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Book Review

Special Functions of Mathematical (Geo-)Physics by Willi Freeden, Martin Gutting, Birkhäuser, 2013; ISBN: 978-3-0348-0562-9

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Mathematics concerned with geoscientific problems is called geomathematics. It deals with qualitative and quantitative properties of the complicated structures of the Earth. In fact, geomathematics may be treated as a crucial discipline enabling scientific understanding of complex Earth processes. In the geosciences, the theory of special functions is extremely important, and the subject has a long and distinguished history. This book is about "useful" functions (terminology of P. Turán, a prominent Hungarian mathematician), their role, purpose, and power.

The book consists of three parts. After the Introduction, the first part contains two chapters dedicated to auxiliary functions. The second part, five chapters in total, is an exhaustive overview of spherically oriented functions. The third part of the book, which comprises three chapters, deals with periodically oriented functions. Each part contains several exercises to illustrate the mathematical techniques just explained and to show their relevance to the geosciences.

Chapter 1, Geomathematical Motivation, briefly describes four (geo)physical fields which are highly important in physics and the geosciences, namely, gravitation, geomagnetism, fluid flow field, and elasticity. The classical approach to gravity field determination is briefly discussed and the concepts of spheriodization and periodization are introduced and explained. Next, the basic system of equations for the magnetic field of the Earth is formulated. In another section the equations of thermodynamics and fluid dynamics used to describe atmospheric and oceanic flow are concisely derived. Mathematical treatment of linear elasticity closes the chapter.

Part I contains basic material concerning auxiliary functions, for example the gamma function and selected classes of orthogonal polynomials.

Chapter 2 deals with the classical gamma function. The basic definitions and properties of the Euler gamma function are formulated. These are followed by two sections dealing with Euler's Beta function and the Stirling formula. The useful properties of the Legendre relationship (or duplication formula) are discussed in the next section. The generalization of the gamma function to complex values (Pochhammer's factorial), together with Euler's constant, are also discussed. The product formulas for the gamma function and for trigonometric functions are derived. For didactic and computational purposes, several problems connected with incomplete gamma and beta functions are presented in the last section.

In Chapter 3, the classical theory of orthogonal polynomials, including elements of necessary material from Fourier analysis, is developed. The basic properties of these polynomials (symmetry, zeros, best approximation, the Christoffel–Darboux formula, etc.), are presented. The n-point quadrature rule is then defined and two examples are described, namely, interpolatory and Gauss quadrature rules. Subsequent sections are concerned with Jacobi polynomials and their special cases (Gegenbauer, Chebyshev, and Legendre polynomials) and Hermite and Laguerre polynomials. For each of these families the basic properties (Rodrigues relationships, three

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term recurrences, expressions as hypergeometric series, differential equations, etc.) are derived and discussed. Some applications in electrostatics, quantum mechanics, and the theory of oscillations are also presented. Finally, some problems in which classical orthogonal polynomials emerge, for example, Gauss–Legendre integration, the Clenshaw algorithm, and error estimates for numerical algorithms, are considered as exercises.

Part II concentrates on spherically structured functions.

Chapter 4 begins by introducing spherical nomenclature and settings. The notion of orthogonal invariance is then explained and the properties of special orthogonal groups are summarized. The theory of scalar spherical harmonics is then formulated. The closure and completeness of spherical harmonics in the space of square-integrable functions is proved by the Abel–Poisson and Bernstein summability methods. The main result in the theory of spherical harmonics, i.e., the Funk–Hecke formula, which establishes the direct connection between the orthogonal invariance of the sphere and the addition theorem, is presented. Subsequently, the Green function with respect to the Beltrami operator is discussed and several integral theorems involving the Green function are formulated. The quantum-mechanical description of the hydrogen atom is discussed briefly as a "canonical" example of the use of spherical harmonics in theoretical physics. Finally, application of the theory of spherical harmonics to several problems is shown by use of carefully selected exercises. These include the spherical low-discrepancy method, locally supported wavelets on a sphere, the idea of the up function, anharmonic functions for the ball, the fast multipole method for the Laplace equation, Wigner matrices, and spherical harmonics in quaternionic representation.

