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On Supersymmetric Fermion Lattice Systems

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Abstract. This note provides a C^*-algebraic framework for supersymmetry. Particularly, we consider fermion lattice models satisfying the simplest supersymmetry relation. Namely, we discuss a restricted sense of supersymmetry without a boson field involved. We construct general supersymmetric C^*-dynamics in terms of a superderivation and a one-parameter group of automorphisms on the CAR algebra. (We do not introduce Grassmann numbers into our formalism.) We show several basic properties of superderivations on the fermion lattice system. Among others, we establish that superderivations defined on the strictly local algebra are norm-closable. We show a criterion of superderivations on the fermion lattice system for being nilpotent. This criterion can be easily checked and hence yields new supersymmetric fermion lattice models.

1. Introduction

It has been expected that supersymmetry will play a crucial role in the unification of fundamental interactions in particle physics, see e.g. [26]. Furthermore, the concept of supersymmetry has been influential and fruitful in wide range of areas in physics and mathematics. Among many topics related to supersymmetry, we shall refer to supersymmetric quantum mechanics (SUSY QM) in which we find remarkable interplays between mathematics and physics. For general account of SUSY QM, let us refer to a comprehensive work [7] and the reference therein. In this note, we study fermion lattice models with hidden supersymmetry. A spinless fermion lattice model that has a hidden supersymmetry was proposed by Nicolai [20] in 1976; this quantum statistical-mechanical model is another (unfortunately not well-known) example of supersymmetric quantum mechanics.[1] Recently, other supersymmetric fermion lattice models have been proposed and investigated by Fendly et al. [8].

[1] The SUSY QM model introduced by Witten [24,25] is a classic in supersymmetry theory. The model by Nicolai can be considered as another pioneer work of non-relativistic SUSY as noted by Junker in [16].

There have been extensive works of C^*-algebraic quantum field theory and quantum statistical mechanics, for general references we refer to [3,4,10]. Although it is straightforward to write down a supersymmetry algebra (without central charges) [11] on a graded C^*-algebra *heuristically* as found in some previous works [14,15,17], it is not clear whether and how supersymmetric models can be formulated in C^*-algebraic quantum theory or its natural extension. It seems that construction of supersymmetry dynamics within a graded C^*-algebra is unexpectedly difficult unless the algebra is finite dimensional.

To make steps toward supersymmetry theory in C^*-algebras, we shall focus on fermion lattice systems. Based on a C^*-algebraic framework for supersymmetric fermion lattice models that we will show we rigorously discuss supersymmetric dynamics and supersymmetric states in the infinite volume limit. We have to emphasize that supersymmetry originally means a symmetry between fermions and bosons. Buchholz–Grundling have invented a C^*-algebra approach to supersymmetry between fermions and bosons in [6]. On the other hand, this note only deals with fermions. By exploiting such a simplified situation with no boson field, we give a general class of C^*-dynamics on the CAR algebra. This corresponds to supersymmetric fermion lattice models of finite-range interactions.

Let us explain the plan of this paper. In Sect. 2, we introduce superderivations of a graded C^*-algebra. We define supersymmetric states and show their basic properties. In Sect. 3, we provide a heuristic overview on supersymmetric fermion lattice systems. In Sect. 4, we formulate supersymmetric fermion models of finite-range interactions as strongly continuous C^*-dynamics on the CAR algebra. A superderivation on the fermion lattice system is determined by assigning local fermionic charges over the lattice. This is analogous to the well-known construction of a time generator generated by local Hamiltonians over the lattice, see [4]. We will show that these superderivations on the CAR algebra are norm-closable when the associated supersymmetry is unbroken. (It is not known, however, whether this statement holds for the case of broken supersymmetry. We conjecture that this is still correct when the supersymmetry is broken.) Using the norm-closability of superderivations mentioned above, we establish a rigorous formulation of supersymmetry that includes commutativity between those superderivations and the global time evolution generated by them. In Sect. 5, we give some concrete supersymmetric fermion lattice models in our C^*-algebraic framework of Sect. 4. These are based on the model by Nicolai [20] and the model by Fendly et al. [8]. Section 6 is a summary of this note. We present a set of axioms for supersymmetric C^*-dynamical systems which is abstracted from the supersymmetric C^*-dynamics on the CAR algebra of Sect. 4.

2. Notation

2.1. Superderivations

We first introduce superderivations in a general C^*-algebra. Let $\mathcal{F}$ denote a graded C^*-algebra with a grading γ, where γ is given by a $\mathbb{Z}_2$-group of $*$-automorphisms of $\mathcal{F}$. The graded structure of $\mathcal{F}$ induced by γ is as follows:

$$\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-, \quad \mathcal{F}_+ := \{F \in \mathcal{F} \mid \gamma(F) = F\},$$
$$\mathcal{F}_- := \{F \in \mathcal{F} \mid \gamma(F) = -F\}. \tag{2.1}$$

The above $\mathcal{F}_+$ and $\mathcal{F}_-$ are called the even and odd parts of $\mathcal{F}$, respectively. The graded commutator $[\ ,\ ]_\gamma$ is defined on $\mathcal{F}$ as

$$[F_+,\, G]_\gamma = [F_+,\, G] = F_+G - GF_+ \quad \text{for} \quad F_+ \in \mathcal{F}_+,\ G \in \mathcal{F},$$
$$[F,\, G_+]_\gamma = [F,\, G_+] = FG_+ - G_+F \quad \text{for} \quad F \in \mathcal{F},\ G_+ \in \mathcal{F}_+,$$
$$[F_-,\, G_-]_\gamma = \{F_-\, G_-\} = F_-G_- + G_-F_- \quad \text{for} \quad F_- \in \mathcal{F}_-,\ G_- \in \mathcal{F}_-. \tag{2.2}$$

Let $\mathcal{A}_\circ$ be a *globally* γ-invariant $*$-subalgebra of $\mathcal{F}$. Namely each element of $\mathcal{A}_\circ$ is not necessarily γ-invariant; however, $\gamma(\mathcal{A}_\circ) = \mathcal{A}_\circ$ holds. A linear map $\delta : \mathcal{A}_\circ \mapsto \mathcal{F}$ is called a *superderivation* of $\mathcal{F}$ with respect to γ if it is *odd* with respect to the grading:

$$\delta \cdot \gamma = -\gamma \cdot \delta \quad \text{on} \quad \mathcal{A}_\circ, \tag{2.3}$$

and it satisfies the *graded Leibniz rule*:

$$\delta(AB) = \delta(A)B + \gamma(A)\delta(B) \quad \text{for every} \quad A, B \in \mathcal{A}_\circ. \tag{2.4}$$

By (2.3)

$$\delta(\mathcal{A}_{\circ+}) \subset \mathcal{F}_-, \quad \delta(\mathcal{A}_{\circ-}) \subset \mathcal{F}_+, \tag{2.5}$$

where

$$\mathcal{A}_{\circ+} := \mathcal{A}_\circ \cap \mathcal{F}_+, \quad \mathcal{A}_{\circ-} := \mathcal{A}_\circ \cap \mathcal{F}_-. \tag{2.6}$$

Assume that $\mathcal{F}$ has a unit element $1 \in \mathcal{F}$ and the subalgebra $\mathcal{A}_\circ$ includes this unit. Since $\delta(1) = \delta(1 \cdot 1) = \delta(1)1 + 1\delta(1) = 2\delta(1)$, we have

$$\delta(1) = 0. \tag{2.7}$$

For a superderivation δ defined on $\mathcal{A}_\circ$ its conjugation superderivation is defined by

$$\delta^*(A) := -\big(\delta\big(\gamma(A^*)\big)\big)^* \quad \text{for every} \quad A \in \mathcal{A}_\circ, \tag{2.8}$$

where * on the right hand side denotes the $*$-operation of the C^*-algebra $\mathcal{F}$. It is easy to see that $\delta^* : \mathcal{A}_\circ \mapsto \mathcal{F}$ is a superderivation and that

$$\delta^{**} = \delta. \tag{2.9}$$

If a superderivation δ_{s} is 'symmetric' with respect to the $*$-operation on superderivations defined in (2.8),

$$\delta_{\mathrm{s}} = \delta_{\mathrm{s}}^* \quad \text{on} \quad \mathcal{A}_\circ, \tag{2.10}$$

then it is said to be *hermite*.

For a general superderivation δ of $\mathcal{F}$ defined on $\mathcal{A}_\circ$, we introduce the following pair of hermite superderivations:

$$\delta_{\mathrm{s},1} := \delta + \delta^*, \quad \delta_{\mathrm{s},2} := i(\delta - \delta^*) \quad \text{on} \ \ \mathcal{A}_\circ. \tag{2.11}$$

Conversely,

$$\delta = \frac{1}{2}(\delta_{\mathrm{s},1} - i\delta_{\mathrm{s},2}), \quad \delta^* = \frac{1}{2}(\delta_{\mathrm{s},1} + i\delta_{\mathrm{s},2}) \quad \text{on} \ \ \mathcal{A}_\circ. \tag{2.12}$$

Remark 2.1. We do not introduce Grassmann numbers (infinitesimal fermionic c-number spinors) into the C^*-algebra unlike the superfield formalism which is a basic language for supersymmetric field theories [26].

Remark 2.2. We denote a general superderivation by δ, and a general hermite superderivation by δ_{s}. Hence δ_{s} stands for both $\delta_{\mathrm{s},1}$ and $\delta_{\mathrm{s},2}$. Later δ is assumed to be nilpotent (accordingly to be non-hermite).

2.2. Supersymmetric States

We specify the meaning of 'supersymmetric states'.

Definition 2.3. Let δ be a superderivation of $\mathcal{F}$ and let $\mathcal{A}_\circ$ denote its domain subalgebra (which is a unital globally γ-invariant $*$-subalgebra of $\mathcal{F}$ by definition). If a state φ on $\mathcal{F}$ is invariant under δ, namely

$$\varphi\left(\delta(A)\right) = 0 \quad \text{for every} \ \ A \in \mathcal{A}_\circ, \tag{2.13}$$

then it is said to be supersymmetric (with respect to δ).

In the above definition, we do not need an actual supersymmetry relation that involves time evolution. It requires only a superderivation. The following two statements are obvious.

Proposition 2.4. *If a state φ on $\mathcal{F}$ is supersymmetric with respect to a superderivation δ defined on $\mathcal{A}_\circ$, then it is also supersymmetric with respect to δ^*:*

$$\varphi\left(\delta^*(A)\right) = 0 \quad \text{for every} \ A \in \mathcal{A}_\circ. \tag{2.14}$$

Proof. By (2.8) and the γ-invariance of $\mathcal{A}_\circ$, Eq. (2.13) implies Eq. (2.14) (and vice versa). □

Proposition 2.5. *Let δ be a superderivation defined on a globally γ-invariant $*$-subalgebra $\mathcal{A}_\circ$ of $\mathcal{F}$. A state φ on $\mathcal{F}$ is supersymmetric with respect to δ if and only if it is invariant under each of $\delta_{\mathrm{s},1}$ and $\delta_{\mathrm{s},2}$, where $\delta_{\mathrm{s},1}$ and $\delta_{\mathrm{s},2}$ denote the hermite superderivations on $\mathcal{A}_\circ$ given in* (2.11).

Proof. By (2.11), (2.12) Proposition 2.4 implies the assertion. □

We then specify 'unbroken–broken supersymmetry'.

Definition 2.6. Let δ be a superderivation of $\mathcal{F}$. If there exists a supersymmetric state on $\mathcal{F}$ with respect to δ as in Definition 2.3, then it is said that the supersymmetry (generated by δ) is unbroken. If no such state exists, then it is said that the supersymmetry is spontaneously broken.

Remark 2.7. In Definition 2.6 we do not require time evolution satisfying the supersymmetry relation. Hence Definition 2.6 is valid not only for the (usual) dynamical supersymmetry, but also for the kinematical supersymmetry which refers to a more general fermion symmetry. In this note we will deal with only the dynamical supersymmetry.

Later the following topological property of superderivations is important.

Definition 2.8. A superderivation δ defined on a norm-dense domain $\mathcal{A}_\circ$ of $\mathcal{F}$ is called norm-closable if for any sequence $\{A_n \in \mathcal{A}_\circ\}$ the convergence $\lim_{n\to\infty} A_n = 0$ and $\lim_{n\to\infty} \delta(A_n) = B$ in norm implies that $B = 0$. If δ is norm-closable, then its least closed extension $\overline{\delta}$ is called the closure of δ.

Proposition 2.9. *If a superderivation δ of $\mathcal{F}$ defined on a norm-dense globally γ-invariant $*$-subalgebra $\mathcal{A}_\circ$ is norm-closable, then its conjugate superderivation δ^* on $\mathcal{A}_\circ$ is also norm-closable.*

Proof. Note that for any $\{A_n \in \mathcal{A}_\circ\}$ convergence $\lim_{n\to\infty} A_n = 0$ and convergence $\lim_{n\to\infty} \gamma(A_n^*) = 0$ are equivalent. Hence by the form of δ^* as given in (2.8) the statement is satisfied. □

The following statement derives norm-closability of superderivations when the associated supersymmetry is unbroken under a general C^*-algebraic setting. (See [2] for the case of derivations.) It is obviously applicable to fermion lattice systems.

Proposition 2.10. *Let $\mathcal{F}$ be a unital graded C^*-algebra and let γ denote its grading automorphism. Let δ be a superderivation defined on a globally γ-invariant $*$-subalgebra $\mathcal{A}_\circ$ of $\mathcal{F}$. Suppose that $\mathcal{A}_\circ$ is norm-dense in $\mathcal{F}$. Suppose that there exists a supersymmetric state φ on $\mathcal{F}$ with respect to δ as in Definition 2.3 and that its GNS representation $\big(\pi_\varphi,\ \mathscr{H}_\varphi,\ \Omega_\varphi\big)$ gives a faithful representation of $\mathcal{F}$. Then δ is norm-closable.*

Proof. As in [5] let us introduce

$$Q\pi_\varphi(A)\Omega_\varphi := \pi_\varphi\big(\delta(A)\big)\Omega_\varphi, \quad A \in \mathcal{A}_\circ. \tag{2.15}$$

This gives a well-defined closable linear operator Q on $\mathscr{H}_\varphi$, as its adjoint Q^* is defined also on the norm-dense subspace $\pi_\varphi(\mathcal{A}_\circ)\Omega_\varphi$ by

$$Q^*\pi_\varphi(A)\Omega_\varphi := \pi_\varphi\big(\delta^*(A)\big)\Omega_\varphi, \quad A \in \mathcal{A}_\circ.$$

It is easy to see that the following operator equality holds for every $A \in \mathcal{A}_\circ$

$$\pi_\varphi\big(\delta(A)\big) = Q\pi_\varphi(A) - \pi_\varphi\big(\gamma(A)\big)Q \quad \text{on} \quad \pi_\varphi(\mathcal{A}_\circ)\Omega_\varphi. \tag{2.16}$$

Let $\{A_n \in \mathcal{A}_\circ\}_{n\in\mathbb{N}}$ be a sequence such that

$$A_n \to 0 \quad \text{and} \quad \delta(A_n) \to B \in \mathcal{F} \quad \text{in norm as} \quad n \to \infty.$$

This obviously yields the convergence $\gamma(A_n) \to 0$ in norm. In the GNS representation $\big(\pi_\varphi,\ \mathscr{H}_\varphi,\ \Omega_\varphi\big)$ for any supersymmetric state φ, we have for every $C, D \in \mathcal{A}_\circ$

$$
\begin{aligned}
&(\pi_\varphi(D)\Omega_\varphi, \pi_\varphi(B)\pi_\varphi(C)\Omega_\varphi)\\
&\quad= \lim_{n\to\infty} \big(\pi_\varphi(D)\Omega_\varphi,\ \pi_\varphi\big(\delta(A_n)\big)\pi_\varphi(C)\Omega_\varphi\big)\\
&\quad= \lim_{n\to\infty} (\pi_\varphi(D)\Omega_\varphi,\ Q\pi_\varphi(A_n)\pi_\varphi(C)\Omega_\varphi)\\
&\qquad - \lim_{n\to\infty} (\pi_\varphi(D)\Omega_\varphi,\ \pi_\varphi(\gamma(A_n))Q\pi_\varphi(C)\Omega_\varphi)\\
&\quad= \lim_{n\to\infty} (Q^*\pi_\varphi(D)\Omega_\varphi,\ \pi_\varphi(A_n)\pi_\varphi(C)\Omega_\varphi)\\
&\qquad - \lim_{n\to\infty} (\pi_\varphi(D)\Omega_\varphi,\ \pi_\varphi(\gamma(A_n))Q\pi_\varphi(C)\Omega_\varphi)\\
&\quad= 0 - 0 = 0,
\end{aligned}
$$

where the operator identity (2.16), and then the norm convergence $\pi_\varphi(A_n) \to 0$ and $\pi_\varphi(\gamma(A_n)) \to 0$ in norm is noted. As $\pi_\varphi(\mathcal{A}_\circ)\Omega_\varphi$ is dense in $\mathscr{H}_\varphi$, this yields $\pi_\varphi(B) = 0$. Since π_φ is injective by the assumption, we conclude that $B = 0$. The proof is completed. □

We are interested in the status of unbroken–broken supersymmetry in the infinite volume limit. Let us quote the following well-known statement from [25].

If supersymmetry is unbroken in an arbitrary finite volume V, this means that the ground-state energy $E(V)$ is zero for every V. Since the large-V limit of zero is zero, this means that the ground-state energy is zero in the infinite-volume limit, and that supersymmetry is unbroken in this limit.

The following proposition provides a rigorous derivation of the above statement. By this one can show the existence of supersymmetric states in the infinite-volume limit for some concrete models.

Proposition 2.11. *Let δ be a superderivation on a graded C^*-algebra $\mathcal{F}$ and let $\mathcal{A}_\circ$ denote its domain. Assume that there exists a sequence of superderivations $\{\delta_n\}$ on the same domain $\mathcal{A}_\circ$ such that*

$$\delta(A) = \lim_n \delta_n(A) \quad \text{in norm for each } A \in \mathcal{A}_\circ. \tag{2.17}$$

Assume that the supersymmetry generated by each δ_n is unbroken, namely for each δ_n there exits an invariant state. Then the supersymmetry generated by δ is unbroken.

