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Results in Mathematics

The Hilbert–Schmidt Analyticity Associated with Infinite-Dimensional Unitary Groups

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Abstract. The article is devoted to the problem of Hilbert–Schmidt type analytic extensions in Hardy spaces over the infinite-dimensional unitary group endowed with an invariant probability measure. Reproducing kernels of Hardy spaces, integral formulas of analytic extensions and their boundary values are considered.

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1. Introduction

The paper deals with the problem of Hilbert–Schmidt type analytic extensions in the Hardy space H^2_χ of complex functions over the infinite-dimensional group $U(\infty)=\bigcup\{U(m)\colon m\in\mathbb{N}\}$ endowed with an invariant probability measure χ where $U(m)$ are subgroups of unitary $m{\times}m$-matrices. The measure χ is defined as a projective limit $\chi=\varprojlim\chi_m$ of the Haar probability measures χ_m on $U(m)$. Moreover, χ is supported by a projective limit $\mathfrak{U}=\varprojlim U(m)$ and is invariant under the right action of $U^2(\infty):=U(\infty)\times U(\infty)$ on $\mathfrak{U}$.

A goal of this work is to find integral formulas for Hilbert–Schmidt analytic extensions of functions from H^2_χ and to describe their radial boundary values on the open unit ball in a Hilbert space E where $U(\infty)$ acts irreducibly.

The measure χ on $\mathfrak{U}$ was described by Olshanski [13] and Neretin [12]. The notion $\mathfrak{U}$ is related to Pickrell's space of a virtual Grassmannian [16]. Hardy spaces in infinite-dimensional settings were discussed in the works of Cole and Gamelin [5], Ørsted and Neeb [14]. Spaces of analytic functions of Hilbert–Schmidt holomorphy types were considered by Dwyer III [6] and Petersson [15].

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More general classes of analytic functions associated with coherent sequences of polynomial ideals were described by Carando et al. [4]. Integral formulas for analytic functions employing Wiener measures on infinite-dimensional Banach spaces were suggested by Pinasco and Zalduendo [17].

Note that spaces of integrable functions with respect to invariant measures over infinite-dimensional groups have been widely applied in stochastic processes [2,3], as well as in other areas.

This paper presents the following results. In Theorem 3.2, we describe an orthogonal basis in the Hardy space H^2_χ indexed by means of Yang diagrams, consisting of χ-essentially bounded functions. Using this basis, in Theorem 4.2 the reproducing kernel of H^2_χ is calculated. It also allows us to define an antilinear isometric isomorphism $\mathcal{J}$ between H^2_χ and the symmetric Fock space Γ generated by E. This isomorphism equips H^2_χ with a suitable infinite-dimensional analytic structure. By means of $\mathcal{J}$, we establish in Theorem 6.2 an integral formula for Hilbert–Schmidt analytic extensions of functions from H^2_χ on the open unit ball $\mathsf{B} \subset \mathsf{E}$. The radial boundary values of these analytic extensions are described in Theorem 7.1.

2. Background on Invariant Measure

Let $U(m)$ $(m \in \mathbb{N})$ be the group of unitary $(m \times m)$-matrices. We endow $U(\infty) = \bigcup U(m)$ with the inductive topology under every continuous inclusion $U(m) \looparrowright U(\infty)$ which assigns to any $u_m \in U(m)$ the matrix $\begin{bmatrix} u_m & 0 \\ 0 & \mathbb{1} \end{bmatrix} \in U(\infty)$. The right action over $U(\infty)$ is defined via

$$u.g = w^{-1}uv, \qquad u \in U(\infty), \quad g = (v, w) \in U^2(\infty) \tag{2.1}$$

(the right action over $U(m)$ is defined similarly with $u \in U(m)$ and $g = (v, w) \in U^2(m)$ where $U^2(m) := U(m) \times U(m)$).

Following [12,13], every $u_m \in U(m)$ with $m > 1$ can be written as $u_m = \begin{bmatrix} z_{m-1} & a \\ b & t \end{bmatrix}$ so that z_{m-1} is a $(m-1) \times (m-1)$-matrix and $t \in \mathbb{C}$. It was proven that the Livšic-type mapping (which is not a group homomorphism)

$$\pi^m_{m-1} \colon u_m \longmapsto u_{m-1} := \begin{cases} z_{m-1} - [a(1+t)^{-1}b] : & t \neq -1 \\ z_{m-1} : & t = -1 \end{cases} \tag{2.2}$$

from $U(m)$ onto $U(m-1)$ is Borel and surjective.

Consider the projective limit $\mathfrak{U} = \varprojlim U(m)$ taken with respect to π^m_{m-1}. The embedding $\rho \colon U(\infty) \looparrowright \mathfrak{U}$ assigns to every $u_m \in U(m)$ the stabilized sequence $u = (u_k)_{k \in \mathbb{N}}$ (see [13, n.4]) so that

$$\rho\colon U(m) \ni u_m \longmapsto (u_k) \in \mathfrak{U}, \qquad u_k = \begin{cases} \pi_k^m(u_m) : k < m, \\ \quad u_m \quad : k = m, \\ \begin{bmatrix} u_m & 0 \\ 0 & \mathbb{1} \end{bmatrix} : k > m \end{cases} \tag{2.3}$$

where the projections $\pi_m\colon \mathfrak{U} \ni u \longrightarrow u_m \in U(m)$ such that $\pi_{m-1}^m \circ \pi_m = \pi_{m-1}$ are surjective and $\pi_k^m := \pi_k^{k+1} \circ \cdots \circ \pi_{m-1}^m$ for $k < m$. Using (2.1), the right action of $U^2(\infty)$ over $\mathfrak{U}$ can be defined as

$$\pi_m(u.g) = w^{-1}\pi_m(u)v, \qquad u \in \mathfrak{U} \tag{2.4}$$

where m is so large that $g = (v, w) \in U^2(m)$ (see [13, Def 4.5]).

We endow every group $U(m)$ with the probability Haar measure χ_m. It is known [12, Thm 1.6] that the pushforward of χ_m to $U(m-1)$ under π_{m-1}^m is the probability Haar measure χ_{m-1} on $U(m)$. Let $U'(m)$ be the subset in $U(m)$ of matrices which do not have -1 as an eigenvalue. Then $U'(m)$ is open in $U(m)$ and $U(m) \setminus U'(m)$ is χ_m-negligible. Moreover, the restriction $\pi_{m-1}^m\colon U'(m) \longrightarrow U'(m-1)$ is continuous and surjective [13, Lem. 3.11].

Following [13, Lem. 4.8], [12, n.3.1], via of the Kolmogorov consistency theorem we uniquely define on $\mathfrak{U}$ the probability measure χ which is the projective limit under the mapping (2.2), i.e., we put

$$\chi = \varprojlim \chi_m \quad \text{with} \quad \chi_m = \chi \circ \pi_m^{-1} \quad \text{for all} \quad m \in \mathbb{N}. \tag{2.5}$$

If $\mathfrak{U}' = \varprojlim U'(m)$ is the projective limit with respect to $\pi_{m-1}^m|_{U'(m)}$ then $\mathfrak{U}\backslash\mathfrak{U}'$ is χ-negligible, because χ_m is zero on $U(m) \setminus U'(m)$ for any m.

A complex-valued function on $\mathfrak{U}$ is called cylindrical if it has the form $f = f_m \circ \pi_m$ for a certain $m \in \mathbb{N}$ and a complex function f_m on $U(m)$ [13, Def. 4.5]. By L_χ^∞ we denote the closed linear hull of all cylindrical χ-essentially bounded Borel functions endowed with the norm $\|f\|_{L_\chi^\infty} = \operatorname{ess\,sup}_{u\in\mathfrak{U}} |f(u)|$.