In Chapter 5, the vector theory of spherical harmonics is developed in coherence with its scalar variant. Basic notations and the necessary differential operators are introduced first. Next, the three operators mapping scalar functions to vectorial functions, together with their properties, are defined and discussed. The Helmholtz decomposition theorem for spherical vector fields by use of the Green function with respect to the Beltrami operator, which motivates the choice of these operators, is formulated and proved. The closure and completeness of vector spherical harmonics based on vectorial variants of the scalar zonal Bernstein kernels are then proved. Subsequently, the interconnections between vector spherical harmonics and homogeneous harmonic vector polynomials are examined, and a vectorial analog of the Beltrami operator is constructed. The generalization of the addition theorem and the Funk–Hecke formulas to the vectorial case, and the vectorial counterparts of the Legendre polynomial, are the themes of subsequent sections. The polynomial solutions of the Cauchy–Navier equation are analyzed in some detail. Finally, several exercises relating to different practical aspects of vector spherical harmonics are suggested, namely, the application of the uncertainty principle in the theory of zonal kernel functions on the sphere and use of Wigner symbols for operating on coupling terms and integrals in the nonlinear Galerkin method for solution of the Navier–Stokes equation.

Chapter 6 deals with the theory of spherical harmonics in R^q. After an introduction to the nomenclature, the fundamental solutions for the Laplace operator in R^q in terms of harmonic and metaharmonic functions, are discussed. The corresponding integral theorems for the Laplace–Beltrami operator are then formulated. In another section the theory of homogeneous harmonic polynomials of dimension q is presented. Particular focus is on the Legendre polynomial of degree n and dimension q. Furthermore, the addition theorem, the Funk–Hecke formula, the closure and completeness theorems, and the eigensolutions of the Beltrami operator are presented and discussed. Moreover, the associated Legendre functions of dimension q are introduced, the pointwise expansion theorem is proved, and asymptotic relationships for the spherical harmonic coefficients are investigated. Finally, the Helmholtz-Beltrami operator in terms of the Green function and the corresponding integral theorems are described. The chapter concludes with fourteen exercises on different computational aspects of spherical harmonics.

Chapter 7 contains a brief presentation of the elements of the theory of classic Bessel functions. Some general properties of these functions are also discussed, including integral and series representations, recursive formulas, and orthogonality relationships. Two exercises deal with discontinuous integrals requiring Bessel functions and the modeling of electrons in a periodically changing magnetic field.

Chapter 8 is devoted to the study of the Bessel function of dimension q. A family of solutions dealing with Helmholtz equation [$(\Delta + \lambda)U = 0$, in standard notation], under the assumption that $\lambda \in \mathrm{R}\backslash\{0\}$, are considered, namely regular Bessel functions, modified Bessel functions, Hankel functions, Neumann functions, and Kelvin functions. Three types of solution of the Helmholtz equation, depending on the space domain, by use of suitable expansion theorems, are considered. The entire solutions of the Helmholtz equation are covered in the exercises.

Part III presents the basic concepts of lattice function theory.

In Chapter 9, the foundations of one-dimensional lattice theory are developed. Background information on Bernoulli polynomials, Bernoulli numbers, and the Bernoulli function is then presented. The notion of Z-periodic polynomials is then introduced. After these preliminaries, the definition of the Z-lattice function with respect to the one-dimensional Laplacian is introduced. The classical one-dimensional Euler summation formula is presented, with the generalization of Stirling's formula and an extension of the Euler summation to periodic boundary conditions. The famous one-dimensional Riemann Zeta function is also discussed. The equivalence of the Euler and Poisson summation formulas for finite intervals is proved, and some remarks concerning the generalization of the Poisson summation formula for R are presented. The construction of the theta function (of degree 0 and dimension 1) and several computational exercises closes the chapter.

Chapter 10 concerns the q-dimensional generalization of lattice point theory. The lattice, Λ, in R^q and the fundamental cell of the lattice, Λ, are defined first. Next, Λ-periodic polynomials are examined. This enables introduction of the Λ-lattice function for the Laplace operator in R^q. The multi-dimensional Euler summation formulas on arbitrary lattices are constituted. The zeta function of dimension $q \geq 3$ and degree n, together with the suitable functional equation, are constructed. Finally, asymptotic relationships for Euler and Poisson summations in R^q, the multi-dimensional Poisson summation formula, and the multi-dimensional counterpart of the theta function are discussed. Very interesting (and difficult) exercises concentrate on different aspects of spline interpolation methods and techniques for calculating lattice sums.

The final chapter, 11, is a terse summary of the key points developed in the book, and suggestions of topics for further research.

To conclude, this is an erudite and competent book of great value, not only to specialists in the field. It should be very useful for both advanced students and active researchers in the field of geomathematics, geophysics, and geoengineering, and also for geo-oriented applied mathematicians. The style is superb and clear, and the organization is very well planned. I believe this excellent book should be required reading for all earth-oriented scientists.