Ann. Henri Poincaré 17 (2016), 3237–3254
© 2016 Springer International Publishing
1424-0637/16/113237-18
published online May 7, 2016
DOI 10.1007/s00023-016-0490-9

Annales Henri Poincaré

CrossMark

The Jacobian Conjecture, a Reduction of the Degree to the Quadratic Case

Axel de Goursac, Andrea Sportiello and Adrian Tanasa

Abstract. The Jacobian Conjecture states that any locally invertible polynomial system in $\mathbb{C}^n$ is globally invertible with polynomial inverse. Bass et al. (Bull Am Math Soc 7(2):287–330, 1982) proved a reduction theorem stating that the conjecture is true for any degree of the polynomial system if it is true in degree three. This degree reduction is obtained with the price of increasing the dimension n. We prove here a theorem concerning partial elimination of variables, which implies a reduction of the generic case to the quadratic one. The price to pay is the introduction of a supplementary parameter $0 \leq n' \leq n$, parameter which represents the dimension of a linear subspace where some particular conditions on the system must hold. We first give a purely algebraic proof of this reduction result and we then expose a distinct proof, in a Quantum Field Theoretical formulation, using the *intermediate field method*.

1. Introduction

The *Jacobian conjecture* has been formulated in [11], as a strikingly simple and natural conjecture concerning the global invertibility of polynomial systems. Later on, it also appeared connected to questions in non-commutative algebra, in particular the conjecture has been shown to be stably equivalent to the Dixmier Conjecture (see [4]), which concerns endomorphisms of the Weyl algebra. Despite several efforts, and various promising partial results, it remains unsolved. An introduction to the problem, the context, and the state of the art up to 1982, can be found in the paper [3], which provides both a clear review, and among the most relevant advances on the problem.

The function $F : \mathbb{C}^n \to \mathbb{C}^n$ is said to be a *polynomial system* if all the coordinate functions F_j's are polynomials. Let us call $\mathcal{P}_n$ the set of such functions that are locally invertible at the origin. For a function F, define

$$J_F(z) = \left(\frac{\mathrm{d}}{\mathrm{d}z_i} F_j(z) \right)_{1 \leq i,j \leq n},$$

the corresponding Jacobian matrix. Then, $\det J_F(z)$ is itself a polynomial, and it is nowhere zero iff it is a non-zero constant. Local invertibility in $y = F(z)$ is related to the invertibility of $J_F(z)$ as a matrix, and thus to the vanishing of its determinant. Depending on the space of functions under analysis, local invertibility may or may not be sufficient to imply global invertibility. The question here is what is the case, for the space $\mathcal{P}_n$ of (locally invertible in 0) polynomial systems over an algebraically closed field of characteristic zero (such as $\mathbb{C}$). For this reason, we define the two subspaces of $\mathcal{P}_n$

Definition 1.1.

$$\mathcal{J}_n^{\mathrm{lin}} := \{F \in \mathcal{P}_n \mid \det J_F(z) = c \in \mathbb{C}^\times\},$$
$$\mathcal{J}_n := \{F \in \mathcal{P}_n \mid F \text{ is invertible}\}.$$

For the questions at hand, it will often be sufficient to analyze the subset of $\mathcal{J}^{\mathrm{lin}}$ such that $\det J_F(z) = 1$.

More precisely w.r.t. what anticipated above, one can see, e.g., from [3, Theorem 2.1, p. 294], that $F \in \mathcal{J}_n^{\mathrm{lin}}$ is a necessary condition for the invertibility of F, and, if F is invertible, the (set theoretic) inverse is automatically polynomial and unique. The question is whether the condition on the Jacobian is also sufficient, i.e.,

Conjecture 1.2 (Jacobian Conjecture [11]).

$$\mathcal{J}_n^{\mathrm{lin}} = \mathcal{J}_n \quad \forall n\,.$$

Define the *total degree* of F, $\deg(F)$, as $\max_j \deg(F_j(z))$, and introduce the subspaces

$$\mathcal{P}_{n,d} = \{F \in \mathcal{P}_n \mid \deg(F) \leq d\}\,;$$

and similarly for $\mathcal{J}$ and $\mathcal{J}^{\mathrm{lin}}$. We mention two positive results on the conjecture. First, a theorem for the quadratic case ($d = 2$) established first in [17], and then through a much simpler proof, in [12] (see also [18, Lemma 3.5] and [3, Thm. 2.4, pag. 298]).

Theorem 1.3 ([17]).

$$\mathcal{J}_{n,2}^{\mathrm{lin}} = \mathcal{J}_{n,2} \quad \forall n\,.$$

Then, a reduction theorem, from the general case to the cubic case, established by Bass, Connell and Wright [3, Sect. II]).

Theorem 1.4 ([3]).

$$\mathcal{J}_{n,3}^{\mathrm{lin}} = \mathcal{J}_{n,3} \quad \forall n \quad \Longrightarrow \quad \mathcal{J}_n^{\mathrm{lin}} = \mathcal{J}_n \quad \forall n\,.$$

The proof of the above theorem involves manipulations under which the dimension n of the system is increased, thus this proof does *not* imply the corresponding statement without the "$\forall n$" quantifier, i.e., that $\mathcal{J}_{n,3}^{\mathrm{lin}} = \mathcal{J}_{n,3} \Rightarrow \mathcal{J}_n^{\mathrm{lin}} = \mathcal{J}_n$.

Our result is a reduction theorem, in the line of the one above, that tries to 'fill the gap' between the cases of degree two and three. However, for this, we have to introduce an adaptation of the statement of the Jacobian conjecture with one more parameter.

For $n' \leq n$ and $F \in \mathcal{P}_{n,d}$, we write $z = (z_1, z_2)$ and $F = (F_1, F_2)$ to distinguish components in the two subspaces $\mathbb{C}^{n'} \times \mathbb{C}^{n-n'} \equiv \mathbb{C}^n$. We set $R(z_2; z_1)$ to be equal to $F_2(z_1, z_2)$, emphasizing that, in R, we consider z_2 as the variables in a polynomial system, and z_1 as parameters. The invertibility of R, denoted by $R(\cdot; z_1) \in \mathcal{J}_{n-n',d}$, for a fixed z_1, means that there exists a polynomial R^{-1} with variables $y_2 \in \mathbb{C}^{n-n'}$, and depending on z_1, such that

$$\forall z_2 \in \mathbb{C}^{n-n'}, \quad R^{-1}(R(z_2; z_1); z_1) = z_2.$$

We are now ready to define the main objects involved in the reduction theorem of this paper.

Definition 1.5. We define the subspaces of $\mathcal{P}_{n,d}$

$$\begin{aligned} \mathcal{J}_{n,d;n'} &:= \{F \in \mathcal{P}_{n,d} \,|\, R(\cdot; z_1) \in \mathcal{J}_{n-n',d} \ \forall z_1 \in \mathbb{C}^{n'} \\ &\qquad\qquad \text{and } F^{-1} \text{ restricted to } \mathbb{C}^{n'} \times \{0\} \text{ is in } \mathcal{P}_{n'}\} \\ \mathcal{J}^{\mathrm{lin}}_{n,d;n'} &:= \{F \in \mathcal{P}_{n,d} \,|\, R(\cdot; z_1) \in \mathcal{J}_{n-n',d} \ \forall z_1 \in \mathbb{C}^{n'} \\ &\qquad\qquad \text{and } (\det J_F)(z_1, R^{-1}(0, z_1)) = c \in \mathbb{C}^\times, \ \forall z_1 \in \mathbb{C}^{n'}\} \end{aligned}$$

The first definition is a clear and natural generalization of $\mathcal{J}_{n,d}$, which corresponds to $n' = n$. Note that the local invertibility of F in 0 (see the Definition of $\mathcal{P}_n$) was needed to consider the formal inverse F^{-1} of F. The second definition evaluates the Jacobian on a suitable algebraic variety, candidate image of $\mathbb{C}^{n'} \times \{0\}$ under F^{-1} (the reason for this choice will become clear only in the following, see in particular Sect. 4). As a consequence, we have that $\mathcal{J}^{\mathrm{lin}}_{n,d;n'} \subseteq \mathcal{J}_{n,d;n'}$, similarly to the original $n' = n$ case.