Proof. We will provide a state on $\mathcal{F}$ which is invariant under the superderivation δ. Let ω_n denote a state on $\mathcal{F}$ which is invariant under δ_n. As the state space of a C^*-algebra is compact in the weak-$*$ topology, there exists a state which is a cluster point of the sequence $\{\omega_n\}$ in the weak-$*$ topology. Let ω denote any of such cluster points of $\{\omega_n\}$. We will show that ω gives a desired state.

Take an arbitrary $A \in \mathcal{A}_\circ$. We fix an arbitrary $\varepsilon > 0$. By the assumption (2.17) there exists k_0 such that for every $k \geq k_0$

$$\|\delta(A) - \delta_k(A)\| < \varepsilon/2.$$

As ω is a cluster point of $\{\omega_n\}$ in the weak-$*$ topology, we can take a subsequence $\{n'\}$ of $\{n\}$ such that

$$\omega(\delta(A)) = \lim_{n'} \omega_{n'}(\delta(A)).$$

Hence by passing to the subsequence $\{n'\}$ from the original sequence $\{n\}$ there exists k_1 such that for every $k \geq k_1$

$$|\omega(\delta(A)) - \omega_k(\delta(A))| < \varepsilon/2.$$

For any n (in the chosen sequence $\{n'\}$) such that $n \geq \max\{k_0, k_1\}$, we have

$$\begin{aligned}\Big|\omega(\delta(A)) - \omega_n(\delta_n(A))\Big| &= \Big|\omega(\delta(A)) - \omega_n(\delta(A)) + \omega_n(\delta(A)) - \omega_n(\delta_n(A))\Big| \\ &\leq |\omega(\delta(A)) - \omega_n(\delta(A))| + |\omega_n(\delta(A) - \delta_n(A))| \\ &\leq |\omega(\delta(A)) - \omega_n(\delta(A))| + \|\delta(A) - \delta_n(A)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Since $\omega_n(\delta_n(A)) = 0$ by the assumption, we have $|\omega(\delta(A))| < \varepsilon$. As $\varepsilon > 0$ is arbitrary, we have $\omega(\delta(A)) = 0$. Therefore, ω is supersymmetric with respect to δ. □

Remark 2.12. The convergence condition (2.17) in Proposition 2.11 may denote the infinite-volume limit of finite subsystems. Proposition 2.11 can be applied to fermion lattice systems and quantum spin lattice systems.

3. Overview

We shall provide a heuristic overview on supersymmetric fermion lattice models. For simplicity let us consider the one-dimensional lattice $\mathbb{Z}$. Let a_i and a_i^* denote the annihilation operator and the creation operator of a spinless fermion at a site $i \in \mathbb{Z}$. Those satisfy the canonical anticommutation relations:

$$\begin{aligned}\{a_i^*, a_j\} &= \delta_{i,j}\, 1,\\ \{a_i^*, a_j^*\} &= \{a_i, a_j\} = 0.\end{aligned}$$

In [20, Sect. 3][2] the following supercharge

$$Q := \sum_{i\in\mathbb{Z}} a_{2i+1} a_{2i}^* a_{2i-1}, \tag{3.1}$$

and the Hamiltonian

$$\begin{aligned}H &:= \{Q,\ Q^*\}\\ &= \sum_{i\in\mathbb{Z}} \big\{ a_{2i}^* a_{2i-1} a_{2i+2} a_{2i+3}^* + a_{2i-1}^* a_{2i} a_{2i+3} a_{2i+2}^* \\ &\quad + a_{2i}^* a_{2i} a_{2i+1} a_{2i+1}^* + a_{2i-1}^* a_{2i-1} a_{2i} a_{2i}^* - a_{2i-1}^* a_{2i-1} a_{2i+1} a_{2i+1}^* \big\}\end{aligned} \tag{3.2}$$

are introduced. Let N denote the total fermion number operator:

$$N := \sum_{i\in\mathbb{Z}} a_i^* a_i.$$

[2] In [20, Sect. 3] the half-sided lattice $\mathbb{N}$ is considered instead of the lattice $\mathbb{Z}$.

We see that

$$\{(-1)^N,\ Q\} = \{(-1)^N,\ Q^*\} = 0. \tag{3.3}$$

Namely Q and Q^* are fermionic. It is straightforward to see that the supercharge Q is nilpotent:

$$Q^2 = 0. \tag{3.4}$$

From (3.2) and (3.4) the commutativity between the Hamiltonian and the supercharges follows:

$$[H,\ Q] = [H,\ Q^*] = 0. \tag{3.5}$$

The pair of self-adjoint (real) supercharges are given by[3]

$$\mathcal{Q}_{\mathrm{s},1} := Q + Q^*, \quad \mathcal{Q}_{\mathrm{s},2} := i(Q - Q^*). \tag{3.6}$$

Inverting the above relations we have

$$Q = \frac{1}{2}\left(\mathcal{Q}_{\mathrm{s},1} - i\mathcal{Q}_{\mathrm{s},2}\right), \quad Q^* = \frac{1}{2}\left(\mathcal{Q}_{\mathrm{s},1} + i\mathcal{Q}_{\mathrm{s},2}\right). \tag{3.7}$$

By definition

$$\mathcal{Q}_{\mathrm{s},1}{}^* = \mathcal{Q}_{\mathrm{s},1}, \quad \mathcal{Q}_{\mathrm{s},2}{}^* = \mathcal{Q}_{\mathrm{s},2}, \tag{3.8}$$

$$\{(-1)^N,\ \mathcal{Q}_{\mathrm{s},1}\} = \{(-1)^N,\ \mathcal{Q}_{\mathrm{s},2}\} = 0, \tag{3.9}$$

$$\{\mathcal{Q}_{\mathrm{s},1},\ \mathcal{Q}_{\mathrm{s},2}\} = 0, \tag{3.10}$$

$$H = \mathcal{Q}_{\mathrm{s},1}^2 = \mathcal{Q}_{\mathrm{s},2}^2, \tag{3.11}$$

and

$$[H,\ \mathcal{Q}_{\mathrm{s},1}] = [H,\ \mathcal{Q}_{\mathrm{s},2}] = 0. \tag{3.12}$$

The above set of relations in terms of $\mathcal{Q}_{\mathrm{s},1}$, $\mathcal{Q}_{\mathrm{s},2}$ and H gives an equivalent expression of the supersymmetry algebra.

Note that the supercharges Q, Q^*, $\mathcal{Q}_{\mathrm{s},1}$, $\mathcal{Q}_{\mathrm{s},2}$ and the Hamiltonian H given above are not well defined as linear operators. However, they determine well-defined infinitesimal generators. Let $\mathcal{A}_\circ$ denote the local algebra which is generated by all local elements. Then let us define the superderivation

$$\delta(A) := [Q,\ A]_\Gamma \quad \text{for every} \quad A \in \mathcal{A}_\circ, \tag{3.13}$$

where the symbol $[\ ,\ ]_\Gamma$ denotes the graded commutator as in (2.2) with the grading automorphism $\Gamma := \mathrm{Ad}(-1)^N$. We note that

$$\delta^*(A) := [Q^*,\ A]_\Gamma \quad \text{for every} \quad A \in \mathcal{A}_\circ. \tag{3.14}$$

By using the (usual) commutator we define the derivation which gives an infinitesimal time-generator

$$d_0(A) := [H,\ A] \quad \text{for every} \quad A \in \mathcal{A}_\circ. \tag{3.15}$$

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[3] The subscript 's' of $\mathcal{Q}_{\mathrm{s},1}$ and $\mathcal{Q}_{\mathrm{s},2}$ will denote 'self-adjoint' or 'symmetric' operators on a Hilbert space.

From (3.1), (3.2), (3.4), (3.13)–(3.15), we obtain the supersymmetry algebra in terms of the superderivations and the time-derivation:

$$\delta \cdot \delta = \mathbf{0} \quad \text{on} \quad \mathcal{A}_\circ, \tag{3.16}$$

$$d_0 = \delta^* \cdot \delta + \delta \cdot \delta^* \quad \text{on} \quad \mathcal{A}_\circ. \tag{3.17}$$

Here we note that

$$\delta(\mathcal{A}_\circ) \subset \mathcal{A}_\circ, \quad \delta^*(\mathcal{A}_\circ) \subset \mathcal{A}_\circ, \quad d_0(\mathcal{A}_\circ) \subset \mathcal{A}_\circ. \tag{3.18}$$

Let

$$\begin{aligned} \delta_{\mathrm{s},1}(A) &:= [\mathcal{Q}_{\mathrm{s},1},\ A]_\Gamma \quad \text{for every} \quad A \in \mathcal{A}_\circ, \\ \delta_{\mathrm{s},2}(A) &:= [\mathcal{Q}_{\mathrm{s},2},\ A]_\Gamma \quad \text{for every} \quad A \in \mathcal{A}_\circ. \end{aligned} \tag{3.19}$$

Then the supersymmetry algebra expressed by Eqs. (3.16), (3.17) is rewritten as

$$\delta_{\mathrm{s},1} \cdot \delta_{\mathrm{s},2} + \delta_{\mathrm{s},2} \cdot \delta_{\mathrm{s},1} = \mathbf{0} \quad \text{on} \quad \mathcal{A}_\circ, \tag{3.20}$$

$$s d_0 = \delta_{\mathrm{s},1}^{\ 2} = \delta_{\mathrm{s},2}^{\ 2} \quad \text{on} \quad \mathcal{A}_\circ. \tag{3.21}$$

Here we note that

$$\delta_{\mathrm{s},1}(\mathcal{A}_\circ) \subset \mathcal{A}_\circ, \quad \delta_{\mathrm{s},2}(\mathcal{A}_\circ) \subset \mathcal{A}_\circ, \quad d_0(\mathcal{A}_\circ) \subset \mathcal{A}_\circ. \tag{3.22}$$

From (3.5), (3.13)–(3.15) the following commutativity relations hold:

$$\delta \cdot d_0 = d_0 \cdot \delta, \quad \delta^* \cdot d_0 = d_0 \cdot \delta^* \quad \text{on} \quad \mathcal{A}_\circ. \tag{3.23}$$

Similarly from (3.12), (3.19) and (3.15) the following commutativity relations hold:

$$\delta_{\mathrm{s},1} \cdot d_0 = d_0 \cdot \delta_{\mathrm{s},1}, \quad \delta_{\mathrm{s},2} \cdot d_0 = d_0 \cdot \delta_{\mathrm{s},2} \quad \text{on} \quad \mathcal{A}_\circ. \tag{3.24}$$

Remark 3.1. If a global supersymmetry is spontaneously broken, then the corresponding supercharge does not exist as a densely defined linear operator, cf. [26, Chapter 29.1]. If the supersymmetry generated by a superderivation is unbroken, then the corresponding supercharge is given as a closable linear operator on the GNS space for any supersymmetric state by (2.15) in Proposition 2.10.

4. Supersymmetric Fermion Lattice Systems

4.1. Fermion Lattice Systems on the CAR Algebra

We shall formulate fermion lattice systems on a quasi-local C^*-algebra. For simplicity we will consider the ν-dimensional cubic integer lattice $\mathbb{Z}^\nu$. However, it will be clear that our setup is easily extended to other lattices. For $x = (x_i),\ y = (y_i) \in \mathbb{Z}^\nu$, let $|x - y| := \max_{1 \le i \le \nu} |x_i - y_i|$. For a subset I of $\mathbb{Z}^\nu$, $|\mathrm{I}|$ denotes the volume, i.e., the number of sites in I. When the volume of $\mathrm{I} \subset \mathbb{Z}^\nu$ is finite, we will denote $\mathrm{I} \Subset \mathbb{Z}^\nu$. For each $l \in \mathbb{N} \cup \{0\} \equiv \{0, 1, 2, 3, \ldots\}$, let us take the following lattice cube with its edge length l

$$\mathrm{C}_l := \{x = (x_1, \ldots, x_\nu) \in \mathbb{Z}^\nu \ ; \ 0 \le x_i \le l,\ i = 1, \ldots, \nu\}. \tag{4.1}$$

By definition its volume $|\mathrm{C}_l|$ is $(l+1)^\nu$.

We consider interacting spinless fermions on $\mathbb{Z}^\nu$. It is automatic to extend the following formulation to the case of fermions with spins. Let a_i and a_i^* denote the annihilation operator and the creation operator of a spinless fermion at $i \in \mathbb{Z}^\nu$, respectively. The canonical anticommutation relations (CARs) are

$$\begin{aligned} \{a_i^*, a_j\} &= \delta_{i,j}\, 1, \\ \{a_i^*, a_j^*\} &= \{a_i, a_j\} = 0. \end{aligned} \tag{4.2}$$

For each $\mathrm{I} \Subset \mathbb{Z}^\nu$, take the finite-dimensional algebra $\mathcal{F}(\mathrm{I})$ generated by $\{a_i^*,\, a_i\,;\ i \in \mathrm{I}\}$. It is isomorphic to $\mathrm{M}_{2^{|\mathrm{I}|}}(\mathbb{C})$, the algebra of all $2^{|\mathrm{I}|} \times 2^{|\mathrm{I}|}$ complex matrices. For $\mathrm{I} \subset \mathrm{J} \Subset \mathbb{Z}^\nu$, $\mathcal{F}(\mathrm{I})$ is imbedded into $\mathcal{F}(\mathrm{J})$ as a subalgebra. Let

$$\mathcal{A}_\circ := \bigcup_{\mathrm{I} \Subset \mathbb{Z}^\nu} \mathcal{F}(\mathrm{I}). \tag{4.3}$$

The total C^*-system $\mathcal{F}$ is given by the norm completion of this normed $*$-algebra $\mathcal{A}_\circ$. It is the CAR algebra. The dense $*$-subalgebra $\mathcal{A}_\circ$ in $\mathcal{F}$ is called the local algebra. Let γ denote the automorphism on the C^*-algebra $\mathcal{F}$ determined by

$$\gamma(a_i) = -a_i, \quad \gamma(a_i^*) = -a_i^*, \quad i \in \mathbb{Z}^\nu. \tag{4.4}$$

This γ gives a grading on $\mathcal{F}$:

$$\mathcal{F}_+ := \{F \in \mathcal{F} \mid \gamma(F) = F\}, \quad \mathcal{F}_- := \{F \in \mathcal{F} \mid \gamma(F) = -F\}, \tag{4.5}$$

$$\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-. \tag{4.6}$$

For each $\mathrm{I} \Subset \mathbb{Z}^\nu$

$$\mathcal{F}_+(\mathrm{I}) := \mathcal{F}(\mathrm{I}) \cap \mathcal{F}_+, \quad \mathcal{F}_-(\mathrm{I}) := \mathcal{F}(\mathrm{I}) \cap \mathcal{F}_-, \tag{4.7}$$

$$\mathcal{F}(\mathrm{I}) = \mathcal{F}_+(\mathrm{I}) \oplus \mathcal{F}_-(\mathrm{I}). \tag{4.8}$$

By the CARs (4.2) the following γ-locality holds:

$$[A,\ B]_\gamma = 0 \quad \text{for every} \quad A \in \mathcal{F}(\mathrm{I}) \text{ and } \ B \in \mathcal{F}(\mathrm{J}) \quad \text{if} \ \ \mathrm{I} \cap \mathrm{J} = \emptyset,\ \mathrm{I}, \mathrm{J} \Subset \mathbb{Z}^\nu. \tag{4.9}$$

4.2. Superderivations Made by Local Fermionic Charges

We shall provide a superderivation by ‘local fermionic charges’ over the lattice. This is analogous to the well-known construction of a time generator by local Hamiltonians on fermion (or quantum spin) lattice systems, see [1] and [4, Sect. 6.2]. Let Ψ be a map from the set of finite regions $\{\mathrm{I};\ \mathrm{I} \Subset \mathbb{Z}^\nu\}$ to the local algebra $\mathcal{A}_\circ$ such that

$$\Psi : \mathrm{I} \longmapsto \Psi(\mathrm{I}) \in \mathcal{F}_-(\mathrm{I}) \quad \text{for every } \mathrm{I} \Subset \mathbb{Z}^\nu. \tag{4.10}$$

The above map Ψ will be called *an assignment of local fermionic charges* (over $\mathbb{Z}^\nu$). Suppose that Ψ and Φ are two assignments of local fermionic charges. We can consider their linear combination:

$$(c\Psi + d\Phi)(\mathrm{I}) := c\Psi(\mathrm{I}) + d\Phi(\mathrm{I}), \quad c, d \in \mathbb{C} \quad \text{for every } \mathrm{I} \Subset \mathbb{Z}^\nu. \tag{4.11}$$

For any $c, d \in \mathbb{C}$, $c\Psi + d\Phi$ is also an assignment of local fermionic charges. The *conjugate* of Ψ is defined as

$$\Psi^* : \mathrm{I} \longmapsto \Psi(\mathrm{I})^* \in \mathcal{F}_-(\mathrm{I}) \quad \text{for every } \mathrm{I} \Subset \mathbb{Z}^\nu. \tag{4.12}$$

If $\Psi = \Psi^*$, namely

$$\Psi(\mathrm{I}) = \Psi(\mathrm{I})^* \quad \text{for every} \quad \mathrm{I} \Subset \mathbb{Z}^\nu, \tag{4.13}$$

then Ψ is said to be hermite. In the following, Ψ is generically *not* hermite.

For a general assignment of local fermionic charges Ψ, let

$$\Psi_{\mathrm{s},1} := \Psi + \Psi^*, \quad \Psi_{\mathrm{s},2} := i(\Psi - \Psi^*). \tag{4.14}$$

These are hermite by definition. Conversely

$$\Psi = \frac{1}{2}(\Psi_{\mathrm{s},1} - i\Psi_{\mathrm{s},2}), \quad \Psi^* = \frac{1}{2}(\Psi_{\mathrm{s},1} + i\Psi_{\mathrm{s},2}). \tag{4.15}$$

We shall provide assumptions upon Ψ. First assume that there exists an $r \geq 0$ such that

$$\Psi(\mathrm{I}) = 0 \quad \text{whenever } \mathrm{diam}(\mathrm{I}) \equiv \max_{x,y \in \mathrm{I}} |x - y| > r. \tag{4.16}$$

The minimum non-negative integer r satisfying the condition (4.16) is called the range of Ψ. Obviously Ψ^* has the same range r as that of Ψ. Each of $\Psi_{\mathrm{s},1}$ and $\Psi_{\mathrm{s},2}$ has its finite range which is equal to or less than r.

