Renormalization in String-Localized Field Theories: A Microlocal Analysis

Christian Gaß

Abstract. Using methods of microlocal analysis, we prove that the regularization of divergent amplitudes stays a pure ultraviolet problem in string-localized field theories, despite the weaker localization. Thus, power counting does not lose its significance as an indicator for renormalizability. It also follows that standard techniques can be used to regularize divergent amplitudes in string-localized field theories.

1. Introduction

The foundations of string-localized field theory (SLFT) have been developed thoroughly by Mund et al. [24,25] in the mid 2000's and since then, SLFT has been under constant investigation and advancement. At the heart of SLFT is a weaker localization of the potentials of the field strength tensors (for arbitrary masses and spins respectively helicities) along a semi-infinite line referred to as *string*.[1] These string-localized potentials replace the usual, point-localized, gauge potentials in quantum field theory (QFT) and in that sense, SLFT is not a separate theory. It rather is a different setting within the framework of QFT that exhibits many desirable properties, which we will briefly sketch in the following.

String-localization has been known for a long time before the works of Mund, Schroer and Yngvason, and has been observed and described more or less explicitly at many occasions in the past: In old works of Jordan [16] and later also Dirac [8] on gauge invariant formulations of quantum electrodynamics (QED), the string-localized nature of the dressing factor of the electron field is clearly visible. A derivation of that dressing factor within SLFT as well

---

[1]Note that the term string refers to a weaker localization of certain quantum fields and is not to be confused with the strings of string theory.

as an investigation of its consequences on QED has been worked out recently [20] with the emphasis on the infrared problems of QED.

Also Mandelstam [18] studied QED by employing expressions that clearly resemble the modern string-localized fields. String-locality later reappeared in considerations of Buchholz and Fredenhagen [5] and also Steinmann [33,34].

The modern formulation of Mund, Schroer and Yngvason has served as starting point for further investigation of SLFT in the last one-and-a-half decades. It has become clear that string-localized fields bear manifold conceptual and practical advantages. First, the string-localized potential for the massless field strength of helicity $s \in \mathbb{N}$ is a rank-s tensor field that lives on Hilbert space and not on an indefinite Krein space like its point-localized gauge field equivalents [22,25]. Also, string-localization allows for the construction of infinite spin fields [24]. Moreover, the decoupling of helicities in the massless limit of massive tensor fields is explained by SLFT [22]; stress-energy tensors that yield the correct Poincaré generators can be constructed for massless fields of arbitrary finite and infinite spin/helicity [22,23,28], circumventing the Weinberg–Witten theorem [39]; the DVZ discontinuity [35,40] in the massless limit of massive gravitons is removed [22,23]; the Velo–Zwanziger problem [36] has been resolved [30]; Gauss' law has been implemented and investigated within SLFT [21]; there is no strong CP problem in string-localized QCD [12].

To summarize, extensive research on conceptual aspects of SLFT has revealed many benefits. On the other hand, the implementation of string-localized perturbation theory is only in its beginnings. Besides some conceptual considerations [6,19], only computations in low orders and at tree level have been performed. The Lie algebra structure of pure massless Yang–Mills theory and of the weak interaction as well as the chirality of the latter have be derived at second order and tree-level of perturbation theory in a bottom-up approach—the structure of these interactions is constrained by the requirement that the scattering matrix be string independent [12,14].

Calculations at higher orders of perturbation theory as well as computations of loop graphs involving internal string-localized potentials have not yet been attacked. The main reason for this is the most evident disadvantage of SLFT: The analytic structure of propagators of string-localized potentials is highly complicated. Consequently, an extension of the causal renormalization procedure as described by Epstein and Glaser [11] naively seems very involved and is currently not at hand. In this article, we make a step towards an Epstein-Glaser renormalization scheme in SLFT by proving that the string-localization does actually not affect the singularity structure of the propagators—provided that care is taken of how the string-localized version of the scattering operator is defined.

The paper is organized as follows. Section 2 is a concise introduction to the interrelation of microlocal analysis and renormalization. We also list some basic theorems about wavefront sets, which will be important for our later proofs. The reader familiar with this may skip Sect. 2 and directly proceed to Sect. 3. There, we investigate the distributional nature of string-integration and string-integrated propagators and outline a proper setup of perturbation

theory in the string-localized setting. Section 4 contains proofs regarding the existence and extension of products of string-localized propagators. Our results and their connections to other approaches are discussed in Sect. 5.

Before continuing, we fix the conventions used in this paper. We employ the mostly negative Minkowski metric $\eta = \mathrm{diag}(1, -1, -1, -1)$. If x, y are Minkowski vectors, we generically denote their Minkowski product by $(xy) := \eta_{\mu\nu} x^\mu y^\nu$ and use x^2 for the Minkowski square of x. The Fourier transform $\hat{f}(p) = \mathcal{F}f(p)$ of a function $f(x)$ over Euclidean space $\mathbb{R}^n$ is defined with negative sign in the exponent, the back transform has a positive sign. All factors of 2π are absorbed in the back transform. In order to match the physics conventions, the signs of the phase factors in the Fourier transform are inverted over Minkowski space $\mathbb{R}^{1+3}$ (in addition to the duality pairing being induced by η). That is,

$$\hat{f}(p) := \int d^4x \, e^{i(px)} f(x), \quad f(x) := \int \frac{d^4p}{(2\pi)^4} \, e^{-i(px)} \hat{f}(p) \qquad (1.1)$$

for a generic f living on $\mathbb{R}^{1+3}$. When it is relevant, we shall always specify whether statements pertain to $\mathbb{R}^n$ or $\mathbb{R}^{1+3}$.

2. Elements of Microlocal Analysis Needed for Renormalization

In the standard approaches to quantum field theory, perturbation theory is typically formulated by writing matrix elements of the scattering operator as products of numerical distributions—the propagators of the quantum fields involved in a certain model—with the help of Wick's theorem [11]. However, products or higher powers of distributions make no sense in general and also the products of propagators in the Wick expansion for the scattering operator are divergent. At nth order of perturbation theory, they only make sense outside the thin diagonal $\{x_1 = \cdots = x_n\} \subset (\mathbb{R}^{1+3})^n$, or after exploiting translation invariance, outside the origin $\{z = 0\} \subset (\mathbb{R}^{1+3})^{n-1}$, where $z = (x_1 - x_n, \ldots, x_{n-1} - x_n)$. In momentum space, the non-existence of these products manifests itself in the well-known ultraviolet (UV) divergences of loop integrals contributing to scattering amplitudes. Renormalization in a mathematically rigorous sense is the extension of non-existent products of distributions in configuration space across the origin $\{z = 0\}$ [4,11].

Once the existence of *some* extension across the origin has been established, one must address the question of uniqueness. On the one hand, two extensions can only differ by a distribution supported at the origin, i.e., by a linear combination of derivatives of the Dirac delta, since both extensions must be equal to the original distribution outside the origin. On the other hand, adding an arbitrary linear combination of derivatives of the Dirac delta to a particular extension gives another extension. These ambiguities are called *renormalization freedom*. They can be controlled via constraints on the short-distance scaling behavior of the extensions, i.e., the scaling behavior with respect to $z = 0$ [4] (cf. also [32]), by requiring that the extension does not scale

worse than the original distribution. This type of constraint is often referred to as *power counting*.

Example 2.1. Consider the massless scalar Feynman propagator $D := \frac{1}{x^2 - i0} \in S'(\mathbb{R}^{1+3})$. We will see in Example 2.8 that the square of D is defined on $\mathbb{R}^{1+3} \setminus 0$ but not on the full space $\mathbb{R}^{1+3}$. For now, we are only interested in constructing an extension. First, note that D is homogeneous, $D(\lambda x) = \lambda^{-2} D(x)$ for all $\lambda > 0$. Correspondingly, the square $(D|_{\mathbb{R}^{1+3} \setminus 0})^2$ scales as λ^{-4}. Power counting is the requirement that any admissible extension does not scale worse than the non-extended distribution, i.e., one requires that $\lim_{\lambda \downarrow 0} \lambda^{4+\omega} w(\lambda x) = 0$ for any admissible extension w of $(D|_{\mathbb{R}^{1+3} \setminus 0})^2$ and for all $\omega > 0$.

It is a simple task to verify that on $\mathbb{R}^{1+3} \setminus 0$, the square of D coincides with the divergence of the vector-valued distribution

$$v^\mu := \frac{1}{2} \frac{x^\mu \ln(x^2 - i0)}{(x^2 - i0)^2}. \tag{2.1}$$

Since v^μ is locally integrable with respect to x at $x = 0$, it is a well-defined distribution[2] on the full space $\mathbb{R}^{1+3}$ and thus, the divergence $\overline{D^2} := \partial_\mu v^\mu$ defines an extension of $(D|_{\mathbb{R}^{1+3} \setminus 0})^2$. It is also admissible by power counting since $\lim_{\lambda \downarrow 0} \lambda^\omega \ln(\lambda^2) = 0$ for all $\omega > 0$.

An arbitrary extension w of $(D|_{\mathbb{R}^{1+3} \setminus 0})^2$ can only differ from $\overline{D^2}$ by a linear combination of derivatives of the Dirac delta. Power counting introduces an upper bound on the number of derivatives appearing in said linear combination. In the case at hand,

$$w - \overline{D^2} = c_0 \delta(x) \tag{2.2}$$

for some constant c_0 and any admissible extension w since the Dirac delta already scales like λ^{-4}. The free parameter c_0 in Eq. (2.2) introduces a renormalization freedom to the model under consideration. It usually needs to be fixed by physical reasoning.

The method to obtain the special extension $\overline{D^2}$ is called *differential renormalization* but there are also other well-established methods (see for example [9] for an introduction or [4,7] for more abstract considerations).

Remark 2.2. The massless Feynman propagator D from Example 2.1 is homogeneous. Therefore, it is obvious how to define the scaling behavior with respect to the origin. A more general definition can for example be found in [4].

A definition of a product of two distributions, which satisfies the known rules of calculus, as well as a criterion for its existence was found by Hörmander [15, Theorem 8.2.10.] (as a special case of Lemma 2.4 below). If u and v are distributions over an open subset $X \subset \mathbb{R}^n$, their product uv can be defined as

---

[2]The reader may try to verify that the logarithm does not cause any trouble by using the tools that we present in the remaining part of the section.

the pullback of the tensor product $u \otimes v$ by the diagonal map $\Delta : X \to X \times X$, $\Delta(x) = (x, x)$ if

$$(x; p) \in \mathrm{WF}\, u \quad \text{implies} \quad (x; -p) \notin \mathrm{WF}\, v, \tag{2.3}$$

where the wavefront set $\mathrm{WF}\, u$ of a distribution u is a subset of the cotangent bundle $\dot{T}^*(X)$ over X deprived of the elements $(x; 0)$ (as indicated by the dot). $\mathrm{WF}\, u$ gives a refined characterization of the singularities of u:

Definition 2.3. (see Ch. 8 in [15]) Let $u \in \mathcal{D}'(X)$ for $X \subset \mathbb{R}^n$ open. Then the *singular support* $\mathrm{singsupp}\, u$ of u is the set of points in X that have no open neighborhood where u is smooth. The *frequency set* $\Sigma_x(u)$ of u over a point $x \in X$ is defined as an intersection

$$\Sigma_x(u) := \bigcap_{\substack{\phi \in C_c^\infty(X) \\ \phi(x) \neq 0}} \Sigma(\phi u), \tag{2.4}$$

where $\Sigma(\phi u)$ is the cone of directions in $\mathbb{R}^n \setminus 0$ having no conic neighborhood in which the Fourier transform of the compactly supported distribution ϕu is rapidly decaying. Finally, the *wavefront set* $\mathrm{WF}\, u$ of u is the closed subset of $\dot{T}^*(X)$ defined by

$$\mathrm{WF}\, u := \{\, (x; p) \in \dot{T}^*(X) \mid p \in \Sigma_x(u) \,\} \tag{2.5}$$

so that the projection of $\mathrm{WF}\, u$ onto the first component yields the singular support.

Thus, the wavefront set does not only encode the information about the singularities of a distribution but also about the high frequencies that are responsible for their appearance. It is easy to verify that the wavefront set is a closed and conic subset of $\dot{T}^*(X)$, where *conic* means that the wavefront set is invariant under scaling the second variable with positive scalars.