The measure (2.5) is a probability measure and is $U^2(\infty)$-invariant under the right actions (2.4) over $\mathfrak{U}$ [12, Prop. 3.2]. Moreover, this measure is Radon so that

$$\int_{\mathfrak{U}} f(u.g)\, d\chi(u) = \int_{\mathfrak{U}} f(u)\, d\chi(u), \qquad g \in U^2(\infty), \quad f \in L_\chi^\infty \tag{2.6}$$

and it satisfies the property: $(\chi \circ \pi_m^{-1})(K) = \chi_m(K)$ for any compact set K in $U(m)$ [11, Lem. 1]. Using the invariance property (2.6) and the Fubini theorem (see [11, Lem. 2]), we obtain

$$\int_{\mathfrak{U}} f\, d\chi = \int_{\mathfrak{U}} d\chi(u) \int_{U^2(m)} f(u.g)\, d(\chi_m \otimes \chi_m)(g), \tag{2.7}$$

$$\int_{\mathfrak{U}} f\, d\chi = \frac{1}{2\pi} \int_{\mathfrak{U}} d\chi(u) \int_{-\pi}^{\pi} f\left[\exp(\mathbb{i}\vartheta)u\right]\, d\vartheta \tag{2.8}$$

for all $f \in L^\infty_\chi$. The closed linear hull of cylindrical complex functions endowed with the norm $\|f\|_{L^2_\chi} = \left(\int_{\mathfrak{U}} |f|^2\, d\chi\right)^{1/2}$ is denoted by L^2_χ. It is clear that $L^\infty_\chi \hookrightarrow L^2_\chi$ and $\|f\|_{L^2_\chi} \le \|f\|_{L^\infty_\chi}$ for all $f \in L^\infty_\chi$.

3. Hardy Spaces

Throughout the paper E is a separable complex Hilbert space with an orthonormal basis $\{\mathfrak{e}_k : k \in \mathbb{N}\}$, scalar product $\langle \cdot \mid \cdot \rangle$ and norm $\|\cdot\| = \langle \cdot \mid \cdot \rangle^{1/2}$. So, for any element $x \in \mathsf{E}$ the following Fourier decomposition holds,

$$x = \sum \mathfrak{e}_k \hat{x}_k, \qquad \hat{x}_k = \langle x \mid \mathfrak{e}_k \rangle. \tag{3.1}$$

In what follows, let $\mathsf{B} = \{x \in \mathsf{E} : \|x\| < 1\}$ and $\mathsf{S} = \{x \in \mathsf{E} : \|x\| = 1\}$.

Let $\mathsf{E}^{\otimes n}$ be the complete nth tensor power of E endowed with the scalar product and norm

$$\left\langle \psi \mid \phi \right\rangle = \langle x_1 \mid y_1 \rangle \cdots \langle x_n \mid y_n \rangle, \qquad \|\psi\| = \langle \psi \mid \psi \rangle^{1/2}$$

for all $\psi = x_1 \otimes \cdots \otimes x_n$, $\phi = y_1 \otimes \cdots \otimes y_n \in \mathsf{E}^{\otimes n}$ with $x_i, y_i \in \mathsf{E}$ $(i = 1, \ldots, n)$. As $\sigma : \{1, \ldots, n\} \longmapsto \{\sigma(1), \ldots, \sigma(n)\}$ runs through all n-elements permutations, the symmetric complete nth tensor power $\mathsf{E}^{\odot n}$ is defined to be a codomain of the orthogonal projector

$$\mathsf{E}^{\otimes n} \ni \psi \longmapsto x_1 \odot \cdots \odot x_n := \frac{1}{n!} \sum_\sigma x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \in \mathsf{E}^{\odot n}.$$

Note that $x^{\otimes n} = x \otimes \cdots \otimes x = x \odot \cdots \odot x = x^{\odot n}$. Put $\mathsf{E}^{\otimes 0} = \mathsf{E}^{\odot 0} = \mathbb{C}$.

Let $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m$ be a partition of an integer $n \in \mathbb{N}$ with $m \le n$ and $\lambda_1 \ge \lambda_2 \ge \cdots \lambda_m > 0$, i.e., $|\lambda| = n$ where $|\lambda| := \lambda_1 + \cdots + \lambda_m$. We identify partitions with Young diagrams. By $\ell(\lambda) = m$ we denote the length of λ defined as the number of rows in λ. Let $\mathbb{Y}$ denote all Young diagrams and $\mathbb{Y}_n := \{\lambda \in \mathbb{Y} : |\lambda| = n\}$. Assume that $\mathbb{Y}$ includes the empty partition $\emptyset = (0, 0, \ldots)$.

An orthogonal basis in $\mathsf{E}^{\odot n}$ is formed by the system of symmetric tensor products (see e.g. [1, Sec. 2.2.2])

$$\mathfrak{e}^{\odot \mathbb{Y}_n} = \bigcup_{\lambda \in \mathbb{Y}_n} \left\{ \mathfrak{e}_\imath^{\odot \lambda} := \mathfrak{e}_{\imath_1}^{\otimes \lambda_1} \odot \cdots \odot \mathfrak{e}_{\imath_m}^{\otimes \lambda_m} : \imath \in \mathbb{N}^m_*,\ m = \ell(\lambda) \right\}, \quad \mathfrak{e}_\imath^{\odot \emptyset} = 1$$

where $\mathbb{N}^m_* := \{\imath = (\imath_1, \ldots, \imath_m) \in \mathbb{N}^m : \imath_j \neq \imath_k, \forall\, j \neq k\}$. As is well known,

$$\left\|\mathfrak{e}_\imath^{\odot \lambda}\right\|^2 = \frac{\lambda!}{|\lambda|!}, \qquad \lambda! := \lambda_1! \cdot \cdots \cdot \lambda_m!. \tag{3.2}$$

In what follows, we will use the fact that for every $\psi \in \mathsf{E}^{\odot n}$ one can uniquely define the so-called *Hilbert–Schmidt n-homogenous polynomial*

$$\psi^*(x) := \left\langle x^{\otimes n} \mid \psi \right\rangle, \qquad x \in \mathsf{E}.$$

In fact, the polarization formula for symmetric tensor products (see [8, 1.5])

$$z_1 \odot \cdots \odot z_n = \frac{1}{2^n n!} \sum_{\theta_1,\ldots,\theta_n=\pm 1} \theta_1 \ldots \theta_n \, x^{\otimes n}, \quad x = \sum_{k=1}^{n} \theta_k z_k \tag{3.3}$$

$(z_1, \ldots, z_n \in \mathsf{E})$ implies that the n-homogenous polynomial $\langle x^{\otimes n} \mid \psi \rangle$ is uniquely determines ψ, because the set of all $z_1 \odot \cdots \odot z_n$ is total in $\mathsf{E}^{\odot n}$.