Let Ψ be an assignment of local fermionic charges satisfying the finite-range condition (4.16). Then for every $\mathrm{I} \Subset \mathbb{Z}^\nu$, we can define a γ-graded infinitesimal transformation on $\mathcal{F}(\mathrm{I})$ as

$$\delta_\Psi(A) := \sum_{\mathrm{X} \cap \mathrm{I} \neq \emptyset,\, \mathrm{X} \Subset \mathbb{Z}^\nu} [\Psi(\mathrm{X}),\ A]_\gamma \quad \text{for every} \quad A \in \mathcal{F}(\mathrm{I}), \tag{4.17}$$

where the summation is taken over all finite subsets $\{\mathrm{X}; \mathrm{X} \Subset \mathbb{Z}^\nu\}$ that have a non-trivial intersection with I. We may suppress '$\mathrm{X} \Subset \mathbb{Z}^\nu$' for simplicity when it is clear from the context. If $\mathrm{I} \subset \mathrm{J} \Subset \mathbb{Z}^\nu$, then it follows from the γ-locality (4.9) and the defining formula (4.17) that for every $A \in \mathcal{F}(\mathrm{I})$

$$\begin{aligned}\sum_{\mathrm{X} \cap \mathrm{J} \neq \emptyset} [\Psi(\mathrm{X}),\ A]_\gamma &= \sum_{\mathrm{X} \cap \mathrm{I} \neq \emptyset} [\Psi(\mathrm{X}),\ A]_\gamma + \sum_{\mathrm{X} \cap \mathrm{I} = \emptyset,\, \mathrm{X} \cap \mathrm{J} \neq \emptyset} [\Psi(\mathrm{X}),\ A]_\gamma \\ &= \sum_{\mathrm{X} \cap \mathrm{I} \neq \emptyset} [\Psi(\mathrm{X}),\ A]_\gamma.\end{aligned}$$

Therefore δ_Ψ is a well-defined linear map on $\mathcal{A}_\circ$, and the set of formulas (4.17) for $\{\mathrm{I}, \mathrm{I} \Subset \mathbb{Z}^\nu\}$ uniquely yields

$$\delta_\Psi(A) = \sum_{\mathrm{X}} [\Psi(\mathrm{X}),\ A]_\gamma \quad \text{for every} \quad A \in \mathcal{A}_\circ. \tag{4.18}$$

In the above formula (4.18), for each $A \in \mathcal{A}_\circ$ only a finite number of $\{\mathrm{X}; \mathrm{X} \Subset \mathbb{Z}^\nu\}$ give non-zero contributions. We verify that this δ_Ψ satisfies the desiderata (2.3), (2.4) for superderivations. Thus the assignment of local fermionic charges provides a fermionic generator. It is similar to the assignment of local Hamiltonians that provides a time-generator.

We note the following nice property:

$$\delta_{\Psi}(\mathcal{A}_{\circ}) \subset \mathcal{A}_{\circ}. \tag{4.19}$$

More specifically,

$$\delta_{\Psi}(\mathcal{F}_{\pm}(\mathrm{I})) \subset \mathcal{F}_{\mp}(\hat{\mathrm{I}}_r) \quad \text{for every} \quad \mathrm{I} \Subset \mathbb{Z}^{\nu}, \tag{4.20}$$

where $\hat{\mathrm{I}}_r \Subset \mathbb{Z}^{\nu}$ denotes the extended finite region of I by the range of Ψ:

$$\hat{\mathrm{I}}_r := \left\{ x \in \mathbb{Z}^{\nu} \ ; \ \min_{y \in \mathrm{I}} |x-y| \le r \right\}. \tag{4.21}$$

The conjugate superderivation δ_{Ψ}^{*} as in (2.8) can be written by the conjugate assignment Ψ^{*} of (4.12). Namely for each $\mathrm{I} \Subset \mathbb{Z}^{\nu}$

$$\delta_{\Psi}^{*}(A) = \delta_{\Psi^{*}}(A) = \sum_{\mathrm{X} \cap \mathrm{I} \neq \emptyset} [\Psi^{*}(\mathrm{X}),\ A]_{\gamma} \quad \text{for every} \quad A \in \mathcal{F}(\mathrm{I}). \tag{4.22}$$

Similarly to (4.19) (4.20) we have

$$\delta_{\Psi}^{*}(\mathcal{A}_{\circ}) \subset \mathcal{A}_{\circ}, \tag{4.23}$$

and

$$\delta_{\Psi}^{*}(\mathcal{F}_{\pm}(\mathrm{I})) \subset \mathcal{F}_{\mp}(\hat{\mathrm{I}}_r) \quad \text{for every} \quad \mathrm{I} \Subset \mathbb{Z}^{\nu}. \tag{4.24}$$

For the superderivation δ_{Ψ} associated with Ψ, which is supposed to be non-hermite, the pair of hermite superderivations are given as in (2.11):

$$\delta_{\Psi_{\mathrm{s},1}} := \delta_{\Psi} + \delta_{\Psi}^{*}, \quad \delta_{\Psi_{\mathrm{s},2}} := i(\delta_{\Psi} - \delta_{\Psi}^{*}) \quad \text{on} \quad \mathcal{A}_{\circ}. \tag{4.25}$$

As in (2.12)

$$\delta_{\Psi} = \frac{1}{2}(\delta_{\Psi_{\mathrm{s},1}} - i\delta_{\Psi_{\mathrm{s},2}}), \quad \delta_{\Psi}^{*} = \frac{1}{2}(\delta_{\Psi_{\mathrm{s},1}} + i\delta_{\Psi_{\mathrm{s},2}}) \text{ on } \mathcal{A}_{\circ}. \tag{4.26}$$

By (4.14) (4.25) for each $\mathrm{I} \Subset \mathbb{Z}^{\nu}$

$$\begin{aligned} \delta_{\Psi_{\mathrm{s},1}}(A) &= \sum_{\mathrm{X} \cap \mathrm{I} \neq \emptyset} [\Psi_{\mathrm{s},1}(\mathrm{X}),\ A]_{\gamma} = \sum_{\mathrm{X}} [\Psi_{\mathrm{s},1}(\mathrm{X}),\ A]_{\gamma} \quad \text{for every} \quad A \in \mathcal{F}(\mathrm{I}), \\ \delta_{\Psi_{\mathrm{s},2}}(A) &= \sum_{\mathrm{X} \cap \mathrm{I} \neq \emptyset} [\Psi_{\mathrm{s},2}(\mathrm{X}),\ A]_{\gamma} = \sum_{\mathrm{X}} [\Psi_{\mathrm{s},2}(\mathrm{X}),\ A]_{\gamma} \quad \text{for every} \quad A \in \mathcal{F}(\mathrm{I}). \end{aligned} \tag{4.27}$$

We immediately see that

$$\delta_{\Psi_{\mathrm{s},1}}(\mathcal{A}_{\circ}) \subset \mathcal{A}_{\circ}, \quad \delta_{\Psi_{\mathrm{s},2}}(\mathcal{A}_{\circ}) \subset \mathcal{A}_{\circ}, \tag{4.28}$$

and more specifically,

$$\delta_{\Psi_{\mathrm{s},1}}(\mathcal{F}_{\pm}(\mathrm{I})) \subset \mathcal{F}_{\mp}(\hat{\mathrm{I}}_r), \quad \delta_{\Psi_{\mathrm{s},2}}(\mathcal{F}_{\pm}(\mathrm{I})) \subset \mathcal{F}_{\mp}(\hat{\mathrm{I}}_r) \quad \text{for every} \quad \mathrm{I} \Subset \mathbb{Z}^{\nu}. \tag{4.29}$$

Second, assume that Ψ is uniformly bounded,

$$\|\Psi\|_{\infty} := \sup_{\mathrm{I} \Subset \mathbb{Z}^{\nu}} \|\Psi(\mathrm{I})\| < \infty. \tag{4.30}$$

Then due to (4.12)

$$\|\Psi^{*}\|_{\infty} = \|\Psi\|_{\infty} < \infty. \tag{4.31}$$

The uniform boundedness of Ψ and Ψ^* together with (4.14) implies the uniform boundedness of $\Psi_{\mathrm{s},1}$ and $\Psi_{\mathrm{s},2}$:

$$\|\Psi_{\mathrm{s},1}\|_\infty \leq 2\|\Psi\|_\infty < \infty, \quad \|\Psi_{\mathrm{s},2}\|_\infty \leq 2\|\Psi\|_\infty < \infty. \tag{4.32}$$

Third, assume that the superderivation δ_Ψ associated with Ψ satisfies the nilpotent condition:

$$\delta_\Psi \cdot \delta_\Psi = \mathbf{0} \quad \text{on} \ \ \mathcal{A}_\circ. \tag{4.33}$$

We may say that Ψ is 'nilpotent' if its associated superderivation δ_Ψ is nilpotent (4.33). It is a strange terminology, but may be a convenient shorthand.

Let us summarize the terminologies given so far.

Definition 4.1. A map $\Psi : \mathrm{I} \longmapsto \Psi(\mathrm{I}) \in \mathcal{F}_-(\mathrm{I})$ defined on $\{\mathrm{I};\ \mathrm{I} \Subset \mathbb{Z}^\nu\}$ is called an assignment of local fermionic charges on the fermion lattice system. If Ψ satisfies the condition (4.16) with some finite r, then it is said to be finite range. For a finite-range assignment of local fermion charges Ψ, the linear map δ_Ψ from the local algebra $\mathcal{A}_\circ$ into $\mathcal{A}_\circ$ by the formula (4.17) is called the superderivation associated with Ψ. The set of all bounded finite-range assignments of local fermion charges is denoted by $\mathscr{C}$. The set of all $\Psi \in \mathscr{C}$ whose associated superderivation δ_Ψ satisfies the nilpotent condition (4.33) is denoted by $\mathscr{C}^\sharp$.

Remark 4.2. We do not assume translation covariance for Ψ in Definition 4.1. Actually we will consider an example which is periodic, but not translationally covariant.

Remark 4.3. To make a concrete example of $\mathscr{C}^\sharp$, the nilpotent condition (4.33) is the most non-trivial requirement. Later we will show a criterion of the nilpotent condition which can be easily checked for $\Psi \in \mathscr{C}$.

Remark 4.4. It is obvious that $\mathscr{C}$ is a $\mathbb{C}$-linear space. However, $\mathscr{C}^\sharp$ is not a linear space, since the nilpotent condition (4.33) is not generally preserved under linear summation.

The following is due to Definition 2.6.

Definition 4.5. Let Ψ be an element of $\mathscr{C}^\sharp$ given in Definition 4.1. If there exists a supersymmetric state φ with respect to the associated superderivation δ_Ψ, then it is said that $\Psi \in \mathscr{C}^\sharp$ gives an unbroken-supersymmetry model, or in short $\Psi \in \mathscr{C}^\sharp$ is unbroken supersymmetry. The set of all $\Psi \in \mathscr{C}^\sharp$ giving an unbroken-supersymmetry model is denoted by $\mathscr{C}^\sharp_{unbroken}$.

4.3. Supersymmetry Formula on the Local Algebra

The following lemma is obvious.

Lemma 4.6. *Suppose that a finite-range assignment of local fermion charges Ψ is given. If the associated superderivation δ_Ψ is nilpotent, namely the condition (4.33) is satisfied, then the conjugate superderivation $\delta_\Psi^* = \delta_{\Psi^*}$ is also nilpotent*:

$$\delta_\Psi^* \cdot \delta_\Psi^* = \mathbf{0} \quad \text{on} \ \mathcal{A}_\circ. \tag{4.34}$$

Proof. By noting (2.8), (2.3) we see that Eqs. (4.33) and (4.34) are equivalent. □

Remark 4.7. It is quite obvious that $\mathscr{C}$ is closed under the $*$-operation by (4.12). By Lemma 4.6 $\mathscr{C}^\sharp$ is also closed under the $*$-operation.

We *define* a derivation which has the required supersymmetric form.

Definition 4.8. For any $\Psi \in \mathscr{C}^\sharp$ define the following derivation

$$d_{\Psi\,0} := \delta_\Psi^* \cdot \delta_\Psi + \delta_\Psi \cdot \delta_\Psi^* \quad \text{on} \quad \mathcal{A}_\circ. \tag{4.35}$$

Since both $\delta_\Psi^* \cdot \delta_\Psi$ and $\delta_\Psi \cdot \delta_\Psi^*$ can be defined on $\mathcal{A}_\circ$ due to (4.19), (4.23), the above definition makes sense. Furthermore,

$$d_{\Psi\,0}(\mathcal{A}_\circ) \subset \mathcal{A}_\circ, \tag{4.36}$$

more specifically, due to (4.20), (4.24)

$$d_{\Psi\,0}(\mathcal{F}_\pm(\mathrm{I})) \subset \mathcal{F}_\pm(\hat{\mathrm{I}}_{2r}) \quad \text{for every} \quad \mathrm{I} \Subset \mathbb{Z}^\nu. \tag{4.37}$$

It is easy to see that $d_{\Psi\,0}$ is a derivation (satisfying the Leibniz rule). From the oddness of superderivations (2.3) and the definition of $d_{\Psi\,0}$ it follows that

$$d_{\Psi\,0} \cdot \gamma = \gamma \cdot d_{\Psi\,0} \quad \text{on} \quad \mathcal{A}_\circ. \tag{4.38}$$

It is possible to rewrite the infinitesimal supersymmetry formula of Definition 4.8 in terms of the pair of (independent) hermite superderivations $\delta_{\Psi_{s,1}}$ and $\delta_{\Psi_{s,2}}$.

Proposition 4.9. *The set of relations* (4.33) *and* (4.35) *is equivalent to the following set of relations in terms of the hermite superderivations* $\delta_{\Psi_{s,1}}$ *and* $\delta_{\Psi_{s,2}}$ *in Eq.* (4.25):

$$\delta_{\Psi_{s,1}} \cdot \delta_{\Psi_{s,2}} + \delta_{\Psi_{s,2}} \cdot \delta_{\Psi_{s,1}} = \mathbf{0} \quad \text{on } \mathcal{A}_\circ, \tag{4.39}$$

$$d_{\Psi\,0} = {\delta_{\Psi_{s,1}}}^2 = {\delta_{\Psi_{s,2}}}^2 \quad \text{on } \mathcal{A}_\circ. \tag{4.40}$$

Proof. First we will derive the formulas (4.39), (4.40) from (4.33) and (4.35). By noting (4.25) and Lemma 4.6 we have

$$\begin{aligned}\delta_{\Psi_{s,1}} \cdot \delta_{\Psi_{s,2}} &= i(\delta_\Psi \cdot \delta_\Psi - \delta_\Psi^* \cdot \delta_\Psi^*) + i(\delta_\Psi^* \cdot \delta_\Psi - \delta_\Psi \cdot \delta_\Psi^*)\\ &= \mathbf{0} + i(\delta_\Psi^* \cdot \delta_\Psi - \delta_\Psi \cdot \delta_\Psi^*)\\ &= i(\delta_\Psi^* \cdot \delta_\Psi - \delta_\Psi \cdot \delta_\Psi^*),\end{aligned}$$

and

$$\begin{aligned}\delta_{\Psi_{s,2}} \cdot \delta_{\Psi_{s,1}} &= i(\delta_\Psi \cdot \delta_\Psi - \delta_\Psi^* \cdot \delta_\Psi^*) - i(\delta_\Psi^* \cdot \delta_\Psi - \delta_\Psi \cdot \delta_\Psi^*)\\ &= \mathbf{0} - i(\delta_\Psi^* \cdot \delta_\Psi - \delta_\Psi \cdot \delta_\Psi^*)\\ &= -i(\delta_\Psi^* \cdot \delta_\Psi - \delta_\Psi \cdot \delta_\Psi^*).\end{aligned}$$

These yield

$$\delta_{\Psi_{s,1}} \cdot \delta_{\Psi_{s,2}} + \delta_{\Psi_{s,2}} \cdot \delta_{\Psi_{s,1}} = \mathbf{0}.$$

We have

$$\delta_{\Psi_{s,1}} \cdot \delta_{\Psi_{s,1}} = (\delta_\Psi^* \cdot \delta_\Psi + \delta_\Psi \cdot \delta_\Psi^*) + \delta_\Psi \cdot \delta_\Psi + \delta_\Psi^* \cdot \delta_\Psi^* = d_{\Psi\,0} + \mathbf{0} + \mathbf{0} = d_{\Psi\,0},$$

and similarly

$$\delta_{\Psi_{s,2}} \cdot \delta_{\Psi_{s,2}} = d_{\Psi\,0}.$$

We will show the converse direction. Assume that Eqs. (4.39) and (4.40) are satisfied. By noting (4.26) we have

$$\begin{aligned}\delta_{\Psi} \cdot \delta_{\Psi} &= \frac{1}{4}(\delta_{\Psi_{s,1}} \cdot \delta_{\Psi_{s,1}} - \delta_{\Psi_{s,2}} \cdot \delta_{\Psi_{s,2}}) - \frac{i}{4}(\delta_{\Psi_{s,1}} \cdot \delta_{\Psi_{s,2}} + \delta_{\Psi_{s,2}} \cdot \delta_{\Psi_{s,1}}) \\ &= \frac{1}{4}(d_{\Psi\,0} - d_{\Psi\,0}) + \mathbf{0} = \mathbf{0}.\end{aligned}$$

We have

$$\delta_{\Psi}^{*} \cdot \delta_{\Psi} + \delta_{\Psi} \cdot \delta_{\Psi}^{*} = \frac{1}{4}(\delta_{\Psi_{s,1}} \cdot \delta_{\Psi_{s,1}} + \delta_{\Psi_{s,2}} \cdot \delta_{\Psi_{s,2}}) \cdot 2 = \frac{1}{2}(d_{\Psi\,0} + d_{\Psi\,0}) = d_{\Psi\,0}.$$

Thus we have shown the assertion. □

At this stage, the derivation d_0 defined in Definition 4.8 is not related to time evolution. We have not yet obtained a supersymmetric dynamics which will be shown in the next subsection. (A similar problem arises to formulate supersymmetry in a fermion–boson C^* system. See [6, 19] for the detail.)