The proofs in Sects. 3 and 4 will be based on several standard statements about properties of the wavefront set. For convenience of the reader, we will now concisely list the statements on which we will rely later.

The Hörmander product of two distributions u and v is defined as a pullback of their tensor product, provided that the criterion (2.3) is satisfied. One can then also give a bound on the wavefront set of the product [15, Theorem 8.2.10.], namely

$$\mathrm{WF}(uv) \subset \{\, (x; p + k) \mid (x; p) \in \mathrm{WF}\, u \text{ or } p = 0, \ (x; k) \in \mathrm{WF}\, v \text{ or } k = 0 \,\}. \tag{2.6}$$

The Hörmander product of two distributions is an important special case of the pullback of distributions but we will also need to consider other pullbacks in order to examine the wavefront set of string-localized propagators.

Lemma 2.4. (Thm. 8.2.4. in [15]) *The pullback f^*u of a distribution $u \in \mathcal{D}'(Y)$ by a smooth map $f : X \to Y$, where $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ are open, can be*

defined such that it coincides with the pullback of smooth maps if $u \in C^\infty(Y)$, provided that $N_f \cap \mathrm{WF}\, u = \emptyset$, where

$$N_f := \{\, (f(x); p) \in Y \times \mathbb{R}^n \mid {}^t f'(x) p = 0 \,\}. \tag{2.7}$$

is the set of normals of the map f. Moreover, we have

$$\mathrm{WF}(f^* u) \subset f^* \mathrm{WF}\, u := \{\, (x; {}^t f'(x) p) \mid (f(x); p) \in \mathrm{WF}\, u \,\}. \tag{2.8}$$

The distributions that appear in quantum field theory are often solutions of partial differential equations. For such distributions, one can give bounds on their wavefront set:

Lemma 2.5. (Eq. (8.1.11) and Thm. 8.3.1. in [15]) *Let $u \in \mathcal{D}'(X)$ for $X \subset \mathbb{R}^n$ open and let $P = \sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha$ be a differential operator of order m on X with smooth coefficients. Then*

$$\mathrm{WF}(Pu) \subset \mathrm{WF}\, u \subset \mathrm{WF}(Pu) \cup \operatorname{char} P, \tag{2.9}$$

where the characteristic set $\operatorname{char} P$ is defined in terms of the principal symbol $P_m(x, p) := \sum_{|\alpha| = m} a_\alpha(x) p^\alpha$ of P via

$$\operatorname{char} P := \{\, (x; p) \in \dot{T}^*(X) \mid P_m(x, p) = 0 \,\}. \tag{2.10}$$

In particular, if u solves $Pu = 0$, then $\mathrm{WF}\, u \subset \operatorname{char} P$.

We will also deal with several homogeneous distributions. These are automatically tempered [15, Theorem 7.1.18.] and the wavefront set of a homogeneous distribution is closely related to the wavefront set of its Fourier transform:

Lemma 2.6. (Thm. 8.1.8. in [15]) *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be homogeneous in $\mathbb{R}^n \setminus 0$. Then*

$$(x; p) \in \mathrm{WF}\, u \Leftrightarrow (p; -x) \in \mathrm{WF}\, \hat{u} \qquad \text{if} \quad x \ne 0 \quad \text{and} \quad p \ne 0,$$
$$x \in \operatorname{supp} u \Leftrightarrow (0; -x) \in \mathrm{WF}\, \hat{u} \qquad \text{if} \quad x \ne 0,$$
$$p \in \operatorname{supp} \hat{u} \Leftrightarrow (0; p) \in \mathrm{WF}\, u \qquad \text{if} \quad p \ne 0.$$

Remark 2.7. The statements from [15] displayed in this section are formulated over Euclidean space with the sign convention of the Fourier transform described in the end of the introduction. The mentioned change of the sign convention due to physical reasons when working over Minkowski space implies that the covector components of wavefront sets over Minkowski space get an additional sign.

Example 2.8. We show that the wavefront set of the massless Feynman propagator D from Example 2.1 is given by

$$\mathrm{WF}\, D = \{\, (x; \lambda x) \mid x^2 = 0,\ x \ne 0,\ \lambda > 0 \,\} \cup \dot{T}_0^*. \tag{2.11}$$

First, we have $\dot{T}_0^* \subset \mathrm{WF}\, D$ by Lemma 2.5 since D is a fundamental solution of the wave equation and since $\mathrm{WF}\, \delta(x) = \dot{T}_0^*$. The latter wavefront set can be computed by using that $\widehat{\varphi \delta}(p) = \varphi(0)$ for $\varphi \in C_c^\infty(\mathbb{R}^{1+3})$. When $x \ne 0$, D is the pullback of the homogeneous distribution $[t - i0]^{-1} \in \mathcal{S}'(\mathbb{R})$ by

the map $f : \mathbb{R}^{1+3} \setminus 0 \to \mathbb{R}$ with $f(x) = x^2$. To verify this, note that the Fourier transform of $[t \pm i0]^{-1}$ is a multiple of the Heaviside distribution $\theta(\pm\lambda)$ and thus, by Lemma 2.6,

$$\mathrm{WF}[t \pm i0]^{-1} = \{ (0; \lambda) \mid \lambda \gtrless 0 \} \tag{2.12}$$

and $N_f \cap \mathrm{WF}[t - i0]^{-1} = \emptyset$. Hence, the pullback is defined by Lemma 2.4. The wavefront set of the pullback is thus contained in the right-hand side of Eq. (2.11) by Lemma 2.4, where the inverted sign of λ comes from the fact that we work over Minkowski space, as explained in Remark 2.7. Since the wavefront set is conic and the projection onto the first component yields the singular support, $\mathrm{WF}\, D$ cannot be smaller than the right-hand side of Eq. (2.11).

Since λ has a fixed sign, the Hörmander square of D exists when $x \neq 0$ but because the wavefront set over $x = 0$ contains any direction, the square is not defined at $x = 0$.

Examples 2.1 resp. 2.8 are prototypical for an extension problem in point-localized gauge theories. The situation becomes much more complex in string-localized field theories. There, the propagators are not only distributions in the variables x and x' but also in spacelike string directions e and e'. The string-localization can induce new singularities to the propagator and moreover, the structure of these singularities depends on the formulation of a string-localized perturbation theory, as we shall investigate in Sect. 3.3. We will then prove in Sect. 4 that in a proper setup of string-localized perturbation theory, the wavefront sets of string-localized propagators are actually contained in the wavefront sets of certain point-localized propagators. That is to say, the singularity structure is not worse in SLFT than it is in point-localized QFT despite the delocalization.

To prove the latter statement, another standard theorem from microlocal analysis about partially smeared distributions will play a central role:

Lemma 2.9. (Thm. 8.2.12. in [15]) *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open and let $K \in \mathcal{D}'(X \times Y)$ with the corresponding linear transformation $\mathcal{K}$ from $\mathcal{D}(Y)$ to $\mathcal{D}'(X)$, i.e.,*

$$[\mathcal{K}\varphi](\phi) = K(\phi \otimes \varphi). \tag{2.13}$$

Then

$$\mathrm{WF}(\mathcal{K}\varphi) \subset \{ (x; p) \mid (x, y; p, 0) \in \mathrm{WF}\ K \ \text{for some}\ y \in \mathrm{supp}\, \varphi \}. \tag{2.14}$$

3. String-Localized Potentials for Finite Spin/Helicity

There is a price to pay for the conceptual advantages of string-localized fields that we have listed in the introduction. String-localized fields do not only depend on the spacetime variable x but also on a spacelike string direction $e \in H$, where $H \subset \mathbb{R}^{1+3}$ denotes the open subset of spacelike vectors in Minkowski space. In both the massless and the massive case, the string-localized potential

$A^{\mu_1\cdots\mu_s}(x,e)$ of the field strength tensor $F_{[\mu_1\nu_1]\cdots[\mu_s\nu_s]}(x)$ for helicity respectively spin $s \in \mathbb{N}$ can be defined as an s-fold integral in string direction via [19,22,25]

$$A_{\mu_1\cdots\mu_s}(x,e) := I_e^s F_{[\mu_1\nu_1]\cdots[\mu_s\nu_s]}(x)e^{\nu_1}\cdots e^{\nu_s}, \tag{3.1}$$

where we have introduced the string-integration operator $I_e X(x) := \int_0^\infty \mathrm{d}s\, X(x+se)$. It is straightforward to verify that $A_{\mu_1\cdots\mu_s}(x,e)$ is indeed a potential for the field strength by exploiting the Bianchi identity for the latter and $e_\mu \partial^\mu I_e = -1$. By the causal commutation relations for $F_{[\mu_1\nu_1]\cdots[\mu_s\nu_s]}(x)$, two string-localized potentials $A_{\mu_1\cdots\mu_s}(x,e)$ and $A_{\kappa_1\cdots\kappa_s}(x',e')$ as in Eq. (3.1) commute if their strings are causally disjoint, i.e.,

$$[A_{\mu_1\cdots\mu_s}(x,e), A_{\kappa_1\cdots\kappa_s}(x',e')] = 0 \quad \text{if} \quad (x+se-x'-s'e')^2 < 0 \; \forall s,s' \geq 0. \tag{3.2}$$

Remark 3.1. The commutation relations (3.2) are in principle also meaningful for lightlike strings and indeed, lightlike string variables have been considered by some [14]. We will discuss in Appendix A.1, why they are no reasonable option in our context. Timelike string directions are excluded if the commutation relations (3.2) are to remain meaningful.

The integrations in Eq. (3.1) improve the ultraviolet (UV) scaling behavior of the string-localized potentials in both massive and massless case: They have the same scaling behavior as a scalar field for arbitrary $s \in \mathbb{N}$. Mund, Schroer and Yngvason [25] conjectured that this improved UV behavior also has positive effects on renormalizability. However, it is not a priori clear what that means, for string-localized potentials as in Eq. (3.1) depend not only on the spacetime variable x but also on the string variable e and so do their propagators. Thus, before one can give meaning to the notion of renormalizability, one must answer a few questions:

1. Of what nature are the products of distributions appearing in a string-localized perturbation theory?
2. What is the singularity structure of string-localized propagators?

We will give answers to these questions in the following.

3.1. Distributional Properties of String-Integration

In momentum space, string integration as in Eq. (3.1) becomes a multiplication with factor a factor $-i[(pe)-i0]^{-1}$ since

$$\widehat{I_e f}(p) = \int \mathrm{d}^4 x \int_0^\infty \mathrm{d}s\, e^{i(p[x-se])}f(x) = \hat{f}(p)\int_0^\infty \mathrm{d}s\, e^{-is(pe)} := \lim_{\varepsilon \downarrow 0} \frac{-i\hat{f}(p)}{(pe)-i\varepsilon}. \tag{3.3}$$

Multiplication with such a factor produces additional singularities when $(pe) = 0$. Already when setting up their framework for SLFT, Mund, Schroer and Yngvason conjectured that the difficulties coming from these singularities can be cured if the string-localized fields are treated as distributions in both x and e [25]: "This opens up the possibility of a perturbative, covariant, implementation of interaction, where the weaker localization (in space-like cones)

requires new techniques but promises better UV behavior." In Sect. 4, we make a first step towards proving their conjecture by showing that the regularization of divergent loop graph amplitudes in SLFT stays a pure short distance problem and that hence the UV scaling behavior remains a meaningful notion.

Let us start our investigations by characterizing the new singularities in detail for general directions $e \in \mathbb{R}^{1+3}$.