Apparently, the most natural and intrinsic generalization would have been to consider an arbitrary linear subspace of dimension $n - n'$, on which z shall vanish, instead of the last $n - n'$ variables. However, it will be notationally simpler to restrict to our choice, and cause no loss of generality for the problem at hand, which is clearly $\mathrm{GL}(n, \mathbb{C})$-invariant.

Let us now state our reduction theorem:

Theorem 1.6. *For $n \in \mathbb{N}$ and $d \geq 3$, there exists an injective map $\Phi : \mathcal{P}_{n,d} \to \mathcal{P}_{n(n+1),d-1}$ satisfying*

$$\Phi(\mathcal{J}^{\mathrm{lin}}_{n,d}) \equiv \mathcal{J}^{\mathrm{lin}}_{n(n+1),d-1;n} \cap \mathrm{Im}(\Phi)\,; \quad \Phi(\mathcal{J}_{n,d}) \equiv \mathcal{J}_{n(n+1),d-1;n} \cap \mathrm{Im}(\Phi)\,,$$

where $\mathrm{Im}(\Phi) = \Phi(\mathcal{P}_{n,d})$.

Combining Theorem 1.4 and the theorem above, the full Jacobian Conjecture reduces to the question whether

$$\mathcal{J}^{\mathrm{lin}}_{n(n+1),2;n} \cap \mathrm{Im}(\Phi) = \mathcal{J}_{n(n+1),2;n} \cap \mathrm{Im}(\Phi).$$

This question seems at a first stage as difficult as the original Jacobian conjecture. However, it involves only a quadratic degree, and this might simplify the resolution, in the light of Wang Theorem (Theorem 1.3).

It is also natural, at this point, to formulate a stronger version of the Jacobian Conjecture

Conjecture 1.7. *For all $n \geq n' \geq 0$, and all $d \geq 1$,*

$$\mathcal{J}^{\mathrm{lin}}_{n,d;n'} = \mathcal{J}_{n,d;n'}\,.$$

As we have seen, the original Conjecture 1.2 follows from the cases. $(n, d; n') \in \{(m(m+1), 2, m)\}_{m\geq 0}$ of the above conjecture, restricted to $\mathrm{Im}(\Phi)$.

As already mentioned above, this paper proves Theorem 1.6. We found this restriction of degree originally in the Quantum Field Theory formulation of the Jacobian conjecture, by applying the intermediate field method. So, in this paper, we first prove the theorem algebraically in Sect. 2, and then we expose the equivalent proof using combinatorial Quantum Field Theory (QFT) in Sect. 3.3. For general references on QFT for combinatorists, we refer the interested reader to [2] or [16], while a rederivation of the QFT analogs of quantities pertinent here is done in Sect. 3.1, at a heuristic level, and in Sect. 3.2, more formally.

For the sake of completeness, let us recall for the interested reader that various purely combinatorial approaches to the Jacobian Conjecture were given. Thus, the first paper of this series was the one of Zeilbelger [21], which proposes the Joyal method of combinatorial species as an appropriate tool to tackle the conjecture. His work has been followed by the one of Wright's [19], which used trees to reformulate the conjecture and then by the one of Singer [14], which used rooted trees (see also [15,20]).

2. Algebraic Proof of the Reduction Theorem

In this section, we use algebraic methods to prove Theorem 1.6. Let $\mathbb{K}$ be a field of characteristic $\neq 2$. Let us first prove the following lemmas.

Lemma 2.1 (Partial elimination, linearized version). *Let $N = n_1 + n_2$, and $S \in \mathcal{P}_N$. Write $z = (z_1, z_2)$ for $z \in \mathbb{K}^N = \mathbb{K}^{n_1} \times \mathbb{K}^{n_2}$ and so on. Call $R(z_2; z_1) = S_2(z_1, z_2)$ the system in $\mathcal{P}_{n_2}$, where z_1 coordinates are intended as parameters. Assume by hypothesis that $R(\cdot; z_1) \in \mathcal{J}_{n_2}$ for all $z_1 \in \mathbb{K}^{n_1}$. Define $H(z_1; y_2) = S_1(z_1, R^{-1}(y_2; z_1)) \in \mathcal{P}_{n_1}$. We have*

$$S \in \mathcal{J}^{\mathrm{lin}}_{N;n_1} \quad \text{iff} \quad H(\cdot; 0) \in \mathcal{J}^{\mathrm{lin}}_{n_1}\,.$$

Proof. We actually prove that

$$\det J_S(z_1, R^{-1}(y_2; z_1)) = (\det J_{R(\cdot;z_1)}) \det J_{H(\cdot;y_2)}(z_1),$$

while $\det J_{R(\cdot;z_1)} \in \mathbb{K}^\times$ is fixed by hypothesis and independent of z_1. Indeed, if $\det J_{R(\cdot;z_1)}$ were depending on z_1, this dependence would be polynomial and there would be a zero z_1 for this function. The result then follows from the previous equation with $y_2 = 0$.

To prove this equation, let us start by calculating explicitly $\det J_S(z)$. Expressing $S(z)$ in terms of $S_{1,2}$ and $z_{1,2}$ gives the block decomposition

$$\frac{\mathrm{d}}{\mathrm{d}z} S(z) = \left(\begin{array}{c|c} \frac{\mathrm{d}}{\mathrm{d}z_1} S_1(z_1, z_2) & \frac{\mathrm{d}}{\mathrm{d}z_1} S_2(z_1, z_2) \\ \hline \frac{\mathrm{d}}{\mathrm{d}z_2} S_1(z_1, z_2) & \frac{\mathrm{d}}{\mathrm{d}z_2} S_2(z_1, z_2) \end{array}\right).$$

We express the determinant of the matrix above by mean of the Schur complement formula.[1] Recognize that $\frac{\mathrm{d}}{\mathrm{d}z_2} S_2(z_1, z_2) = \frac{\mathrm{d}}{\mathrm{d}z_2} R(z_2; z_1) = J_{R(\cdot; z_1)}$. Thus

$$\begin{aligned} &\det \left(\frac{\mathrm{d}S(z)}{\mathrm{d}z}\right) \\ &\quad = (\det J_{R(\cdot;z_1)}) \ \det \left(\frac{\mathrm{d}S_1(z_1, z_2)}{\mathrm{d}z_1} - \frac{\mathrm{d}S_2(z_1, z_2)}{\mathrm{d}z_1} (J_{R(\cdot;z_1)})^{-1} \frac{\mathrm{d}S_1(z_1, z_2)}{\mathrm{d}z_2} \right). \end{aligned}$$

Recall that $z_2 = R^{-1}(y_2; z_1)$, however, if one wants to substitute one for the other in the expression above, some care is required in how to interpret derivatives w.r.t. z_1 and z_2. We pass instead to the second evaluation

$$\det J_{H(\cdot;y_2)}(z_1) = \det \left(\frac{\mathrm{d}H(z_1; y_2)}{\mathrm{d}z_1} \right) = \det \left(\frac{\mathrm{d}S_1(z_1; R^{-1}(y_2; z_1))}{\mathrm{d}z_1} \right). \tag{1}$$