4.4. Supersymmetric C^*-Dynamics on the CAR Algebra

The supersymmetry formula (4.35) in Definition 4.8 made by a nilpotent superderivation (4.33) expresses the same supersymmetry algebra as described in Sect. 3. However, to make a more complete form global time evolution should be involved as well as its infinitesimal generator. We immediately see that there exists a strongly continuous one-parameter group of $*$-automorphisms generated by the derivation $d_{\Psi\,0}$ on the CAR algebra $\mathcal{F}$, since $d_{\Psi\,0}$ defined on the local algebra is of finite range as noted in (4.37). We shall provide its detailed construction and discuss its characteristic properties due to hidden supersymmetry.

4.4.1. Expansion of the Iteration of Superderivations. Take any nilpotent finite-range assignment of local fermion charges $\Psi \in \mathscr{C}^{\sharp}$ of Definition 4.1. We will denote the hermite assignments of local fermion charges $\Psi_{s,1}$ and $\Psi_{s,2}$ in (4.14) simply by Ψ_s. There will arise no essential difference between $\Psi_{s,1}$ and $\Psi_{s,2}$ in the following. Of course we have to use '1' and '2' consistently.

Consider $\delta_{\Psi_s}^{n} \equiv \underbrace{\delta_{\Psi_s} \cdot \delta_{\Psi_s} \cdots \delta_{\Psi_s}}_{n \text{ times}}$ for $n \in \mathbb{N}$, i.e., the nth iterate of the map δ_{Ψ_s}. Due to the finite-range condition (4.29) for δ_{Ψ_s}, $\delta_{\Psi_s}^{n}$ is well-defined and finite range for each $n \in \mathbb{N}$. Particularly, for $n = 2m - 1$ ($m \in \mathbb{N}$) $\delta_{\Psi_s}^{n}$ is a superderivation from $\mathcal{A}_\circ$ into $\mathcal{A}_\circ$, while for $n = 2m$ ($m \in \mathbb{N}$) it is a derivation from $\mathcal{A}_\circ$ into $\mathcal{A}_\circ$. Take any $\mathrm{I} \Subset \mathbb{Z}^{\nu}$. For every $A \in \mathcal{F}(\mathrm{I})$ by using the formula (4.27) repeatedly we have

$$\delta_{\Psi_s}^{n}(A) = \sum_{X_1, X_2, \ldots, X_n} \Big[\Psi_s(X_n), \ldots \Big[\Psi_s(X_2), \ \Big[\Psi_s(X_1), \ A\Big]_\gamma\Big]_\gamma \cdots\Big]_\gamma$$
$$= \sum_{\substack{\{(X_1, X_2, \ldots, X_n) \,|\, X_1 \cap V_0 \neq \emptyset, \\ X_2 \cap V_1 \neq \emptyset, \ldots \text{ and } X_n \cap V_{n-1} \neq \emptyset\}}} \Big[\Psi_s(X_n), \ldots \Big[\Psi_s(X_2), \ \Big[\Psi_s(X_1), \ A\Big]_\gamma\Big]_\gamma \cdots\Big]_\gamma$$
$$\in \mathcal{F}(\hat{I}_{nr}), \tag{4.41}$$

where we have defined

$$V_0 := I, \quad V_j := X_j \cup X_{j-1} \cup \cdots \cup X_1 \cup V_0 \text{ for } j \in \{1, 2, 3, \ldots\}. \tag{4.42}$$

The condition upon the multiplets $(X_1, X_2, \ldots, X_n)$ in the second equality of (4.41) is due to the γ-locality (4.9); the graded commutator of $\Psi_s(X)$ with any element outside of X vanishes, and $\delta_{\Psi_s}^{n}(A) \in \mathcal{F}(\hat{I}_{nr})$ is due to (4.29). Hence the finite regions V_j $(j \in \{1, 2, 3, \ldots\})$ which may give non-zero contribution to the formula (4.41) should satisfy

$$V_j \subset \hat{I}_{jr}, \quad |V_j| \le |I| + j(|C_r| - 1) < |I| + j \cdot (r+1)^\nu. \tag{4.43}$$

We will consider the derivation $d_{\Psi\,0}$ of Definition 4.8. By the identity $d_{\Psi\,0} = {\delta_{\Psi_s}}^2$ on $\mathcal{A}_\circ$ given in Proposition 4.9, Eq. (4.41) gives a similar expansion formula for the nth iteration of $d_{\Psi\,0}$ $(n \in \mathbb{N})$: For each $A \in \mathcal{F}(I)$

$${d_{\Psi\,0}}^n(A) = {\delta_{\Psi_s}}^{2n}(A)$$
$$= \sum_{\substack{\{(X_1, X_2, \ldots, X_{2n}) \,|\, X_1 \cap V_0 \neq \emptyset, \\ X_2 \cap V_1 \neq \emptyset, \ldots \text{ and } X_{2n} \cap V_{2n-1} \neq \emptyset\}}} \Big[\Psi_s(X_{2n}), \ldots \Big[\Psi_s(X_2), \Big[\Psi_s(X_1), \ A\Big]_\gamma\Big]_\gamma \cdots\Big]_\gamma$$
$$\in \mathcal{F}(\hat{I}_{2nr}), \tag{4.44}$$

where the notation (4.42) is used.

Let us introduce a set of bounded superderivations indexed by $\Lambda \Subset \mathbb{Z}^\nu$ as

$$\delta_{\Psi_s,\,\Lambda}(F) := \sum_{X \cap \Lambda \neq \emptyset} [\Psi_s(X), \ F]_\gamma \quad \text{for every} \quad F \in \mathcal{F}. \tag{4.45}$$

As the range of Ψ_s is equal to or less than r, for each Λ fixed, the finite subsets $X \Subset \mathbb{Z}^\nu$ which may give non-zero contribution to the formula (4.45) are all included in $\hat{\Lambda}_r$. If $I \subset \Lambda \Subset \mathbb{Z}^\nu$, then

$$\delta_{\Psi_s}(A) = \delta_{\Psi_s,\,\Lambda}(A) \quad \text{for every} \quad A \in \mathcal{F}(I).$$

Hence

$$\delta_{\Psi_s}(A) = \lim_{\Lambda \nearrow \mathbb{Z}^\nu} \delta_{\Psi_s,\,\Lambda}(A) \quad \text{for each} \quad A \in \mathcal{A}_\circ, \tag{4.46}$$

where '$\Lambda \nearrow \mathbb{Z}^\nu$' means that the net $\{\Lambda; \ \Lambda \Subset \mathbb{Z}^\nu\}$ tends to the whole $\mathbb{Z}^\nu$, i.e., it eventually contains any finite subset of $\mathbb{Z}^\nu$.

For each $\Lambda \Subset \mathbb{Z}^\nu$ define the bounded derivation associated with Λ by

$$d_{\Psi\,0,\,\Lambda}(F) := \delta_{\Psi_s,\,\Lambda} \cdot \delta_{\Psi_s,\,\Lambda}(F) \quad \text{for every} \quad F \in \mathcal{F}. \tag{4.47}$$

Note that the above $d_{\Psi\,0,\,\Lambda}$ actually depends on the choice of $\Psi_{\mathrm{s},1}$ and $\Psi_{\mathrm{s},2}$. However, this choice does not change the argument in what follows. By (4.45) it is written in terms of Ψ_{s} as

$$d_{\Psi\,0,\,\Lambda}(F)=\sum_{\mathrm{X}_1\cap\Lambda\neq\emptyset,\ \mathrm{X}_2\cap\Lambda\neq\emptyset}\Big[\Psi_{\mathrm{s}}(\mathrm{X}_2),\ \Big[\Psi_{\mathrm{s}}(\mathrm{X}_1),\ F\Big]_{\gamma}\Big]_{\gamma}. \tag{4.48}$$

Let us consider its nth iterate map ($n\in\mathbb{N}$), i.e., $d_{\Psi\,0,\,\Lambda}^{\;n}\equiv\underbrace{d_{\Psi\,0,\,\Lambda}\cdot d_{\Psi\,0,\,\Lambda}\cdots d_{\Psi\,0,\,\Lambda}}_{n\text{ times}}$, which is obviously a bounded derivation of $\mathcal{F}$. For every $A\in\mathcal{F}(\mathrm{I})$ with a fixed $\mathrm{I}\Subset\mathbb{Z}^{\nu}$ we have

$$\begin{aligned}
&d_{\Psi\,0,\,\Lambda}^{\;n}(A)\\
&\quad=\sum_{\substack{\mathrm{X}_1\cap\Lambda\neq\emptyset,\\ \mathrm{X}_2\cap\Lambda\neq\emptyset,\ldots,\mathrm{X}_{2n}\cap\Lambda\neq\emptyset}}\Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n}),\ldots\Big[\Psi_{\mathrm{s}}(\mathrm{X}_2),\ \Big[\Psi_{\mathrm{s}}(\mathrm{X}_1),\ A\Big]_{\gamma}\Big]_{\gamma}\cdots\Big]_{\gamma}\\
&\quad=\sum_{\substack{\big\{(\mathrm{X}_1,\mathrm{X}_2,\ldots,\mathrm{X}_{2n})\,|\,\mathrm{X}_i\cap\mathrm{V}_{i-1}\neq\emptyset\\ \text{and }\mathrm{X}_i\cap\Lambda\neq\emptyset\text{ for each }i\in\{1,2,\ldots,2n\}\big\}}}\Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n}),\ldots\Big[\Psi_{\mathrm{s}}(\mathrm{X}_2),\Big[\Psi_{\mathrm{s}}(\mathrm{X}_1),A\Big]_{\gamma}\Big]_{\gamma}\cdots\Big]_{\gamma}\\
&\quad\in\mathcal{F}(\mathrm{I}\cup\hat{\Lambda}_r)\cap\mathcal{F}(\hat{\mathrm{I}}_{2nr}),
\end{aligned}\tag{4.49}$$

where the notation (4.42) is used again. By comparing Eq. (4.49) with Eq. (4.44) there exists a sufficiently large $\Lambda_{\circ}\Subset\mathbb{Z}^{\nu}$ which depends on the given $n\in\mathbb{N}$ and $\mathrm{I}\Subset\mathbb{Z}^{\nu}$ such that for any finite $\Lambda\supset\Lambda_{\circ}$

$$d_{\Psi\,0,\,\Lambda}^{\;k}(A)=d_{\Psi\,0}^{\;k}(A),\quad k=1,2,\ldots,n-1,n,\quad\text{for every}\ \ A\in\mathcal{F}(\mathrm{I}). \tag{4.50}$$

For each $A\in\mathcal{A}_{\circ}$ and $n\in\mathbb{N}$

$$\lim_{\Lambda\nearrow\mathbb{Z}^{\nu}}d_{\Psi\,0,\,\Lambda}^{\;k}(A)=d_{\Psi\,0}^{\;k}(A),\quad k=1,2,\ldots,n-1,n. \tag{4.51}$$

We can rewrite the above expansion formulas of $d_{\Psi\,0,\,\Lambda}$ and $d_{\Psi\,0}$ in terms of Ψ_{s} as follows.

Lemma 4.10. *Let Ψ_{s} denote $\Psi_{\mathrm{s},1}$ or $\Psi_{\mathrm{s},2}$ given in (4.14). Let $d_{\Psi\,0,\,\Lambda}$ ($\Lambda\Subset\mathbb{Z}^{\nu}$) denote the bounded derivation of $\mathcal{F}$ defined in (4.47). Let $d_{\Psi\,0}$ denote the derivation on $\mathcal{A}_{\circ}$ as in (4.40). Then for each $F\in\mathcal{F}$,*

$$\begin{aligned}
d_{\Psi\,0,\,\Lambda}(F)&=\frac{1}{2}\sum_{\substack{\mathrm{X}_1\Subset\mathbb{Z}^{\nu},\,\mathrm{X}_2\Subset\mathbb{Z}^{\nu}\\ \mathrm{X}_1\cap\Lambda\neq\emptyset,\,\mathrm{X}_2\cap\Lambda\neq\emptyset\\ \mathrm{X}_1\cap\mathrm{X}_2\neq\emptyset}}\Big[\big\{\Psi_{\mathrm{s}}(\mathrm{X}_2),\ \Psi_{\mathrm{s}}(\mathrm{X}_1)\big\},\ F\Big]\\
&=\sum_{\substack{\mathrm{X}_1\Subset\mathbb{Z}^{\nu},\,\mathrm{X}_2\Subset\mathbb{Z}^{\nu}\\ \mathrm{X}_1\cap\Lambda\neq\emptyset,\,\mathrm{X}_2\cap\Lambda\neq\emptyset\\ \mathrm{X}_1\cap\mathrm{X}_2\neq\emptyset}}\Big[\Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1),\ F\Big].
\end{aligned}\tag{4.52}$$

For each $A \in \mathcal{A}_\circ$,

$$d_{\Psi\,0}(A) = \frac{1}{2} \sum_{\substack{\mathrm{X}_1 \Subset \mathbb{Z}^\nu,\, \mathrm{X}_2 \Subset \mathbb{Z}^\nu \\ \mathrm{X}_1 \cap \mathrm{X}_2 \neq \emptyset}} \Big[\big\{ \Psi_{\mathrm{s}}(\mathrm{X}_2),\ \Psi_{\mathrm{s}}(\mathrm{X}_1) \big\},\ A \Big]$$

$$= \sum_{\substack{\mathrm{X}_1 \Subset \mathbb{Z}^\nu,\, \mathrm{X}_2 \Subset \mathbb{Z}^\nu \\ \mathrm{X}_1 \cap \mathrm{X}_2 \neq \emptyset}} \Big[\Psi_{\mathrm{s}}(\mathrm{X}_2) \Psi_{\mathrm{s}}(\mathrm{X}_1),\ A \Big], \tag{4.53}$$

where the summation is a finite sum.

Proof. For a given $F \in \mathcal{F}$ we have its unique decomposition $F = F_+ + F_-$, where $F_+ = \frac{1}{2}(F + \gamma(F)) \in \mathcal{F}_+$ and $F_- = \frac{1}{2}(F - \gamma(F)) \in \mathcal{F}_-$. From (4.48) we have

$$\begin{aligned}
&d_{\Psi\,0,\,\Lambda}(F) \\
&\quad = d_{\Psi\,0,\,\Lambda}(F_+) + d_{\Psi\,0,\,\Lambda}(F_-) \\
&\quad = \sum_{\substack{\mathrm{X}_1 \cap \Lambda \neq \emptyset, \\ \mathrm{X}_2 \cap \Lambda \neq \emptyset}} \Big\{ \Psi_{\mathrm{s}}(\mathrm{X}_2),\ \ \Psi_{\mathrm{s}}(\mathrm{X}_1) F_+ - F_+ \Psi_{\mathrm{s}}(\mathrm{X}_1) \Big\} \\
&\qquad + \Big[\Psi_{\mathrm{s}}(\mathrm{X}_2),\ \Psi_{\mathrm{s}}(\mathrm{X}_1) F_- + F_- \Psi_{\mathrm{s}}(\mathrm{X}_1) \Big] \\
&\quad = \sum_{\substack{\mathrm{X}_1 \cap \Lambda \neq \emptyset, \\ \mathrm{X}_2 \cap \Lambda \neq \emptyset}} \Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1) F_+ + \Psi_{\mathrm{s}}(\mathrm{X}_1) F_+ \Psi_{\mathrm{s}}(\mathrm{X}_2) - \Psi_{\mathrm{s}}(\mathrm{X}_2) F_+ \Psi_{\mathrm{s}}(\mathrm{X}_1) \\
&\qquad - F_+ \Psi_{\mathrm{s}}(\mathrm{X}_1)\Psi_{\mathrm{s}}(\mathrm{X}_2) \\
&\qquad + \sum_{\substack{\mathrm{X}_1 \cap \Lambda \neq \emptyset, \\ \mathrm{X}_2 \cap \Lambda \neq \emptyset}} \Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1) F_- - \Psi_{\mathrm{s}}(\mathrm{X}_1) F_- \Psi_{\mathrm{s}}(\mathrm{X}_2) + \Psi_{\mathrm{s}}(\mathrm{X}_2) F_- \Psi_{\mathrm{s}}(\mathrm{X}_1) \\
&\qquad - F_- \Psi_{\mathrm{s}}(\mathrm{X}_1)\Psi_{\mathrm{s}}(\mathrm{X}_2) \\
&\quad = \sum_{\substack{\mathrm{X}_1 \cap \Lambda \neq \emptyset, \\ \mathrm{X}_2 \cap \Lambda \neq \emptyset}} \Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1) F_+ + \Big(\Psi_{\mathrm{s}}(\mathrm{X}_1) F_+ \Psi_{\mathrm{s}}(\mathrm{X}_2) - \Psi_{\mathrm{s}}(\mathrm{X}_2) F_+ \Psi_{\mathrm{s}}(\mathrm{X}_1) \Big) \\
&\qquad - F_+ \Psi_{\mathrm{s}}(\mathrm{X}_1)\Psi_{\mathrm{s}}(\mathrm{X}_2) \\
&\qquad + \sum_{\substack{\mathrm{X}_1 \cap \Lambda \neq \emptyset, \\ \mathrm{X}_2 \cap \Lambda \neq \emptyset}} \Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1) F_- + \Big(-\Psi_{\mathrm{s}}(\mathrm{X}_1) F_- \Psi_{\mathrm{s}}(\mathrm{X}_2) + \Psi_{\mathrm{s}}(\mathrm{X}_2) F_- \Psi_{\mathrm{s}}(\mathrm{X}_1) \Big) \\
&\qquad - F_- \Psi_{\mathrm{s}}(\mathrm{X}_1)\Psi_{\mathrm{s}}(\mathrm{X}_2) \\
&\quad = \sum_{\substack{\mathrm{X}_1 \cap \Lambda \neq \emptyset, \\ \mathrm{X}_2 \cap \Lambda \neq \emptyset}} \Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1) F_+ - F_+ \Psi_{\mathrm{s}}(\mathrm{X}_1)\Psi_{\mathrm{s}}(\mathrm{X}_2) \\
&\qquad + \Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1) F_- - F_- \Psi_{\mathrm{s}}(\mathrm{X}_1)\Psi_{\mathrm{s}}(\mathrm{X}_2) \\
&\quad = \sum_{\substack{\mathrm{X}_1 \cap \Lambda \neq \emptyset, \\ \mathrm{X}_2 \cap \Lambda \neq \emptyset}} \Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1) F_+ - F_+ \Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1) \\
&\qquad + \Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1) F_- - F_- \Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1)
\end{aligned}$$

$$= \sum_{\substack{X_1\cap\Lambda\neq\emptyset,\\ X_2\cap\Lambda\neq\emptyset}} \Big[\Psi_{s}(X_2)\Psi_{s}(X_1),\ F_{+}+F_{-}\Big] = \sum_{\substack{X_1\cap\Lambda\neq\emptyset,\\ X_2\cap\Lambda\neq\emptyset}} \Big[\Psi_{s}(X_2)\Psi_{s}(X_1),\ F\Big]$$
$$= \frac{1}{2}\sum_{\substack{X_1\cap\Lambda\neq\emptyset,\\ X_2\cap\Lambda\neq\emptyset}} \Big[\Psi_{s}(X_2)\Psi_{s}(X_1)+\Psi_{s}(X_1)\Psi_{s}(X_2),\ F\Big]$$
$$= \frac{1}{2}\sum_{\substack{X_1\cap\Lambda\neq\emptyset,\\ X_2\cap\Lambda\neq\emptyset}} \Big[\big\{\Psi_{s}(X_2),\ \Psi_{s}(X_1)\big\},\ F\Big].$$