Using the embedding (2.3), we define the E-valued mapping

$$\zeta \colon \mathfrak{U} \ni u \longmapsto \rho^{-1}(u)\mathfrak{e}_1$$

which do not depend on the choice of $\mathfrak{e}_1$ in

$$\mathsf{S}(\infty) := \{\zeta(u) \colon u \in \mathfrak{U}\} = \bigcup \{\mathsf{S}(m) \colon m \in \mathbb{N}\}$$

where $\mathsf{S}(m)$ is the m-dimensional unit sphere. In fact, for each stabilized sequence $u = (u_k) \in \mathfrak{U}$ there exists an index m such that $\rho^{-1}(u)\mathfrak{e}_1 = u_k\mathfrak{e}_1$ belongs to $\mathsf{S}(m)$ for all $k \geq m$. On the other hand, for each $\mathfrak{e} \in \mathsf{S}(k)$ there exists $v \in U(k)$ such that $v\mathfrak{e} = \mathfrak{e}_1$. Defining $u.g \in \mathfrak{U}$ with $g = (1, v) \in U^2(k)$ by means of (2.3)–(2.4), we have $\rho^{-1}(u.g)\mathfrak{e} = \pi_k(u.g)\mathfrak{e} = \pi_k(u)\mathfrak{e}_1 = \rho^{-1}(u)\mathfrak{e}_1$.

Consider the following system of cylindrical Borel functions

$$\varepsilon_k(u) := \left\langle \zeta(u) \;\middle|\; \mathfrak{e}_k \right\rangle, \qquad k \in \mathbb{N}$$

where $\varepsilon_k := \mathfrak{e}_k^* \circ \zeta$. Using ζ, we may define the $\mathsf{E}^{\odot n}$-valued Borel mapping

$$\zeta^{\otimes n} \colon \mathfrak{U} \ni u \longmapsto \underbrace{\zeta(u) \otimes \cdots \otimes \zeta(u)}_{n}, \qquad \zeta^{\otimes 0} \equiv 1.$$

The following assertion, which is a consequence of the polarization formula (3.3), is proved in [11, Lem. 3].

Lemma 3.1. *The equality* $\mathsf{S}(\infty) = \{\zeta(u) \colon u \in \mathfrak{U}'\}$ *holds. As a consequence, to every* $\psi \in \mathsf{E}_\imath^{\odot n}$ *there uniquely corresponds the function in* L_χ^∞

$$\psi_\zeta(u) := \left\langle \zeta^{\otimes n}(u) \;\middle|\; \psi \right\rangle, \qquad u \in \mathfrak{U}$$

given by continuous restriction to $\mathfrak{U}'$. *In particular, to every* $\mathfrak{e}_\imath^{\odot\lambda} \in \mathfrak{e}^{\odot \mathbb{Y}_n}$ *there corresponds in* L_χ^∞ *the cylindrical function in the variable* $u \in \mathfrak{U}$,

$$\varepsilon_\imath^\lambda(u) := \left\langle \zeta^{\otimes n}(u) \;\middle|\; \mathfrak{e}_\imath^{\odot\lambda} \right\rangle = \prod_{k=1}^{\ell(\lambda)} \left\langle \zeta(u) \;\middle|\; \mathfrak{e}_{\imath_k} \right\rangle^{\lambda_k}. \tag{3.4}$$

Lemma 3.1 straightforwardly implies that the system $\mathfrak{e}^{\odot\mathbb{Y}} := \bigcup \mathfrak{e}^{\odot\mathbb{Y}_n}$ of tensor products $\mathfrak{e}_\imath^{\odot\lambda} = \mathfrak{e}_{\imath_1}^{\otimes\lambda_1} \odot \cdots \odot \mathfrak{e}_{\imath_m}^{\otimes\lambda_m}$, indexed by $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Y}$ and $\imath = (\imath_1, \ldots, \imath_m) \in \mathbb{N}_*^m$ with $m = \ell(\lambda)$, uniquely defines the appropriate system

$$\varepsilon^{\mathbb{Y}} := \bigcup_{\lambda \in \mathbb{Y}} \left\{ \varepsilon_\imath^\lambda := \varepsilon_{\imath_1}^{\lambda_1} \cdot \cdots \cdot \varepsilon_{\imath_m}^{\lambda_m} \colon \imath \in \mathbb{N}_*^m, \; m = \ell(\lambda) \right\}, \quad \varepsilon_\imath^{\emptyset} \equiv 1,$$

of χ-essentially bounded cylindrical functions in the variable $u \in \mathfrak{U}$ that possess continuous restrictions to $\mathfrak{U}'$.

Theorem 3.2. *For any $\imath \in \mathbb{N}_*^m$ and $\psi, \phi \in \mathsf{E}_\imath^{\odot n}$, the following equality holds,*

$$\binom{n+m-1}{n} \int_{\mathfrak{U}} \phi_\zeta \, \bar{\psi}_\zeta \, d\chi = \langle \psi \mid \phi \rangle . \tag{3.5}$$

As a consequence, given $(\lambda, \imath) \in \mathbb{Y} \times \mathbb{N}_*^m$ with $m = \ell(\lambda)$, the system $\varepsilon^{\mathbb{Y}}$ of functions $\varepsilon_\imath^\lambda$ is orthogonal in the space L_χ^2 and

$$\left\| \varepsilon_\imath^\lambda \right\|_{L_\chi^2} = \left(\frac{(m-1)!\lambda!}{(m-1+|\lambda|)!} \right)^{1/2} . \tag{3.6}$$

Proof. Let $\mathsf{E}_\imath$ with $\imath = (\imath_1, \ldots, \imath_m) \in \mathbb{N}_*^m$ be the m-dimensional subspace in E spanned by $\{\mathfrak{e}_{\imath_1}, \ldots, \mathfrak{e}_{\imath_m}\}$ and $U(\imath)$ be the unitary subgroup of $U(\infty)$ acting in $\mathsf{E}_\imath$. The symbol $\mathsf{E}_\imath^{\odot n}$ means the nth symmetric tensor power of $\mathsf{E}_\imath$. Briefly denote $\psi_\dagger[v\zeta(u)] := \left\langle \left([v\rho^{-1}(u)]\mathfrak{e}_1 \right)^{\otimes n} \,\middle|\, \psi \right\rangle$ with $\psi \in \mathsf{E}_\imath^{\odot n}$ for all $v \in U(\imath)$ and $u \in \mathfrak{U}$. Using (2.7) with $U(\imath)$ instead of $U(m)$, we have

$$\int_{\mathfrak{U}} \phi_\zeta \, \bar{\psi}_\zeta \, d\chi = \int_{\mathfrak{U}} d\chi(u) \int_{U(\imath)} \phi_\dagger[v\zeta(u)] \cdot \bar{\psi}_\dagger[v\zeta(u)] \, d\chi_\imath(v) \tag{3.7}$$

for all $\psi, \phi \in \mathsf{E}_\imath^{\odot n}$. It is clear that

$$\Big| \int_{U(\imath)} \phi_\dagger \, \bar{\psi}_\dagger \, d\chi_\imath \Big| \le \sup_{v \in U(\imath)} \left| \phi_\dagger[v\zeta(u)] \right| \left| \psi_\dagger[v\zeta(u)] \rangle \right| \le \|\phi\| \, \|\psi\|$$

for all $u \in \mathfrak{U}$. Hence, the corresponding sesquilinear form in (3.7) is continuous on $\mathsf{E}_\imath^{\odot n}$. Thus, there exists a linear bounded operator A over $\mathsf{E}_\imath^{\odot n}$ such that

$$\langle A\psi \mid \phi \rangle = \int_{U(\imath)} \phi_\dagger \, \bar{\psi}_\dagger \, d\chi_\imath .$$