We have to evaluate the total derivative w.r.t. the components of z_1. In the rightmost matrix, we have two contributions, coming from derivations acting on the first and second arguments, namely

$$\begin{aligned} &\frac{\mathrm{d}S_1(z_1; R^{-1}(y_2; z_1))}{\mathrm{d}z_1} \\ &\quad = \left(\frac{\mathrm{d}S_1(z_1; R^{-1}(y_2; u_1))}{\mathrm{d}z_1} + \frac{\mathrm{d}S_1(u_1; R^{-1}(y_2; z_1))}{\mathrm{d}z_1} \right)\Bigg|_{u_1 = z_1} \\ &\quad = \left(\frac{\mathrm{d}S_1(z_1; z_2)}{\mathrm{d}z_1} + \frac{\mathrm{d}R^{-1}(y_2; z_1)}{\mathrm{d}z_1} \frac{\mathrm{d}S_1(z_1; z_2)}{\mathrm{d}z_2} \right) \end{aligned} \tag{2}$$

and finally recognize that (here I_n is the n-dimensional identity matrix)

$$0 = \frac{\mathrm{d}\, I_{n_2}}{\mathrm{d}z_1} = \frac{\mathrm{d}R(R^{-1}(y_2; z_1); z_1)}{\mathrm{d}z_1} = \frac{\mathrm{d}R(z_2; z_1)}{\mathrm{d}z_1} + \frac{\mathrm{d}R^{-1}(y_2; z_1)}{\mathrm{d}z_1} \frac{\mathrm{d}R(z_2; z_1)}{\mathrm{d}z_2}$$

from which

$$\frac{\mathrm{d}R^{-1}(y_2; z_1)}{\mathrm{d}z_1} = -\frac{\mathrm{d}R(z_2; z_1)}{\mathrm{d}z_1} \left(\frac{\mathrm{d}R(z_2; z_1)}{\mathrm{d}z_2} \right)^{-1} = -\frac{\mathrm{d}S_2(z_1, z_2)}{\mathrm{d}z_1} (J_{R(\cdot;z_1)})^{-1}. \tag{3}$$

Substituting (3) into (2), and comparing to (1), leads to the conclusion. $\square$

---

[1] For a block matrix $M = \begin{pmatrix} AB \\ CD \end{pmatrix}$, the *Schur complement formula* states $\det M = (\det D) \det(A - BD^{-1}C)$.

Lemma 2.2 (Partial elimination, invertible version). *Let S, R and H as in Lemma 2.1. In particular, assume by hypothesis that $R(\cdot;z_1)\in\mathcal{J}_{n_2}$ for all $z_1\in\mathbb{K}^{n_1}$. We have*

$$S\in\mathcal{J}_{N;n_1}\quad\text{iff}\quad H(\cdot;0)\in\mathcal{J}_{n_1}\,.$$

Proof. We shall prove that $S^{-1}(\cdot;0)\in\mathcal{P}_{n_1}$ if and only if $H^{-1}(\cdot;0)\in\mathcal{P}_{n_1}$.

For the direct implication, we start by assuming to have

$$z_1=(S^{-1})_1(y_1,y_2)\,;\quad z_2=(S^{-1})_2(y_1,y_2)\,;$$

but we already know that $z_2=R^{-1}(y_2;z_1)$. Thus, from the unicity of the inverse, we obtain that

$$(S^{-1})_2(y_1,y_2)=R^{-1}(y_2;(S^{-1})_1(y_1,y_2))\,. \tag{4}$$

From the definition of H in terms of S, we get

$$H(z_1;y_2)=S_1(z_1,R^{-1}(y_2;z_1))\,.$$

Calculate

$$\begin{aligned}H\big((S^{-1})_1(y_1,y_2);y_2\big)&=S_1\big((S^{-1})_1(y_1,y_2),R^{-1}(y_2;(S^{-1})_1(y_1,y_2))\big)\\&=S_1\big((S^{-1})_1(y_1,y_2),(S^{-1})_2(y_1,y_2)\big)\\&=S_1(S^{-1}(y_1,y_2))=y_1;\end{aligned}$$

where we used Eq. (4). From the unicity of the inverse (when it exists), and its characterizing equation $H\big(H^{-1}(y_1;y_2);y_2\big)=y_1$, we can identify

$$H^{-1}(y_1;y_2)=(S^{-1})_1(y_1,y_2).$$

Setting $y_2=0$, the expression $H^{-1}(y_1;0)=(S_1^{-1}(y_1,0)$ is polynomial by hypothesis: $S\in\mathcal{J}_{N;n_1}$.

For the opposite implication, we set S^{-1} as:

$$(S^{-1})_1(y_1,y_2)=H^{-1}(y_1;y_2)\,;\quad (S^{-1})_2(y_1,y_2)\quad=R^{-1}(y_2;H^{-1}(y_1;y_2))\,.$$

Then, we can see directly that $S_1\big((S^{-1})_1(y_1,y_2),(S^{-1})_2(y_1,y_2)\big)=y_1$ and $S_2\big((S^{-1})_1(y_1,y_2),(S^{-1})_2(y_1,y_2)\big)=y_2$, so that S^{-1} is the formal inverse of S. Since $H^{-1}(y_1;0)$ is polynomial by hypothesis, $(S^{-1})_1(y_1,0)$ is also polynomial. Let us eventually show that

$$(S^{-1})_2(y_1,0)=R^{-1}(0;H^{-1}(y_1;0))$$

is polynomial. Indeed, $R^{-1}(z_2;z_1)$ is the formal inverse of $R(z_2;z_1)$ with respect to z_2 and with coefficients in $\mathbb{K}[z_1]$. So, due to the expression of the formal inverse [3], each term in z_2 has also a coefficient in $\mathbb{K}[z_1]$. Since $R(z_2;z_1)$ is invertible for any $z_1\in\mathbb{K}^{n_1}$, the degree of $R^{-1}(z_2;z_1)$ in z_2 is finite and bounded by d^{n_2-1} [3, Cor. 1.4]. So, all coefficients of $R^{-1}(z_2;z_1)$ in z_2 for a degree greater than this bound d^{n_2-1} uniformly vanish in z_1, hence these coefficients are 0 in $\mathbb{K}[z_1]$. This implies that $R^{-1}(z_2;z_1)$ is also polynomial in z_1. □

As mentioned above, we now prove Theorem 1.6.

Proof. Let us outline the proof. From $F \in \mathcal{P}_{n,d+1}$, we will construct a function $\tilde{F} = \Phi(F) \in \mathcal{P}_{n(n+1),d}$ with $F(z^{(1)}) = H(z^{(1)}; 0)$. Using Lemma 2.1, $F \in \mathcal{J}^{\text{lin}}_{n,d+1}$ if and only if $\tilde{F} \in \mathcal{J}^{\text{lin}}_{n(n+1),d,n}$, so

$$\Phi(\mathcal{J}^{\text{lin}}_{n,d+1}) = \mathcal{J}^{\text{lin}}_{n(n+1),d;n} \cap \text{Im}(\Phi).$$

Using Lemma 2.2 in a similar way, we also have $\Phi(\mathcal{J}_{n,d+1}) = \mathcal{J}_{n(n+1),d;n} \cap \text{Im}(\Phi)$.

We start from $F(z^{(1)}) \in \mathcal{P}_{n,d+1}$. Trivial arguments allow to establish that $F \in \mathcal{J}_n$ iff $F - F(0) \in \mathcal{J}_n$, i.e., we can drop the part of degree zero in F (see e.g., [3, Proposition 1.1, p. 303]). Thus, a generic $F \in \mathcal{P}_{n,d+1}$ has the form

$$F(z^{(1)}) = \sum_{c=1}^{d+1} F_c(z^{(1)})\,;$$

where F_c is homogeneous of degree c.

From $F \in \mathcal{P}_{n,d+1}$, we will construct a $\tilde{F} = \Phi(F) \in \mathcal{P}_{n(n+1),d}$ such that we can identify $F(z^{(1)}) = H(z^{(1)}; 0)$, where $H(z^{(1)}; y^{(2)})$ is associated with $\tilde{F}$ in the way this is done in Lemma 2.1. We indeed use here notations similar to those of Lemma 2.1, with $S = \tilde{F}$ and $N = n(n+1)$, except that, as we use explicit component indices, we use upper-scripts for blocks (e.g., $\tilde{F}^{(1)}(z^{(1)}, z^{(2)})$, instead of $F_1(z_1, z_2)$, and $z^{(1)} = \{z^{(1)}_i\}_{1\le i\le n}$). Clearly, we have $n_1 = n$ and $n_2 = n^2$. As our construction is structured, we use double indices for components in the second block, i.e., $z^{(2)} = \{z^{(2)}_{ij}\}_{1\le i,j\le n}$, instead of $z^{(2)} = \{z^{(2)}_\ell\}_{1\le \ell\le n^2}$.