By noting the vanishing anti-commutator $\{\Psi_{s}(X_2),\ \Psi_{s}(X_1)\}=0$ if $X_1\cap X_2=\emptyset$ we get

$$d_{\Psi\,0,\,\Lambda}(F) = \frac{1}{2}\sum_{\substack{X_1\cap\Lambda\neq\emptyset,\ X_2\cap\Lambda\neq\emptyset\\ X_1\cap X_2\neq\emptyset}} \Big[\big\{\Psi_{s}(X_2),\ \Psi_{s}(X_1)\big\},\ F\Big]$$
$$= \frac{1}{2}\sum_{\substack{X_1\cap\Lambda\neq\emptyset,\ X_2\cap\Lambda\neq\emptyset\\ X_1\cap X_2\neq\emptyset}} \Big[\Psi_{s}(X_2)\Psi_{s}(X_1)+\Psi_{s}(X_1)\Psi_{s}(X_2),\ F\Big]$$
$$= \sum_{\substack{X_1\cap\Lambda\neq\emptyset,\ X_2\cap\Lambda\neq\emptyset\\ X_1\cap X_2\neq\emptyset}} \Big[\Psi_{s}(X_2)\Psi_{s}(X_1),\ F\Big].$$

Hence Eq. (4.52) is shown. We obtain (4.53) from (4.52) for any $A\in\mathcal{A}_{\circ}$ due to the asymptotic formula (4.51) by taking $\Lambda\nearrow\mathbb{Z}^{\nu}$. Let $\mathrm{I}\Subset\mathbb{Z}^{\nu}$ be the least subset such that $A\in\mathcal{F}(\mathrm{I})$. As $\Psi_{s}(X_2)\Psi_{s}(X_1)\in\mathcal{A}_{\circ+}$, if $X_1\cup X_2$ is included in I^{c} (i.e., the complement of I), then the commutator for such X_1 and X_2 appeared in (4.53) vanishes. Hence Eq. (4.53) is actually a finite sum. □

Remark 4.11. In Lemma 4.10 the finite-range assignment of local fermion charges Ψ needs not to be nilpotent. This fact will be exploited later to show Proposition 4.15.

4.4.2. Global Time Evolution. We will construct a global time evolution from $\Psi\in\mathscr{C}^{\sharp}$ of Definition 4.1. For each $\Lambda\Subset\mathbb{Z}^{\nu}$ by taking the exponential of the bounded derivation $d_{\Psi\,0,\,\Lambda}$ let

$$\alpha_t^{\Lambda}(F) := \exp(itd_{\Psi\,0,\,\Lambda})(F) \quad \text{for every} \quad t\in\mathbb{R} \quad \text{and} \quad F\in\mathcal{F}. \tag{4.54}$$

By (4.49) for any $\mathrm{I}\Subset\mathbb{Z}^{\nu}$

$$\alpha_t^{\Lambda}(A)\in\mathcal{F}(\mathrm{I}\cup\hat{\Lambda}_r) \quad \text{for every} \quad t\in\mathbb{R} \quad \text{and} \quad A\in\mathcal{F}(\mathrm{I}). \tag{4.55}$$

We note its expansion formula:

$$\alpha_t^{\Lambda}(F) = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}d_{\Psi\,0,\,\Lambda}^{\ n}(F) \quad \text{for every} \quad t\in\mathbb{R} \quad \text{and} \quad F\in\mathcal{F}. \tag{4.56}$$

We will get a time evolution on the whole system $\mathcal{F}$ by taking infinite volume limit of the inner one-parameter group of $*$-automorphisms $\{\alpha_t^{\Lambda};\ t\in\mathbb{R}\}$ of $\mathcal{F}$. It has been known that the generator $d_{\Psi\,0}$ of a finite-range interaction is analytic on the local algebra $\mathcal{A}_\circ$, see [21] and Sect. 7.3 of [22]. We shall recapture its proof in the following lemma.

Lemma 4.12. *Let $\Psi \in \mathscr{C}^\sharp$ of Definition 4.1 and let r denote the range of Ψ. Let Ψ_{s} denote $\Psi_{\mathrm{s},1}$ or $\Psi_{\mathrm{s},2}$ given in (4.14). Let $d_{\Psi\,0}$ denote the derivation defined on $\mathcal{A}_\circ$ given in (4.35) in Definition 4.8. For each $\Lambda \Subset \mathbb{Z}^\nu$ let $d_{\Psi\,0,\,\Lambda}$ denote the bounded derivation of $\mathcal{F}$ given in (4.47). Then for each $A \in \mathcal{A}_\circ$ and $n \in \mathbb{N}$,*

$$\frac{1}{n!}\left\|{d_{\Psi\,0}}^n(A)\right\| \leq k_{\{A,\,\Psi_{\mathrm{s}}\}} \cdot {m_{\Psi_{\mathrm{s}}}}^n \tag{4.57}$$

and

$$\frac{1}{n!}\left\|{d_{\Psi\,0,\,\Lambda}}^n(A)\right\| \leq k_{\{A,\,\Psi_{\mathrm{s}}\}} \cdot {m_{\Psi_{\mathrm{s}}}}^n, \tag{4.58}$$

where $\mathrm{I} \Subset \mathbb{Z}^\nu$ denotes the least subset such that $A \in \mathcal{F}(\mathrm{I})$, and

$$k_{\{A,\,\Psi_{\mathrm{s}}\}} := \|A\| \cdot \exp\Big(4\big\{|\mathrm{I}| - 2(r+1)^\nu\big\} l_{\Psi_{\mathrm{s}}}\Big), \quad m_{\Psi_{\mathrm{s}}} := \exp\Big(8(r+1)^\nu l_{\Psi_{\mathrm{s}}}\Big),$$
$$l_{\Psi_{\mathrm{s}}} := \sup_{i\in\mathbb{Z}^\nu} \sum_{\mathrm{Y};\,\mathrm{Y}\cap\mathrm{X}\neq\emptyset}\ \sum_{\mathrm{X};\,\mathrm{X}\ni i} \|\Psi_{\mathrm{s}}(\mathrm{Y})\| \cdot \|\Psi_{\mathrm{s}}(\mathrm{X})\| < \infty. \tag{4.59}$$

(*In the above defining formula of $l_{\Psi_{\mathrm{s}}}$, first fix $i \in \mathbb{Z}^\nu$, second take a finite subset* X *that contains the fixed site i, and then take* Y *such that it has a non-trivial intersection with the fixed finite subset* X.)

Proof. First we will see that $l_{\Psi_{\mathrm{s}}}$ is finite.

$$\begin{aligned}
l_{\Psi_{\mathrm{s}}} &= \sup_{i\in\mathbb{Z}^\nu} \sum_{\substack{\mathrm{Y};\,\mathrm{Y}\cap\mathrm{X}\neq\emptyset\\ \Psi_{\mathrm{s}}(\mathrm{X})\neq 0}}\ \sum_{\substack{\mathrm{X};\,\mathrm{X}\ni i\\ \Psi_{\mathrm{s}}(\mathrm{X})\neq 0}} \|\Psi_{\mathrm{s}}(\mathrm{Y})\| \cdot \|\Psi_{\mathrm{s}}(\mathrm{X})\| \\
&\leq \sup_{\mathrm{X};\,\Psi_{\mathrm{s}}(\mathrm{X})\neq 0} \Big|\big\{\mathrm{Y};\ \mathrm{Y}\cap\mathrm{X}\neq\emptyset,\ \Psi_{\mathrm{s}}(\mathrm{Y})\neq 0\big\}\Big| \\
&\qquad \cdot \sup_{i\in\mathbb{Z}^\nu} \Big|\big\{\mathrm{X};\ \mathrm{X}\ni i,\ \Psi_{\mathrm{s}}(\mathrm{X})\neq 0\big\}\Big| \cdot \|\Psi_{\mathrm{s}}\|_\infty \cdot \|\Psi_{\mathrm{s}}\|_\infty,
\end{aligned}$$

where $\big|\{\ \}\big|$ denotes the cardinality of the set. By using (4.16), (4.32) we further estimate $l_{\Psi_{\mathrm{s}}}$ as follows.

$$\begin{aligned}
l_{\Psi_{\mathrm{s}}} &\leq \sup_{\mathrm{X};\,\mathrm{diam}(\mathrm{X})\leq r} \Big|\big\{\mathrm{Y};\ \mathrm{Y}\cap\mathrm{X}\neq\emptyset,\ \Psi_{\mathrm{s}}(\mathrm{Y})\neq 0\big\}\Big| \\
&\qquad \cdot \sup_{i\in\mathbb{Z}^\nu} \Big|\big\{\mathrm{X};\ \mathrm{X}\ni i,\ \mathrm{diam}(\mathrm{X})\leq r\big\}\Big| \cdot \|\Psi_{\mathrm{s}}\|_\infty^2 \\
&\leq \sup_{\mathrm{X};\,\mathrm{diam}(\mathrm{X})\leq r} \sum_{j\in\mathrm{X}} \Big|\big\{\mathrm{Y};\ \mathrm{Y}\ni j,\ \mathrm{diam}(\mathrm{Y})\leq r\big\}\Big| \\
&\qquad \cdot \sup_{i\in\mathbb{Z}^\nu} \Big|\big\{\mathrm{X};\ \mathrm{X}\ni i,\ \mathrm{diam}(\mathrm{X})\leq r\big\}\Big| \cdot \|\Psi_{\mathrm{s}}\|_\infty^2 \\
&\leq \sup_{\mathrm{X};\,\mathrm{diam}(\mathrm{X})\leq r} |\mathrm{X}| \cdot \max_{j\in\mathrm{X}} \Big|\big\{\mathrm{Y};\ \mathrm{Y}\ni j,\ \mathrm{diam}(\mathrm{Y})\leq r\big\}\Big| \\
&\qquad \cdot \sup_{i\in\mathbb{Z}^\nu} \Big|\big\{\mathrm{X};\ \mathrm{X}\ni i,\ \mathrm{diam}(\mathrm{X})\leq r\big\}\Big| \cdot \|\Psi_{\mathrm{s}}\|_\infty^2
\end{aligned}$$

$$
\begin{aligned}
&\le |\mathrm{C}_r| \cdot \sup_{j\in\mathbb{Z}^\nu} \Big|\big\{\mathrm{Y};\ \mathrm{Y}\ni j,\ \mathrm{diam}(\mathrm{Y})\le r\big\}\Big| \\
&\quad \cdot \sup_{i\in\mathbb{Z}^\nu} \Big|\big\{\mathrm{X};\ \mathrm{X}\ni i,\ \mathrm{diam}(\mathrm{X})\le r\big\}\Big| \cdot \|\Psi_{\mathrm{s}}\|_\infty^2 \\
&= \left(\sup_{i\in\mathbb{Z}^\nu} \Big|\big\{\mathrm{X};\ \mathrm{X}\ni i,\ \mathrm{diam}(\mathrm{X})\le r\big\}\Big|\right)^2 \cdot |\mathrm{C}_r| \cdot \|\Psi_{\mathrm{s}}\|_\infty^2 \\
&\le \Big|\big\{\mathrm{X};\ |\mathrm{X}|\le |\mathrm{C}_{2r}|\big\}\Big|^2 \cdot |\mathrm{C}_r| \cdot \|\Psi_{\mathrm{s}}\|_\infty^2 \\
&= \left(2^{|\mathrm{C}_{2r}|}\right)^2 \cdot |\mathrm{C}_r| \cdot \|\Psi_{\mathrm{s}}\|_\infty^2 = 4^{|\mathrm{C}_{2r}|} \cdot |\mathrm{C}_r| \cdot \|\Psi_{\mathrm{s}}\|_\infty^2 \\
&\le 4^{|\mathrm{C}_{2r}|} \cdot |\mathrm{C}_r| \cdot (2\|\Psi\|_\infty)^2 = 4^{(2r+1)^\nu} \cdot (r+1)^\nu \cdot 4\|\Psi\|_\infty^2 \\
&= 4^{(2r+1)^\nu+1} \cdot (r+1)^\nu \cdot \|\Psi\|_\infty^2 \\
&< \infty.
\end{aligned} \tag{4.60}
$$

Let both $n \in \mathbb{N}$ and $\mathrm{I} \Subset \mathbb{Z}^\nu$ be fixed. For every $A \in \mathcal{F}(\mathrm{I})$ by using (4.53) in Lemma 4.10 repeatedly we have

$$
\begin{aligned}
&{d_{\Psi}}_0^n(A) \\
&\quad = \sum_{\mathrm{X}_{2n-1}\cap \mathrm{X}_{2n}\neq\emptyset} \cdots \sum_{\mathrm{X}_3\cap\mathrm{X}_4\neq\emptyset} \sum_{\mathrm{X}_1\cap\mathrm{X}_2\neq\emptyset} \\
&\qquad \Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n})\Psi_{\mathrm{s}}(\mathrm{X}_{2n-1}), \ldots \Big[\Psi_{\mathrm{s}}(\mathrm{X}_4)\Psi_{\mathrm{s}}(\mathrm{X}_3),\ \Big[\Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1),\ A\Big]\Big]\cdots\Big] \\
&\quad = \sum_{\substack{\{(\mathrm{X}_1,\mathrm{X}_2,\ldots,\mathrm{X}_{2n-1},\mathrm{X}_{2n})\,|\,(\mathrm{X}_{2i-1}\cup\mathrm{X}_{2i})\cap\mathrm{V}_{2i-2}\neq\emptyset \\ \text{and } \mathrm{X}_{2i-1}\cap\mathrm{X}_{2i}\neq\emptyset \text{ for every } i\in\{1,2,\ldots,n\}\}}} \\
&\qquad \Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n})\Psi_{\mathrm{s}}(\mathrm{X}_{2n-1}), \ldots \Big[\Psi_{\mathrm{s}}(\mathrm{X}_4)\Psi_{\mathrm{s}}(\mathrm{X}_3),\ \Big[\Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1),\ A\Big]\Big]\cdots\Big],
\end{aligned} \tag{4.61}
$$

where the second summation formula is due to the locality and the notation (4.42) is used. Similarly, for each $\Lambda \Subset \mathbb{Z}^\nu$, by using (4.52) in Lemma 4.10 we obtain for every $A \in \mathcal{F}(\mathrm{I})$

$$
\begin{aligned}
{d_{\Psi}}_{0,\Lambda}^n(A) = &\sum_{\substack{\{(\mathrm{X}_1,\mathrm{X}_2,\ldots,\mathrm{X}_{2n-1},\mathrm{X}_{2n})\,|\,(\mathrm{X}_{2i-1}\cup\mathrm{X}_{2i})\cap\mathrm{V}_{2i-2}\neq\emptyset, \\ \mathrm{X}_{2i-1}\cap\mathrm{X}_{2i}\neq\emptyset,\ \mathrm{X}_{2i-1}\cap\Lambda\neq\emptyset,\ \mathrm{X}_{2i}\cap\Lambda\neq\emptyset \text{ for } \forall i\in\{1,2,\ldots,n\}\}}} \\
&\Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n})\Psi_{\mathrm{s}}(\mathrm{X}_{2n-1}), \ldots \Big[\Psi_{\mathrm{s}}(\mathrm{X}_4)\Psi_{\mathrm{s}}(\mathrm{X}_3),\ \Big[\Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1),\ A\Big]\Big]\cdots\Big].
\end{aligned} \tag{4.62}
$$

Let us estimate the norm of ${d_{\Psi}}_0^n(A)$ by induction of $n \in \mathbb{N}$. Take any $2(n-1)$ finite subsets $(\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_{2n-3}, \mathrm{X}_{2n-2})$ that satisfy the required conditions $(\mathrm{X}_{2i-1} \cup \mathrm{X}_{2i}) \cap \mathrm{V}_{2i-2} \neq \emptyset$ and $\mathrm{X}_{2i-1} \cap \mathrm{X}_{2i} \neq \emptyset$ for every $i \in$

$\{1, 2, \ldots, n-1\}$. For such fixed $(\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_{2n-3}, \mathrm{X}_{2n-2})$ let us define the following multiple commutator[4]

$$\begin{aligned} & B_{n-1}(A) \\ & \quad := \Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n-2})\Psi_{\mathrm{s}}(\mathrm{X}_{2n-3}), \ldots \Big[\Psi_{\mathrm{s}}(\mathrm{X}_4)\Psi_{\mathrm{s}}(\mathrm{X}_3),\ \Big[\Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1),\ A\Big]\Big] \cdots\Big] \\ & \qquad \in \mathcal{F}(\mathrm{V}_{2n-2}), \\ & B_0(A) := A \in \mathcal{F}(\mathrm{I}), \end{aligned} \tag{4.63}$$

where we use the notation (4.42)

$$\mathrm{V}_{2n-2} \equiv \mathrm{X}_{2n-2} \cup \mathrm{X}_{2n-3} \cup \cdots \cup \mathrm{X}_1 \cup \mathrm{I}.$$

Note that $d_{\Psi\,0}^{\,n-1}(A)$ is the sum of $B_{n-1}(A)$s over all the finite subsets $(\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_{2n-3}, \mathrm{X}_{2n-2})$ satisfying the required conditions stated above.