Next we show that A commutes with all operators $w^{\otimes n} \in \mathscr{L}\left(\mathsf{E}_\imath^{\odot n}\right)$ with $w \in U(\imath)$ acting as $w^{\otimes n} x^{\otimes n} = (wx)^{\otimes n}$, $(x \in \mathsf{E}_\imath)$. Invariance properties (2.6) of $\chi_\imath$ under the right action (2.4) yield

$$\begin{aligned}
&\left\langle (A \circ w^{\otimes n})\psi \mid \phi \right\rangle \\
&= \int_{U(\imath)} \left\langle [v\zeta(u)]^{\otimes n} \mid \phi \right\rangle \overline{\langle [v\zeta(u)]^{\otimes n} \mid w^{\otimes n}\psi \rangle} d\chi_\imath(v) \\
&= \int_{U(\imath)} \left\langle [w^{-1}v\zeta(u)]^{\otimes n} \mid (w^{-1})^{\otimes n}\phi \right\rangle \overline{\langle [w^{-1}v\zeta(u)]^{\otimes n} \mid \psi \rangle} d\chi_\imath(v) \\
&= \int_{U(\imath)} \left\langle [v\zeta(u)]^{\otimes n} \mid (w^{-1})^{\otimes n}\phi \right\rangle \overline{\langle [v\zeta(u)]^{\otimes n} \mid \psi \rangle} d\chi_\imath(v) \\
&= \left\langle A\psi \mid (w^{-1})^{\otimes n}\phi \right\rangle = \left\langle (w^{\otimes n} \circ A)\psi \mid \phi \right\rangle ,
\end{aligned}$$

where $w^{-1} \in U(\imath)$ is the hermitian adjoint matrix of w. Hence, the equality

$$A \circ w^{\otimes n} = w^{\otimes n} \circ A, \qquad w \in U(\imath) \tag{3.8}$$

holds. Let us check that the operator A, satisfying the condition (3.8), is proportional to the identity operator on $\mathsf{E}_\imath^{\otimes n}$. To this end we form the nth tensor power of the unitary group $U(\imath)$,

$$[U(\imath)]^{\otimes n} = \left\{ w^{\otimes n} \in \mathscr{L}\left(\mathsf{E}_\imath^{\odot n}\right) : w \in U(\imath) \right\}, \qquad [U(\imath)]^{\otimes 0} = 1.$$

Clearly, $[U(\imath)]^{\otimes n}$ is a unitary group over $\mathsf{E}_\imath^{\odot n}$. Let us check that the corresponding unitary representation

$$U(\imath) \ni w \longmapsto w^{\otimes n} \in \mathscr{L}\left(\mathsf{E}_\imath^{\odot n}\right) \tag{3.9}$$

is irreducible. This means that there is no subspace in $\mathsf{E}_\imath^{\odot n}$ other than $\{0\}$ and the whole space which is invariant under the action of $[U(\imath)]^{\otimes n}$.

Suppose, on the contrary, that there is an element $\psi \in \mathsf{E}_\imath^{\odot n}$ such that the equality $\left\langle \left([w\rho^{-1}(u)]\mathfrak{e}_1\right)^{\otimes n} \mid \psi \right\rangle = 0$ holds for all $w \in U(\imath)$ and $u \in U(\infty)$. By Lemma 3.1 the elements $w\rho^{-1}(u)$ act transitively on $\mathsf{S}(\infty)$. Hence, by n-homogeneity, we obtain $\langle x^{\otimes n} \mid \psi \rangle = 0$ for all $x \in \mathsf{E}_\imath$. Applying the polarization formula (3.3), we get $\psi = 0$. Hence, (3.9) is irreducible.

Thus, we can apply to (3.9) the Schur lemma [10, Thm 21.30]: a non-zero matrix which commutes with all matrices of an irreducible representation is a constant multiple of the unit matrix. As a result, we obtain that the operator A, satisfying (3.8), is proportional to the identity operator on $\mathsf{E}_\imath^{\odot n}$ i.e. $A = \alpha_{(n,\imath)} \mathbb{1}_{\mathsf{E}_\imath^{\odot n}}$ with a constant $\alpha_{(n,\imath)} > 0$. It follows that

$$\int_{U(\imath)} \phi_\dagger \, \bar{\psi}_\dagger \, d\chi_\imath = \alpha_{(n,\imath)} \langle \psi \mid \phi \rangle, \qquad \phi, \psi \in \mathsf{E}_\imath^{\odot n}. \tag{3.10}$$

In particular, the subsystem of cylindrical functions $\varepsilon_\imath^\lambda$ with a fixed $\imath \in \mathbb{N}_*^m$ is orthogonal in L^2_χ, because the corresponding system of tensor products $\mathfrak{e}_\imath^{\odot \lambda}$ indexed by $\lambda \in \mathbb{Y}_n$ with $\ell(\lambda) = m$ forms an orthogonal basis in $\mathsf{E}_\imath^{\odot n}$.

It remains to note that the set of all indices $\imath = (\imath_1, \ldots, \imath_m) \in \mathbb{N}_*^m$ with all $m = \ell(\lambda)$ is directed with respect to the set-theoretic embedding, i.e., for any $\imath, \imath'$ there exists $\imath''$ so that $\imath \cup \imath' \subset \imath''$. This fact and the above reasoning imply that the whole system $\varepsilon^{\mathbb{Y}}$ is also orthogonal in L^2_χ.

Taking into account (3.2), we can choose $\phi_n = \psi_n = \varepsilon_\imath^\lambda \sqrt{n!/\lambda!}$ in (3.10). As a result, we obtain

$$\alpha_{(n,\imath)} = \frac{n!}{\lambda!} \int_{U(\imath)} \left|\varepsilon_\imath^\lambda\right|^2 d\chi_\imath = \frac{n!}{\lambda!} \left\|\varepsilon_\imath^\lambda\right\|^2_{L^2_\chi}.$$

The well known formula [18, 1.4.9] for the unitary m-dimensional group gives

$$\int_{U(\imath)} \left|\varepsilon_\imath^\lambda\right|^2 d\chi_\imath = \frac{\lambda!(m-1)!}{(n+m-1)!}, \qquad |\lambda| = n, \quad \ell(\lambda) = m.$$

Using the last two formulas, we arrive at the relation

$$\alpha_{(n,\imath)} = \frac{n!}{\lambda!} \int_{U(\imath)} \left|\varepsilon_\imath^\lambda\right|^2 d\chi_\imath = \frac{n!}{\lambda!} \, \frac{\lambda!(m-1)!}{(n+m-1)!} = \frac{n!(m-1)!}{(n+m-1)!}. \tag{3.11}$$

Combining (3.7) and (3.11), we get (3.5) and, as a consequence, (3.6). $\square$

Definition 3.3. By H^2_χ we denote the Hardy space over $U(\infty)$ defined as the L^2_χ-closure of the complex linear span of the orthogonal system $\varepsilon^{\mathbb{Y}}$.

Let the space $H^{2,n}_\chi$ be the L^2_χ-closure of the complex linear span of the subsystem $\varepsilon^{\mathbb{Y}_n} := \left\{\varepsilon^\lambda_\imath \in \varepsilon^{\mathbb{Y}} \colon (\lambda,\imath) \in \mathbb{Y}_n \times \mathbb{N}_*^{\ell(\lambda)}\right\}$ with a fixed $n \in \mathbb{Z}_+$.