Set now $\tilde{F}$ of degree at most d and block dimensions n and n^2, with explicit expression

$$\tilde{F}^{(1)}_i(z^{(1)}, z^{(2)}) = (F_{c=1})_i(z^{(1)}) + \sum_j z^{(2)}_{ij} z^{(1)}_j\,;$$

$$\tilde{F}^{(2)}_{ij}(z^{(1)}, z^{(2)}) = z^{(2)}_{ij} - \sum_{c\ge 2} \frac{1}{c} \frac{\mathrm{d}}{\mathrm{d}z^{(1)}_j} \big(F_c\big)_i(z^{(1)})\,.$$

First, we can easily check that $\tilde{F}$ is locally invertible in 0, so $\tilde{F} \in \mathcal{P}_{n(n+1),d}$. The complicated rightmost summand in $\tilde{F}^{(2)}_{ij}$ only depends on $z^{(1)}$, so that in fact $R(z^{(2)}; z^{(1)})$ is linear, and its invertibility is trivially established, namely

$$R^{-1}(y^{(2)}; z^{(1)}) = y^{(2)}_{ij} + \sum_{c\ge 2} \frac{1}{c} \frac{\mathrm{d}}{\mathrm{d}z^{(1)}_j} \big(F_c\big)_i(z^{(1)})\,.$$

We only have to check that H, specialized to $y^{(2)} = 0$, is equal to F, i.e., that

$$F(z^{(1)}) = H(z^{(1)}; 0) := \tilde{F}^{(1)}(z^{(1)}, R^{-1}(0; z^{(1)}))\,.$$

Dropping the now useless superscripts, this reads

$$\begin{aligned}\left(\tilde{F}^{(1)}\right)_i(z, R^{-1}(0;z)) &= (F_{c=1})_i(z) + \sum_j z_j \left(0 + \sum_{c\geq 2} \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}z_j}\left(F_c\right)_i(z)\right)\\ &= (F_{c=1})_i(z) + \sum_{c\geq 2}\frac{1}{c}\sum_j z_j \frac{\mathrm{d}}{\mathrm{d}z_j}\left(F_c\right)_i(z) = F_i(z)\,,\end{aligned}$$

as was to be proven.[2] □

3. QFT Proof of the Reduction Theorem

3.1. Heuristic QFT Proof

We consider in this section QFT arguments which are *heuristic*, as they involve rewritings of the involved algebraic quantities, in terms of formal integrals, deformations of Gaussian integrals, that do not generally converge. The reasonings implying (some of) the facts that can be derived by these methods are illustrated in [2]. Although, in our case, it would just be easier to translate our procedure, step by step, into a purely algebraic one. We did not perform this here, as we find that the QFT formalism provides a notational shortcut and a useful visualization of the algebraic derivation.

Let us start by briefly recalling what we shall call the *Abdesselam–Rivasseau model* (see [1] for details). This model is a 'zero-dimensional QFT model'. Here, the 'dimension' D refers to the fact that QFT models are stated in terms of functional integrals, for field $\phi_i(x)$ depending on a discrete index i and a continuous coordinate $x \in \mathbb{R}^D$. Here, we are in the much simpler case $D = 0$, i.e., we have only discrete indices, this fact being in part responsible for the possibility of producing rigorous proofs within this formalism (in such a situation, some authors refer to a *combinatorial QFT*). Note that D shall not be confused with the dimension n of the linear system $F(z)$. We anticipate that our fields will be complex variables, in holomorphic basis, and the associated integrals will be on $\mathbb{C}^n$ (i.e., with measure $\mathrm{d}\phi\,\mathrm{d}\phi^\dagger$).

Now, let $n, d \geq 1$, and let $F \in \mathcal{P}_{n,d}$. Invertibility of $F_1(z)$ is equivalent to the invertibility of $F_2(z) := F_1(Rz+u)$, for $R \in \mathrm{GL}(n,\mathbb{C})$ and $u \in \mathbb{C}^n$, and F_1, F_2 have the same degree, thus w.l.o.g. we can assume that $F(z) = z + \mathcal{O}(z^2)$. In such a case, the coordinate functions of F can be written as:

$$F_i(z) = z_i - \sum_{k=2}^{d}\sum_{j_1,\ldots,j_k=1}^{n} w^{(k)}_{i,j_1\ldots j_k} z_{j_1}\ldots z_{j_k} =: z_i - \sum_{k=2}^{d} W_i^{(k)}(z)\,,$$

[2] We used the obvious fact that if $A(x_1,\ldots,x_n)$ is a homogeneous polynomial of degree d, one has

$$\frac{1}{d}\sum_{i=1}^{n} x_i \frac{\mathrm{d}}{\mathrm{d}x_i} A(x) = A(x)\,.$$

for $i \leq n$ and $w^{(k)}_{i,j_1\dots j_k}$ some coefficients.[3] We introduce the inhomogeneous extension of the QFT model of [1].

$$Z(J,K) = \int_{\mathbb{C}^n} \mathrm{d}\varphi \mathrm{d}\varphi^\dagger e^{-\varphi^\dagger\varphi + \varphi^\dagger \sum_{k=2}^d W^{(k)}(\varphi) + J^\dagger\varphi + \varphi^\dagger K},$$

where J, K are vectors in $\mathbb{C}^n$. The full expression is called *partition function*, the expression in the exponential is called *action*, the coefficients w are called *coupling constants*, while J and K are called *external sources*. When the coupling constants are set to zero, the integral is calculated by Gaussian integration:

$$\int_{\mathbb{C}^n} \mathrm{d}\varphi \mathrm{d}\varphi^\dagger e^{-\varphi^\dagger\varphi + J^\dagger\varphi + \varphi^\dagger K} = e^{J^\dagger K}. \tag{5}$$

We can then express the unique formal inverse G of F. Indeed, for H_i an analytic function and $u \in \mathbb{C}^n$,

$$\begin{aligned}
&\int \mathrm{d}\varphi \mathrm{d}\varphi^\dagger H_i(\varphi) e^{-\varphi^\dagger F(\varphi) + \varphi^\dagger u} \\
&\quad = \int \mathrm{d}\tilde{\varphi} \mathrm{d}\varphi^\dagger H_i(G(\tilde{\varphi}+u)) e^{-\varphi^\dagger\tilde{\varphi}} \det(\partial G(\tilde{\varphi}+u)) \\
&\quad = \int \mathrm{d}\tilde{\varphi} H_i(G(\tilde{\varphi}+u)) \delta(\tilde{\varphi}) \det(\partial G(\tilde{\varphi}+u)) = H_i(G(u)) \det(\partial G(u)),
\end{aligned}$$

with the change of variables: $\tilde{\varphi} = F(\varphi) - u$. Taking the ratio of such expressions, for $H_i(z) = z_i$ at numerator, and $H_i(z) = 1$ at denominator, we obtain that the formal inverse corresponds to the one-point (outgoing) correlation function:

$$G_i(u) = \frac{\int_{\mathbb{C}^n} \mathrm{d}\varphi \mathrm{d}\varphi^\dagger \varphi_i e^{-\varphi^\dagger\varphi + \varphi^\dagger \sum_{k=2}^d W^{(k)}(z) + \varphi^\dagger u}}{\int_{\mathbb{C}^n} \mathrm{d}\varphi \mathrm{d}\varphi^\dagger e^{-\varphi^\dagger\varphi + \varphi^\dagger \sum_{k=2}^d W^{(k)}(z) + \varphi^\dagger u}} \tag{6}$$

Moreover, the partition function coincides with the inverse of the Jacobian:

$$Z(0,u) = \det(\partial G(u)) = JG(u) = \frac{1}{JF(G(u))}.$$

The sets of polynomial functions involved in the Jacobian Conjecture can be rephrased in this framework:

$$\begin{aligned}
\mathcal{J}^{\mathrm{lin}}_{n,d} &= \{F \in \mathcal{P}_{n,d} \,|\, Z(0,u) = 1 \,\forall u \in \mathbb{C}^n\}, \\
\mathcal{J}_{n,d} &= \{F \in \mathcal{P}_{n,d} \,|\, G_i(u) \text{ given by (6) is in } \mathcal{P}_n\}.
\end{aligned}$$