Lemma 3.2. *The expressions* $U_\pm(p, e) := [(pe) \pm i0]^{-1}$ *are tempered distributions on* $(\mathbb{R}^{1+3})^2$ *with*

$$\mathrm{WF}\, U_\pm = \{\, (p, e; \lambda e, \lambda p) \mid \lambda \lessgtr 0,\ (pe) = 0,\ (p, e) \neq (0, 0) \,\} \cup \dot{T}^*_{(0,0)}, \quad (3.4)$$

where $\dot{T}^*_{(0,0)}$ *is the cotangent space at* $(p, e) = (0, 0)$ *deprived of the zero-covector.*

Proof. First note that if $U_\pm$ are well-defined distributions, they are also tempered because they are homogeneous. When $(p, e) \neq (0, 0)$, $U_\pm$ are the pull-backs of the distributions $[t \pm i0]^{-1} \in \mathcal{S}'(\mathbb{R})$ by the map $f : (\mathbb{R}^{1+3})^2 \backslash (0, 0) \to \mathbb{R}$, $f(p, e) = (pe)$ with set of normals

$$N_f = \{\, ((pe); \lambda) \in \mathbb{R}^2 \mid \lambda e = \lambda p = 0,\ (p, e) \neq (0, 0) \,\} = \{\, (t; 0) \mid t \in \mathbb{R} \,\} \quad (3.5)$$

so that $N_f \cap \mathrm{WF}[t \pm i0]^{-1} = \emptyset$. Thus, by Lemma 2.4, Remark 2.7 and the form of $\mathrm{WF}[t \pm i0]^{-1}$ given in Eq. (2.12), we have

$$\mathrm{WF}\, U_\pm|_{(p,e) \neq (0,0)} \subset f^* \mathrm{WF}[t \pm i0]^{-1} = \{\, (p, e; \lambda e, \lambda p) \mid (pe) = 0,\ \lambda \lessgtr 0 \,\}. \quad (3.6)$$

Equation (3.6) must actually be an equality since the wavefront set is conic and the projection onto the first component must yield the singular support. Since $U_\pm$ are locally integrable at $(p, e) = (0, 0)$, we have established their existence as tempered distributions.

It remains to show that the wavefront set over $(p, e) = (0, 0)$ is the whole cotangent space (deprived of the zero-covector). To do so, we introduce the bilinear form

$$A := \frac{1}{2} \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} \quad (3.7)$$

on $(\mathbb{R}^{1+3})^2$ such that $A(p, e) = (pe)$ and $4A^2 = \mathbb{I}$. By [15, Theorem 6.2.1.],

$$(\partial_p \partial_e) [(pe) \pm i0]^{-3} = a_\pm \delta(p, e), \quad (3.8)$$

where $a_\pm$ are non-vanishing constants that are unimportant for the following arguments. Moreover, we have

$$(\partial_p \partial_e)^2 U_\pm = 4 [(pe) \pm i0]^{-3} \quad \Rightarrow \quad (\partial_p \partial_e)^3 U_\pm = 4 a_\pm \delta(p, e). \quad (3.9)$$

Consequently $\mathrm{WF}\, \delta(p, e) = \dot{T}^*_{(0,0)} \subset \mathrm{WF}\, U_\pm$ by Lemma 2.5 and the proof is completed. $\qquad \square$

The distributions $U_\pm$ in Lemma 3.2 depend on a general string direction $e \in \mathbb{R}^{1+3}$. In SLFT, however, the string directions are usually restricted to a set of spacelike (or lightlike) directions. Within our derivations, they are elements of the *open* subset $H \subset \mathbb{R}^{1+3}$ of spacelike directions, as explained in the beginning of the current section. The restriction of a distribution to an open subset always exists and it follows immediately from Definition 2.3 that the wavefront set of the restricted distribution is the restriction of the wavefront set. We therefore define:

Definition 3.3. Let $u_\pm(p, e) := U_\pm(p, e)|_{\mathbb{R}^{1+3} \times H}$ denote the restriction of the distributions $U_\pm$ over $(\mathbb{R}^{1+3})^2$ from Lemma 3.2 to the open subset $\mathbb{R}^{1+3} \times H$ of spacelike string directions with

$$\mathrm{WF}\, u_\pm = \{\, (p, e; x, \xi) \mid (p, e; x, \xi) \in \mathrm{WF}\, U_\pm,\ e \in H \,\} \qquad (3.10)$$

by definition of the wavefront set.

Lemma 3.2 has the following important consequence for the restricted distributions $u_\pm$.

Corollary 3.4. *Hörmander products $(u_+)^k$ and $(u_-)^k$ of the restrictions to spacelike string variables do exist for arbitrary $k \in \mathbb{N}$, but the Hörmander product $u_+ \cdot u_-$ with opposite imaginary shift does not exist. Moreover,*

$$\mathrm{WF}\,\big[(u_\pm)^k\big] = \mathrm{WF}\, u_\pm. \qquad (3.11)$$

Proof. If $e \in H$, then $(p, e) \neq (0, 0)$ and

$$\mathrm{WF}\, u_\pm = \{\, (p, e; \lambda e, \lambda p) \mid e \in H,\ \lambda \lessgtr 0,\ (pe) = 0 \,\}. \qquad (3.12)$$

The Hörmander product of two distributions exists if Eq. (2.3) is satisfied. Since the sign of λ in Eq. (3.12) is fixed by the sign of the imaginary shift, $(u_\pm)^2$ are defined but $u_+ \cdot u_-$ is not. It also follows immediately from the shape of $\mathrm{WF}\, u_\pm$ and Eq. (2.6) that

$$\mathrm{WF}\,\big[(u_\pm)^2\big] \subset \mathrm{WF}\, u_\pm \qquad (3.13)$$

and both sides must be equal since the wavefront set is conic and the projection onto the first component must yield the singular support. By induction, we get the statement for arbitrary powers. $\qquad \square$

Remark 3.5. In the literature, the string variables are usually considered as elements of the *closed* subset $H_{-1} \subset \mathbb{R}^{1+3}$ of spacelike vectors with Minkowski square $e^2 = -1$ (as for example in [12,22]). The restriction of a distribution to a closed subset is much more involved than the restriction to an open subset. It does not always exist and even if it does, it may affect the form of the wavefront set [15]. We will briefly sketch in Appendix A.2 why the restriction to H_{-1} is indeed unproblematic. For our purposes, however, the simpler case of the restriction to the open subset H is sufficient.

Lemma 3.2 and Corollary 3.4 are the starting point for the full analysis of the singularities of string-localized propagators that we will subsequently perform.

3.2. String-Localized Propagators for All Spins and Helicities

For two arbitrary point-localized fields $X(x)$ and $X'(x')$ of mass $m \geq 0$, we introduce the notation

$$\langle\!\langle X(x)X'(x')\rangle\!\rangle := \int \mathrm{d}\mu_m(p)\, e^{-i(p(x-x'))}\, {}_mM^{X,X'}(p) \qquad (3.14)$$

for the two-point function of X and X', with the measure $\mathrm{d}\mu_m(p) = \frac{d^4p}{(2\pi)^3}\delta(p^2 - m^2)\theta(p^0)$ on the mass shell and where ${}_mM^{X,X'}(p)$ is a polynomial in p. Furthermore, we write

$$\langle\!\langle T_0 X(x)X'(x')\rangle\!\rangle := \int \frac{d^4p}{(2\pi)^4}\frac{e^{-i(p(x-x'))}}{p^2 - m^2 + i0}\, {}_mM^{X,X'}(p) \qquad (3.15)$$

for the corresponding *kinematic* propagator. Using translation invariance $x - x' \to x$, we sometimes also use the notation

$$\langle\!\langle XX'\rangle\!\rangle(x) \quad \text{resp.} \quad \langle\!\langle T_0 XX'\rangle\!\rangle(x). \qquad (3.16)$$

The kinematic propagator from Eq. (3.15) is in general only a specific choice for a propagator since the transition from Eq. (3.14) to Eq. (3.15) might be non-unique: Dependent on the scaling behavior of ${}_mM^{X,X'}(p)$, there can arise ambiguities in the definition of time-ordering at $x = x'$ [11, 29]. Any other propagator can only differ from Eq. (3.15) by a linear combination

$$\sum_{|\alpha|\leq n} b_\alpha \partial^\alpha \delta(x - x'), \qquad (3.17)$$

where α is a multi-index, b_α are constants and $n \in \mathbb{N}_0$ is restricted by the scaling behavior in a similar manner to the restrictions from power counting displayed in Example 2.1. The ambiguity (3.17) can be understood more easily by a momentum space consideration. Adding a term

$$(p^2 - m^2)\tilde{M}(p) \qquad (3.18)$$

to ${}_mM^{X,X'}(p)$, where $\tilde{M}(p)$ is another polynomial, does not contribute to the two-point function (3.14) but yields a contribution of the form (3.17) to the propagator (3.15).

Remark 3.6. We will frequently refer to the expression ${}_mM^{X,X'}(p)$ as *kernel* of a propagator or two-point function and hope that this usage does not cause confusion with distribution kernels that will be used implicitly in Sect. 4.

String-integrating $X(x)$ in Eq. (3.14) gives an additional factor $-iu_-(p, e)$ in momentum space, while string integrating $X'(x')$ yields a factor $iu_+(p, e') = -iu_-(p, -e')$. A natural choice of a propagator involving a string-integrated field is thus given by inserting the appropriate powers of $-iu_-(p, e)$ and $iu_+(p, e')$ into Eq. (3.15). Again, the propagator might not be unique but two propagators can at most differ by a linear combination of string-integrated Dirac deltas. We will investigate these ambiguities in Sect.4.2. For now, we

prove the well-definedness of the relevant class of momentum space representations of string-localized *kinematic* propagators.[3]

Lemma 3.7. (massless case)   *Let $m = 0$ and let $M_\times(p)$ be a polynomial in p such that $\omega \in \mathbb{N}_0$ is the smallest power of p appearing in $M_\times$, where the subscript $\times$ is a placeholder for possible Lorentz indices. Let further $k, k' \in \mathbb{N}_0$ and $\omega - k - k' - 2 > -4$. Then the expression*

$$\frac{[u_-(p,e)]^k [u_+(p,e')]^{k'} M_\times(p)}{p^2 + i0} \tag{3.19}$$

is a well-defined (and possibly tensor-valued) distribution on $\mathbb{R}^{1+3} \times H^2$.

Proof. By Corollary 3.4, the powers $[u_-(p,e)]^k$ and $[u_+(p,e')]^{k'}$ exist on $\mathbb{R}^{1+3} \times H$ and their wavefront set is given by Eq. (3.12). We promote them to distributions on $\mathbb{R}^{1+3} \times H^2$ by tensoring with the constant distribution in the missing string variable, so that

$$\mathrm{WF}\left([u_-(p,e)]^k \otimes 1_{e'}\right) = \{\, (p,e,e'; \lambda e, \lambda p, 0) \mid \lambda > 0,\ (pe) = 0 \,\}, \tag{3.20a}$$

$$\mathrm{WF}\left([u_+(p,e')]^{k'} \otimes 1_e\right) = \{\, (p,e,e'; \kappa e', 0, \kappa p) \mid \kappa < 0,\ (pe') = 0 \,\}. \tag{3.20b}$$

By Eq. (2.11) and Lemma 2.6,

$$\mathrm{WF}\left([p^2 + i0]^{-1} \otimes 1_{e,e'}\right) = \{\, (p,e,e'; \lambda p, 0, 0) \mid p^2 = 0,\ p \neq 0,\ \lambda < 0 \,\}$$
$$\cup \{\, (0, e, e'; x, 0, 0) \mid x \in \mathbb{R}^{1+3} \setminus 0 \,\}. \tag{3.21}$$

Hence, the covector components of the three wavefront sets (3.20a), (3.20b) and (3.21) cannot add up to zero when $p \neq 0$ and thus the Hörmander product exists on $(\mathbb{R}^{1+3} \setminus 0) \times H^2$. The requirement that $\omega - k - k' - 2 > -4$ ensures that (3.19) is locally integrable at $p = 0$. Therefore it is a well-defined distribution on $\mathbb{R}^{1+3} \times H^2$. $\square$

Lemma 3.7 has an analogue for $m > 0$, where a weaker constraint on the smallest power of p appearing in the polynomial $M_\times(p)$ is sufficient to guarantee local integrability at $p = 0$ because the denominator $[p^2 - m^2 + i0]^{-1}$ has a better behaved wavefront set for $m > 0$ than for $m = 0$.

Lemma 3.8. (massive case)   *Let $m > 0$ and let $M_\times(p)$ be a polynomial in p such that $\omega \in \mathbb{N}_0$ is the smallest power of p appearing in $M_\times$, where the subscript $\times$ is a placeholder for possible Lorentz indices. Let further $k, k' \in \mathbb{N}_0$ and $\omega - k - k' > -4$. Then the expression*

$$\frac{[u_-(p,e)]^k [u_+(p,e')]^{k'} M_\times(p)}{p^2 - m^2 + i0} \tag{3.22}$$

is a well-defined (and possibly tensor-valued) distribution on $\mathbb{R}^{1+3} \times H^2$.

---

[3] There are similar statements for the two-point functions but in this paper, we are interested in scattering theory only. Therefore, we only consider propagators. The interested reader may carry through the existence proof for the two-point functions as an exercise, using our findings as a guide.