Corollary 3.4. *For any positive integers $n \neq k$ the orthogonality $H^{2,n}_\chi \perp H^{2,k}_\chi$ holds in L^2_χ. As a consequence, the following orthogonal decomposition holds,*

$$H^2_\chi = \mathbb{C} \oplus H^{2,1}_\chi \oplus H^{2,2}_\chi \oplus \cdots . \tag{3.12}$$

Proof. The orthogonal property $\varepsilon^\mu_\jmath \perp \varepsilon^\lambda_\imath$ with $|\mu| \neq |\lambda|$ for any $\imath \in \mathbb{N}_*^{\ell(\lambda)}$ and $\jmath \in \mathbb{N}_*^{\ell(\mu)}$ follows from (2.8), since

$$\begin{aligned}\int_{\mathfrak{U}} \varepsilon^\mu_\jmath \,\bar{\varepsilon}^\lambda_\imath \,d\chi &= \int_{\mathfrak{U}} \varepsilon^\mu_\jmath\big(\exp(\mathbb{i}\vartheta)u\big)\,\bar{\varepsilon}^\lambda_\imath\big(\exp(\mathbb{i}\vartheta)u\big)d\chi(u)\\ &= \frac{1}{2\pi}\int_{\mathfrak{U}} \varepsilon^\mu_\jmath \,\bar{\varepsilon}^\lambda_\imath \,d\chi \int_{-\pi}^{\pi} \exp\big(\mathbb{i}(|\mu|-|\lambda|)\vartheta\big)\,d\vartheta = 0\end{aligned}$$

for all $\lambda \in \mathbb{Y}$ and $\mu \in \mathbb{Y} \setminus \{\emptyset\}$. This yields $H^{2,|\mu|}_\chi \perp H^{2,|\lambda|}_\chi$ in the space L^2_χ. $\square$

4. Reproducing Kernels

Let us construct the reproducing kernel of H^2_χ. We refer to [19] for the basic definitions and properties of reproducing kernels.

Lemma 4.1. *For every $u, v \in \mathfrak{U}$ there exists a $q \in \mathbb{N}$ such that the reproducing kernel of the subspace $H^{2,n}_\chi$ in L^2_χ has the form*

$$\begin{aligned}\mathfrak{h}_n(v,u) &= \sum_{m\le q}\binom{n+m-1}{n}\langle\zeta(v)\mid\zeta(u)\rangle^n\\ &= \sum_{(\lambda,\imath)\in\mathbb{Y}_n\times\mathbb{N}_*^{\ell(\lambda)}} \frac{\varepsilon^\lambda_\imath(v)\,\bar{\varepsilon}^\lambda_\imath(u)}{\|\varepsilon^\lambda_\imath\|^2_{L^2_\chi}}, \qquad u,v\in\mathfrak{U}.\end{aligned} \tag{4.1}$$

Proof. Note that $\mathfrak{h}_0 \equiv 1$. From (2.3) it follows that for each stabilized sequence $u \in \mathfrak{U}$ there exists $u_m \in U(m)$ with a certain $m = m(u)$ such that $u = \rho(u_m)$. So, the element $\zeta(u) = \rho^{-1}(u)\mathfrak{e}_1$ is located on the m-dimensional sphere $\mathsf{S}(m)$. It means that its Fourier series $\zeta(u) = \sum \mathfrak{e}_k\varepsilon_k(u)$ has $m(u)$ terms. The tensor multinomial theorem yields the Fourier decomposition

$$[\zeta(u)]^{\otimes n} = \Big(\sum \mathfrak{e}_k\varepsilon_k(u)\Big)^{\otimes n} = \sum_{(\lambda,\imath)\in\mathbb{Y}_n\times\mathbb{N}_*^{\ell(\lambda)}} \frac{n!}{\lambda!}\mathfrak{e}_\imath^{\odot\lambda}\,\varepsilon^\lambda_\imath(u)$$

in the space $\mathsf{E}^{\odot n}$. Using the formula (3.2), we obtain

$$\begin{aligned}\langle\zeta(v)\mid\zeta(u)\rangle^n &= \left\langle[\zeta(v)]^{\otimes n}\mid[\zeta(u)]^{\otimes n}\right\rangle\\ &= \sum_{(\lambda,\imath)\in\mathbb{Y}_n\times\mathbb{N}_*^{\ell(\lambda)}}\left(\frac{n!}{\lambda!}\right)^2\left\langle\mathfrak{e}_\imath^{\odot\lambda}\mid\mathfrak{e}_\imath^{\odot\lambda}\right\rangle\varepsilon_\imath^\lambda(v)\,\bar{\varepsilon}_\imath^\lambda(u)\\ &= \sum_{(\lambda,\imath)\in\mathbb{Y}_n\times\mathbb{N}_*^{\ell(\lambda)}}\frac{\varepsilon_\imath^\lambda(v)\,\bar{\varepsilon}_\imath^\lambda(u)}{\|\mathfrak{e}_\imath^{\odot\lambda}\|^2}\end{aligned}$$

where $\langle\zeta(v)\mid\zeta(u)\rangle$ is decomposed into $q=\min\{m(u),m(v)\}$ summands in virtue of orthogonality. Multiplying both sides by $\binom{n+m-1}{n}$ and summing over all $m\le q$, we get (4.1). It follows that $\int_{\mathfrak{U}}\mathfrak{h}_n(v,u)\varepsilon_\imath^\lambda(u)\,d\chi(u)=\varepsilon_\imath^\lambda(v)$ for each $v\in\mathfrak{U}$. Via Theorem 3.1 the system $\varepsilon^{\mathbb{Y}_n}$ of functions $\varepsilon_\imath^\lambda$ forms an orthogonal basis in $H^{2,n}_\chi$. So, the integral operator

$$\int_{\mathfrak{U}}\mathfrak{h}_n(v,u)\psi_\zeta(u)\,d\chi(u)=\psi_\zeta(v),\qquad \psi_\zeta\in H^{2,n}_\chi \tag{4.2}$$

acts identically on $H^{2,n}_\chi$. Thus, the kernel (4.1) is reproducing in $H^{2,n}_\chi$. □

Let us consider the complex-valued kernel

$$\mathfrak{h}(z;v,u)=\prod_{m\le\min\{m(u),m(v)\}}\left[\,1-z\,\langle\zeta(v)\mid\zeta(u)\rangle\right]^{-m},\quad u,v\in\mathfrak{U},\quad |z|<1$$

where $m(u)$ is the number of terms in the Fourier series $\zeta(u)=\sum\mathfrak{e}_k\varepsilon_k(u)$.

Theorem 4.2. *The expansion $\mathfrak{h}(z;v,u)=\sum z^n\mathfrak{h}_n(v,u)$ holds for any $u,v\in\mathfrak{U}$ and $|z|<1$. The kernel $\mathfrak{h}(1;v,u)=\sum\mathfrak{h}_n(v,u)$ is reproducing in H^2_χ in the sense that*

$$\int_{\mathfrak{U}}\mathfrak{h}(1;v,u)f(u)\,d\chi(u)=f(v),\qquad f\in H^2_\chi,\quad v\in\mathfrak{U}. \tag{4.3}$$

Proof. Let $q=\min\{m(u),m(v)\}$ and $m\le q$. As is well known [18, 1.4.10],

$$\left[\,1-z\,\langle\zeta(v)\mid\zeta(u)\rangle\right]^{-m}=\sum_{n\in\mathbb{Z}_+}\binom{n+m-1}{n}\langle z\zeta(v)\mid\zeta(u)\rangle^n \tag{4.4}$$

for all $|z|<1$. By the Vandermonde identity, we have

$$\begin{aligned}\binom{n+m-1}{n}\langle z\zeta(v)\mid\zeta(u)\rangle^n &= \binom{r+k+p+l-2}{r+k}\langle z\zeta(v)\mid\zeta(u)\rangle^{r+k}\\ &= \sum_{r=0}^{n}\binom{r+p-1}{r}\binom{n-r+l-1}{n-r}\langle z\zeta(v)\mid\zeta(u)\rangle^{r+k}\end{aligned}$$

for all $n = r + k$ and $m = p + l - 1$. Applying recursively this identity to the series (4.4) with any $m \le q$ and using Lemma 4.1, we obtain

$$\begin{aligned}\mathfrak{h}(z; v, u) &= \prod_{m\le q}\sum_{n\in\mathbb{Z}_+}\binom{n+m-1}{n}\langle z\zeta(v) \mid \zeta(u)\rangle^n \\ &= \sum_{n\in\mathbb{Z}_+} z^n \sum_{(\lambda,\imath)\in\mathbb{Y}_n\times\mathbb{N}_*^{\ell(\lambda)}} \frac{\varepsilon_\imath^\lambda(v)\,\bar{\varepsilon}_\imath^\lambda(u)}{\|\varepsilon_\imath^\lambda\|_{L_\chi^2}^2} = \sum_{n\in\mathbb{Z}_+} z^n\mathfrak{h}_n(v, u).\end{aligned}$$