Let us now introduce the *intermediate field method* to reduce the degree d of F. We will thus add n^2 "intermediate fields" σ to the model. Indeed, we have, from the general formula (5) of Gaussian integration,

[3] Although only the symmetrized quantities $\sum_{\sigma\in\mathfrak{S}_k} w^{(k)}_{i,j_{\sigma(1)}\cdots j_{\sigma(k)}}$ contribute, it is convenient to keep this redundant notation.

$$e^{(\varphi_i^\dagger \varphi_j)\left(\sum_{j_2,\ldots,j_d=1}^n w^{(d)}_{i,j,j_2\ldots j_d}\varphi_{j_2}\cdots\varphi_{j_d}\right)} = \int_{\mathbb{C}^{n^2}} \mathrm{d}\sigma_{i,j}\mathrm{d}\sigma^\dagger_{i,j} e^{-\sigma^\dagger_{i,j}\sigma_{i,j}+\sigma^\dagger_{i,j}\left(\sum_{j_2,\ldots,j_d=1}^n w^{(d)}_{i,j,j_2\ldots j_d}\varphi_{j_2}\cdots\varphi_{j_d}\right)+(\varphi_i^\dagger\varphi_j)\sigma_{i,j}} \tag{7}$$

We now use the identity (7), for each pair (i,j), in the partition function of the model with n dimensions and degree d, in order to re-express the monomials of degree d in the fields φ. This leads to

$$Z(J,K) = \int_{\mathbb{C}^n} \mathrm{d}\varphi \mathrm{d}\varphi^\dagger \int_{\mathbb{C}^{n^2}} \mathrm{d}\sigma\mathrm{d}\sigma^\dagger e^{-\varphi^\dagger\varphi+\varphi^\dagger\sum_{k=2}^{d-1}W^{(k)}(\varphi)+J^\dagger\varphi+\varphi^\dagger K} \times e^{\sum_{i,j=1}^n\left(-\sigma^\dagger_{i,j}\sigma_{i,j}+\sigma^\dagger_{i,j}\sum_{j_2,\ldots,j_d=1}^n w^{(d)}_{i,j,j_2\ldots j_d}\varphi_{j_2}\cdots\varphi_{j_d}+\varphi_i^\dagger\varphi_j\sigma_{i,j}\right)}.$$

We define the new vector ϕ of $\mathbb{C}^{n+n^2}$ by $\phi = (\varphi_1,\ldots,\varphi_n,\sigma_{1,1},\ldots,\sigma_{1,n}, \ldots,\sigma_{n,1},\ldots,\sigma_{n,n})$. We further define the interaction coupling constants $\tilde{w}$ as:

- for $k=d-1$, we set $\tilde{w}^{(d-1)}_{i,j,j_2\ldots j_d} := w^{(d-1)}_{i,j,j_2\ldots j_d}$ and $\tilde{w}^{(d-1)}_{i\cdot n+j,j_2\ldots j_d} = w^{(d)}_{i,j,j_2\ldots j_d}$ with $i,j,j_2,\ldots j_n \leq n$.
- for $k\in\{3,\ldots,d-2\}$, we set $\tilde{w}^{(k)}_{i,j,j_2\ldots j_k} := w^{(k)}_{i,j,j_2\ldots j_k}$ with $i,j,j_2,\ldots j_n\leq n$.
- for $k=2$, we set $\tilde{w}^{(2)}_{i,j,j_2} := w^{(2)}_{i,j,j_2}$ and $\tilde{w}^{(2)}_{i,j,i\cdot n+j}=1$ with $i,j,j_2\leq n$.

The remaining coefficients of $\tilde{w}$ are set to 0.

In the same way, the external sources are defined to be $\tilde{J}=(J,0)$ and $\tilde{K}=(K,0)$, where, of course, the number of extra vanishing coordinates is n^2. It is important to note that these external sources have fewer degrees of freedom than coordinates (n vs. $n(n+1)$). We also remark that, for generic d, in order to have a relation adapted to induction, it is crucial to consider an inhomogeneous model, since the intermediate field method originates terms of degrees $d-1$ and 3.

One now has

$$Z(J,K) = \int_{\mathbb{C}^{n+n^2}} \mathrm{d}\phi\mathrm{d}\phi^\dagger e^{-\phi^\dagger\phi+\phi^\dagger\sum_{k=2}^{d-1}\tilde{W}^{(k)}(\phi)+\tilde{J}^\dagger\phi+\phi^\dagger\tilde{K}}$$

and

$$G_i(u) = \frac{\int_{\mathbb{C}^{n+n^2}}\mathrm{d}\phi\mathrm{d}\phi^\dagger\phi_i e^{-\phi^\dagger\phi+\phi^\dagger\sum_{k=2}^{d-1}\tilde{W}^{(k)}(\phi)+\phi^\dagger\tilde{u}}}{\int_{\mathbb{C}^{n+n^2}}\mathrm{d}\phi\mathrm{d}\phi^\dagger e^{-\phi^\dagger\phi+\phi^\dagger\sum_{k=2}^{d-1}\tilde{W}^{(k)}(\phi)+\phi^\dagger\tilde{u}}},$$

for $i\in\{1,\ldots,n\}$.

We have thus showed in a heuristic way that the partition function (resp. the one-point correlation function) of the model with dimension $n\in\mathbb{N}$ and degree $d\in\mathbb{N}\backslash\{1,2\}$ is equal to the partition function (resp. the n first coordinates of the one-point correlation function) of the model with dimension $n(n+1)$ and degree $d-1$, up to a redefinition of the coupling constant $w\mapsto\tilde{w}$ and a trivial redefinition of the external sources. Since the partition function corresponds to the inverse of the Jacobian (resp. the one-point correlation function corresponds to the formal inverse), this gives, as mentioned above, a heuristic proof of Theorem 1.6.

3.2. Formal Inverse in QFT

In this section, we adopt notations as above, but we proceed at a more formal level. In particular, the coefficients w are considered as formal indeterminates in (multi-dimensional) power series. To project more easily to the simpler context of univariate power series, we introduce a further, redundant, indeterminate θ, by replacing each coefficient $w^{(k)}_{i,j_1\ldots j_k}$ by $\theta^{k-1}w^{(k)}_{i,j_1\ldots j_k}$. We denote by $\mathbb{C}[[\theta]]$ the ring of formal power series in θ. The exponent of θ in the w's measures the "spin" of the associated monomial, i.e., the action is invariant under the transformation $\phi_j \to \phi_j e^{i\omega}$, $\phi^\dagger_j \to \phi^\dagger_j e^{-i\omega}$, $\theta \to \theta e^{-i\omega}$.

The polynomial function F now is extended naturally to a function from $\mathbb{C}[[\theta]]^n$ to itself, although we are ultimately interested on invertibility on $\mathbb{C}^n$. The integrals of the previous subsection are now well defined as a formal expansion in θ

$$Z(J,K) := \int_{\mathbb{C}^n} \mathrm{d}\varphi \mathrm{d}\varphi^\dagger e^{-\varphi^\dagger\varphi + J^\dagger\varphi + \varphi^\dagger K} \sum_{r=0}^{\infty} \frac{1}{r!}\Big(\varphi^\dagger \sum_{k=2}^{d} (\theta^{k-1}W^{(k)}(\varphi)\Big)^r, \quad (8)$$

where, as in the previous section, J and K can be considered as vectors in $\mathbb{C}^n$ (as they enter the quadratic part of the action, there is no need of promoting them to formal indeterminates).