Proof. The numerator of (3.22) is the same as in Lemma 3.7. Since $\frac{1}{p^2-m^2+i0}$ is smooth at $p = 0$ for $m > 0$, it is enough to require $\omega - k - k' > -4$ for local integrability at $p = 0$ instead of $\omega - k - k' - 2 > -4$, which was necessary in the massless case.

Similar to the procedure in the proof of Lemma 3.2, the distribution $[p^2 - m^2 + i0]^{-1}$ can be seen as the pullback of $[t + i0]^{-1} \in \mathcal{S}'(\mathbb{R})$ by the map $f : \mathbb{R}^{1+3} \to \mathbb{R}$ with $f(p) = p^2 - m^2$ with set of normals

$$N_f = \{ (p^2 - m^2; \xi) \in \mathbb{R}^2 \mid \xi p = 0 \} \tag{3.23}$$

because $N_f \cap \mathrm{WF}[t + i0]^{-1} = \emptyset$. Then

$$\mathrm{WF}[p^2 - m^2 + i0]^{-1} \subset f^* \, \mathrm{WF}[t + i0]^{-1} = \{ (p; \lambda p) \mid p^2 = m^2, \ \lambda < 0 \}, \tag{3.24}$$

where the inverted sign of λ again comes from the fact that we work over Minkowski space, as explained in Remark 2.7. Clearly, the covector components of $\mathrm{WF}\left([p^2 - m^2 + i0]^{-1} \otimes 1_{e,e'}\right)$ and the wavefront sets (3.20a), (3.20b) cannot add up to zero when $p \neq 0$, giving the well-definedness of (3.22) as a distribution on $\mathbb{R}^{1+3} \times H^2$. $\qquad\square$

Remark 3.9. The conditions $\omega - k - k' - 2 > -4$ in the massless and $\omega - k - k' > -4$ in the massive case ensure local integrability with respect to $p = 0$. It does not automatically follow that the respective distributions are ill-defined if these integrability conditions are not satisfied. However, when investigating the position space representation of the doubly string-integrated massless Feynman propagator $I_e I_{-e'} [x^2 - i0]^{-1}$, the singularity is explicitly observed as an infrared effect [13].

Remark 3.10. In configuration space, one might be tempted to circumvent the integrability conditions in Lemmas 3.7 and 3.8 by shifting the string-integration operation to the x-part of the test function. Since the latter is a Schwartz function, application of any finite number of string-integrations to it remains finite. However, the result is no Schwartz function anymore. In direction of the string, it converges to a constant that is in general non-zero. Therefore, the integrability conditions are necessary if one does not wish to leave the regime of distribution theory.

The string-localized potentials defined in Eq. (3.1) are homogeneous of degree 0 in the string variable because each string-integration I_e is accompanied by a factor e^μ. Therefore it proves useful to replace the distributions $u_\pm(p, e)$ by the vector-valued distributions

$$q_\pm^\mu(p, e) := \pm i u_\pm(p, e) \, e^\mu. \tag{3.25}$$

The wavefront set of a vector-valued distribution is defined as the union of the wavefront sets of the components. But since each component of $q_\pm^\mu$ is nothing but $u_\pm$ times a smooth function, we have

$$\mathrm{WF} \, q_\pm^\mu \subset \mathrm{WF} \, u_\pm. \tag{3.26}$$

As long as $e \neq 0$, in particular if e is spacelike, both wavefront sets in Eq. (3.26) are equal. Thus, if

$$_m M^{F,F}_{[\mu_1 \nu_1] \cdots [\mu_s \nu_s][\kappa_1 \lambda_1] \cdots [\kappa_s \lambda_s]}(p) \tag{3.27}$$

denotes the kernel of the kinematic spin/helicity-s field strength propagator, then the kinematic propagator for the string-localized potential is given by replacing $_m M^{F,F}$ by

$$_m M^{A,A}_{\mu_1 \cdots \mu_s \kappa_1 \cdots \kappa_s}(p, e, e') := {}_m M^{F,F}_{[\mu_1 \nu_1] \cdots [\mu_s \nu_s][\kappa_1 \lambda_1] \cdots [\kappa_s \lambda_s]} \prod_{i=1}^{s} q_-^{\nu_i}(p, e) q_+^{\lambda_i}(p, e') \tag{3.28}$$

in Eq. (3.15), provided that the requirements of Lemmas 3.7 or 3.8, respectively, are satisfied (cf. also [19]).

For both $m > 0$ and $m = 0$, the kernel $_m M^{F,F}_{\mu\nu,\kappa\lambda}$ of field strength $F_{\mu\nu}$ of spin/helicity $s = 1$ reads

$$_m M^{F,F}_{\mu\nu,\kappa\lambda}(p) = -\eta_{\mu\kappa} p_\nu p_\lambda + \eta_{\mu\lambda} p_\nu p_\kappa + \eta_{\nu\kappa} p_\mu p_\lambda - \eta_{\nu\lambda} p_\mu p_\kappa \tag{3.29}$$

and is homogeneous of degree $\omega = 2$. Since for $s = 1$, the string-localized potential is $A_\mu(x, e) = I_e F_{\mu\nu}(x) e^\nu$, string-integration of both F's in the propagator gives a total factor $q_-^\nu(p, e) \cdot q_+^\lambda(p, e')$ so that $\omega - k - k' = 0$. Then the requirements for the respective Lemmas 3.7 and 3.8 are met and

$$_m M^{A,A}_{\mu\kappa}(p, e, e') = -\eta_{\mu\kappa} + \frac{e_\kappa p_\mu}{(pe) - i0} + \frac{e'_\mu p_\kappa}{(pe') + i0} - \frac{(ee') p_\mu p_\kappa}{[(pe) - i0][(pe') + i0]}$$
$$=: -E_{\mu\kappa}(p, e, e'). \tag{3.30}$$

It turns out that the quantity $E_{\mu\kappa}(p, e, e')$ from Eq. (3.30), which is the kernel for the kinematic propagator of $s = 1$ string-localized potentials is enough to describe the kernel of the kinematic propagator for string-localized potentials of arbitrary spin/helicity [22]. By symmetry of the field strengths and therefore also of the potentials in Eq. (3.1), one does not lose any information if one contracts all indices of $A_{\mu_1 \cdots \mu_s}(x, e)$ with the same (arbitrary) four-vector f^μ and defines

$$A_f^{(s)}(x, e) := f^{\mu_1} \cdots f^{\mu_s} A_{\mu_1 \cdots \mu_s}(x, e). \tag{3.31}$$

The authors of [22] were able to prove—without considering questions of well-definedness—that the kernel of the two-point function of $A^{(s)}(x, f)$ and $A^{(s)}(x', f')$ for all $m \geq 0$ is given by

$$_m M^{A_f^{(s)}, A_{f'}^{(s)}}(p, e, e') = (-1)^s \sum_{2n \leq s} \beta_n^s (E_{ff})^n (E_{f'f'})^n (E_{ff'})^{s-2n}, \tag{3.32}$$

with coefficients β_n^s that are of no interest here. We have used the notation

$$E_{ff} := f^\mu E_{\mu\nu}(p, e, -e) f^\nu, \quad E_{f'f'} := f'^\mu E_{\mu\nu}(p, -e', e') f'^\nu, \quad \text{and}$$
$$E_{ff'} := f^\mu E_{\mu\nu}(p, e, e') f'^\nu. \tag{3.33}$$

The signs in the arguments of Eq. (3.33) ensure that each (pe) is accompanied by a shift $-i0$ and each (pe') is accompanied by a shift $+i0$. Moreover,

it is clear that the kernels from Eq. (3.32) arise from an s-fold e-integration and an s-fold $-e'$-integration times a polynomial in p which is homogeneous of degree $\omega = 2s$. Therefore, Lemmas 3.7 and 3.8 apply to the kinematic propagators with kernels as in Eq. (3.32) for all s and have the following Theorem as a corollary.

Theorem 3.11. *The kinematic propagators of the string-localized potentials for arbitrary mass $m \geq 0$ and spin/helicity $s \in \mathbb{N}$ defined by*

$$\langle\!\langle T_0 A_f^{(s)} A_{f'}^{(s)} \rangle\!\rangle(x) := \int \frac{d^4p}{(2\pi)^4} \frac{e^{-i(px)}}{p^2 - m^2 + i0} \, {}_m M^{A_f^{(s)}, A_{f'}^{(s)}}(p, e, e') \qquad (3.34)$$

are well-defined distributions on $\mathbb{R}^{1+3} \times H^2$.

3.3. Perturbation Theory with String-Localized Fields

At the heart of perturbation theory in quantum field theory is the construction of the scattering operator, or S-matrix $\mathbb{S}$ as a formal power series (Dyson series) of time-ordered products of an interaction Lagrangian $\mathcal{L}_{\text{int}}$ describing a certain model [2].

In the usual point-localized theories, $\mathbb{S} = \mathbb{S}[g]$ is considered as a functional of a multiplet of Schwartz functions g, which are interpreted as large distance cutoffs of coupling constants, and tend to constants in the adiabatic limit. For a rigorous construction, one usually axiomatizes certain properties of the S-matrix, such as its unitarity, Lorentz and translation invariance and causality [2,11]. Additionally, it can be subject to (sometimes model-dependent) internal and discrete symmetries [38, Sec. 3.3]. In gauge theories, its form is further constrained by the requirement of perturbative gauge invariance [1,10,29]. There is no concept of gauge in string-localized field theories and the gauge invariance principle is replaced by the requirement for string independence of the S-matrix (see for example [12,14]).

The time-ordered products of operator-valued distributions (i.e., of $\mathcal{L}_{\text{int}}$) that appear in the Dyson series for $\mathbb{S}[g]$ are usually reduced to products of numerical distributions by taking expectation values and employing Wick's theorem. These products are ill-defined at a diagonal set and renormalization is the extension of these products of distributions to the whole space [4,11], as we have illustrated in Example 2.1. In the following, we sketch a possible transition of these notions, which are well-established in point-localized theories, to SLFT.

As a first step for this transition, one must declare the nature of the string-localization. Is string-localization a feature of the potentials, the Lagrangian or the S-matrix? That is to say: Does each field come with its own string variable, do the fields in the interaction Lagrangian depend on the same string variable or do *all* appearing fields depend on the same string variable:

$$\mathcal{L}_{\text{int}} = \mathcal{L}_{\text{int}}(x, e_1, \dots, e_k) \text{(each SL field has its own string variable)}, \quad (3.35a)$$

$$\mathcal{L}_{\text{int}} = \mathcal{L}_{\text{int}}(x, e) \quad \text{(all SL fields in } \mathcal{L}_{\text{int}} \text{ depend on the same } e), \qquad (3.35b)$$

$$\mathbb{S} = \mathbb{S}[g; e] \quad \text{(there is only a single string variable)}. \qquad (3.35c)$$

In a generic model, the three alternatives (3.35a), (3.35b) and (3.35c) result in completely different analytic properties of the corresponding perturbation theory. Note, however, that the alternatives (3.35a) and (3.35b) are equivalent if $\mathcal{L}_{\mathrm{int}}$ is at most linear in the string-localized potentials, as is the case in QED.

Alternative (3.35c) is desirable if one wants to keep the delocalization as small as possible. However, it is in general not realizable. If there is only a single string variable, the kernels (3.32) of the propagators of the involved string-localized potentials need to be pulled back to the e-diagonal and consequently will contain ill-defined products $u_+ \cdot u_-$. Now, the Hörmander criterion Eq. (2.3) is only sufficient but not necessary, meaning that it is not fully excluded that one can make sense of $u_+ \cdot u_-$ although the Hörmander criterion is not met.[4] In the case at hand, however, the divergence can be observed explicitly [13]. This rules out alternative (3.35c) for spacelike strings.

Remark 3.12. There have been approaches that employ alternative (3.35c) for *lightlike* string variables and massive string-localized potentials [14]. But such an approach comes with other drawbacks, which we will describe in Appendix A.1.

The interaction Lagrangian $\mathcal{L}_{\mathrm{int}}$ depends only on a single x-variable. One can therefore argue that alternative (3.35b) is a natural choice to set up perturbation theory in SLFT. In this case, loop graph contributions would consist of products of propagators in x and e and one must expect renormalization to become very complicated. Recent observations in the string-localized equivalent of massless Yang–Mills theory suggest that alternative (3.35b) does not reproduce the standard model of particle physics [12].[5] This observation rules out alternative (3.35b) for phenomenological reasons.