Hence, the required expansion holds. By (3.12) we have $f = \sum_n f_n$ for any $f \in H_\chi^2$ where $f_n \in H_\chi^{2,n}$ is the orthogonal projection of f. Observing that $\mathfrak{h}_k(z; \cdot, u) \perp f_n(\cdot)$ with $n \neq k$ holds in L_χ^2, we obtain

$$\int_{\mathfrak{U}} \mathfrak{h}(1; v, u) f(u)\, d\chi(u) = \sum \int_{\mathfrak{U}} \mathfrak{h}_n(v, u) f_n(v)\, d\chi(u) = \sum f_n(v) = f(v)$$

for all $v \in \mathfrak{U}$ and $f \in H_\chi^2$. Hence, (4.3) is valid. □

5. The Hilbert–Schmidt Analyticity

Recall (see e.g. [7]) that a function f on an open domain in a Banach space is said to be analytic if it is Gâteaux analytic and norm continuous. Similarly to [6,15], we say that f is *Hilbert–Schmidt analytic* if its Taylor coefficients are Hilbert–Schmidt polynomials. Now we describe a space H^2 of Hilbert–Schmidt analytic complex functions on the open ball B.

The symmetric Fock space is defined to be the orthogonal sum

$$\Gamma = \bigoplus_{n\in\mathbb{Z}_+} \mathsf{E}^{\odot n}, \qquad \langle \psi \mid \phi\rangle = \sum_{n\in\mathbb{Z}_+}\langle \psi_n \mid \phi_n\rangle$$

for all elements $\psi = \bigoplus_n \psi_n$, $\phi = \bigoplus_n \phi_n \in \Gamma$ with $\psi_n, \phi_n \in \mathsf{E}^{\odot n}$. The subset $\{x^{\otimes n} : x \in \mathsf{B}\}$ is total in $\mathsf{E}^{\odot n}$ by virtue of (3.3). This provides the total property of the subsets $\big\{(1-x)^{-\otimes 1} : x \in \mathsf{B}\big\}$ in Γ where we denote

$$(1-x)^{-\otimes 1} := \sum x^{\otimes n}, \qquad x^{\otimes 0} = 1.$$

The Γ-valued function $(1-x)^{-\otimes 1}$ in the variable $x \in \mathsf{B}$ is analytic, since

$$\big\|(1-x)^{-\otimes 1}\big\|^2 = \sum \|x\|^{2n} = \big(1 - \|x\|^2\big)^{-1} < \infty. \tag{5.1}$$

Let us define the Hilbert space of analytic complex functions in the variable $x \in \mathsf{B}$, associated with the Fock space Γ, as follows

$$H^2 = \big\{\psi^*(x) = \big\langle (1-x)^{-\otimes 1} \mid \psi\big\rangle : \psi \in \Gamma\big\}, \qquad \|\psi^*\|_{H^2} := \|\psi\|$$

for all $x \in \mathsf{B}$. This description is correct, because each function ψ^* in the variable $x \in \mathsf{B}$ is analytic by virtue of [9, Prop. 2.4.2], as a composition of the analytic Γ-valued function $(1-x)^{-\otimes 1}$ in the variable $x \in \mathsf{B}$ and the linear functional $\langle \cdot \mid \psi\rangle$ on Γ.

Similarly, we define the closed subspace in H^2 of n-homogenous Hilbert–Schmidt polynomials ψ_n^* in the variable $x \in \mathsf{E}$ as

$$H_n^2 = \left\{ \psi_n^*(x) = \left\langle x^{\otimes n} \mid \psi_n \right\rangle : \psi_n \in \mathsf{E}^{\odot n} \right\}.$$

Differentiating at zero any function $\psi^* = \bigoplus \psi_n^* \in H^2$ with $\psi_n^* \in H_n^2$, we obtain that its Taylor coefficients at zero $(n!)^{-1} d_0^n \psi^* = \psi_n^*$ are Hilbert–Schmidt polynomials. Hence, every function from H^2 is Hilbert–Schmidt analytic. Clearly, the following orthogonal decomposition holds,

$$H^2 = \mathbb{C} \oplus H_1^2 \oplus H_2^2 \oplus \cdots. \tag{5.2}$$

One can show that $\left(H_n^2\right)_n$ is a coherent sequence of polynomial ideals over E in the meaning of [4, Def. 1.1].

For each pair $(\lambda, \imath) \in \mathbb{Y}_n \times \mathbb{N}_*^{\ell(\lambda)}$, we can uniquely assign the Hilbert–Schmidt n-homogenous polynomial

$$\hat{x}_\imath^\lambda := \left\langle x^{\otimes n} \;\middle|\; \mathfrak{e}_\imath^{\odot \lambda} \right\rangle, \qquad x \in \mathsf{E},$$

defined via the Fourier coefficients $\hat{x}_k := \mathfrak{e}_k^*(x) = \langle x \mid \mathfrak{e}_k \rangle$ of an element $x \in \mathsf{E}$. Taking into account (3.2), the tensor multinomial theorem yields the following orthogonal decompositions with respect to the basis $\mathfrak{e}^{\odot \mathbb{Y}}$ in Γ,

$$(1 - x)^{-\otimes 1} = \sum_{(\lambda, \imath) \in \mathbb{Y} \times \mathbb{N}_*^{\ell(\lambda)}} \frac{\hat{x}_\imath^\lambda \mathfrak{e}_\imath^{\odot \lambda}}{\|\mathfrak{e}_\imath^{\odot \lambda}\|^2}, \qquad x \in \mathsf{B}. \tag{5.3}$$

Hence, any function $\psi^* \in H^2$ has the orthogonal expansion

$$\psi^*(x) = \left\langle (1 - x)^{-\otimes 1} \mid \psi \right\rangle = \sum_{(\lambda, \imath) \in \mathbb{Y} \times \mathbb{N}_*^{\ell(\lambda)}} \hat{\psi}_{(\lambda, \imath)} \hat{x}_\imath^\lambda, \qquad x \in \mathsf{B} \tag{5.4}$$

where $\hat{\psi}_{(\lambda, \imath)} := \langle \mathfrak{e}_\imath^{\odot \lambda} \mid \psi \rangle \|\mathfrak{e}_\imath^{\odot \lambda}\|^{-2}$ are the Fourier coefficients of $\psi \in \Gamma$ with respect to the basis $\mathfrak{e}^{\odot \mathbb{Y}}$ and, moreover, $\|\psi^*\|_{H^2}^2 = \sum_{(\lambda, \imath)} |\langle \mathfrak{e}_\imath^{\odot \lambda} \mid \psi \rangle|^2 \|\mathfrak{e}_\imath^{\odot \lambda}\|^{-2}$. Thus, $\|\psi^*\|_{H^2}$ is a Hilbert–Schmidt type norm on H^2.

