n(\infty) \to 0$, we can assume $1 + a_n(r) > \frac{1}{2}$, so that

$$\begin{aligned}
&\pi \int_0^{+\infty} k^2(1+\theta_n'(r)^2)\frac{\sin^2(\theta_n(r))}{r}\,dr \\
&\quad\le 4\pi \int_0^{+\infty} k^2(1+a_n(r))^2(1+\theta_n'(r)^2)\frac{\sin^2(\theta_n(r))}{r}\,dr \\
&\quad\le 4F(\theta_n,a_n) \le 4C,
\end{aligned}$$

where $C > 0$ is the constant of Lemma 4.1. Then, from Lemma 4.2 we have

$$\begin{aligned}
0 < F(\theta_n,0) - F(\theta_n,a_n) &= \pi \int_0^{+\infty} k^2(-a_n(r))(1+\theta_n'(r)^2)\frac{\sin^2(\theta_n(r))}{r}\,dr \\
&< -4a_n(\infty)C \to 0,
\end{aligned}$$

therefore $F(\theta_n,0) \to \inf_{\theta\in\bar X} F(\theta,0)$ and the claim is proved. But, for the ungauged functional $F(\theta,0)$, it is well known (from [4], or by using the arguments of Theorem 1.1) that there exists $\hat\theta \in \bar X$, such that $F(\hat\theta,0) = \inf_{\theta\in\bar X} F(\theta,0)$, and (up a subsequence) $\theta_n \to \hat\theta$ weakly in $W^{1,2}([r_1,r_2])$ and uniformly on every compact subinterval $[r_1,r_2] \subset]0,+\infty[$, and the proof is complete. □

5. The Strong Coupling Regime

Let us consider a sequence $(\gamma_n)_n$ such that $\gamma_n \to 0$, and set again $(\theta_{\gamma_n}, a_{\gamma_n}) = (\theta_n, a_n)$. From (7) and the fact that $-1 \le a_n(r) \le 0$, we get that for every $[r_1,r_2] \subset]0,+\infty[$, $(a_n)_n$ is bounded in $W^{1,2}([r_1,r_2])$. Then, there exists a continuous function $\bar a\colon]0,+\infty[\to \mathbb{R}$ such that, up subsequences, $a_n \to \bar a$ weakly in $W^{1,2}([r_1,r_2])$ and uniformly on $[r_1,r_2]$ for every $[r_1,r_2] \subset]0,+\infty[$; of course we have $-1 \le \bar a(r) \le 0$ and $\bar a(r)$ is decreasing.

Since $F(\theta_n,a_n) \le C$ (see Lemma 4.1) implies that the sequence $\left(\int_0^{+\infty}(\theta_n(r)^2 + \theta_n'(r)^2)r\,dr\right)_n$ is bounded, we get in the same way a continuous function $\bar\theta\colon]0,+\infty[\to \mathbb{R}$ such that, up subsequences, $\theta_n \to \bar\theta$ weakly in $W^{1,2}([r_1,r_2])$ and uniformly on $[r_1,r_2]$ for every $[r_1,r_2] \subset]0,+\infty[$; clearly $0 \le \bar\theta(r) \le \pi$, and moreover $\int_0^{+\infty}(\bar\theta(r)^2 + \bar\theta'(r)^2)r\,dr < +\infty$, so that we have $\bar\theta(\infty) = 0$.

We aim to show that $\bar a(r) \equiv -1$ and $\bar\theta(r) \equiv 0$. We start by proving the following lemma.

Lemma 5.1. *There exists $R > 0$ such that $\bar a(R) = -1$.*

Proof. Let us suppose, by contradiction, that $\bar a(R) > -1$ for every $R > 0$, and fix $R > 0$; since $a_n(r)$ is strictly decreasing, from (8) we have

$$\begin{aligned}
\int_0^R \sin^2(\theta_n(r))r\,dr &\le \frac{1}{1+a_n(R)}\int_0^R (1+a_n(r))(1+\theta_n'(r)^2)\sin^2(\theta_n(r))r\,dr \\
&\le -\frac{2\gamma_n a_n(\infty)}{1+a_n(R)} \le \frac{2\gamma_n}{1+a_n(R)} \to 0,
\end{aligned}$$

and since for every r with $0 < r < R$ we have $\theta_n \to \bar{\theta}$ uniformly on $[r, R]$, we obtain $\int_r^R \sin^2(\bar{\theta}(r))r\,dr = 0$, so that $\bar{\theta}(r) \equiv 0$ or $\bar{\theta}(r) \equiv \pi$ on $]0, R]$. On the other hand, from Lemma 4.1 we have

$$\begin{aligned}\int_0^R \theta_n'(r)^2 \frac{\sin^2(\theta_n(r))}{r}\,dr &\leq \int_0^R (1+\theta_n'(r)^2)\frac{\sin^2(\theta_n(r))}{r}\,dr \\ &\leq \frac{1}{\pi k^2(1+a_n(R))^2}\pi \int_0^R k^2(1+a_n(r))^2(1+\theta_n'(r)^2) \\ &\quad \frac{\sin^2(\theta_n(r))}{r}\,dr \\ &\leq \frac{C}{\pi k^2(1+a_n(R))^2},\end{aligned}$$