We are thus left with the alternative (3.35a), i.e., $\mathcal{L}_{\mathrm{int}} = \mathcal{L}_{\mathrm{int}}(x, e_1, \ldots, e_k)$ $=: \mathcal{L}_{\mathrm{int}}(x, \boldsymbol{e})$, which is also employed in the cited work [12]. The analyses therein additionally require a symmetry under exchange of all string variables that appear in a fixed order of perturbation theory. This symmetry can be achieved by smearing all string variables with the same averaging function $c \in \mathcal{D}(H)$ with $\int d^4 e \, c(e) = 1$.[6] With this at hand, we are finally able to write down a candidate for the string-localized S-matrix,

$$\mathbb{S}[g; c] := 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \prod_{j=1}^{n} \prod_{l=1}^{k} \int d^4 x_j \int d^4 e_{j,l} \, g(x_j) c(e_{j,l}) \, S_n(x_1, \boldsymbol{e}_1; \ldots; x_n, \boldsymbol{e}_n),$$

$$(3.36)$$

---

[4] The result may then have unwanted properties such as that the Leibniz rule is not applicable.

[5] The cited work considers our alternative (3.35a) from the beginning without mentioning other alternatives but one can verify without too much effort that the Lie algebra structure of gluon self-interactions is not compatible with alternative (3.35b) by adjusting Section 2.3 in [12] according to $\mathcal{L}_{\mathrm{int}}(x, e_1, e_2) \to \mathcal{L}_{\mathrm{int}}(x, e)$.

[6] The test function c needs to have integral equal to 1 if the string-localized potential is to remain a potential for the field strength after smearing out the e-variable.

where the first-order coupling $S_1(x, \boldsymbol{e}) = \, :\!\mathcal{L}_{\mathrm{int}}(x, \boldsymbol{e})\!:$ is the Wick-ordered inter-action Lagrangian. The property that c integrates to unity ensures consistency if $\mathcal{L}_{\mathrm{int}}$ is a sum of terms where different powers of string-localized potentials appear. The higher-order couplings S_n need to be constructed recursively as time-ordered products of the first-order coupling.

However, the construction of time-ordered products in a string-localized field theory is a non-trivial task, for one needs to make sense of how to order several semi-infinite strings in time. An axiomatic framework comparable to the one that is available in point-localized QFT has not yet been formulated.

One approach towards a construction of time-ordered products in string-localized QFT is called *string-chopping* [6]. It proceeds by time-ordering segments of string integrals wherever possible and taking account of the singularity structure where time-ordering is ambiguous. String chopping has been implemented for certain models [6,12] but a proof of its general validity has not yet been given and it is still unclear how a generalization could work.

The formulation of a fully self-contained and comprehensive axiomatic framework for the construction of the time-ordered products in Eq. (3.36) is beyond the scope of this paper. Instead, our aim is to show that there exist finite solutions (in the sense of operator-valued distributions) for the string-localized S_n if they exist in the point-localized equivalent. To prove the assertion, we write down the formal Wick-expansion

$$
\begin{aligned}
S_n &= T[\,:\!\mathcal{L}_{\mathrm{int}}(x_1, \boldsymbol{e}_1)\!:\, \ldots \,:\!\mathcal{L}_{\mathrm{int}}(x_n, \boldsymbol{e}_n)\!:\,] \\
&= \, :\!\mathcal{L}_{\mathrm{int}}(x_1, \boldsymbol{e}_1) \ldots \mathcal{L}_{\mathrm{int}}(x_n, \boldsymbol{e}_n)\!: \\
&\quad + \sum_{\phi,\chi} :\!\frac{\partial \mathcal{L}_{\mathrm{int}}(x_1, \boldsymbol{e}_1)}{\partial \phi} \frac{\partial \mathcal{L}_{\mathrm{int}}(x_2, \boldsymbol{e}_2)}{\partial \chi} \ldots \mathcal{L}_{\mathrm{int}}(x_n, \boldsymbol{e}_n)\!: \langle\!\langle T\phi\chi \rangle\!\rangle + \ldots \\
&\quad + \sum_{\substack{\phi_1,\phi_2, \\ \chi_1,\chi_2}} :\!\frac{\partial^2 \mathcal{L}_{\mathrm{int}}(x_1, \boldsymbol{e}_1)}{\partial \phi_1 \phi_2} \frac{\partial^2 \mathcal{L}_{\mathrm{int}}(x_2, \boldsymbol{e}_2)}{\partial \chi_1 \chi_2} \ldots \mathcal{L}_{\mathrm{int}}(x_n, \boldsymbol{e}_n)\!: \langle\!\langle T\phi_1\chi_1 \rangle\!\rangle \langle\!\langle T\phi_2\chi_2 \rangle\!\rangle \\
&\quad + \ldots
\end{aligned}
\tag{3.37}
$$

as a sum containing a priori ill-defined products of propagators, some of which may be string-localized. An important property of the Dyson series Eq. (3.36) is that each string-localized potential comes with its own string variable. This property has the consequence that the products of propagators in Eq. (3.37) are products *only* in the x-variables but tensor products in the string variables. It is therefore enough to regularize Eq. (3.37) *after* integrating out the string dependence of the propagators with the test function c. In Sect. 4, we prove that, after smearing out the string variables, the wavefront set of a relevant class of string-localized propagators is contained in the wavefront set of the ordinary Feynman propagator. As a consequence, the products of propagators appearing in Eq. (3.37) are well-defined whenever they are well-defined in the point-localized case and the regularization of divergent amplitudes by extension of the products of propagators across the remaining points stays a pure short distance problem in SLFT. This statement is due to the fact that the

singularities of string-localized propagators, which are the building blocks of the Wick expansion, are better behaved than one might naively expect.

Our considerations do not yield a full classification of the Epstein-Glaser-like freedom of renormalization of the S_n. However, because string-localized fields have an improved scaling behavior compared to their point-localized counterparts, these ambiguities are not expected to exceed the ones in point-localized theories.

4. Products of String-Localized Propagators

The heuristic considerations in Sect. 3.3 led us to the conclusion that each string-localized potential has its own string variable and that all string variables are smeared with the same test function. Therefore, we can make a transition from Eq. (3.1) to the smeared potentials

$$A_{c,\mu_1\cdots\mu_s}(x) := \int d^4e\, A_{\mu_1\cdots\mu_s}(x,e)\, c(e) \tag{4.1}$$

for $c \in \mathcal{D}(H)$ *before* plugging them into the S-matrix. Similarly, after using translation invariance of the propagator, we can define a map

$$K : \mathcal{D}(H) \to \mathcal{S}'(\mathbb{R}^{1+3}), \ c \mapsto \langle\!\langle T_0 A_{c,\mu_1\cdots\mu_s} A_{c,\kappa_1\cdots\kappa_s}\rangle\!\rangle(x). \tag{4.2}$$

The right-hand side of Eq. (4.2) is a distribution only in x but its singularities might depend on the test function c. By Lemma 2.9, we have

$$\mathrm{WF}[K(c)] \subset \{\, (x;p) \mid (x,e,e';p,0,0) \in \mathrm{WF}\langle\!\langle T_0 A_{\mu_1\cdots\mu_s}(e)A_{\kappa_1\cdots\kappa_s}(e')\rangle\!\rangle,$$
$$e, e' \in \mathrm{supp}\, c \,\}. \tag{4.3}$$

The estimate (4.3) is the key to proving that string-integration does not introduce new singularities to the propagators of string-localized fields.

4.1. Products of Kinematic Propagators

We investigate the effect of the transition from distributions over $\mathbb{R}^{1+3} \times H^2$ to distributions over $\mathbb{R}^{1+3}$ on the kinematic string-localized propagators described in Sect. 3.2, starting with the following lemma.

Lemma 4.1. *For $c \in \mathcal{D}(H)$, we define*

$$q_{c,\pm}^{\mu_1\cdots\mu_s}(p) := \int d^4e\, c(e) \frac{(\pm i)^s e^{\mu_1}\cdots e^{\mu_s}}{[(pe)\pm i0]^s}. \tag{4.4}$$

Then $q_{c,\pm}^{\mu_1\cdots\mu_s}(p) \in \mathcal{S}'(\mathbb{R}^{1+3})$ with $\mathrm{WF}\, q_{c,\pm}^{\mu_1\cdots\mu_s}(p) = \{\, (0;\lambda e) \mid \lambda \lessgtr 0, e \in \mathrm{supp}\, c \,\}$.

Proof. The expressions $q_{c,\pm}^{\mu_1\cdots\mu_s}(p)$ are the results of smearing distributions of the form appearing in Corollary 3.4 times a smooth (tensor-valued) function in the string variable. Therefore, they are well-defined distributions. By homogeneity, they are also tempered.

Since $e \in H$ is spacelike and hence non-zero, the wavefront set of $q_{c,\pm}^{\mu_1 \cdots \mu_s}(p)$ must be contained in $\{ (p; x) \mid (p, e; x, 0) \in \mathrm{WF}\, u_\pm \}$ by Lemma 2.9, with $u_\pm$ as in Corollary 3.4 and $\mathrm{WF}\, u_\pm$ as in Eq. (3.12). This yields

$$(p; x) \in \mathrm{WF}\, q_{c,\pm}^{\mu_1 \cdots \mu_s}(p) \quad \Rightarrow \quad p = 0 \text{ and } x = \lambda e \text{ for some } \lambda \lessgtr 0 \text{ and } e \in \mathrm{supp}\, c. \tag{4.5}$$

To show that any such element $(0; \lambda e)$ is in the wavefront set, note that the Fourier transform of $q_{c,\pm}^{\mu_1 \cdots \mu_s}(p)$ is

$$\int d^4 p\, e^{i(px)} q_{c,\pm}^{\mu_1 \cdots \mu_s}(p) \sim \int d^4 e\, c(e)\, e^{\mu_1} \cdots e^{\mu_s} I_{\pm e}^s \delta(x) \tag{4.6}$$

with support $\{\, x = \lambda e \mid \lambda \lessgtr 0,\ e \in \mathrm{supp}\, c \,\}$. By homogeneity and Lemma 2.6, $(0; x)$ is an element of $\mathrm{WF}\, q_{c,\pm}^{\mu_1 \cdots \mu_s}(p)$ if and only if x is in the support of the Fourier transform. This proves the claim. $\qquad \square$

With Lemma 4.1 at hand, it is straightforward to adjust the proofs for the existence of the string-integrated kinematic propagators given in Sect. 3.2 to the expressions which are smeared in the string variables. When $p \neq 0$, the expressions $q_{c,\pm}^{\mu_1 \cdots \mu_s}(p)$ are smooth and therefore, they can at most contribute to the wavefront set over $p = 0$. We hence arrive at the following statement.

Lemma 4.2. *The kernels*

$$\mathcal{F}\left[\langle\!\langle T_0 A_{c,f}^{(s)} A_{c,f'}^{(s)} \rangle\!\rangle(x) \right](p) = \int d^4 e \int d^4 e'\, c(e) c(e') \frac{m M^{A_f^{(s)}, A_{f'}^{(s)}}(p, e, e')}{p^2 - m^2 + i0} \tag{4.7}$$

of the smeared kinematic string-localized propagators for all masses $m \geq 0$ and all spins respectively helicities $s \in \mathbb{N}$ are tempered distributions with

$$\mathrm{WF}\left(\mathcal{F}\left[\langle\!\langle T_0 A_{c,f}^{(s)} A_{c,f'}^{(s)} \rangle\!\rangle(x) \right](p) \right) \subset \mathrm{WF}\, \frac{1}{p^2 - m^2 + i0} \cup \dot{T}_0^*. \tag{4.8}$$

In the massless case, the Fourier transform (4.7) is homogeneous. Therefore, the wavefront set of the massless kinematic propagator in configuration space can be determined easily from Eq. (4.8) by use of Lemma 2.6. We obtain our first main result.

Theorem 4.3. (massless case) *At $m = 0$, the wavefront set of the smeared string-localized kinematic propagator*

$$\langle\!\langle T_0 A_{c,f}^{(s)} A_{c,f'}^{(s)} \rangle\!\rangle(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-i(px)} \int d^4 e \int d^4 e'\, c(e) c(e') \frac{m M^{A_f^{(s)}, A_{f'}^{(s)}}(p, e, e')}{p^2 + i0} \tag{4.9}$$

is contained in the wavefront set of the massless point-localized Feynman propagator from Eq. (2.11). In particular, products of massless string-localized kinematic propagators and their product with the propagators of point-localized fields are well-defined on $\mathbb{R}^{1+3} \setminus 0$.