We shall estimate each term $d_{\Psi\,0}(B_{n-1}(A))$ that appears in the summation formula (4.61) of $d_{\Psi\,0}^{\,n}(A)$. By (4.53) and (4.63) we have

$$\begin{aligned} & d_{\Psi\,0}(B_{n-1}(A)) \\ & \quad = \sum_{\mathrm{X}_{2n-1}\cap \mathrm{X}_{2n}\neq\emptyset} \Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n})\Psi_{\mathrm{s}}(\mathrm{X}_{2n-1}),\ B_{n-1}(A)\Big] \\ & \quad = \sum_{\substack{\{(\mathrm{X}_{2n-1},\mathrm{X}_{2n})\,|\,(\mathrm{X}_{2n-1}\cup\mathrm{X}_{2n})\cap\mathrm{V}_{2n-2}\neq\emptyset \\ \text{and } \mathrm{X}_{2n-1}\cap\mathrm{X}_{2n}\neq\emptyset\}}} \Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n})\Psi_{\mathrm{s}}(\mathrm{X}_{2n-1}),\ B_{n-1}(A)\Big], \end{aligned} \tag{4.64}$$

where $\Psi_{\mathrm{s}}(\mathrm{X}_{2n})\Psi_{\mathrm{s}}(\mathrm{X}_{2n-1}) \in \mathcal{A}_{\circ+}$ and the locality are noted in the second equality. By using (4.63) and (4.64), we proceed the estimate as

$$\begin{aligned} & \big\| d_{\Psi\,0}(B_{n-1}(A)) \big\| \\ & \quad \le 2\|B_{n-1}(A)\| \cdot \sum_{\substack{\{(\mathrm{X}_{2n-1},\mathrm{X}_{2n})\,|\,(\mathrm{X}_{2n-1}\cup\mathrm{X}_{2n})\cap\mathrm{V}_{2n-2}\neq\emptyset \\ \text{and } \mathrm{X}_{2n-1}\cap\mathrm{X}_{2n}\neq\emptyset\}}} \|\Psi_{\mathrm{s}}(\mathrm{X}_{2n})\| \cdot \|\Psi_{\mathrm{s}}(\mathrm{X}_{2n-1})\| \\ & \quad \le 2\|B_{n-1}(A)\| \\ & \qquad \cdot \Bigg\{ \sum_{\mathrm{X}_{2n};\, \mathrm{X}_{2n}\cap\mathrm{X}_{2n-1}\neq\emptyset} \ \sum_{\mathrm{X}_{2n-1};\, \mathrm{X}_{2n-1}\cap\mathrm{V}_{2n-2}\neq\emptyset} \|\Psi_{\mathrm{s}}(\mathrm{X}_{2n})\| \cdot \|\Psi_{\mathrm{s}}(\mathrm{X}_{2n-1})\| \\ & \qquad + \sum_{\mathrm{X}_{2n-1};\, \mathrm{X}_{2n-1}\cap\mathrm{X}_{2n}\neq\emptyset} \ \sum_{\mathrm{X}_{2n};\, \mathrm{X}_{2n}\cap\mathrm{V}_{2n-2}\neq\emptyset} \|\Psi_{\mathrm{s}}(\mathrm{X}_{2n})\| \cdot \|\Psi_{\mathrm{s}}(\mathrm{X}_{2n-1})\| \Bigg\} \\ & \quad \le 2\|B_{n-1}(A)\| \cdot \big|\mathrm{V}_{2n-2}\big| \end{aligned}$$

[4] The notation $B_{n-1}(A)$ should be made precise by noting explicitly its dependence on the set of $2(n-1)$ finite subsets, like $B_{n-1}(A; \{\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_{2n-3}, \mathrm{X}_{2n-2}\})$. However, for simplicity we will employ this short-hand notation as in the cited literature.

$$
\begin{aligned}
&\cdot \Bigg\{ \max_{i \in \mathrm{V}_{2n-2}} \sum_{\mathrm{X}_{2n};\, \mathrm{X}_{2n} \cap \mathrm{X}_{2n-1} \neq \emptyset} \ \sum_{\mathrm{X}_{2n-1} \ni i} \|\Psi_{\mathrm{s}}(\mathrm{X}_{2n})\| \cdot \|\Psi_{\mathrm{s}}(\mathrm{X}_{2n-1})\| \\
&+ \max_{i \in \mathrm{V}_{2n-2}} \sum_{\mathrm{X}_{2n-1};\, \mathrm{X}_{2n-1} \cap \mathrm{X}_{2n} \neq \emptyset} \ \sum_{\mathrm{X}_{2n} \ni i} \|\Psi_{\mathrm{s}}(\mathrm{X}_{2n})\| \cdot \|\Psi_{\mathrm{s}}(\mathrm{X}_{2n-1})\| \Bigg\} \\
\leq\ & 2\|B_{n-1}(A)\| \cdot \big|\mathrm{V}_{2n-2}\big| \\
&\cdot 2 \Bigg\{ \sup_{i \in \mathbb{Z}^{\nu}} \sum_{\mathrm{X}_{2n};\, \mathrm{X}_{2n} \cap \mathrm{X}_{2n-1} \neq \emptyset} \ \sum_{\mathrm{X}_{2n-1} \ni i} \|\Psi_{\mathrm{s}}(\mathrm{X}_{2n})\| \cdot \|\Psi_{\mathrm{s}}(\mathrm{X}_{2n-1})\| \Bigg\} \\
=\ & 2\|B_{n-1}(A)\| \cdot \big|\mathrm{V}_{2n-2}\big| \cdot 2 \cdot l_{\Psi_{\mathrm{s}}} = 4 l_{\Psi_{\mathrm{s}}} \big|\mathrm{V}_{2n-2}\big| \cdot \|B_{n-1}(A)\| \\
\leq\ & 4 l_{\Psi_{\mathrm{s}}} \Big(|\mathrm{I}| + 2(n-1) \cdot (r+1)^{\nu} \Big) \cdot \|B_{n-1}(A)\|, \qquad (4.65)
\end{aligned}
$$

where we recall the notation $l_{\Psi_{\mathrm{s}}}$ defined in (4.59) and the estimate of $\big|\mathrm{V}_{2n-2}\big|$ given in (4.43). Note that $B_{n-1}(A)$ of (4.63) denotes an arbitrary term in the summation formula of $d_{\Psi\,0}{}^{n-1}(A)$. Hence by iteration we get

$$
\begin{aligned}
\big\| d_{\Psi\,0}{}^{n}(A) \big\| &\leq (4 l_{\Psi_{\mathrm{s}}})^{n} \|B_0(A)\| \prod_{k=1}^{n} \Big(|\mathrm{I}| + 2(k-1) \cdot (r+1)^{\nu} \Big) \\
&\leq (4 l_{\Psi_{\mathrm{s}}})^{n} \|A\| \Big(|\mathrm{I}| + 2(n-1) \cdot (r+1)^{\nu} \Big)^{n} \qquad (4.66)
\end{aligned}
$$

By noting the inequality $\frac{x^n}{n!} \leq \mathrm{e}^{x}$ for $x \geq 0$, we have the estimate

$$
\begin{aligned}
&\frac{1}{n!} \Big(4^{n} l_{\Psi_{\mathrm{s}}}{}^{n} \big\{ |\mathrm{I}| + 2(n-1) \cdot (r+1)^{\nu} \big\}^{n} \Big) = \frac{1}{n!} \Big(4 \big\{ |\mathrm{I}| + 2(n-1) \cdot (r+1)^{\nu} \big\} l_{\Psi_{\mathrm{s}}} \Big)^{n} \\
&\quad \leq \exp\Big(4 \big\{ |\mathrm{I}| + 2(n-1) \cdot (r+1)^{\nu} \big\} l_{\Psi_{\mathrm{s}}} \Big) \\
&\quad = \exp\Big(4 |\mathrm{I}| l_{\Psi_{\mathrm{s}}} \Big) \exp\Big(8(n-1)(r+1)^{\nu} l_{\Psi_{\mathrm{s}}} \Big) \\
&\quad = \exp\Big(4 \big\{ |\mathrm{I}| - 2(r+1)^{\nu} \big\} l_{\Psi_{\mathrm{s}}} \Big) \exp\Big(n \big\{ 8(r+1)^{\nu} l_{\Psi_{\mathrm{s}}} \big\} \Big). \qquad (4.67)
\end{aligned}
$$

From the above estimates (4.66), (4.67), we obtain the estimate (4.57) by setting the real numbers $k_{\{A,\, \Psi_{\mathrm{s}}\}}$ and $m_{\Psi_{\mathrm{s}}}$ as in (4.59).

In the same way as we have shown the estimate (4.66), from the formula (4.62), we obtain the following norm estimate of $d_{\Psi\,0,\,\Lambda}{}^{n}(A)$ independently of $\Lambda \Subset \mathbb{Z}^{\nu}$:

$$
\begin{aligned}
\big\| d_{\Psi\,0,\,\Lambda}{}^{n}(A) \big\| &\leq (4 l_{\Psi_{\mathrm{s}}})^{n} \|A\| \prod_{k=1}^{n} \Big(|\mathrm{I}| + 2(k-1) \cdot (r+1)^{\nu} \Big) \\
&\leq (4 l_{\Psi_{\mathrm{s}}})^{n} \|A\| \Big(|\mathrm{I}| + 2(n-1) \cdot (r+1)^{\nu} \Big)^{n}. \qquad (4.68)
\end{aligned}
$$

This leads to the desired estimate (4.58) by repeating a similar argument given above. □

We now give a global time evolution.

Proposition 4.13. *Let $\Psi \in \mathscr{C}^\sharp$, i.e., any uniformly bounded nilpotent finite-range assignment of local fermion charges on the fermion lattice system of Definition 4.1. Let $d_{\Psi\,0}$ denote the derivation defined on $\mathcal{A}_\circ$ as in Definition 4.8. Then there exists a strongly continuous one-parameter group of $*$-automorphisms $\{\alpha_t^\Psi\,;\, t \in \mathbb{R}\}$ on $\mathcal{F}$ given by*

$$\alpha_t^\Psi(F) := \lim_{\Lambda \nearrow \mathbb{Z}^\nu} \alpha_t^\Lambda(F) \quad \text{in norm for each } F \in \mathcal{F} \text{ and } t \in \mathbb{R}, \tag{4.69}$$

where $\alpha_t^\Lambda \equiv e^{itd_{\Psi\,0,\,\Lambda}}$ as defined in (4.54). For each $F \in \mathcal{F}$ the convergence of (4.69) is uniform with respect to the parameter $t \in \mathbb{R}$ on any compact subset of $\mathbb{R}$. Furthermore, $d_{\Psi\,0}$ is a pre-generator of $\{\alpha_t^\Psi\,;\, t \in \mathbb{R}\}$ given above. Namely, the norm closure of $d_{\Psi\,0}$ is exactly equal to the generator of $\{\alpha_t^\Psi\,;\, t \in \mathbb{R}\}$.

Proof. By (4.51) $\lim_{\Lambda \nearrow \mathbb{Z}^\nu} d_{\Psi\,0,\,\Lambda}(A) = d_{\Psi\,0}(A)$ holds for every $A \in \mathcal{A}_\circ$. Let $A \in \mathcal{A}_\circ$ and let I be a finite subset of $\mathbb{Z}^\nu$ such that $A \in \mathcal{F}(\mathrm{I})$. Let

$$t_\circ := \frac{1}{m_{\Psi_{\mathrm{s}}}}, \tag{4.70}$$

where $m_{\Psi_{\mathrm{s}}}$ denotes the positive number given in (4.59) in Lemma 4.12. By (4.57) the norm of $\sum_{n=0}^{\infty} \frac{t^n}{n!} {d_{\Psi\,0}}^n(A)$ is estimated from the above by $C \sum_{n=0}^{\infty} \left|\frac{t}{t_0}\right|^n$, where C is some constant determined solely by $A \in \mathcal{A}_\circ$ and $\Psi \in \mathscr{C}^\sharp$. So it is convergent for $|t| < t_\circ$. Hence any element of $\mathcal{A}_\circ$ is analytic for $d_{\Psi\,0}$. Using the argument given in [4, Proposition 6.2.3., Theorem 6.2.4], we obtain the statement. □

We now redefine '$d_{\Psi\,0}$' as the generator of the strongly continuous one-parameter group of $*$-automorphisms $\{\alpha_t^\Psi\,;\, t \in \mathbb{R}\}$ of $\mathcal{F}$.

Definition 4.14. For each $\Psi \in \mathscr{C}^\sharp$, the infinitesimal generator of the strongly continuous one-parameter group of $*$-automorphisms $\{\alpha_t^\Psi\,;\, t \in \mathbb{R}\}$ on $\mathcal{F}$ given in Proposition 4.13 is denoted as $d_{\Psi\,0}$. (From now on $d_{\Psi\,0}$ denotes the norm closure of the pre-generator defined on $\mathcal{A}_\circ$ given in Definition 4.8.)

4.4.3. Basic Properties of Superderivations on the Fermion Lattice System. In the proof of Lemma 4.10 we do not actually use the nilpotent condition (4.33) of δ_Ψ and the hermite property of $\delta_{\Psi_{\mathrm{s}}}$, either. Only the identity $d_{\Psi\,0} = {\delta_{\Psi_{\mathrm{s}}}}^2$ is used there. By noting this fact, we can invent a criterion for the nilpotent condition of δ_Ψ in terms of Ψ.

Proposition 4.15. *Suppose that Ψ is a finite-range assignment of local fermion charges on the fermion lattice system of Definition 4.1. Let δ_Ψ denote the superderivation associated with Ψ. Then for each $A \in \mathcal{A}_\circ$,*

$$\delta_\Psi \cdot \delta_\Psi(A) = \frac{1}{2} \sum_{\substack{\mathrm{X} \Subset \mathbb{Z}^\nu,\, \mathrm{Y} \Subset \mathbb{Z}^\nu \\ \mathrm{X} \cap \mathrm{Y} \neq \emptyset}} \Big[\big\{\Psi(\mathrm{X}),\ \Psi(\mathrm{Y})\big\},\ A\Big] = \sum_{\substack{\mathrm{X} \Subset \mathbb{Z}^\nu,\, \mathrm{Y} \Subset \mathbb{Z}^\nu \\ \mathrm{X} \cap \mathrm{Y} \neq \emptyset}} \Big[\Psi(\mathrm{X})\Psi(\mathrm{Y}),\ A\Big]. \tag{4.71}$$

If $\Psi(\mathrm{X})$ and $\Psi(\mathrm{Y})$ anti-commute for any (non-disjoint) pair of $\mathrm{X} \Subset \mathbb{Z}^\nu$ and $\mathrm{Y} \Subset \mathbb{Z}^\nu$, then δ_Ψ is nilpotent.

Proof. We can derive the formula (4.71) in the same way as the formula (4.53) in Lemma 4.10 by taking Ψ in the place of Ψ_{s}. If $\{\Psi(\mathrm{X}),\ \Psi(\mathrm{Y})\} = 0$ for any $\mathrm{X} \Subset \mathbb{Z}^\nu$ and $\mathrm{Y} \Subset \mathbb{Z}^\nu$, then by the formula (4.71) $\delta_\Psi \cdot \delta_\Psi = \mathbf{0}$ on $\mathcal{A}_\circ$. □

Remark 4.16. Proposition 4.15 is valid when the spin of fermions exists. It is valid for other lattices as well.

Remark 4.17. A condition of superderivations on the fermion lattice system for being nilpotent has been given under the periodic boundary condition imposed in [12]. Proposition 4.15 does not require any specific boundary condition on finite systems. The superderivation δ_Ψ and the finite-range assignment of local fermion charges Ψ are linked together by the relation (4.17).

We will show norm-closability of δ_Ψ, as it is a fundamental property required for reasonable supersymmetric C^*-dynamics. For some technical reason, we assume that supersymmetry is unbroken, namely Ψ is in $\mathscr{C}^\sharp_{unbroken}$.

Proposition 4.18. *Let $\Psi \in \mathscr{C}^\sharp_{unbroken}$. Then the associated nilpotent superderivations δ_Ψ and $\delta_\Psi{}^*$ defined on $\mathcal{A}_\circ$ are norm-closable. Similarly, the hermite superderivations $\delta_{\Psi_{\mathrm{s},1}}$ and $\delta_{\Psi_{\mathrm{s},2}}$ defined on $\mathcal{A}_\circ$ are norm-closable.*

Proof. By the assumption there exists a supersymmetric state φ with respect to δ_Ψ. By Proposition 2.4, it is invariant under $\delta_\Psi{}^*$ on $\mathcal{A}_\circ$ as well. Similarly by Proposition 2.5, it is invariant under $\delta_{\Psi_{\mathrm{s},1}}$ and $\delta_{\Psi_{\mathrm{s},2}}$ on $\mathcal{A}_\circ$. We note that the total system $\mathcal{F}$ is simple, as it is the CAR algebra. Hence by applying Proposition 2.10 to the superderivations δ_Ψ, $\delta_\Psi{}^*$, $\delta_{\Psi_{\mathrm{s},1}}$ and $\delta_{\Psi_{\mathrm{s},2}}$ it is concluded that they are all norm-closable. □

4.4.4. Global Time Evolution Commutes with Superderivations. Based on the commutativity between supersymmetric Hamiltonian and supercharges as in (3.5) in Sect. 3, it is usually said that supersymmetry is a symmetry.[5] Since it is a direct consequence of the supersymmetry algebra, we expect a similar commutativity relation for supersymmetric fermion lattice models. In fact, we will show below that the global time evolution $\{\alpha_t^\Psi\, ;\ t \in \mathbb{R}\}$ commutes with the superderivations δ_Ψ and $\delta_\Psi{}^*$. It requires non-trivial analysis.