so that, for every $r \in [0, R]$:

$$\begin{aligned}\cos(\theta_n(r)) + 1 &\leq \int_0^r \frac{|\sin(\theta_n(s))||\theta_n'(s)|}{\sqrt{s}}\sqrt{s}\,ds \\ &\leq \sqrt{\int_0^R \theta_n'(s)^2 \frac{\sin^2(\theta_n(s))}{s}\,ds}\,\frac{r}{\sqrt{2}} \leq \frac{\sqrt{C}}{\sqrt{2\pi}|k|(1+a_n(R))}r.\end{aligned}$$

Passing to the limit we get $\cos(\bar{\theta}(r)) + 1 \leq (\sqrt{C}/\sqrt{2\pi}|k|(1+\bar{a}(R)))r$ for every $r \in [0, R]$, and then we must have $\bar{\theta}(r) \equiv \pi$ on $]0, R]$. Since R is arbitrary, we have $\bar{\theta}(r) \equiv \pi$ on $]0, +\infty[$, and this is impossible, since $\bar{\theta}(\infty) = 0$. □

From now on we set $R_0 = \inf\{r > 0 \mid \bar{a}(R) = -1\}$. Of course $R_0 \geq 0$ and $\bar{a}(r) \equiv -1$ on $[R_0, +\infty[$; moreover, if $R_0 > 0$, we have, from the proof of the lemma above, $\bar{\theta}(r) \equiv \pi$ on $]0, R_0]$.

Lemma 5.2. *For every* $[r_1, r_2] \subset]R_0, +\infty[$ *we have* $\lim_{n\to+\infty} a_n'(r) = 0$ *uniformly on* $[r_1, r_2]$.

Proof. Let $r > R_0$ be fixed; for every $s \in [R_0, r]$ we have $a_n'(s) < (a_n'(r)/r)s$; then, by integration, $a_n(r) - a_n(R_0) < a_n'(r)(r^2 - R_0^2)/2r < 0$, so that, since $a_n(r) - a_n(R_0) \to 0$, we have $a_n'(r) \to 0$ for every $r > R_0$. Therefore $a_n'(r)/r$ goes to zero uniformly on every $[r_1, +\infty[\subset]R_0, +\infty[$, and so $a_n'(r) \to 0$ on every $[r_1, r_2] \subset]R_0, +\infty[$. □

Lemma 5.3. *For every* $[r_1, r_2] \subset]0, +\infty[$, *the sequences* $(\theta_n')_n$ *and* $(\theta_n'')_n$ *are bounded in* $L^\infty([r_1, r_2])$.

Proof. Let $[r_1, r_2] \subset]0, +\infty[$; for every n there exists $R_n \in [r_1, r_2]$ such that $\theta_n(r_2) - \theta_n(r_1) = \theta_n'(R_n)(r_2 - r_1)$, and since $\theta_n(r_2) - \theta_n(r_1) \to \bar{\theta}(r_2) - \bar{\theta}(r_1)$, the sequence $(\theta_n'(R_n))_n$ is bounded. Moreover $\theta_n(r)$ satisfies the Eq. (5), namely

$$A_n(r)\theta_n''(r) = B_n(r)\theta_n'(r)^2 + C_n(r)\theta_n'(r) + D_n(r),$$

where the coefficients depend on n. Recalling that $-1 < a_n(r) < 0$ and $-2 \leq ra_n'(r) < 0$ because of (7), it is easy to check that $B_n(r)$, $C_n(r)$ and $D_n(r)$ are bounded in $L^\infty([r_1, r_2])$, so that

$$|\theta_n''(r)| \leq C_1 \theta_n'(r)^2 + C_2 |\theta_n'(r)| + C_3 \tag{9}$$

on $[r_1, r_2]$, where C_i, $i = 1, 2, 3$, does not depend on n; Lemma 4.1 implies that $(\theta_n')_n$ is bounded in $L^2([r_1, r_2])$, so that $(\theta_n'')_n$ is bounded in $L^1([r_1, r_2])$. Since clearly $|\theta_n'(r) - \theta_n'(R_n)| \leq \int_{R_n}^r |\theta_n''(r)|\, dr$, we get the boundness of $(\theta_n')_n$ in $L^\infty([r_1, r_2])$ and, by (9), also $(\theta_n'')_n$ is bounded in $L^\infty([r_1, r_2])$ as claimed. □