In the massive case, homogeneity is lost and a transition from momentum to configuration space needs more effort. Before proving a similar statement to Theorem 4.3 for $m > 0$, we prove an auxiliary lemma.

Lemma 4.4. *Let $u, v \in \mathcal{S}'(\mathbb{R}^{1+3})$. Suppose further that $\hat{u}$ is polynomially bounded, that $\mathrm{WF}\,\hat{u} \subset \dot{T}_0^*$, that $\hat{v}$ is smooth on a neighborhood of $p = 0$ and that the Hörmander product $\hat{u}\hat{v}$ is an element of $\mathcal{S}'(\mathbb{R}^{1+3})$. Then $\mathrm{WF}\left[\mathcal{F}^{-1}(\hat{u}\hat{v})\right] \subset \mathrm{WF}\, v$.*

Proof. We have to investigate the decay properties of the Fourier transforms of $\phi\mathcal{F}^{-1}(\hat{u}\hat{v})$ for $\phi \in C_c^\infty(\mathbb{R}^{1+3})$. Since ϕ is compactly supported and smooth, we know that its Fourier transform is a Schwartz function, $\hat{\phi} \in \mathcal{S}(\mathbb{R}^{1+3})$. Moreover, $\mathcal{F}^{-1}(\hat{u}\hat{v})$ is a tempered distribution since by assumption also $\hat{u}\hat{v}$ is tempered. Then

$$\left[\mathcal{F}\left(\phi\mathcal{F}^{-1}(\hat{u}\hat{v})\right)\right](p) = \hat{\phi} * \hat{u}\hat{v}(p) = \hat{u}\hat{v}(\hat{\phi}(p - \cdot)) \tag{4.10}$$

is smooth and polynomially bounded [27, Thm. IX.4].

To investigate the decay of $\hat{u}\hat{v}(\hat{\phi}(p - \cdot))$, we introduce a second cutoff function $\chi \in C_c^\infty(\mathbb{R}^{1+3})$ with $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $B_r(0)$ and $\operatorname{supp}\chi \subset B_R(0)$, where $B_\varrho(p)$ is the closed ball of *Euclidean* radius ϱ and center $p \in \mathbb{R}^{1+3}$, and where $0 < r < R$ such that $B_R(0) \cap \operatorname{singsupp}\hat{v} = \emptyset$.

Then $\hat{u}$ is smooth on $\operatorname{supp}(1 - \chi)$, $\hat{v}$ is smooth on $\operatorname{supp}\chi$ and $\hat{u}\hat{v} = \chi\hat{u}\hat{v} + (1 - \chi)\hat{u}\hat{v}$. The first term is unproblematic, for there are constants N, C and C' such that

$$|\chi\hat{u}\hat{v}(\hat{\phi}(p - \cdot))| \leq C \sum_{|\alpha + \beta| \leq N} \sup_{k \in B_R(0)} \left|k^\alpha \partial_k^\beta \hat{\phi}(p - k)\right|$$

$$\leq C'(1 + |p|)^N \sum_{|\alpha + \beta| \leq N} \sup_{k \in B_R(p)} \left|k^\alpha \partial_k^\beta \hat{\phi}(k)\right|, \tag{4.11}$$

where $|p|$ is the Euclidean norm of p. The right-hand side of Eq. (4.11) is rapidly decaying since $\hat{\phi} \in \mathcal{S}(\mathbb{R}^{1+3})$ and since the supremum is taken over a compact set around p.

To estimate the second term $(1 - \chi)\hat{u}\hat{v}$, note that the smooth function $(1 - \chi)\hat{u}$ is polynomially bounded and thus

$$[(1 - \chi)\hat{u}\hat{v}]\left(\hat{\phi}(p - \cdot)\right) = \hat{v}\left((1 - \chi)\hat{u}(\cdot)\,\hat{\phi}(p - \cdot)\right) \tag{4.12}$$

falls off rapidly if $\hat{v}\left(\hat{\phi}(p - \cdot)\right) = \widehat{\phi v}$ falls off rapidly. Thus also the full expression falls off rapidly if $\widehat{\phi v}$ does, which proves the lemma. $\square$

The desired statement about the wavefront sets of string-localized propagators for $m > 0$ then follows as a corollary of Lemma 4.4:

Theorem 4.5. (massive case) *At $m > 0$, the wavefront set of the smeared string-localized kinematic propagator*

$$\langle\!\langle T_0 A_{c,f}^{(s)} A_{c,f'}^{(s)} \rangle\!\rangle(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-i(px)} \int d^4 e \int d^4 e'\, c(e)c(e') \frac{m M^{A_f^{(s)}, A_{f'}^{(s)}}(p, e, e')}{p^2 - m^2 + i0} \tag{4.13}$$

is contained in the wavefront set of the massive point-localized Feynman propagator. In particular, products of massive string-localized kinematic propagators,

their product with massless string-localized kinematic propagators and with the propagators of point-localized fields are well-defined on $\mathbb{R}^{1+3} \setminus 0$.

Proof. We define the distributions

$$\hat{u}(p) = \int d^4 e \int d^4 e'\, c(e)c(e')\, {}_m M^{A_f^{(s)}, A_{f'}^{(s)}}(p, e, e') \tag{4.14}$$

and $\hat{v}(p) = [p^2 - m^2 + i0]^{-1}$ with $m > 0$. $\hat{u}(p)$ is homogeneous of degree 0 in p and arises from contraction of the distributions $q_{c,\pm}^{\mu_1 \cdots \mu_s}$ from Lemma 4.1 with a polynomial in p. Local integrability of $\hat{u}(p)$ at $p = 0$ ensures its existence as a tempered distribution and by Lemma 4.1, WF $\hat{u} \subset \dot{T}_0^*$. $\hat{v}$ is smooth at $p = 0$ because $m > 0$ and hence $\hat{u}$ and $\hat{v}$ satisfy the assumptions of Lemma 4.4. Therefore, Theorem 4.5 is a special case of Lemma 4.4 and

$$\text{WF} \langle\!\langle T_0 A_{c,f}^{(s)} A_{c,f'}^{(s)} \rangle\!\rangle \subset \text{WF}\, v = \text{WF} \left[\int \frac{d^4 p}{(2\pi)^4} \frac{e^{-i(px)}}{p^2 - m^2 + i0} \right], \tag{4.15}$$

where v is the massive scalar Feynman propagator, whose wavefront set is the same as for the massless scalar Feynman propagator given by Eq. (2.11) [3]. □

Remark 4.6. Note that Lemma 4.4 is only helpful at $m > 0$ since wavefront set of the massless kernel $[p^2 + i0]^{-1}$ contains $\dot{T}_0^*$, so that the Hörmander product $\hat{u}\hat{v}$ of the respective $\hat{u}$ and $\hat{v}$ does not exist at $p = 0$. For the same reason, there is no straightforward generalization of Lemma 4.4 to the massless case.

Theorems 4.3 and 4.5 show that the problem of regularizing divergent products of string-localized propagators is not posed worse than in point-localized QFT, provided that kinematic propagators are employed. However, the transition from the two-point functions to the propagators might not be unique as we have argued in Sect. 3.2. In the following section, we will show that Theorems 4.3 and 4.5 are generalizable to a large class of propagators.

4.2. Products of Non-kinematic Propagators

Dependent on the scaling behavior at $x = 0$ (see e.g. [4,29] for the corresponding power counting arguments), there arise ambiguities in defining the propagator for the point-localized spin/helicity-s field strength,

$$\langle\!\langle T F_\times^{(s)} F_\times^{(s)} \rangle\!\rangle - \langle\!\langle T_0 F_\times^{(s)} F_\times^{(s)} \rangle\!\rangle = \sum_{|a| \leq 2(s-1)} C_\times^a \partial_\times^{|a|} \delta(x), \tag{4.16}$$

with a generic time-ordering recipe T, the kinematic time-ordering T_0, some constants $C_\times^a$ and where $\times$ is a placeholder for possible Lorentz indices.

In order to not lose the connection between field strength and string-localized potential given by Eq. (3.1), it is a natural requirement that the freedom of choosing a time-ordering recipe for the string-localized potentials arises from the freedom (4.16) by appropriate string-integration of the right-hand side. Then

$$\langle\!\langle T A_{c,\times}^{(s)} A_{c,\times}^{(s)} \rangle\!\rangle - \langle\!\langle T_0 A_{c,\times}^{(s)} A_{c,\times}^{(s)} \rangle\!\rangle = \sum_{|a| \leq 2(s-1)} C_\times^a I_{c,+\times}^s I_{c,-\times}^s \partial_\times^{|a|} \delta(x), \tag{4.17}$$

where we have introduced the smeared s-fold string-integration

$$I^{\mathrm{s}}_{\mathrm{c},\pm\mu_1\cdots\mu_{\mathrm{s}}}f(x) := \int d^4e\,c(e)\,e_{\mu_1}\cdots e_{\mu_{\mathrm{s}}}\,I^{\mathrm{s}}_{\pm e}f(x). \qquad (4.18)$$

The right-hand side of Eq. (4.17) is a tempered distribution because it is a linear combination of Fourier transforms of

$$(-ip)^{|a|}_{\times}q^{\mu_1\cdots\mu_{\mathrm{s}}}_{\mathrm{c},+}(p)q^{\kappa_1\cdots\kappa_{\mathrm{s}}}_{\mathrm{c},-}(p), \qquad (4.19)$$

provided that the latter is locally integrable at $p = 0$, i.e., if $|a| > 2s - 4$.

Thus, the scaling behavior of the field strengths gives an upper bound on $|a|$, while the requirement for local integrability in momentum space gives a lower bound. In the massless case, the two-point functions of field strength and string-localized potential are homogeneous. The requirement that the propagators are homogeneous of the same degree then restricts the freedom to $|a| = 2s - 2$ at $m = 0$. In summary, we demand

$$\langle\!\langle TA^{(s)}_{\mathrm{c},\times}A^{(s)}_{\mathrm{c},\times}\rangle\!\rangle(x) - \langle\!\langle T_0A^{(s)}_{\mathrm{c},\times}A^{(s)}_{\mathrm{c},\times}\rangle\!\rangle(x) = \sum_{|a|=2s-2}C^a_{\times}\partial^{|a|}_{\times}I^{\mathrm{s}}_{\mathrm{c},\times}I^{\mathrm{s}}_{-c,\times}\delta(x) \quad \text{at} \quad m = 0,$$

$$\qquad (4.20\mathrm{a})$$

$$\langle\!\langle TA^{(s)}_{\mathrm{c},\times}A^{(s)}_{\mathrm{c},\times}\rangle\!\rangle(x) - \langle\!\langle T_0A^{(s)}_{\mathrm{c},\times}A^{(s)}_{\mathrm{c},\times}\rangle\!\rangle(x) = \sum_{|a|=2s-3}^{2s-2}C^a_{\times}\partial^{|a|}_{\times}I^{\mathrm{s}}_{\mathrm{c},\times}I^{\mathrm{s}}_{-c,\times}\delta(x) \quad \text{at} \quad m > 0$$

$$\qquad (4.20\mathrm{b})$$

for a general time-ordering recipe T ordering string-localized potentials.

Remark 4.7. We want to stress that the requirements $|a| = 2s - 2$ for $m = 0$ and $2s - 3 \leq |a| \leq 2s - 2$ for $m > 0$ are stronger than the power counting constraints. The latter would only imply $|a| \leq 2s - 2$ but the integrability condition—an *infrared effect*—forbids all $|a| \leq 2s - 4$. At $m = 0$, homogeneity gives an even stronger constraint and excludes all $|a| < 2s - 2$.

For time-ordering recipes T that are subject to Eq.s (4.20a) and (4.20b), the renormalization problem is the same as for T_0:

Theorem 4.8. *Let T denote a time-ordering recipe that is subject to Eq. (4.20a) if $m = 0$ and Eq. (4.20b) if $m > 0$. Then*

$$\mathrm{WF}\langle\!\langle TA^{(s)}_{\mathrm{c},\times}A^{(s)}_{\mathrm{c},\times}\rangle\!\rangle \subset \mathrm{WF}\langle\!\langle T_0A^{(s)}_{\mathrm{c},\times}A^{(s)}_{\mathrm{c},\times}\rangle\!\rangle \qquad (4.21)$$

and consequently, products of $\langle\!\langle TA^{(s)}_{\mathrm{c},\times}A^{(s)}_{\mathrm{c},\times}\rangle\!\rangle$ as well as their products with propagators of point-localized fields exist on $\mathbb{R}^{1+3} \setminus 0$.