Any term of this power series, i.e., $[\theta^r]Z(J,K)$ for $p \in \mathbb{N}$, can be calculated as a finite linear combination of terms of the following form, with $r = q - p$

$$\int_{\mathbb{C}^n} \mathrm{d}\varphi \mathrm{d}\varphi^\dagger e^{-\varphi^\dagger\varphi + J^\dagger\varphi + \varphi^\dagger K} \varphi^\dagger_{i_1} \ldots \varphi^\dagger_{i_p} \varphi_{j_1} \ldots \varphi_{j_q} \quad (9)$$

which, of course, is also given by

$$\frac{\partial^p}{\partial K_{i_1} \ldots \partial K_{i_p}} \frac{\partial^q}{\partial J^\dagger_{j_1} \ldots \partial J^\dagger_{j_q}} \int_{\mathbb{C}^n} \mathrm{d}\varphi \mathrm{d}\varphi^\dagger e^{-\varphi^\dagger\varphi + J^\dagger\varphi + \varphi^\dagger K} \quad (10)$$

An analysis of such an expression, in light of (5), leads to the *Wick Theorem* (see, for example, textbooks such as [5]): the integral in (9) is equal to the sum over all possible substitutions, in the monomial $\varphi^\dagger_{i_1} \ldots \varphi^\dagger_{i_p} \varphi_{j_1} \ldots \varphi_{j_q}$ of the patterns $\varphi^\dagger_i \to J^\dagger_i$, $\varphi_j \to K_j$, and $\varphi^\dagger_i \varphi_j \to \delta_{ij}$, up to have no φ's and $\varphi^\dagger$'s left.

One can then associate graphical representations Γ to these expressions, going under the name of *Feynman graphs*. For the model analyzed here, the set of graphs and their associated weights are obtained through the following rules: vertices in the graph have indices $i \in \{1,\ldots,n\}$ attached to the incident edges; a term δ_{ij} is represented as a directed edge, with index i at its endpoints; a term $J^\dagger_i$ is represented as a vertex of in-degree one and out-degree zero, incident to an edge of index i; similarly, a term K_j is represented as a vertex of in-degree zero and out-degree one, incident to an edge of index i; a weight $\theta^{k-1}w^{(k)}_{i,j_1\ldots j_k}$ is associated with a vertex with out-degree one (and index i), and in-degree k (and indices $\{j_1,\ldots,j_k\}$) (The incident edges have a cyclic ordering). Finally, a symmetry factor $1/|\mathrm{Aut}(\Gamma)|$ appears overall, as a combination of the $1/r!$ factor and of multiple counting for the same diagram in the expansion of (8).

Note that, in general, several but finitely many graphs contribute to a given expression associated with a monomial $\varphi^\dagger_{i_1} \dots \varphi^\dagger_{i_p} \varphi_{j_1} \dots \varphi_{j_q}$.

For the problem at hand here, both when evaluating the Jacobian and the formal inverse of a component, we can restrict to the case $J = 0$. In order to conform to notations in the literature, we shall also rename K_j's into u_j's (indeed, when $J = 0$, the source K induces a formal translation of the components ϕ's, without translating the $\phi^\dagger$'s). It is easy to see that, in a theory with such a constrained set of vertex out-degrees, all contributing Feynman graphs with no J-leaves contain exactly one cycle[4] per connected component while connected graphs with a unique J-leaf correspond to directed trees rooted at this leaf.

We can now state the expression for the formal inverse which in fact coincides with the heuristically derived (6), as well as the expression of the partition function (8). Since they both have been already derived in [1], we do not give the proofs here.

Theorem 3.1. *Define $\mathcal{A}_i(T)(u)$ the amplitude of a tree T with exactly one outgoing edge, of index i. The formal inverse of the function F is the function G with coordinates*

$$G_i(u) = \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{T:|T|=r} \mathcal{A}_i(T)(u),$$

where T denotes a tree with one outgoing edge, and $|T|$ is the number of vertices in T.

Note an abuse of notation here, as i is not an index pertinent to the function $\mathcal{A}$, but to the variable T. We can equivalently think that T has its root unlabeled (and the vertex adjacent to the root comes with no weight), and the function $\mathcal{A}_i$ completes the weight of T by including the appropriate factor $w^{(k)}_{i,j_1\dots j_k}$.

Remark 3.2. From the aforementioned homogeneity of θ, we also have $G_i(u,\theta) = \lambda G_i(\lambda u, \lambda^{-1}\theta)$ for $\lambda \in \mathbb{C}[[\theta]]^\times$. This essentially allows to eliminate θ, and, adapting an argument of [1], show that G_i is actually analytic on a certain domain of convergence in the variables u_j.

One has:

Proposition 3.3. *The partition function (8) of the above theory is given by*

$$Z(0,u) = \frac{1}{\det[J(F)(G(u))]}.$$

3.3. Proof of the Theorem

We give in this section the combinatorial QFT proof of our main result, Theorem 1.6.

[4] Cycles are commonly called "loops" in QFT.

Proof. Consider a directed tree T, constructed from the Feynman rules defined in Sect. 3.2, in dimension n (i.e., with edge indices in $\{1,\ldots,n\}$), and degree d (i.e., with vertices of in-degree at most d). From Theorem 3.1, we know that $\mathcal{A}_i(T)(u)$ is used to compute the formal inverses $G_i(u)$.

Let us now define the Feynman rules for the model in dimension $n(n+1)$, obtained using the intermediate field method described in Sect. 3.1. For $i,j\in\{1,\ldots,n(n+1)\}$, one has:

- propagators, i.e., directed edges with index i correspond to the term δ_{ij} (obtained by the $\phi_i^\dagger\phi_j$ substitution in Wick Theorem);
- leaves with in-degree 1 correspond to the term $\tilde{u}_j$, and in particular, as $\tilde{u}_j=0$ for $j>n$, contributing diagrams have all leaf-indices in the range $\{1,\ldots,n\}$;
- vertices of coordination $k+1$, for $k\in\{1,\ldots,d-1\}$, with one outgoing edge of index i, and k incoming edges of indices $j_1,\ldots,j_k$, correspond to the term $\theta^{k-1}\tilde{w}^{(k)}_{i,j_1\ldots j_k}$. Indices of the vertices are summed on.

For a graphical representation of the intermediate field method leading to the model in dimension $n(n+1)$, see Fig. 1. As stated in Sect. 3.1, $\tilde{u}=(u_1,\ldots,u_n,0,\ldots,0)$, and the coefficients $\tilde{w}$ are

- for $k=d-1$, set $\tilde{w}^{(d-1)}_{i,j,j_2\ldots j_d}:=w^{(d-1)}_{i,j,j_2\ldots j_d}$; and $\tilde{w}^{(d-1)}_{i\cdot n+j,j_2\ldots j_d}=w^{(d)}_{i,j,j_2\ldots j_d}$ with $i,j,j_2,\ldots j_n\le n$.
- for $k\in\{3,\ldots,d-2\}$, set $\tilde{w}^{(k)}_{i,j,j_2\ldots j_k}:=w^{(k)}_{i,j,j_2\ldots j_k}$ with $i,j,j_2,\ldots j_n\le n$.
- for $k=2$, set $\tilde{w}^{(2)}_{i,j,j_2}:=w^{(2)}_{i,j,j_2}=0$; and $\tilde{w}^{(2)}_{i,j,i\cdot n+j}=1$ with $i,j,j_2\le n$.

The other components of $\tilde{w}$ are set to 0 by definition. Note that this corresponds to consider the new polynomial map $\tilde{F}:\mathbb{C}^{n(n+1)}\to\mathbb{C}^{n(n+1)}$ given by

$$\tilde{F}_i(z)=z_i-\sum_{k=3}^{d-1}\sum_{j_1,\ldots,j_k=1}^{n}w^{(k)}_{i,j_1\ldots j_k}z_{j_1}\ldots z_{j_k}-\sum_{j=1}^{n}z_jz_{i\cdot n+j}\qquad\text{for }1\le i\le n,$$

$$\tilde{F}_{i\cdot n+j}(z)=z_{i\cdot n+j}-\sum_{j_2,\ldots,j_d=1}^{n}w^{(d)}_{i,j,j_2,\ldots j_k}z_{j_2}\ldots z_{j_k}\qquad\text{for }1\le i,j\le n.$$

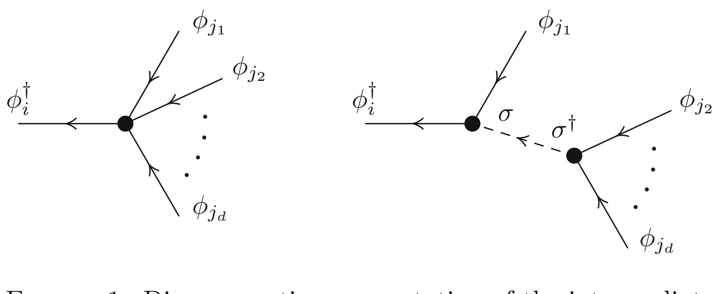

FIGURE 1. Diagrammatic representation of the intermediate field method

To each tree T in the theory of dimension n, and a choice of incoming edge per each vertex of degree $d+1$, we can associate canonically a tree $\tilde{T}$, constructed from these new Feynman rules. Propagators, leaves and vertices of coordination less than or equal to d in the tree T are identically transposed in the tree $\tilde{T}$, while a vertex in T of coordination $d+1$ with one outgoing edge i and d incoming edges of indices $j_1, \ldots, j_d$ is split into two vertices in $\tilde{T}$, connected by an edge of index $n < i \leq n(n+1)$. Edges of such indices are never adjacent on the tree. The precise construction is depicted in Fig. 1.