Lemma 5.4. *The function $\bar{\theta}(r)$ is differentiable on $]0, +\infty[$, twice differentiable on $]R_0, +\infty[$, and $\lim_{n\to+\infty} \theta_n'(r) = \bar{\theta}'(r)$ uniformly on every $[r_1, r_2] \subset]0, +\infty[$, $\lim_{n\to+\infty} \theta_n''(r) = \bar{\theta}''(r)$ uniformly on every $[r_1, r_2] \subset]R_0, +\infty[$. Moreover $\bar{\theta}(r)$ satisfies the equation $\partial_r(r\bar{\theta}'(r)) = \sin(\bar{\theta}(r))r$ on $]R_0, +\infty[$.*

Proof. Let us consider the sequence $(\theta_n')_n$; since $(\theta_n'')_n$ is bounded over the compact subsets of $]0, +\infty[$, by the Ascoli-Arzelà theorem there exists a continuous function $\eta\colon]0, +\infty[\to \mathbb{R}$ such that, up subsequences, $\theta_n' \to \eta$ uniformly on every $[r_1, r_2] \subset]0, +\infty[$. But we know that $\theta_n \to \bar{\theta}$, so that $\bar{\theta}$ is differentiable and $\bar{\theta}' = \eta$, and the first claim is proved.

We observe now that, since $\theta_n \to \bar{\theta}$, $a_n \to -1$ and $a_n' \to 0$ uniformly on every $[r_1, r_2] \subset]R_0, +\infty[$, by the Eq. (3) the same holds true for the sequence $(\theta_n'')_n$; but $\theta_n' \to \bar{\theta}'$, so that $\bar{\theta}'$ is differentiable on $]R_0, +\infty[$, and $\theta_n''(r) \to \bar{\theta}''(r)$ uniformly on every $[r_1, r_2] \subset]R_0, +\infty[$. Moreover, from (3) we get also the equation $\partial_r(r\bar{\theta}'(r)) = \sin(\bar{\theta}(r))r$. □

We can prove now Theorem 1.3.

Proof of Theorem 1.3. First of all, we claim that $R_0 = 0$; in fact, let us suppose $R_0 > 0$; then $\bar{\theta}(r) \equiv \pi$ on $]0, R_0]$; since $\bar{\theta}(r)$ is differentiable at R_0, we must have $\bar{\theta}'(R_0) = 0$, and, from the equation $\partial_r(r\bar{\theta}'(r)) = \sin(\bar{\theta}(r))r$ we get $\bar{\theta}(r) \equiv \pi$; but this is impossible since $\bar{\theta}(\infty) = 0$. Therefore $R_0 = 0$, so that $\bar{a}(r) \equiv -1$, and, since $a_n(r)$ is strictly decreasing, $a_n(r) \to -1$ uniformly on $[R, +\infty[$ for every $R > 0$; in particular, $a_n(\infty) \to -1$.

We want to show now that $\bar{\theta}(r) \equiv 0$; in fact, since $0 \leq \bar{\theta}(r) \leq \pi$, from equation $\partial_r(r\bar{\theta}'(r)) = \sin(\bar{\theta}(r))r$, we deduce that $r\bar{\theta}'(r)$ is not decreasing; clearly we must have $\bar{\theta}'(r) \leq 0$; for if not $0 < r_0\bar{\theta}'(r_0)/r \leq \bar{\theta}'(r)$ on some interval $[r_0, +\infty[$, and, by integration, $\bar{\theta}(\infty) = +\infty$. Therefore the function $r\bar{\theta}'(r)$ is ≤ 0 and not decreasing. In particular, there exists the limit

$$\lim_{r\to 0} r\bar{\theta}'(r) \leq 0.$$

We claim that the above limit is equal to zero. For if not, there exists $K \in \mathbb{R}$ with $K < 0$, and a neighbourhood $]0, r_0[$ of zero such that $\bar{\theta}'(r) < K/r$ on $]0, r_0[$ and, by integration, $\bar{\theta}(r_0) - \bar{\theta}(r) < K \log(r_0/r)$ for every $r \in]0, r_0[$, so that $\bar{\theta}(r) \to +\infty$ as $r \to 0$, whereas $0 \le \bar{\theta}(r) \le \pi$. Then $\bar{\theta}'(r) \equiv 0$, and so $\bar{\theta}(r) \equiv 0$ as claimed, and therefore $\theta_n(r) \to 0$ on $]0, +\infty[$.

Finally, let $R > 0$ be fixed; for every n there exists $R_n > 0$ such that $\theta_n(R_n) = \pi/2$, and $\theta_n(r)$ is strictly decreasing on $[R_n, +\infty[$ (see the proof of Theorem 1.1). From Lemmas 2.4 and 4.1 we have

$$R_n \le \max\left(1, \frac{4}{\pi^2}\left(\pi^2 + 2C\right)\right);$$

on the other hand, $\theta_n(r) \to 0$ uniformly on every $[r_1, r_2] \subset]0, +\infty[$, so that we must have $R_n \to 0$. Then, for n large enough, we have $R_n < R$, so $\theta_n(r)$ is strictly decreasing on $[R, +\infty[$, and, since $\theta_n(R) \to 0$, we get $\theta_n \to 0$ uniformly on $[R, +\infty[$. □

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