6. Integral Formulas

The one-to-one correspondence $\mathfrak{e}_\imath^{\odot \lambda} \leftrightarrow \varepsilon_\imath^\lambda$ allows us to construct an antilinear isometric isomorphism $\mathcal{J} \colon \Gamma \longrightarrow H_\chi^2$ and its adjoint $\mathcal{J}^* \colon H_\chi^2 \longrightarrow \Gamma$ by the following change of orthonormal bases

$$\mathcal{J} \colon \Gamma \ni \mathfrak{e}_\imath^{\odot \lambda} \left\| \mathfrak{e}_\imath^{\odot \lambda} \right\|^{-1} \longmapsto \varepsilon_\imath^\lambda \left\| \varepsilon_\imath^\lambda \right\|_{L_\chi^2}^{-1} \in H_\chi^2, \qquad \lambda \in \mathbb{Y}, \quad \imath \in \mathbb{N}_*^{\ell(\lambda)}.$$

Clearly, $\mathcal{J}^* \colon \varepsilon_\imath^\lambda \left\| \varepsilon_\imath^\lambda \right\|_{L_\chi^2}^{-1} \longmapsto \mathfrak{e}_\imath^{\odot \lambda} \left\| \mathfrak{e}_\imath^{\odot \lambda} \right\|^{-1}$, because $\left\langle \mathcal{J} \mathfrak{e}_\imath^{\odot \lambda} \;\middle|\; f \right\rangle_{L_\chi^2} = \left\langle \mathfrak{e}_\imath^{\odot \lambda} \;\middle|\; \mathcal{J}^* f \right\rangle$ for any $f \in H_\chi^2$. Using Theorem 3.2, for any element $\psi \in \Gamma$ with the Fourier coefficients $\hat{\psi}_{(\lambda, \imath)} = \langle \mathfrak{e}_\imath^{\odot \lambda} \mid \psi \rangle \|\mathfrak{e}_\imath^{\odot \lambda}\|^{-2}$, we obtain

$$\mathfrak{J}\psi=\sum_{(\lambda,\imath)\in\mathbb{Y}\times\mathbb{N}_*^{\ell(\lambda)}}\hat{\psi}_{(\lambda,\imath)}\frac{\|\mathfrak{e}_\imath^{\odot\lambda}\|^2}{\|\varepsilon_\imath^\lambda\|^2_{L^2_\chi}}\varepsilon_\imath^\lambda\quad\text{where}\quad\frac{\|\mathfrak{e}_\imath^{\odot\lambda}\|^2}{\|\varepsilon_\imath^\lambda\|^2_{L^2_\chi}}=\frac{(\ell(\lambda)-1+|\lambda|)!}{(\ell(\lambda)-1)!|\lambda|!}.$$

In particular, $\mathfrak{J}x=\sum\hat{x}_k\varepsilon_k$ for any elements $x\in\mathsf{E}$ with the Fourier coefficients $\hat{x}_k=\langle x\mid\mathfrak{e}_k\rangle$. Moreover, $\|\mathfrak{J}x\|^2_{L^2_\chi}=\sum\|\hat{x}_k\|^2=\|x\|^2$.

In what follows, we assign to each $x\in\mathsf{E}$ the L^2_χ-valued function

$$x_{\mathfrak{J}}\colon\mathfrak{U}\ni u\longmapsto(\mathfrak{J}x)(u).$$

Lemma 6.1. *The function $\mathfrak{J}(1-x)^{-\otimes1}=(1-x_{\mathfrak{J}})^{-1}$ in the variable $u\in\mathfrak{U}$ takes values in L^2_χ for all $x\in\mathsf{B}$.*

Proof. Applying $\mathfrak{J}$ to the decompositions (3.1) and (5.3), we obtain

$$\begin{aligned}\mathfrak{J}(1-x)^{-\otimes1}&=\sum_{(\lambda,\imath)\in\mathbb{Y}\times\mathbb{N}_*^{\ell(\lambda)}}\frac{\hat{x}_\imath^\lambda\varepsilon_\imath^\lambda}{\|\mathfrak{e}_\imath^{\odot\lambda}\|^2}\\&=\sum_{n\in\mathbb{Z}_+}\Big(\sum_{k\in\mathbb{N}}\hat{x}_k\varepsilon_k\Big)^n=(1-x_{\mathfrak{J}})^{-1}\end{aligned}\tag{6.1}$$

where the following orthogonal series with a fixed $n\in\mathbb{N}$,

$$x_{\mathfrak{J}}^n=\Big(\sum_{k\in\mathbb{N}}\hat{x}_k\varepsilon_k\Big)^n=\sum_{(\lambda,\imath)\in\mathbb{Y}_n\times\mathbb{N}_*^{\ell(\lambda)}}\frac{\hat{x}_\imath^\lambda\varepsilon_\imath^\lambda}{\|\mathfrak{e}_\imath^{\odot\lambda}\|^2},\tag{6.2}$$

is convergent in L^2_χ. Moreover, taking into account the orthogonality, we get

$$\begin{aligned}\big\|(1-x_{\mathfrak{J}})^{-1}\big\|^2_{L^2_\chi}&=\sum_{n\in\mathbb{Z}_+}\sum_{(\lambda,\imath)\in\mathbb{Y}_n\times\mathbb{N}_*^{\ell(\lambda)}}\frac{|\hat{x}_\imath^\lambda|^2}{\|\mathfrak{e}_\imath^{\odot\lambda}\|^2}\\&=\sum_{n\in\mathbb{Z}_+}\Big(\sum_{k\in\mathbb{N}}|\hat{x}_k|^2\Big)^n=\big(1-\|x\|^2\big)^{-1}.\end{aligned}$$

Hence, the function $(1-x_{\mathfrak{J}})^{-1}$ with $x\in\mathsf{B}$ takes values in L^2_χ. □

Let $f=\sum_n f_n\in H^2_\chi$ with $f_n\in H^{2,n}_\chi$. Then $\mathfrak{J}^*f\in\Gamma$ and $\mathfrak{J}^*f_n\in\mathsf{E}^{\odot n}$. Briefly denote $\tilde{f}:=(\mathfrak{J}^*f)^*\in H^2_n$ and $\tilde{f}_n:=(\mathfrak{J}^*f_n)^*\in H^2$. Thus,

$$\begin{aligned}\tilde{f}(x)&=\big\langle(1-x)^{-\otimes1}\mid\mathfrak{J}^*f\big\rangle,&&x\in\mathsf{B},\\\tilde{f}_n(x)&=\big\langle x^{\otimes n}\mid\mathfrak{J}^*f_n\big\rangle,&&x\in\mathsf{E}.\end{aligned}$$

Theorem 6.2. *Each Hilbert–Schmidt analytic function $\tilde{f}\in H^2$ has the integral representation*

$$\tilde{f}(x)=\int_{\mathfrak{U}}\frac{f\,d\chi}{1-x_{\mathfrak{J}}},\qquad x\in\mathsf{B}\tag{6.3}$$

and its Taylor coefficients at zero have the form

$$\frac{d_0^n\tilde{f}(x)}{n!}=\int_{\mathfrak{U}}x_{\mathfrak{J}}^nf_n\,d\chi,\qquad x\in\mathsf{E}.\tag{6.4}$$

The mapping $f \longmapsto \tilde{f}$ *produces a linear isometry* $H^2_\chi \simeq H^2$.