Consider now the Feynman integral $\tilde{\mathcal{A}}_i(\tilde{T})(\tilde{u})$, in the model of dimension $n(n+1)$. For propagators, leaves and vertices of coordination less than or equal to d in the tree T, the contribution to $\tilde{\mathcal{A}}_i(\tilde{T})(\tilde{u})$ is the same as in $\mathcal{A}_i(T)(u)$, because the summation in $i, j_1, \ldots, j_k$ of the vertices reduces to $\{1, \ldots, n\}$, except for $k = d-1$ where i could a priori take value in $\{n+1, \ldots, n(n+1)\}$. However, the outgoing edge of this vertex is either the external edge (with $i \in \{1, \ldots, n\}$) or is adjacent to another vertex (of coordination greater than or equal to four by hypothesis), so we also have $i \in \{1, \ldots, n\}$.

For any vertex of coordination $d+1$ in T, the above coefficients $\tilde{w}^{(d-1)}_{i\cdot n+j, j_2 \ldots j_d}$ and $\tilde{w}^{(2)}_{i,j,i\cdot n+j}$ have been chosen so that the contribution in $\mathcal{A}_i(T)(u)$ coincides with the one in $\tilde{\mathcal{A}}_i(\tilde{T})(\tilde{u})$. The only thing to check is that the index ℓ relating the two new vertices in $\tilde{T}$ is summed over only $\ell \in \{n+1, \ldots, n(n+1)\}$. This is the case because $\tilde{w}^{(2)}_{i,j,\ell} = 0$ for $\ell \leq n$. Note also that the formal factor θ^{d-1} for the vertex in T corresponds to θ for the new vertex of coordination 3 and θ^{d-2} for the one of coordination d in $\tilde{T}$. Due to Feynman rules, only trees $\tilde{T}$ obtained from a tree T can contribute to the formal inverse $\tilde{G}_i(\tilde{u})$ of Theorem 3.1.

One then concludes that

$$\tilde{\mathcal{A}}_i(\tilde{T})(\tilde{u}) = \mathcal{A}_i(T)(u).$$

Due to Theorem 3.1, we have $\tilde{G}_i(\tilde{u}) = G_i(u)$ for $i \leq n$. Moreover, for the one point correlation function with one external leg in the auxiliary field, one has:

$$\tilde{G}_{i\cdot n+j}(\tilde{u}) = \theta^{d-2} \sum_{j_2, \ldots, j_d} w^{(d)}_{i,j,j_2\ldots j_d} G_{j_2}(u) \ldots G_{j_d}(u) = R^{-1}_{i\cdot n+j}(0, \theta G(u)), \tag{11}$$

since $R_{i\cdot n+j}(v, u) = v_{i\cdot n+j} - \sum_{j_2, \ldots, j_d} w^{(d)}_{i,j,j_2\ldots j_d} u_{j_2} \ldots u_{j_d}$ (see Definition 1.5).

In particular, $\tilde{G}_{i\cdot n+j}(\tilde{u})$ is polynomial in u iff the functions $G_j(u)$ are also polynomial. To conclude the first part of the proof, we can associate injectively to a polynomial function $F \in \mathcal{P}_{n,d}$ another function $\tilde{F} = \Phi(F) \in \mathcal{P}_{n(n+1),d-1}$ and we proved that $F \in \mathcal{J}_{n,d} \Leftrightarrow \tilde{F} \in \mathcal{J}_{n(n+1),d-1;n}$.

The same process can be performed for graphs without external edges, which leads to the equality of partition functions in both QFT models, the one of dimension $n(n+1)$ and the one of dimension n:

$$\tilde{Z}(0, \tilde{u}) = Z(0, u).$$

Using Eq. (11) and Proposition 3.3, we then obtain

$$\begin{aligned}\det(J_{\tilde{F}})(\theta G(u), \quad R^{-1}(0, \theta G(u))) &= \det(J_{\tilde{F}})(\theta \tilde{G}(\tilde{u})) \\ &= \tilde{Z}(0, \tilde{u})^{-1} = Z(0, u)^{-1} = \det(J_F)(\theta G(u)).\end{aligned}$$

This proves the second part of the Theorem, namely $F \in \mathcal{J}^{\text{lin}}_{n,d} \Leftrightarrow \tilde{F} \in \mathcal{J}^{\text{lin}}_{n(n+1),d-1;n}$. □

4. Example

Let us illustrate the conjecture in low dimension $n = 2$, $n' = 1$ and arbitrary degree d in this section. This will not be useful for the Jacobian conjecture of course, because it shows the case $n = 1$, $d+1$, which is trivial. But it will give explicit computations involving the definitions introduced above.

We consider the polynomial given by

$$\begin{aligned}F_1(z) &= z_1 - \sum_{k=0}^{d} a_{1,k} z_1^k z_2^{d-k} \\ F_2(z) &= z_2 - \sum_{k=0}^{d} a_{2,k} z_1^k z_2^{d-k},\end{aligned}$$

where the complex coefficients $a_{i,k}$ are fixed. Then, the Jacobian takes the form

$$\begin{aligned}\det(J_F)(z) = 1 &- \sum_{k=0}^{d-1} (a_{1,k+1}(k+1) + a_{2,k}(d-k)) z_1^k z_2^{d-1-k} \\ &+ \sum_{k,l=0}^{d} a_{1,k} a_{2,l} (d-k) l z_1^{k+l-1} z_2^{2d-k-l-1} \qquad (12)\end{aligned}$$

In the standard case $n' = n = 2$, using (12), the equation $\det(J_F) = 1$ leads to the conditions.

- For any $k \in \{0, \dots, d-1\}$, $a_{1,k+1}(k+1) = a_{2,k}(d-k)$.
- For any $m \geq 1$, $\sum_{k=0}^{\min(d,m)} a_{1,k} a_{2,m-k} d(2k-m) = 0$.

If F satisfies these conditions, F lies in $\mathcal{J}^{\text{lin}}_{2,d}$.

Let us describe the polynomials F that belong to $\mathcal{J}^{\text{lin}}_{2,d;1}$ and compare with the above conditions. We will see that they are very different. Moreover, we will see that $\mathcal{J}^{\text{lin}}_{2,d;1} = \mathcal{J}_{2,d;1}$.

For $n' = 1$, we have to set

$$R(z_2; z_1) = z_2 - \sum_{k=0}^{d} a_{2,k} z_1^k z_2^{d-k}.$$

This expression has to be invertible as a polynomial in z_2 and for any parameter $z_1 \in \mathbb{C}$. In particular, the Jacobian of R with respect to z_2 has to be constant, which implies $a_{2,k} = 0$ for any $k < d-1$. So $R(z_2; z_1) = z_2 - a_{2,d-1} z_1^{d-1} z_2 - a_{2,d} z_1^d$. But the invertibility of R for any z_1 also implies that $a_{2,d-1} = 0$.

