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Book Review

The Classical Orthogonal Polynomials. Brian George Spencer Doman, World Scientific, 2016;
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Orthogonal polynomials play a crucial role in pure, applied, and computational mathematics, as well as in engineering, computer sciences, and natural sciences. The theory of orthogonal polynomials was initiated in mathematical analysis in the fifties of the nineteenth century, by the russian mathematician P. L. Chebyshev (1821–1894).

This book is devoted to a systematic and unified discussion of various properties of classical orthogonal polynomials. It consists of 12 short chapters. They are comprehensive and represent necessary material for both novices and experienced scientists. Each chapter ends with selected references.

Chapter 1, *Definitions and General Properties*, deals with some basic notions and facts of the general theory of orthogonal polynomials. After a short introduction of the orthogonality concept, the Gram-Schmidt orthogonalisation procedure is explained. Next, some information regarding the zeros of the *N*th order polynomial and the *n*-point Gaussian quadrature are presented. A brief analysis of recurrence relation together with the terse discussion of Shohat-Favard theorem are the theme of the following sections. The important consequences of the recurrence relation, namely, the Christoffel-Darboux identities and interlacing of zeros, are addressed also. Then, the classical orthogonal polynomials, the Hermite, Associated Laguerre and Jacobi polynomials and its properties are discussed. Included are the three recurrence relations and the differential relation. The step down and the step up operators are defined here and used to alternative derivation of the interlacing zeros properties. By successive application of these operators, the second order linear ordinary differential equations are obtained for three classical polynomials.

Chapter 2, *Hermite Polynomials*, gives a concise treatment of ”physicists’ Hermite polynomials”, named in honor of Charles Hermite (1822–1901). It should be noted that this notion differs slightly from ”probabilists’ Hermite polynomials”, preferred by probabilists (see: Koornwinder T. H. et al. *Orthogonal Polynomials*. In: Olver F. W. J. et al. NIST *Handbook of Mathematical Functions*, NIST and Cambridge University Press, 2010, p. 439). First, the ordinary differential equation associated with the Hermite polynomials H_n is presented. Then, the orthogonality and derivative property of H_n are showed. In the following section one of the most important characterizations for classical orthogonal polynomials in one variable, the so-called Rodrigues formula, is obtained. Subsequently, an explicit power series expansion and the generating function for Hermite polynomials are derived. Three important properties of H_n polynomials, namely, the recurrence relations, addition formulae and step up and step down operators are demonstrated, using the generating function. The relations of H_n to Weber parabolic cylinder functions are also mentioned.

Chapter 3 addresses basic issues of *Associated Laguerre Polynomials*, L_n^α $a > -1$. The Laguerre polynomials ($\alpha = 0$) were introduced by Edmond Laguerre (1834–1886). The suitable construction of these polynomials is considered first and its beautiful connection with the Confluent Hypergeometric functions is explicated. The rest of chapter is

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organized as follows. The differential equation satisfied by L_n^α is discussed and analysed in the next section. Orthogonality, derivative property, Rodrigues formula and explicit form of the L_n^α are briefly presented. By using the generating function method, the number of useful identities are obtained, including recurrence relations and addition formulae and finally, the step up and step down operators are specified. It is noteworthy that Laguerre polynomials were not very popular before their use in quantum mechanics (see very interesting paper by Mawhin and Ronveaux [*Archive for the History of Exact Sciences* 64 (2010), 429–460)].

Chapters 4–6 concentrate on the special cases of the *Jacobi Polynomials* $P_n^{\alpha,\beta}$: *Legendre Polynomials* P_n, ($\alpha = \beta = 0$); *Chebyshev Polynomials of the First Kind* T_n, ($\alpha = \beta = -1/2$), and *Chebyshev Polynomials of the Second Kind* U_n, ($\alpha = \beta = 1/2$). Characteristics of these polynomials include: the form of differential equations corresponding to the polynomials; the orthogonality conditions; the Rodrigues formulae; the explicit representations, and the corresponding generating functions. The recurrence and differential relations as well as the step up and step down operators are also presented. Additionally, the chapter 4 contains the useful appendix (some integral) whereas the chapters 5 and 6 comprise the trigonometric representations and the relations with other Chebyshev polynomials.

There are several kinds of Chebyshev polynomials. Chapter 7 and 8 deal with *Chebyshev Polynomials of the Third Kind* V_n, and *Chebyshev Polynomials of the Fourth Kind* W_n, respectively. These polynomials are sometimes referred to as the "airfoil polynomials" and are, in fact, rescalings of two particular Jacobi polynomials $P_n^{\alpha,\beta}$ with $\alpha = -1/2$, $\beta = 1/2$ and vice versa. A brief summary of basic properties of Chebyshev polynomials of third and fourth kinds is presented. It shows that V_n and W_n are directly related, respectively, to the first- and second-kind Chebyshev polynomials. Although Chebyshev polynomials of third and fourth kinds are explored less than first and second kinds in the literature, their use in improving the performance of numerical algorithms is very promising.

Chapter 9 is devoted to *Gegenbauer Polynomials*. They are named after Leopold Gegenbauer (1849–1903). Gegenbauer (or Ultraspherical) polynomials $C_n^{(\alpha)}$ are particular cases of the Jacobi polynomials $P_n^{\alpha,\beta}$ specified by $\alpha = \beta$. The Gegenbauer polynomials include a number of polynomials as special cases: $C_n^{(\alpha)}$ for $\alpha = 1/2$ gives the Legendre polynomials, $\alpha = 1$ gives the type II Chebyshev polynomials, and the case $\alpha = 0$ gives the Chebyshev polynomials of the first kind. The main properties of Gegenbauer polynomials are listed in the compact manner.

Chapter 10 concerns with *Associated Legendre Functions* P_m^n, where the indices n and m are both integers and referred to as the degree and order of the Associated Legendre function respectively. These functions are closely related to the Legendre polynomials and the Gegenbauer polynomials. For example, when m is zero and n is integer, these functions are identical to the Legendre polynomials. The basic facts from the theory are given and, in particular, the orthogonality relations for the associated Legendre functions are derived in some detail. The methods are based on some known properties of the Gegenbauer polynomials. An appendix contains some interesting integral which leads to the recurrence relations.

Chapter 11 contains a brief survey of the *Jacobi polynomials* $P_n^{\alpha,\beta}$, that are the most general of the classical orthogonal polynomials in the domain [−1, 1]. The definitions and basic properties of the Jacobi polynomials are recalled. Furthermore, alternative derivation of the recurrence relation is presented by using the generating function method.

The last twelfth chapter, *General Appendix*, gives a concise overview of additional results in the theory of special functions, including the Gamma, Beta and Hypergeometric functions. The basic and alternative definitions of these functions are discussed, together with their integral representations.

This book is well written and it is relatively easy to read. For better prepared readers with a strong background in complex analysis, some supplemental references could usefully be added to the book as additional reading in a related spirit (e.g., J. Dereziński, *Hypergeometric Type Functions and Their Symmetries*. Ann. Henri Poincaré 15 (2014), 1569–1653). In conclusion, I recommend it to anyone who is interested not only in orthogonal polynomials theory, but also in its physical applications. The book can be also successfully used by instructors of undergraduate courses.