Proposition 4.19. *Let $\Psi \in \mathscr{C}^\sharp$. Let δ_Ψ denote the superderivation defined on $\mathcal{A}_\circ$ associated with Ψ. Let $\delta_{\Psi_{\mathrm{s}}}$ denote each of the hermite superderivations $\delta_{\Psi_{\mathrm{s},1}}$ and $\delta_{\Psi_{\mathrm{s},2}}$ defined on $\mathcal{A}_\circ$ in (4.27). Let $\{\alpha_t^\Psi\, ;\ t \in \mathbb{R}\}$ denote the strongly continuous one-parameter group of $*$-automorphisms given in Proposition 4.13. If $\delta_{\Psi_{\mathrm{s}}}$ is norm-closable, then*

$$\alpha_t^\Psi(\mathcal{A}_\circ) \subset \mathcal{D}_{\overline{\delta_{\Psi_{\mathrm{s}}}}}, \quad \overline{\delta_{\Psi_{\mathrm{s}}}} \cdot \alpha_t^\Psi = \alpha_t^\Psi \cdot \delta_{\Psi_{\mathrm{s}}} \quad \text{on } \mathcal{A}_\circ \text{ for each } t \in \mathbb{R}, \tag{4.72}$$

where $\overline{\delta_{\Psi_{\mathrm{s}}}}$ denotes the norm closure of $\delta_{\Psi_{\mathrm{s}}}$ and $\mathcal{D}_{\overline{\delta_{\Psi_{\mathrm{s}}}}}$ denotes the domain of $\overline{\delta_{\Psi_{\mathrm{s}}}}$. If the nilpotent superderivation δ_Ψ is norm-closable, then

$$\alpha_t^\Psi(\mathcal{A}_\circ) \subset \mathcal{D}_{\overline{\delta_\Psi}}, \quad \overline{\delta_\Psi} \cdot \alpha_t^\Psi = \alpha_t^\Psi \cdot \delta_\Psi \quad \text{on } \mathcal{A}_\circ \text{ for each } t \in \mathbb{R}, \tag{4.73}$$

[5] Compare this extended usage with the notion of symmetries by Wigner. We refer to, e.g., Sect. 8.1 of [22].

and

$$\alpha_t^{\Psi}(\mathcal{A}_\circ) \subset \mathcal{D}_{\overline{\delta_{\Psi}^*}}, \quad \overline{\delta_{\Psi}^*} \cdot \alpha_t^{\Psi} = \alpha_t^{\Psi} \cdot \delta_{\Psi}^* \quad \text{on } \mathcal{A}_\circ \text{ or each } t \in \mathbb{R}. \tag{4.74}$$

Proof. Let r denote the range of Ψ. Let A denote an arbitrary element of $\mathcal{A}_\circ$ which will be fixed in what follows. Let I denote the smallest finite subset such that $A \in \mathcal{F}(\mathrm{I})$. Take any finite subset Λ that includes I. (Later we will let $\Lambda \nearrow \mathbb{Z}^\nu$.) From (4.49) and $\mathrm{I} \cup \hat{\Lambda}_r = \hat{\Lambda}_r$ it follows that $d_{\Psi\,0,\,\Lambda}^{\ n}(A) \in \mathcal{F}(\hat{\Lambda}_r)$ for every $n \in \mathbb{N}$, and from (4.55) $\alpha_t^{\Lambda}(A) \in \mathcal{F}(\hat{\Lambda}_r)$ for every $t \in \mathbb{R}$. By noting (4.27) (4.49) we have for each $n \in \mathbb{N}$

$$\begin{aligned}
&\delta_{\Psi_{\mathrm{s}}}\big(d_{\Psi\,0,\,\Lambda}^{\ n}(A)\big)\\
&\quad= \sum_{\mathrm{X}} \Big[\Psi_{\mathrm{s}}(\mathrm{X}),\ d_{\Psi\,0,\,\Lambda}^{\ n}(A)\Big]_\gamma = \sum_{\mathrm{X}\cap\hat{\Lambda}_r\neq\emptyset} \Big[\Psi_{\mathrm{s}}(\mathrm{X}),\ d_{\Psi\,0,\,\Lambda}^{\ n}(A)\Big]_\gamma\\
&\quad= \sum_{\mathrm{X}\cap\hat{\Lambda}_r\neq\emptyset}\ \sum_{\mathrm{X}_1\cap\Lambda\neq\emptyset,\ldots,\mathrm{X}_{2n}\cap\Lambda\neq\emptyset} \Big[\Psi_{\mathrm{s}}(\mathrm{X}),\ \Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n}),\ldots\Big[\Psi_{\mathrm{s}}(\mathrm{X}_1),\ A\Big]_\gamma\cdots\Big]_\gamma\Big]_\gamma\\
&\quad= \sum_{\mathrm{X}\cap\Lambda\neq\emptyset}\ \sum_{\mathrm{X}_1\cap\Lambda\neq\emptyset,\ldots,\mathrm{X}_{2n}\cap\Lambda\neq\emptyset} \Big[\Psi_{\mathrm{s}}(\mathrm{X}),\ \Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n}),\ldots\Big[\Psi_{\mathrm{s}}(\mathrm{X}_1),\ A\Big]_\gamma\cdots\Big]_\gamma\Big]_\gamma\\
&\qquad+ \sum_{\substack{\mathrm{X}\cap\hat{\Lambda}_r\neq\emptyset\\ \mathrm{X}\cap\Lambda=\emptyset}}\ \sum_{\mathrm{X}_1\cap\Lambda\neq\emptyset,\ldots,\mathrm{X}_{2n}\cap\Lambda\neq\emptyset} \Big[\Psi_{\mathrm{s}}(\mathrm{X}),\ \Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n}),\ldots\Big[\Psi_{\mathrm{s}}(\mathrm{X}_1),\ A\Big]_\gamma\cdots\Big]_\gamma\Big]_\gamma\\
&\quad= \sum_{\mathrm{X}\cap\Lambda\neq\emptyset}\ \sum_{\mathrm{X}_2\cap\Lambda\neq\emptyset,\ldots,\mathrm{X}_{2n}\cap\Lambda\neq\emptyset}\\
&\qquad \Big[\Psi_{\mathrm{s}}(\mathrm{X}),\ \Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n}),\ldots\Big[\Psi_{\mathrm{s}}(\mathrm{X}_2),\ \sum_{\mathrm{X}_1\cap\Lambda\neq\emptyset}\Big[\Psi_{\mathrm{s}}(\mathrm{X}_1),\ A\Big]_\gamma\Big]_\gamma\cdots\Big]_\gamma\Big]_\gamma\\
&\qquad+ \sum_{\substack{\mathrm{X}\cap\hat{\Lambda}_r\neq\emptyset\\ \mathrm{X}\cap\Lambda=\emptyset}}\ \sum_{\substack{\mathrm{X}_1\cap\Lambda\neq\emptyset,\\ \mathrm{X}_2\cap\Lambda\neq\emptyset,\ldots,\mathrm{X}_{2n}\cap\Lambda\neq\emptyset}}\\
&\qquad \Big[\Psi_{\mathrm{s}}(\mathrm{X}),\ \Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n}),\ldots\Big[\Psi_{\mathrm{s}}(\mathrm{X}_2),\ \Big[\Psi_{\mathrm{s}}(\mathrm{X}_1),\ A\Big]_\gamma\Big]_\gamma\cdots\Big]_\gamma\Big]_\gamma\\
&\quad= \sum_{\substack{\mathrm{X}_2\cap\Lambda\neq\emptyset,\ldots,\mathrm{X}_{2n}\cap\Lambda\neq\emptyset,\\ \mathrm{X}\cap\Lambda\neq\emptyset}} \Big[\Psi_{\mathrm{s}}(\mathrm{X}),\ \Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n}),\ldots\Big[\Psi_{\mathrm{s}}(\mathrm{X}_2),\ \delta_{\Psi_{\mathrm{s}}}(A)\Big]_\gamma\cdots\Big]_\gamma\Big]_\gamma\\
&\qquad+ \sum_{\mathrm{X}\cap\hat{\Lambda}_r\neq\emptyset,\,\mathrm{X}\subset\Lambda^c} \Big[\Psi_{\mathrm{s}}(\mathrm{X}),\ d_{\Psi\,0,\,\Lambda}^{\ n}(A)\Big]_\gamma\\
&\quad= d_{\Psi\,0,\,\Lambda}^{\ n}\big(\delta_{\Psi_{\mathrm{s}}}(A)\big) + b_{(n,\,\Lambda)}(A),
\end{aligned} \tag{4.75}$$

where Λ^c denotes the complement of Λ in $\mathbb{Z}^\nu$ and we have defined

$$b_{(n,\,\Lambda)}(A) := \sum_{\mathrm{X}\cap\hat{\Lambda}_r\neq\emptyset,\,\mathrm{X}\subset\Lambda^c} \Big[\Psi_{\mathrm{s}}(\mathrm{X}),\ d_{\Psi\,0,\,\Lambda}^{\ n}(A)\Big]_\gamma. \tag{4.76}$$

We will show that there exists a constant $k'_{\{A,\,\Psi_{\mathrm{s}}\}}$ depending on $A \in \mathcal{A}_{\circ}$ and Ψ_{s}, and a constant $m_{\Psi_{\mathrm{s}}}$ depending on Ψ_{s} satisfying that

$$\frac{1}{n!}\left\|b_{(n,\,\Lambda)}(A)\right\| \leq k'_{\{A,\,\Psi_{\mathrm{s}}\}} \cdot {m_{\Psi_{\mathrm{s}}}}^{n}. \tag{4.77}$$

By the definition of $b_{(n,\,\Lambda)}$ in (4.76) and the summation formula of $d_{\Psi\,0,\,\Lambda}^{\;n}(A)$ in (4.62), we see that

$$\begin{aligned} &b_{(n,\,\Lambda)}(A)\\ &\quad= \sum_{\mathrm{X}\cap\hat{\Lambda}_r\neq\emptyset,\,\mathrm{X}\cap\Lambda=\emptyset}\ \sum_{\substack{\{(\mathrm{X}_1,\mathrm{X}_2,\ldots,\mathrm{X}_{2n})\,\mid\,(\mathrm{X}_{2i-1}\cup\mathrm{X}_{2i})\cap\mathrm{V}_{2i-2}\neq\emptyset,\ \mathrm{X}_{2i-1}\cap\mathrm{X}_{2i}\neq\emptyset,\\ \mathrm{X}_{2i-1}\cap\Lambda\neq\emptyset,\\ \mathrm{X}_{2i}\cap\Lambda\neq\emptyset\ \text{for}\ \forall i\in\{1,2,\ldots,n\}\}}}\\ &\quad\Big[\Psi_{\mathrm{s}}(\mathrm{X}),\Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n})\Psi_{\mathrm{s}}(\mathrm{X}_{2n-1}),\ldots\Big[\Psi_{\mathrm{s}}(\mathrm{X}_4)\Psi_{\mathrm{s}}(\mathrm{X}_3),\Big[\Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1),\ A\Big]\Big]\cdots\Big]\Big]_{\gamma}, \end{aligned} \tag{4.78}$$

where only the last commutator involving $\Psi_{\mathrm{s}}(\mathrm{X})$ is the γ-graded commutator, while the others are the usual commutator. For each $(\mathrm{X}_1,\mathrm{X}_2,\ldots,\mathrm{X}_{2n-1},\mathrm{X}_{2n})$ that is relevant for the above sum, as in (4.63), let

$$\begin{aligned} B_n(A) &= \Big[\Psi_{\mathrm{s}}(\mathrm{X}_{2n})\Psi_{\mathrm{s}}(\mathrm{X}_{2n-1}),\ldots\Big[\Psi_{\mathrm{s}}(\mathrm{X}_4)\Psi_{\mathrm{s}}(\mathrm{X}_3),\Big[\Psi_{\mathrm{s}}(\mathrm{X}_2)\Psi_{\mathrm{s}}(\mathrm{X}_1),\ A\Big]\Big]\cdots\Big]\\ &\in \mathcal{F}(\mathrm{V}_{2n}),\\ B_0(A) &= A \in \mathcal{F}(\mathrm{I}),\\ &\quad \mathrm{V}_{2n} \equiv \mathrm{X}_{2n}\cup\mathrm{X}_{2n-1}\cup\cdots\cup\mathrm{X}_1\cup\mathrm{I}. \end{aligned} \tag{4.79}$$

Note that the above $B_n(A)$ depends on the set of finite subsets $(\mathrm{X}_1,\mathrm{X}_2,\ldots,\mathrm{X}_{2n-1},\mathrm{X}_{2n})$. Let us estimate $\sum_{\mathrm{X}\cap\hat{\Lambda}_r\neq\emptyset,\,\mathrm{X}\cap\Lambda=\emptyset}\big[\Psi_{\mathrm{s}}(\mathrm{X}),\ B_n(A)\big]_{\gamma}$. By (4.79) together with the γ-locality (4.9)

$$\sum_{\substack{\mathrm{X}\cap\hat{\Lambda}_r\neq\emptyset,\\ \mathrm{X}\cap\Lambda=\emptyset}}\big[\Psi_{\mathrm{s}}(\mathrm{X}),\ B_n(A)\big]_{\gamma} = \sum_{\substack{\mathrm{X}\cap\hat{\Lambda}_r\neq\emptyset,\\ \mathrm{X}\cap\Lambda=\emptyset,\\ \mathrm{X}\cap\mathrm{V}_{2n}\neq\emptyset}}\big[\Psi_{\mathrm{s}}(\mathrm{X}),\ B_n(A)\big]_{\gamma}.$$

Hence we obtain

$$\begin{aligned} &\left\|\sum_{\mathrm{X}\cap\hat{\Lambda}_r\neq\emptyset,\,\mathrm{X}\cap\Lambda=\emptyset}\big[\Psi_{\mathrm{s}}(\mathrm{X}),\ B_n(A)\big]_{\gamma}\right\|\\ &\quad\leq \sum_{\mathrm{X}\cap\hat{\Lambda}_r\neq\emptyset,\,\mathrm{X}\cap\Lambda=\emptyset,\,\mathrm{X}\cap\mathrm{V}_{2n}\neq\emptyset} 2\|\Psi_{\mathrm{s}}(\mathrm{X})\|\cdot\|B_n(A)\|\\ &\quad\leq \sum_{\mathrm{X}\cap\mathrm{V}_{2n}\neq\emptyset} 2\|\Psi_{\mathrm{s}}(\mathrm{X})\|\cdot\|B_n(A)\|\\ &\quad\leq \big|\mathrm{V}_{2n}\big|\cdot\max_{i\in\mathrm{V}_{2n}}\sum_{\mathrm{X}\ni i} 2\|\Psi_{\mathrm{s}}(\mathrm{X})\|\cdot\|B_n(A)\| \end{aligned}$$

$$\leq \left|\mathrm{V}_{2n}\right| \cdot 2^{|\mathrm{C}_{2r}|} \cdot 2\|\Psi_{\mathrm{s}}\|_{\infty} \cdot \|B_n(A)\|$$
$$\leq \left(|\mathrm{I}| + 2n \cdot (r+1)^{\nu}\right) \cdot 2^{(2r+1)^{\nu}} \cdot 2\|\Psi_{\mathrm{s}}\|_{\infty} \cdot \|B_n(A)\|. \tag{4.80}$$

In the same way as (4.65) for $d_{\Psi\,0}$ we can show a similar estimate for $d_{\Psi\,0,\,\Lambda}$. Namely for each $k \in \mathbb{N}$

$$\left\|d_{\Psi\,0,\,\Lambda}\,(B_k(A))\right\| \leq 4l_{\Psi_{\mathrm{s}}}\Big(|\mathrm{I}| + 2k \cdot (r+1)^{\nu}\Big) \cdot \|B_k(A)\|. \tag{4.81}$$

Now recall the multi-summation formula (4.78) of $b_{(n,\,\Lambda)}(A)$. By using (4.81) inductively from $k = 0$ to $k = n-1$ and then applying (4.80) we get

$$\begin{aligned}
&\|b_{(n,\,\Lambda)}(A)\| \\
&\quad \leq \left(|\mathrm{I}| + 2n \cdot (r+1)^{\nu}\right) \cdot 2^{(2r+1)^{\nu}} \cdot 2\|\Psi_{\mathrm{s}}\|_{\infty} \\
&\qquad \cdot (4l_{\Psi_{\mathrm{s}}})^n \|A\| \prod_{k=0}^{n-1}\Big(|\mathrm{I}| + 2k \cdot (r+1)^{\nu}\Big) \\
&\quad = 2^{(2r+1)^{\nu}+1}\|\Psi_{\mathrm{s}}\|_{\infty} \cdot \|A\| \cdot (4l_{\Psi_{\mathrm{s}}})^n \cdot \left(|\mathrm{I}| + 2n \cdot (r+1)^{\nu}\right) \cdot |\mathrm{I}| \\
&\qquad \cdot \prod_{k=1}^{n-1}\Big(|\mathrm{I}| + 2k \cdot (r+1)^{\nu}\Big) \\
&\quad \leq 2^{(2r+1)^{\nu}+1}\|\Psi_{\mathrm{s}}\|_{\infty} \cdot \|A\| \cdot |\mathrm{I}| \cdot (4l_{\Psi_{\mathrm{s}}})^n \Big(|\mathrm{I}| + 2n \cdot (r+1)^{\nu}\Big)^n.
\end{aligned} \tag{4.82}$$

By the inequality $\frac{x^n}{n!} \leq e^x$ for $x \geq 0$ we have

$$\begin{aligned}
&\frac{1}{n!}\Big((4l_{\Psi_{\mathrm{s}}})^n\left(|\mathrm{I}| + 2n \cdot (r+1)^{\nu}\right)^n\Big) = \frac{1}{n!}\Big(4\big\{|\mathrm{I}| + 2n \cdot (r+1)^{\nu}\big\}l_{\Psi_{\mathrm{s}}}\Big)^n \\
&\quad \leq \exp\Big(4\big\{|\mathrm{I}| + 2n \cdot (r+1)^{\nu}\big\}l_{\Psi_{\mathrm{s}}}\Big) = \exp\Big(4|\mathrm{I}|l_{\Psi_{\mathrm{s}}}\Big)\exp\Big(8n(r+1)^{\nu}l_{\Psi_{\mathrm{s}}}\Big) \\
&\quad = \exp\Big(4|\mathrm{I}|l_{\Psi_{\mathrm{s}}}\Big)\exp\Big(n\big\{8(r+1)^{\nu}l_{\Psi_{\mathrm{s}}}\big\}\Big).
\end{aligned} \tag{4.83}$$

By setting

$$k'_{\{A,\,\Psi_{\mathrm{s}}\}} := 2^{(2r+1)^{\nu}+1}\|\Psi_{\mathrm{s}}\|_{\infty} \cdot \|A\| \cdot |\mathrm{I}| \cdot \exp\Big(4|\mathrm{I}|l_{\Psi_{\mathrm{s}}}\Big), \tag{4.84}$$

and

$$m_{\Psi_{\mathrm{s}}} := \exp\Big(8(r+1)^{\nu}l_{\Psi_{\mathrm{s}}}\Big), \tag{4.85}$$

the desired estimate (4.77) holds for any Λ. (The above $m_{\Psi_{\mathrm{s}}}$ and $l_{\Psi_{\mathrm{s}}}$ are both exactly same as those in (4.59). Note that these constants are independent of Λ.)