Proof. By the constraints on $|a|$ coming from Eqs. (4.20a) and (4.20b), the difference between $\langle\!\langle TA^{(s)}_{\mathrm{c},\times}A^{(s)}_{\mathrm{c},\times}\rangle\!\rangle$ and $\langle\!\langle T_0A^{(s)}_{\mathrm{c},\times}A^{(s)}_{\mathrm{c},\times}\rangle\!\rangle$ is a well-defined tempered distribution since it is the Fourier transform of a sum of distributions

$$(-ip)^{|a|}_{\times}q^{\mu_1\cdots\mu_{\mathrm{s}}}_{\mathrm{c},+}(p)q^{\kappa_1\cdots\kappa_{\mathrm{s}}}_{\mathrm{c},-}(p), \qquad (4.22)$$

which are locally integrable at $p = 0$ and homogeneous in p. Note that $q_{c,\pm}$ are smooth when $p \neq 0$ by Lemma 4.1 so that there are no issues with well-definedness of the expression (4.22). Moreover, the same lemma gives

$$\mathrm{WF}\left[(-ip)^{|a|}_{\times} q^{\mu_1 \cdots \mu_s}_{c,+}(p) q^{\kappa_1 \cdots \kappa_s}_{c,-}(p)\right] \subset \dot{T}^*_0 \tag{4.23}$$

so that homogeneity implies

$$\mathrm{WF}\left[\partial^{|a|}_{\times} I^s_{c,\mu_1 \cdots \mu_s} I^s_{-c,\kappa_1 \cdots \kappa_s} \delta(x)\right] = \dot{T}^*_0 \tag{4.24}$$

by Lemma 2.6, where the equality comes from the fact that the Fourier transform (4.22) is supported everywhere. Therefore, since the wavefront set of the sum of two distributions is contained in the union of their wavefront sets, adding a linear combination of derivatives of smeared string-deltas subject to Eq.s (4.20a) or (4.20b), respectively, does not affect the wavefront set over $\mathbb{R}^{1+3} \setminus 0$. Because the kinematic propagators of the field strengths are derivatives of a fundamental solution of the wave equation, $\dot{T}^*_0$ is already contained in their wavefront set and hence also in the wavefront set of the kinematic propagators of the string-localized potentials since the wavefront set of a string-integrated smeared Dirac delta is given by Eq. (4.24).   □

To summarize, Theorem 4.8 gives a generalization of Theorems 4.3 and 4.5 to all propagators that

1. arise from one of the field strength propagators displayed in Eq. (4.16) by appropriate string-integration, and
2. are subject to the constraints of power counting, integrability in momentum space and, at $m = 0$, homogeneity of the same degree as the two-point function.

Note that only the lower bounds on $|a|$ are needed in the proof of Theorem 4.8, but not the constraints coming from power counting. The latter are only an additional requirement in order to reduce the (finite) renormalization freedom.

Remark 4.9. In all our considerations in Sects. 4.1 and 4.2, we have only considered pure string-localized propagators $\langle\!\langle T A^{(s)}_{c,\times} A^{(s)}_{c,\times}\rangle\!\rangle$, whereas also mixed propagators like $\langle\!\langle T F^{(s)}_{\times} A^{(s)}_{c,\times}\rangle\!\rangle$ are non-vanishing in SLFT. However, it should be obvious that such propagators are subject to similar statements as Theorems 4.3, 4.5 and 4.8.

5. Discussion

We have established that the regularization of a priori ill-defined products of propagators remains a pure short distance problem in SLFT, provided that each string-localized potential comes with its own string variable and provided that the pertinent propagators differ from the kinematic propagators only by string-localized Dirac deltas. Let us discuss some consequences of these results.

5.1. A First Step Towards an Epstein–Glaser Construction in SLFT

Since each potential comes with its own string variable, their propagators can (and should) be smeared in the string variables before inserting them into the Dyson series for the scattering operator. As a result, the distributions appearing in the Dyson series only depend on the spacetime variable (and the smearing function for the string variables, of course).

We proved in Sects. 4.1 and 4.2 that the wavefront set of smeared string-localized propagators is contained in the wavefront set of the ordinary point-localized Feynman propagator. Therefore, the problem of regularizing divergent loop graph amplitudes remains an extension problem across an x-diagonal. In particular, string integration does not introduce new singularities to the propagators after smearing out the string variables. In other words, the divergences in SLFT are of the same nature as in point-localized approaches: *they are pure UV divergences.*

Due to the improved ultraviolet scaling behavior of the string-localized potentials, coming from the string integrations as in Eq. (3.1), the freedom of choosing a regularization of divergent products of propagators is reduced compared to point-localized theories. It therefore seems a promising task to investigate whether one can formulate renormalizable string-localized models involving higher spin fields where the point-localized equivalent is non-renormalizable. A prime example may be the graviton self-coupling. However, to carry out such a task, a comprehensive axiomatic framework for the construction of $\mathbb{S}[g;c]$ from Eq. (3.36) needs to be set up. Only then can one hope to give a full classification of the ambiguities of a time-ordering prescription in SLFT.

The ultraviolet behavior—or correspondingly: power counting—is not the only way to constrain the renormalization freedom. Typical additional requirements in point-localized theories are that the renormalized time-ordered products must not be in conflict with gauge invariance [29] or that a refined characterization of Lorentz covariance is preserved during renormalization [26]. Such notions can also serve as guiding principles for a full axiomatic construction of the S-matrix in SLFT, the proper substitute for gauge invariance being the principle of string independence (see for example [12,37] for applications).

5.2. Renormalization in Practice

No new singularities are introduced to string-localized propagators if one sets up perturbation theory as described in this paper, meaning that each string-localized potential in the Dyson series comes with its own string variable. Thus, there is no need to develop new renormalization techniques that are specifically adjusted to SLFT. One can rely on well-known methods such as analytic regularization or differential renormalization (see for example [9] for an introduction) to construct special extensions of products of propagators in SLFT. This is a remarkable statement, for analytic structures in SLFT are quite complicated and this complexity is commonly considered to be one of the main drawbacks of SLFT.

5.3. Connections to Axial Gauges

The analytic structure of propagators in axial or light-cone gauges is similar to the one of string-localized propagators (see for example [17] for an introduction to axial gauges). Axial gauge propagators, however, are usually not treated as distributions in the variable n, which represents the preferred direction and is the analogue of the string variable in SLFT. Hence, the singularities that arise when the Minkowski product of n with the momentum p vanishes are of a different nature than the ones discussed in this paper. The singularities at $(pn) = 0$ were an important reason for the decreasing interest in axial gauges over the past decades.

Adjusting the framework of axial gauges by treating the respective propagators as distributions in n and letting each appearing axial gauge field depend on its own n, our results can be transferred with benefit to axial gauge theories (with n spacelike). Thus, the singularities at $(pn) = 0$ in axial gauges do not cause additional problems for renormalization if they are treated as described in this paper.

Axial gauges suffer from analytic complexity but also offer advantages, in particular if each axial gauge field comes with its own n: They prove useful in the so-called spinor-helicity formalism that drastically reduces the computational effort to determine gluon scattering matrix elements [31, Chapters 25.4.3 and 27]. Due to the close formal connection between axial gauge and string-localized potentials, it is worthwhile to investigate whether the spinor-helicity formalism can be adjusted to the string-localized setup of perturbation theory presented in this paper.

5.4. Connections to the Method of String-Chopping

It is a non-trivial question how the time-ordered products of string-localized fields—or interaction Lagrangians—can be defined. In Sect. 3.3, we have mentioned the method of string-chopping, first described by Cardoso et al. [6] for linear fields and later adjusted to certain models containing self-interactions of string-localized potentials [12]. In a nutshell, string-chopping says that the strings appearing at each order in perturbation theory can be chopped into a finite number of compact segments plus an infinite tail for each string, so that all pieces can be meaningfully ordered in time. Moreover, the result of the time-ordering arising from string-chopping is unique outside an exceptional set, which consists of the configurations where two of the appearing strings intersect. A generalization of that method to arbitrary models has not yet been proven but seems a natural conjecture.

Morally, the formal Wick expansion Eq. (3.37) is a realization of the string chopping method because the kinematic propagator of the string-localized potentials from Theorem 3.11, obtained by inserting the string-localized kernel (3.32) into the kinematic propagator of point-localized fields Eq. (3.15), automatically chops the strings and orders them in time. This can be seen by recalling that it is defined as the string-integral over propagators which are already time-ordered with respect to their arguments. Therefore, string-integration promotes the step functions $\theta(\pm x^0)$, which are responsible for time-ordering

in the point-localized propagators, to

$$\theta(\pm(x^0 + te^0 - t'e'^0)) \tag{5.1}$$

so that an automatic chopping is implicitly achieved. Similar to the point-localized case described by Epstein and Glaser [11], causality as in Eq. (3.2) implies that the propagator is uniquely defined outside the set

$$\{\, x + te - t'e' = 0 \text{ for some } t, t' \geq 0 \,\}, \tag{5.2}$$

which is in accordance with the ambiguities from Eqs. (4.20a) and (4.20b) as well as with the ambiguities observed in the abstract formulation of string chopping [6,12].

Acknowledgements

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

A. Other Choices of String Variables

Throughout this paper, we have always worked with spacelike string variables e living in the open subset $H = \{\, e^2 < 0 \,\} \subset \mathbb{R}^{1+3}$. In the literature, one also finds other choices: lightlike string variables, normalized spacelike string variables with Minkowski square $e^2 = -1$ or purely spacelike string variables $e = (0, \vec{e})$, all of which correspond to restrictions of the string variables to *closed* subsets (or more precisely, closed submanifolds). Such restrictions are much more subtle than the restriction to H used in this paper. We briefly examine the described options.

A.1. Lighlike Strings

Lightlike string directions have been employed in [14] when dealing with massive string-localized potentials, where they promise a computational advantage. The authors of [14] were able to set equal all string variables appearing in the Dyson series for the scattering operator describing the weak interaction by exploiting that the problematic denominator $([(pe) + i0][(pe) - i0])^{-1}$ in $E_{\mu\kappa}(p, e, e')|_{e=e'}$ from Eq. (3.30) drops out when $e^2 = 0$. This simplification of $E_{\mu\kappa}$ yields an essential reduction of the complexity of tree-graph calculations. Similarly, one can check that also the problematic terms in the kernel for $s = 2$ given by Eq. (3.32) drop out, resulting in an even bigger computational simplification than for $s = 1$.[7]

However, the authors of [14] restricted their considerations to tree graph contributions, where no products (or convolution products in momentum space) of several $E_{\mu\kappa}(p, e, e')|_{e=e'}$ appear. It is very likely that this changes when treating loop graph contributions and therefore, the divergent denominators will pop up again in loop amplitudes, resulting in complex renormalization schemes and spoiling the computational advantage that was achieved at tree level.

One can also think of SLFT with lightlike strings where not all string variables are set equal. However, an analysis similar to the one presented in this paper cannot be performed in that case. This is due to the fact that the restriction to the closed set of lightlike string directions causes trouble. Without loss of generality, we can investigate the restriction to lightlike string variables with zero-component equal to 1, which is given by the pullback of the respective inclusion map [15, Corollary 8.2.7.], provided that this pullback exists. Thus, consider the map

$$\iota : \mathbb{R}^{1+3} \times (0, 2\pi) \times (0, \pi) \to (\mathbb{R}^{1+3})^2, \ (p, \varphi, \vartheta) \mapsto (p, e),$$

$$\text{where } e = \begin{pmatrix} 1 \\ \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}, \tag{A.1}$$

so that the desired restriction is the pullback $\iota^* U_\pm$ with $U_\pm$ as in Lemma 3.2.

Remark A.1. The submanifold of elements (p, e), where e is lightlike and has 0-component equal to one is $\mathbb{R}^{1+3} \times \mathbb{S}^2$. To avoid confusion with coordinate-related singularities, one needs several charts. The map ι corresponds to only a single chart but is enough to demonstrate the issues that come with lightlike strings.