Eventually, we get $R(z_2;z_1)=z_2-a_{2,d}z_1^d$, and $R^{-1}(y_2;z_1)=y_2+a_{2,d}z_1^d$. Let us look at the condition $\det(J_F)(z_1,R^{-1}(0;z_1))=1$. Replacing z_2 by $R^{-1}(0;z_1)=a_{2,d}z_1^d$ in (12), we find the following expression:

$$\det(J_F)(z_1,R^{-1}(0;z_1)) \\ =1+\sum_{k=1}^{d}a_{1,k}a_{2,d}^{d-k}(k(d-1)-d^2)z_1^{(d-1)(d-1-k)}+a_{1,0}a_{2,d}^{d}z_1^{(d-1)(d+1)}.$$

Then, the polynomial F belongs to $\mathcal{J}^{\text{lin}}_{2,d;1}$ if and only if $a_{1,d}=0$ and

$$\forall k\in\{0,\ldots,d-1\},\, a_{1,k}=0 \quad \text{or} \quad a_{2,d}=0.$$

We see indeed that these conditions are very different from the one of $\mathcal{J}^{\text{lin}}_{2,d}$. Now, let us show that these polynomials $F\in\mathcal{J}^{\text{lin}}_{2,d;1}$ are also in $\mathcal{J}_{2,d;1}$, so $(F^{-1})_1(z_1,0)$ is polynomial in z_1.

The first case of $\mathcal{J}^{\text{lin}}_{2,d;1}$ corresponds to

$$F_1(z)=z_1,\quad F_2(z)=z_2-a_{2,d}z_1^d.$$

Here, the global inverse is

$$F_1^{-1}(y)=y_1,\quad F_2(y)=y_2+a_{2,d}y_1^d,$$

so the condition of $\mathcal{J}_{2,d;1}$ is trivially satisfied. The second case coincides with polynomials

$$F_1(z)=z_1-\sum_{k=0}^{d-1}a_{1,k}z_1^kz_2^{d-k},\quad F_2(z)=z_2.$$

The global inverse is not polynomial in y_1,y_2. However, by setting $u=(F^{-1})_1(y_1,0)$, we have the following equation $y_1=F_1(u,0)=u$, so

$$(F^{-1})_1(y_1,0)=y_1$$

is polynomial in y_1, and $F\in\mathcal{J}_{2,d;1}$.

5. Concluding Remarks and Perspectives

We thus proved in this paper a reduction theorem to the quadratic case for the Jacobian conjecture, up to the addition of a new parameter n'. Moreover, we did this first using formal algebraic methods and then using QFT methods. This idea of using intermediate field method represents an illustration of how QFT methods can be successfully used to prove "purely" mathematical results.

Recall here that the Jacobian Conjecture is proved in the quadratic case by Wang [17]. The immediate perspective thus appears to be the adaptation of Wang's proof to our particular case, where the parameter n' plays a non-trivial role. An interesting approach for this may be the reformulation of Wang's proof in a QFT language, since we saw here that reduction results can be established in a natural way when using QFT techniques.

Let us end this paper by recalling that the Jacobian Conjecture is stably equivalent to the Dixmier Conjecture for endomorphisms of the Weyl algebra.

It should be interesting to revisit the Dixmier Conjecture from the perspective of Noncommutative QFT (see [6,7,9,10] and references within) on the deformation quantization of the complex plane, which is an extension of the Weyl algebra (see [8,13]).

Acknowledgements

A. Tanasa warmly thanks V. Rivasseau for bringing his attention to the Jacobian Conjecture in general and to the Abdesselam-Rivasseau combinatorial QFT model in particular; several discussions with J. Magnen at an early stage of this work are also warmly acknowledged. A. Tanasa was partially supported by the PN 09370102 and ANR JCJC CombPhysMat2Tens grants. A. de Goursac was partially supported by the ANR JCJC CombPhysMat2Tens grant and by the Belgian Interuniversity Attraction Pole (IAP) within the framework "Dynamics, Geometry and Statistical Physics" (DYGEST). We also thank one of the referees of this paper for his relevant remarks concerning the algebraic proof of Sect. 2.

References

[1] Abdesselam, A.: The Jacobian conjecture as a problem of perturbative quantum field theory. Ann. Henri Poincaré **4**, 199–215 (2003)

[2] Abdesselam, A.: Feynman diagrams in algebraic combinatorics. Sémin. Lotharingien Comb. **49**, B49c (2002)

[3] Bass, C.W., Connell, E.H., Wright, D.: The Jacobian conjecture: reduction of degree and formal expansion of the inverse. Bull. Am. Math. Soc. **7**(2), 287–330 (1982)

[4] Belov-Kanel, A., Kontsevich, M.: The Jacobian Conjecture is stably equivalent to the dixmier conjecture. Moscow Math. J. **7**, 209–218 (2007)

[5] Connes, A., Marcolli, M.: Noncommutative Geometry, Quantum Fields and Motives, 55 edn. Hindustand Book Agency, AMS Colloquium Publications, New York (2008)

[6] de Goursac, A.: Renormalization of the commutative scalar theory with harmonic term to all orders. Ann. Henri Poincaré **14**, 2025–2043 (2013)

[7] de Goursac, A., Tanasa, A., Wallet, J.C.: Vacuum configurations for renormalizable non-commutative scalar models. Eur. Phys. J. C **53**, 459–466 (2008)

[8] Garay, A., de Goursac, A., Straten, D.van : Resurgent deformation quantisation. Ann. Phys. **342**, 83–102 (2014)

[9] Gurau, R., Magnen, J., Rivasseau, V., Tanasa, A.: A translation-invariant renormalizable non-commutative scalar model. Commun. Math. Phys. **287**, 275–290 (2009)

[10] Grosse, H., Wulkenhaar, R.: Renormalisation of phi**4 theory on noncommutative R**4 in the matrix base. Commun. Math. Phys. **256**, 305–374 (2005)

[11] Keller, O. H.: Ganze cremona transformations. Monats. Math. Phys. **47**, 299–306 (1939)

[12] Oda, S.: The Jacobian problem and the simply-connectedness of $\mathbf{a}^n$ over a field k of characteristic zero. Osaka University, Osaka (1980)

[13] Omori, H., Maeda, Y., Miyazaki, N., Yoshioka, A.: Anomalous quadratic exponentials in the star-products. RIMS Kokyuroku **1150**, 128–132 (2000)

[14] Singer, D.: On the catalan trees and the jacobian conjecture. Electron. J. Comb. **8**, R2 (2001)

[15] Singer, D.: Toward a combinatorial proof of the jacobian conjecture! Electron. J. Comb. **18**, P27 (2011)

[16] Tanasa, A.: Some combinatorial aspects of quantum field theory. Sémin. Lotharingien Comb. **65**, B65g (2012)

[17] Wang, S.: A Jacobian criterion for separability. J. Algebra **65**, 453–494 (1980)

[18] Wright, D.: The tree formulas for reversion of power series. J. Pure Appl. Algebra **57**(2), 191–211 (1989)

[19] Wright, D.: Reversion, trees, and the Jacobian conjecture. Contemp. Math. Comb. Comput. Algebra Am. Math. Soc. Providence **264**, 249–267 (1999)

[20] Wright, D.: The Jacobian Conjecture as a problem in combinatorics. In the monograph "Affine Algebraic Geometry", in honour of Masayoshi Miyanishi. Osaka University Press, Osaka (2006)

[21] Zeilberger, D.: Toward a combinatorial proof of the jacobian conjecture? In: Labelle, G., Leroux, P. (eds.) Combinatoire enumerative. Lecture Notes in Mathematics, vol. 1234, pp. 370–380. Springer, Berlin, Heidelberg (1987)

Axel de Goursac
Chargé de Recherche FNRS
Université Catholique de Louvain
Louvain, Belgium
e-mail: axelmg@melix.net

Andrea Sportiello
LIPN, Institut Galilée, CNRS UMR 7030
Université Paris 13
Villetaneuse, France

Adrian Tanasa
LaBRI, UMR 5800
Université de Bordeaux
Talence, France

Adrian Tanasa
Horia Hulubei National Institute for Physics and Nuclear Engineering
Magurele, Romania

Communicated by Abdelmalek Abdesselam.
Received: July 16, 2015.
Accepted: March 3, 2016.