Proof. Consider the Fourier decomposition of f with respect to the basis $\varepsilon^{\mathbb{Y}}$ and its $\mathcal{J}^*$-image, respectively

$$f = \sum_{(\lambda,\imath)\in\mathbb{Y}\times\mathbb{N}_*^{\ell(\lambda)}} \hat{f}_{(\lambda,\imath)}\varepsilon_\imath^\lambda, \qquad \mathcal{J}^* f = \sum_{(\lambda,\imath)\in\mathbb{Y}\times\mathbb{N}_*^{\ell(\lambda)}} \bar{\hat{f}}_{(\lambda,\imath)} \frac{\|\varepsilon_\imath^\lambda\|^2_{L^2_\chi}}{\|\mathfrak{e}_\imath^{\odot\lambda}\|^2}\mathfrak{e}_\imath^{\odot\lambda}$$

where $\hat{f}_{(\lambda,\imath)} = \|\varepsilon_\imath^\lambda\|^{-2}_{L^2_\chi}\int_{\mathfrak{U}} f\,\bar{\varepsilon}_\imath^\lambda\,d\chi$. Substituting $\hat{f}_{(\lambda,\imath)}$ to $\tilde{f} = (\mathcal{J}^* f)^*$ and using the orthogonal property and the relations (5.3) and (6.1), we obtain

$$\begin{aligned}\tilde{f}(x) &= \sum_{(\lambda,\imath)\in\mathbb{Y}\times\mathbb{N}_*^{\ell(\lambda)}} \frac{\hat{f}_{(\lambda,\imath)}\hat{x}_\imath^\lambda \left\langle \mathfrak{e}_\imath^{\odot\lambda} \mid \mathfrak{e}_\imath^{\odot\lambda}\right\rangle \|\varepsilon_\imath^\lambda\|^2_{L^2_\chi}}{\|\mathfrak{e}_\imath^{\odot\lambda}\|^4} \\ &= \int_{\mathfrak{U}} \sum_{(\lambda,\imath)\in\mathbb{Y}\times\mathbb{N}_*^{\ell(\lambda)}} \frac{\hat{x}_\imath^\lambda\varepsilon_\imath^\lambda}{\|\mathfrak{e}_\imath^{\odot\lambda}\|^2} f\,d\chi = \int_{\mathfrak{U}} \frac{f\,d\chi}{1 - x_{\mathcal{J}}}.\end{aligned}$$

Hence, (6.3) holds. Using (6.2), we similarly obtain

$$\tilde{f}_n(x) = \left\langle x^{\otimes n} \mid \mathcal{J}^* f_n\right\rangle = \int_{\mathfrak{U}} x_{\mathcal{J}}^n f_n\,d\chi. \tag{6.5}$$

Taking into account (6.5) and the orthogonal decomposition (3.12), we get

$$\tilde{f}\,(\alpha x) = \left\langle (1-\alpha x)^{-\otimes 1} \mid \mathcal{J}^* f\right\rangle = \sum \alpha^n \int_{\mathfrak{U}} x_{\mathcal{J}}^n f_n d\chi, \quad |\alpha| \le 1. \tag{6.6}$$

Note that $\tilde{f}\,(\alpha x)$ is analytic in α for all $x \in \mathsf{B}$. Differentiating $\tilde{f}\,(\alpha x)$ at $\alpha = 0$ and using the n-homogeneity of derivatives, we obtain

$$\frac{d^n}{d\alpha^n}\sum \alpha^n \int_{\mathfrak{U}} x_{\mathcal{J}}^n f_n\,d\chi\,\Big|_{\alpha=0} = n! \int_{\mathfrak{U}} x_{\mathcal{J}}^n f_n\,d\chi.$$

Hence, the functions (6.4) coincide with the Taylor coefficients at zero of $\tilde{f}$.

Finally, since the image of $\varepsilon^{\mathbb{Y}}$ under $\mathcal{J}^*$ coincides with $\mathfrak{e}^{\odot\mathbb{Y}}$, the mapping $H^2_\chi \ni f \longmapsto \tilde{f} \in H^2$ is an isometry. □

7. Radial Boundary Values

Using (6.3), for each $f = \sum_n f_n \in H^2_\chi$ with $f_n \in H^{2,n}_\chi$ we can rewrite (6.6) as

$$\tilde{f}(rx) = \left\langle (1-rx)^{-\otimes 1} \mid \mathcal{J}^* f\right\rangle = \int_{\mathfrak{U}} \frac{f\,d\chi}{1 - r x_{\mathcal{J}}}, \qquad x \in \mathsf{K}, \quad r \in [0,1)$$

where $\mathsf{K} = \{x \in \mathsf{E} \colon \|x\| \le 1\}$.

Theorem 7.1. *The integral transform* $\mathcal{C}_r \colon f \longmapsto \mathcal{C}_r[f]$, *defined as*

$$\mathcal{C}_r[f](x) := \int_{\mathfrak{U}} \frac{f\,d\chi}{1 - r x_{\mathcal{J}}}, \qquad x \in \mathsf{K}, \quad r \in [0,1), \tag{7.1}$$

belongs to the space of bounded linear operators $\mathscr{L}(H^2_\chi, H^2)$. The radial boundary values of $\mathcal{C}_r[f] \in H^2$ are equal to $\tilde{f} \in H^2$ in the following sense:

$$\lim_{r \nearrow 1} \left\| \mathcal{C}_r[f] - \tilde{f} \right\|_{H^2} = 0. \tag{7.2}$$

Moreover, the following equality holds,

$$\|\tilde{f}\|^2_{H^2} = \sup_{r \in [0,1)} \|\mathcal{C}_r[f]\|^2_{H^2}. \tag{7.3}$$

Proof. Theorem 6.2 and (7.1) imply the equality $\mathcal{C}_r[f] = \sum r^n \tilde{f}_n$ for any $r \in [0,1)$. By (5.2), we have $\tilde{f}_k \perp \tilde{f}_n$ as $n \neq k$ in H^2. It follows that

$$\|\mathcal{C}_r[f]\|^2_{H^2} = \left\| \sum r^n \tilde{f}_n \right\|^2_{H^2} = \sum r^{2n} \|\tilde{f}_n\|^2_{H^2} = \sum r^{2n} \|f_n\|^2_{L^2_\chi},$$

since $\mathcal{J}^*$ acts isometrically from $H^{2,n}_\chi$ onto the space $\mathsf{E}^{\odot n}$ which is antilinear isometric to H^2_n by definition. Similarly, we obtain that

$$\left\| \mathcal{C}_r[f] - \tilde{f} \right\|^2_{H^2} = \sum \left(r^{2n} - 1 \right) \|f_n\|^2_{L^2_\chi} \longrightarrow 0, \qquad r \to 1.$$

Moreover, the Cauchy–Schwarz inequality implies that

$$\|\mathcal{C}_r[f]\|^2_{H^2} \leq \frac{1}{(1-r^2)^{1/2}} \Big(\sum \|f_n\|^2_{L^2_\chi} \Big)^{1/2} = \frac{\|f\|_{L^2_\chi}}{(1-r^2)^{1/2}}$$

for all $f \in H^2_\chi$. Hence, the operator $\mathcal{C}_r$ belongs to $\mathscr{L}(H^2_\chi, H^2)$ for all $r \in [0,1)$.

Finally, the equalities

$$\sup_{r \in [0,1)} \|\mathcal{C}_r[f]\|^2_{H^2} = \sup_{r \in [0,1)} \sum r^{2n} \|\tilde{f}_n\|^2_{H^2} = \sum \|\tilde{f}_n\|^2_{H^2} = \|\tilde{f}\|^2_{H^2}$$

give the required formula (7.3). □

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