---

[7]It is a conjecture of the author that the problematic denominators drop out for any helicity but whether that happens or not is of no interest for our current considerations.

Having a look at Lemma 3.2 and using

$$
{}^t\iota'\begin{pmatrix}\xi\\\eta\end{pmatrix}=\begin{pmatrix}\xi\\\sin\vartheta(\eta_1\sin\varphi-\eta_2\cos\varphi)\\\eta_3\sin\vartheta-\cos\vartheta(\eta_1\cos\varphi+\eta_2\sin\varphi)\end{pmatrix}\tag{A.2}
$$

for $(\xi,\eta)\in(\mathbb{R}^{1+3})^2$, one can easily verify that $\iota^*U_\pm$ is well-defined but that the wavefront set of the pullback contains elements $(p,\varphi,\vartheta;\lambda e,0,0)$ when p becomes proportional to e, the latter being defined as in Eq. (A.1). Note that the singular-support-criterion $(pe)=0$ is met when p is proportional to e only if e is lightlike (or $p=0$).

Hence, there is no immediate analogue of Lemmas 3.7 and 3.8 for the case of lightlike strings and in particular, analyses as performed in Sect. 4, which led to a simple renormalization description, are not feasible for lightlike strings because lightlike strings produce additional singularities also when $0\neq p=\lambda e$. This problem is worse in the massless case than in the massive case, for p^2+i0 is singular when $p=\lambda e$, but p^2-m^2+i0 with $m>0$ is not.

In conclusion, spacelike strings seem preferable over lightlike strings from analytic and heuristic viewpoints. Nevertheless, lightlike strings cannot be fully excluded at the present time.

A.2. Closed Subsets of Spacelike Strings

In Remark 3.5, we claimed that the restriction to the closed submanifold H_{-1} of normalized spacelike string directions with Minkowski square -1 is harmless. In principle, this restriction can cause similar issues as the restriction to the lightlike string directions, but a brief analysis shows that it is indeed much better behaved than the latter. Similar to the case of lightlike strings, we consider an inclusion map

$$
\tilde{\iota}:\mathbb{R}^{1+3}\times\mathbb{R}\times(0,2\pi)\times(0,\pi)\to(\mathbb{R}^{1+3})^2,
$$

$$
(p,\tau,\varphi,\vartheta)\mapsto(p,e),\ \text{where}\ e=\begin{pmatrix}\sinh\tau\\\cosh\tau\sin\vartheta\cos\varphi\\\cosh\tau\sin\vartheta\sin\varphi\\\cosh\tau\cos\vartheta\end{pmatrix},\tag{A.3}
$$

which is again only a single chart but a generalization to cover the full submanifold is straightforward. For $(\xi,\eta)\in(\mathbb{R}^{1+3})^2$, we have

$$
{}^t\tilde{\iota}'\begin{pmatrix}\xi\\\eta\end{pmatrix}=\begin{pmatrix}\xi\\\eta^0\cosh\tau-\sinh\tau\left[\eta_1\sin\vartheta\cos\varphi+\eta_2\sin\vartheta\sin\varphi+\eta_3\cos\vartheta\right]\\\cosh\tau\sin\vartheta(\eta_1\sin\varphi-\eta_2\cos\varphi)\\\cosh\tau\left[\eta_3\sin\vartheta-\cos\vartheta(\eta_1\cos\varphi+\eta_2\sin\varphi)\right]\end{pmatrix}\tag{A.4}
$$

and thus, the pullback $\tilde{\iota}^*U_\pm$ is well-defined by Lemmas 2.4 and 3.2. In contrast to the case of lightlike string variables, the wavefront set of the pullback does not contain elements $(p,\tau,\varphi,\vartheta;\lambda e,0,0,0)$, provided that $p\neq0$. This can be seen by inserting $\eta=\lambda p$, $\lambda\neq0$, into Eq. (A.4) and noting that the pullback

is only singular when

$$(pe) = p^0 \sinh \tau - \cosh \tau \left[p_1 \sin \vartheta \cos \varphi + p_2 \sin \vartheta \sin \varphi + p_3 \cos \vartheta \right] = 0. \quad \text{(A.5)}$$

Consequently, the results in Sects. 3 and 4 remain valid also if one restricts to H_{-1}. We nevertheless chose to consider the restriction to the open set H in the main part of the paper because it is much simpler and also exhibits the practical advantage that one can easily derive with respect to the string variables.

Remark A.2. A qualitative and simpler argument that the restriction to H_{-1} is unproblematic is the homogeneity in the string-variables of all string-localized propagators of degree $\omega = 0$: When one interprets H as $H_{-1} \times \mathbb{R}_{\geq 0}$, the "radial" part is constant and can simply be integrated out with the radial part of the test function.

A.3. Purely Spacelike Strings

Another case appearing in the literature [21] is the case of purely spacelike string variables $e = (0, \vec{e})$, for example with $|\vec{e}| = 1$. It is motivated by the fact that the inner product $-(ee')$ becomes positive definite, which is not the case in H or H_{-1}. This case can be investigated by adjusting the inclusion map (A.1) from the lightlike case by setting the zero-component of e to 0 instead of 1. Then the only— but very important—difference in the wavefront set analysis is the criterion for the singular support, which becomes

$$\vec{p} \cdot \vec{e} = 0 \quad \text{instead of} \quad \vec{p} \cdot \vec{e} = p^0 \quad \text{in the lightlike case.} \quad \text{(A.6)}$$

The wavefront set of the restriction to purely spacelike strings can then only contain elements $(p, \varphi, \vartheta; \lambda e, 0, 0)$ if $\vec{p} = 0$ but not when $p = \kappa e$ for some $\kappa \neq 0$, in contrast to the lightlike case. Nevertheless, the appearance of these critical elements in the wavefront set can happen for arbitrary p^0 and hence, our results from Sects. 3 and 4 cannot be directly transferred to a restriction to purely spacelike strings.

However, in the mentioned application [21], a time-ordering of the string-localized expression is not required because there, the string-localized part is perturbed with a point-localized Lagrangian.

References

[1] Aste, A., Scharf, G.: Non-abelian gauge theories as a consequence of perturbative quantum gauge invariance. Int. J. Mod. Phys. A **14**, 3421–3434 (1999)

[2] Bogoliubov, N.N., Shirkov, D.V.: Introduction to the Theory of Quantized Fields. Interscience (1959)

[3] Brouder, C., Dang, N.V., Hélein, F.: A smooth introduction to the wavefront set. J. Phys. A Math. Theor. **47**, 443001 (2014)

[4] Brunetti, R., Fredenhagen, K.: Microlocal analysis and interacting quantum field theories: renormalization on physical backgrounds. Commun. Math. Phys. **208**, 623–661 (2000)

[5] Buchholz, D., Fredenhagen, K.: Locality and the structure of particle states. Commun. Math. Phys. **84**, 1–54 (1982)

[6] Cardoso, L.T., Mund, J., Várilly, J.C.: String chopping and time-ordered products of linear string-localized quantum fields. Math. Phys. Anal. Geom. **21**, 3 (2018)

[7] Dang, N.V.: The extension of distributions on manifolds, a microlocal approach. Ann. Henri Poincaré **17**, 819–859 (2016)

[8] Dirac, P.A.M.: Gauge-invariant formulation of quantum electrodynamics. Can. J. Phys. **33**, 650–660 (1955)

[9] Dütsch, M.: From Classical Field Theory to Perturbative Quantum Field Theory. Birkhäuser, Basel (2018)

[10] Dütsch, M., Scharf, G.: Perturbative gauge invariance: the electroweak theory. Ann. Phys. (Leipzig) **8**, 359–387 (1999)

[11] Epstein, H., Glaser, V.J.: The role of locality in perturbation theory. Ann. Inst. Henri Poincaré A **19**, 211–295 (1973)

[12] Gaß, C., Gracia-Bondía, J.M., Mund, J.: Revisiting the Okubo–Marshak argument. Symmetry **13**(9), 1645 (2021)

[13] Gaß, C., Rehren, K.-H., Tippner, F.: On the spacetime structure of infrared divergencies in QED. arXiv:2109.10148

[14] Gracia-Bondía, J.M., Mund, J., Várilly, J.C.: The chirality theorem. Ann. Henri Poincaré **19**, 843–874 (2018)

[15] Hörmander, L.: The Analysis of Linear Partial Differential Operators I, 2nd edn. Springer, Berlin (1990)

[16] Jordan, P.: Zur Quantenelektrodynamik. III. Eichinvariante Quantelung und Diracsche Magnetpole. Zeitschrift fur Physik **97**, 535–537 (1935)

[17] Leibbrandt, G.: Introduction to noncovariant gauges. Rev. Mod. Phys. **59**, 1067–1119 (1987)

[18] Mandelstam, S.: Quantum electrodynamics without potentials. Ann. Phys. (N.Y.) **19**, 1–24 (1962)

[19] Mund, J., de Oliveira, E.T.: String-localized free vector and tensor potentials for massive particles with any spin: I. Bosons. Commun. Math. Phys. **355**, 1243–1282 (2017)

[20] Mund, J., Rehren, K.-H., Schroer, B.: Infraparticle quantum fields and the formation of photon clouds. arXiv:2109.10342

[21] Mund, J., Rehren, K.-H., Schroer, B.: Gauss' Law and string-localized quantum field theory. JHEP, **01**:2020, 001

[22] Mund, J., Rehren, K.-H., Schroer, B.: Helicity decoupling in the massless limit of massive tensor fields. Nucl. Phys. B **924**, 699–727 (2017)

[23] Mund, J., Rehren, K.-H., Schroer, B.: Relations between positivity, localization and degrees of freedom: the Weinberg–Witten theorem and the van Dam–Veltman–Zakharov discontinuity. Phys. Lett. B **773**, 625–631 (2017)

[24] Mund, J., Schroer, B., Yngvason, J.: String-localized quantum fields from Wigner representations. Phys. Lett. B **596**, 156–162 (2004)

[25] Mund, J., Schroer, B., Yngvason, J.: String-localized quantum fields and modular localization. Commun. Math. Phys. **268**, 621–672 (2006)

[26] Nikolov, N.M., Stora, R., Todorov, I.: Renormalization of massless Feynman amplitudes in configuration space. Rev. Math. Phys. **26**, 1430002 (2014)

[27] Reed, M., Simon, B.: Methods of Modern Mathematical Physics II. Academic Press, San Diego (1975)

[28] Rehren, K.-H.: Pauli–Lubański limit and stress-energy tensor for infinite-spin fields. JHEP **11**, 130 (2017)

[29] Scharf, G.: Gauge Field Theories: Spin One and Spin Two. Dover, New York (2016)

[30] Schroer, B.: The role of positivity and causality in interactions involving higher spin. Nucl. Phys. B **941**, 91–144 (2019)

[31] Schwartz, M.D.: Quantum Field Theory and the Standard Model. Cambridge University Press, Cambridge (2014)

[32] Steinmann, O.: Perturbation Expansions in Axiomatic Field Theory. Lecture Notes in Physics, vol. 11. Springer, Berlin (1971)

[33] Steinmann, O.: A Jost–Schroer theorem for string fields. Commun. Math. Phys. **87**, 259–264 (1982)

[34] Steinmann, O.: Perturbative QED in terms of gauge invariant fields. Ann. Phys. (N.Y.) **157**, 232–254 (1984)

[35] van Dam, T., Veltman, M.: Massive and massless Yang–Mills and gravitational fields. Nucl. Phys. B **22**, 397–411 (1970)

[36] Velo, G., Zwanziger, D.: Noncausality and other defects of interaction Lagrangians for particles with spin one and higher. Phys. Rev. **188**, 2218 (1969)

[37] Várilly, J.C., Gracia-Bondía, J.M.: Stora's fine notion of divergent amplitudes. Nuclear Phys. B **912**, 28–37 (2016)

[38] Weinberg, S.: The Quantum Theory of Fields I. Cambridge University Press, Cambridge (1995)

[39] Weinberg, S., Witten, E.: Limits on massless particles. Phys. Lett. B **96**, 59–62 (1980)

[40] Zakharov, V.I.: Linearized graviton theory and the graviton mass. JETP Lett. **12**, 312–313 (1970)

Christian Gaß
Institut für Theoretische Physik
Georg-August-Universität Göttingen
37077 Göttingen
Germany
e-mail: cgass@uni-goettingen.de