As noted before $\alpha_t^{\Lambda}(A)$ and $d_{\Psi\,0,\,\Lambda}^{\;n}(A)$ are in the finite subsystem $\mathcal{F}(\hat{\Lambda}_r)$ on which δ_{Ψ_s} is obviously bounded. By using (4.56), (4.75) we have

$$\begin{aligned}
\delta_{\Psi_s}\big(\alpha_t^{\Lambda}(A)\big) &= \delta_{\Psi_s}\left(\sum_{n=0}^{\infty}\frac{(it)^n}{n!}d_{\Psi\,0,\,\Lambda}^{\;n}(A)\right) = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\delta_{\Psi_s}\left(d_{\Psi\,0,\,\Lambda}^{\;n}(A)\right)\\
&= \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\Big(d_{\Psi\,0,\,\Lambda}^{\;n}\big(\delta_{\Psi_s}(A)\big)+b_{(n,\,\Lambda)}(A)\Big)\\
&= \sum_{n=0}^{\infty}\frac{(it)^n}{n!}d_{\Psi\,0,\,\Lambda}^{\;n}\big(\delta_{\Psi_s}(A)\big)+\sum_{n=0}^{\infty}\frac{(it)^n}{n!}b_{(n,\,\Lambda)}(A)\\
&= \alpha_t^{\Lambda}\big(\delta_{\Psi_s}(A)\big)+\sum_{n=0}^{\infty}\frac{(it)^n}{n!}b_{(n,\,\Lambda)}(A).
\end{aligned}\tag{4.86}$$

As before let $t_\circ := \frac{1}{m_{\Psi_s}} > 0$. Due to the estimate (4.77), for an arbitrary $\varepsilon > 0$ there exists an $N_\circ \in \mathbb{N}$ independently of $\Lambda \Subset \mathbb{Z}^\nu$ such that

$$\sum_{n=N_\circ+1}^{\infty}\left\|\frac{b_{(n,\,\Lambda)}(A)}{n!}t^n\right\| < \varepsilon \quad \text{for} \quad |t| < t_\circ. \tag{4.87}$$

Next we will show that $\sum_{n=0}^{N_\circ}\frac{(it)^n}{n!}b_{(n,\,\Lambda)}(A)$ is negligible if we take Λ sufficiently large. By (4.49) we have $d_{\Psi\,0,\,\Lambda}^{\;n}(A) \in \mathcal{F}(\hat{\mathrm{I}}_{2N_\circ r})$ for every $n \in \{0,1,2,\ldots,N_\circ\}$. By the definition of $b_{(n,\,\Lambda)}(A)$ (4.76) and the γ-locality, if $\Lambda \supset \hat{\mathrm{I}}_{2N_\circ r}$, then $b_{(n,\,\Lambda)}(A) = 0$ for all $n \in \{0,1,2,\ldots,N_\circ\}$. Hence

$$\lim_{\Lambda\nearrow\mathbb{Z}^\nu}\sum_{n=0}^{N_\circ}\frac{(it)^n}{n!}b_{(n,\,\Lambda)}(A) = 0 \quad \text{for all} \quad t \in \mathbb{R}. \tag{4.88}$$

By (4.87) and (4.88), letting $\varepsilon \to 0$ we obtain

$$\lim_{\Lambda\nearrow\mathbb{Z}^\nu}\sum_{n=0}^{\infty}\frac{(it)^n}{n!}b_{(n,\,\Lambda)}(A) = 0 \quad \text{for} \quad |t| < t_\circ. \tag{4.89}$$

From the above formulas (4.86) (4.89) combined with (4.69) in Proposition 4.13, it follows that

$$\lim_{\Lambda\nearrow\mathbb{Z}^\nu}\delta_{\Psi_s}\big(\alpha_t^{\Lambda}(A)\big) = \alpha_t^{\Psi}\big(\delta_{\Psi_s}(A)\big) \quad \text{for} \quad |t| < t_\circ. \tag{4.90}$$

If δ_{Ψ_s} is norm-closable, then Eqs. (4.90), (4.69) imply that

$$\alpha_t^{\Psi}(A) \in \mathcal{D}_{\overline{\delta_{\Psi_s}}}, \quad \overline{\delta_{\Psi_s}}\big(\alpha_t^{\Psi}(A)\big) = \alpha_t^{\Psi}\big(\delta_{\Psi_s}(A)\big) \quad \text{for} \quad |t| < t_\circ. \tag{4.91}$$

We shall remove the restriction of $t \in \mathbb{R}$ in the equality (4.91). Let A denote any element of $\mathcal{A}_\circ$ as before. Take $s, t \in \mathbb{R}$ such that $|s| < t_\circ$ and $|t| < t_\circ$. By applying the formula (4.91) directly to $\alpha_s^{\Lambda}(A) \in \mathcal{A}_\circ$ for any fixed $\Lambda \Subset \mathbb{Z}^\nu$ we have

$$\alpha_t^{\Psi}\big(\alpha_s^{\Lambda}(A)\big) \in \mathcal{D}_{\overline{\delta_{\Psi_s}}}, \quad \overline{\delta_{\Psi_s}}\left(\alpha_t^{\Psi}\big(\alpha_s^{\Lambda}(A)\big)\right) = \alpha_t^{\Psi}\left(\delta_{\Psi_s}\big(\alpha_s^{\Lambda}(A)\big)\right) \quad \text{for} \quad |t| < t_\circ. \tag{4.92}$$

As $|s| < t_\circ$, by noting (4.90), taking $\Lambda \nearrow \mathbb{Z}^\nu$ of the right hand side of (4.92) yields

$$\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \alpha_t^\Psi \left(\delta_{\Psi_{\mathrm{s}}}\left(\alpha_s^\Lambda(A)\right)\right) = \alpha_t^\Psi \left(\alpha_s^\Psi\left(\delta_{\Psi_{\mathrm{s}}}(A)\right)\right) = \alpha_{t+s}^\Psi\left(\delta_{\Psi_{\mathrm{s}}}(A)\right). \tag{4.93}$$

Since $\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \alpha_t^\Psi\left(\alpha_s^\Lambda(A)\right) = \alpha_{t+s}^\Psi(A)$ and $\delta_{\Psi_{\mathrm{s}}}$ is norm-closable by the assumption, by (4.92) and (4.93) we have

$$\alpha_{t+s}^\Psi(A) \in \mathcal{D}_{\overline{\delta_{\Psi_{\mathrm{s}}}}}, \ \overline{\delta_{\Psi_{\mathrm{s}}}}\left(\alpha_{t+s}^\Psi(A)\right) = \alpha_{t+s}^\Psi\left(\delta_{\Psi_{\mathrm{s}}}(A)\right) \quad \text{for} \quad |s|,\, |t| < t_\circ. \tag{4.94}$$

Repeating the above extension procedure from Eqs. (4.91) to (4.94) we obtain the assertion (4.72).

Next we will show (4.73) and (4.74). As Eq. (4.75) holds for both $\Psi_{\mathrm{s}} = \Psi_{\mathrm{s},1}$ and $\Psi_{\mathrm{s}} = \Psi_{\mathrm{s},2}$, by noting (4.26) we see that for every $n \in \mathbb{N}$

$$\begin{aligned}
\delta_\Psi\left(d_{\Psi\,0,\,\Lambda}^{\ n}(A)\right) &= d_{\Psi\,0,\,\Lambda}^{\ n}\left(\delta_\Psi(A)\right) + \bar{b}_{(n,\,\Lambda)}(A),\\
\delta_\Psi^*\left(d_{\Psi\,0,\,\Lambda}^{\ n}(A)\right) &= d_{\Psi\,0,\,\Lambda}^{\ n}\left(\delta_\Psi^*(A)\right) + \tilde{b}_{(n,\,\Lambda)}(A),
\end{aligned} \tag{4.95}$$

where we have defined

$$\begin{aligned}
\bar{b}_{(n,\,\Lambda)}(A) &:= \sum_{\mathrm{X}\cap\hat{\Lambda}_r \neq \emptyset,\, \mathrm{X}\subset\Lambda^c} \left[\Psi(\mathrm{X}),\ d_{\Psi\,0,\,\Lambda}^{\ n}(A)\right]_\gamma,\\
\tilde{b}_{(n,\,\Lambda)}(A) &:= \sum_{\mathrm{X}\cap\hat{\Lambda}_r \neq \emptyset,\, \mathrm{X}\subset\Lambda^c} \left[\Psi^*(\mathrm{X}),\ d_{\Psi\,0,\,\Lambda}^{\ n}(A)\right]_\gamma.
\end{aligned} \tag{4.96}$$

Repeating the argument from Eq.(4.75) to Eq.(4.90) we have

$$\begin{aligned}
\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \delta_\Psi\left(\alpha_t^\Lambda(A)\right) &= \alpha_t^\Psi\left(\delta_\Psi(A)\right) \quad \text{for} \quad |t| < t_\circ,\\
\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \delta_\Psi^*\left(\alpha_t^\Lambda(A)\right) &= \alpha_t^\Psi\left(\delta_\Psi^*(A)\right) \quad \text{for} \quad |t| < t_\circ.
\end{aligned} \tag{4.97}$$

If δ_Ψ is norm-closable, then δ_Ψ^* is also norm-closable by Proposition 2.9. Hence for this case, we obtain both

$$\alpha_t^\Psi(A) \in \mathcal{D}_{\overline{\delta_\Psi}}, \quad \overline{\delta_\Psi}\left(\alpha_t^\Psi(A)\right) = \alpha_t^\Psi\left(\delta_\Psi(A)\right) \quad \text{for} \quad |t| < t_\circ, \tag{4.98}$$

and

$$\alpha_t^\Psi(A) \in \mathcal{D}_{\overline{\delta_\Psi^*}}, \quad \overline{\delta_\Psi^*}\left(\alpha_t^\Psi(A)\right) = \alpha_t^\Psi\left(\delta_\Psi^*(A)\right) \quad \text{for} \quad |t| < t_\circ. \tag{4.99}$$

We can remove the restriction of $t \in \mathbb{R}$ in (4.98), (4.99) as before and get the desired formulas (4.73), (4.74). □

Corollary 4.20. *Let $\Psi \in \mathscr{C}^\sharp_{unbroken}$. Then all the commutativity relations between the global time evolution and the superderivations* (4.72)–(4.74) *in Proposition* 4.19 *hold.*

Proof. The statement directly follows from the combination of Proposition 4.18 and Proposition 4.19. □

By collecting the results established in this section, we propose a general class of supersymmetric C^*-dynamics on the fermion lattice system.

Theorem 4.21. *Let* $(\mathcal{F},\ \{\mathcal{F}(\mathrm{I});\ \mathrm{I} \Subset \mathbb{Z}^{\nu}\},\ \gamma)$ *denote the fermion lattice system on* $\mathbb{Z}^{\nu}$*. Let* $\mathcal{A}_{\circ}$ *denote the local algebra of* $\mathcal{F}$*. Take any* $\Psi \in \mathscr{C}^{\sharp}$ *of Definition 4.1, namely any uniformly bounded finite-range assignment of local fermion charges on the fermion lattice system whose associated superderivation* δ_{Ψ} *is nilpotent. Then there exists a strongly continuous one-parameter group of* $*$*-automorphisms* $\{\alpha_t^{\Psi};\ t \in \mathbb{R}\}$ *on the CAR algebra* $\mathcal{F}$ *whose generator* $d_{\Psi\,0}$ *is equal to the norm closure of the supersymmetric derivation in the form of* $\delta_{\Psi}^{*} \cdot \delta_{\Psi} + \delta_{\Psi} \cdot \delta_{\Psi}^{*}$ *on* $\mathcal{A}_{\circ}$*. Furthermore if* $\Psi \in \mathscr{C}^{\sharp}_{unbroken}$ *of Definition 4.5, then the nilpotent superderivations* δ_{Ψ} *and* δ_{Ψ}^{*}*, and the hermite superderivations* $\delta_{\Psi_{\mathrm{s},1}}$ *and* $\delta_{\Psi_{\mathrm{s},2}}$ *all commute with the time evolution* $\{\alpha_t^{\Psi};\ t \in \mathbb{R}\}$ *on* $\mathcal{A}_{\circ}$*.*

Remark 4.22. Consider $\Psi \in \mathscr{C}^{\sharp}_{unbroken}$ of Definition 4.5. Then any supersymmetric state φ with respect to δ_{Ψ} (in the sense of Definition 2.3) is a ground state for $\{\alpha_t^{\Psi};\ t \in \mathbb{R}\}$, because $\mathcal{A}_{\circ}$ is a core for the generator $d_{\Psi\,0}$ of $\{\alpha_t^{\Psi};\ t \in \mathbb{R}\}$ by Definition 4.14, and the required condition to be a ground state (Definition 5.3.18 of [4]) is satisfied as follows:

$$\begin{aligned}\varphi\left(A^{*} d_{\Psi\,0}(A)\right) &= \varphi\left(A^{*}\delta_{\Psi}^{*}(\delta_{\Psi}(A))\right) + \varphi\left(A^{*}\delta_{\Psi}(\delta_{\Psi}^{*}(A))\right)\\ &= \varphi\left(\delta_{\Psi}(A)^{*}\delta_{\Psi}(A)\right) + \varphi\left(\delta_{\Psi}^{*}(A)^{*}\delta_{\Psi}^{*}(A)\right) \geq 0, \quad \forall A \in \mathcal{A}_{\circ}.\end{aligned} \tag{4.100}$$

See Proposition 2.2 of [5] for the detail.

5. Examples

In this section, we present some supersymmetric fermion lattice models in the C^{*}-algebraic formulation stated in Sect. 4. For the first example we will reproduce the Nicolai's model (3.1), (3.2) in Sect. 3 in the C^{*}-algebraic format. Define a map from $\{\mathrm{I};\ \mathrm{I} \Subset \mathbb{Z}\}$ into $\mathcal{A}_{\circ -}$ by

$$\begin{aligned}\Psi_{\mathrm{Nic}}\big(\{2i-1,\, 2i,\, 2i+1\}\big) &:= a_{2i+1}a_{2i}^{*}a_{2i-1} \quad \text{for each } \ \{2i-1,\, 2i,\, 2i+1\}\\ &\qquad (i \in \mathbb{Z}),\\ \Psi_{\mathrm{Nic}}(\mathrm{I}) &:= 0 \quad \text{for any other } \ \mathrm{I} \Subset \mathbb{Z}.\end{aligned} \tag{5.1}$$

Note that $a_{2i+1}a_{2i}^{*}a_{2i-1} \in \mathcal{F}_{-}\big(\{2i-1,\, 2i,\, 2i+1\}\big)$. We see that Ψ_{Nic} is 2-periodic by the lattice translation on $\mathbb{Z}$ and that its range is finite $r = 2$. Therefore, Ψ_{Nic} defined above is a bounded finite-range assignment of local fermion charges on $\mathbb{Z}$, namely Ψ_{Nic} belongs to $\mathscr{C}$.

We will see that the nilpotent condition (4.33) necessary for the supersymmetry is saturated by $\delta_{\Psi_{\mathrm{Nic}}}$. By noting the formula (4.71) of Proposition 4.15, we have to look for only the pairs $\mathrm{X}_1 \Subset \mathbb{Z}$ and $\mathrm{X}_2 \Subset \mathbb{Z}$ such that $\mathrm{X}_1 \cap \mathrm{X}_2 \neq \emptyset$ and $\Psi_{\mathrm{Nic}}(\mathrm{X}_2)\Psi_{\mathrm{Nic}}(\mathrm{X}_1) \neq 0$. By the formula (5.1) and the CARs (4.2), we verify that such pair does not exist. Thus $\delta_{\Psi_{\mathrm{Nic}}} \cdot \delta_{\Psi_{\mathrm{Nic}}}(A) = 0$ holds for every $A \in \mathcal{A}_{\circ}$. Accordingly $\Psi_{\mathrm{Nic}} \in \mathscr{C}^{\sharp}$.

We will show the existence of supersymmetric states with respect to $\delta_{\Psi_{\mathrm{Nic}}}$. Let φ_{emp} denote the unique state of $\mathcal{F}$ determined by

$$\varphi_{\mathrm{emp}}(a_j^{*}a_j) = 0 \quad \text{for all } \ j \in \mathbb{Z}. \tag{5.2}$$

This is a Fock state. Similarly, one may take the fully occupied state φ_{occup} determined by

$$\varphi_{\mathrm{occup}}(a_j a_j^*) = 0 \quad \text{for all} \quad j \in \mathbb{Z}. \tag{5.3}$$

For any $A \in \mathcal{A}_\circ$ and $j \in \mathbb{Z}$,

$$0 = \varphi_{\mathrm{emp}}(A a_{2j+1} a_{2j}^* a_{2j-1}) = \varphi_{\mathrm{emp}}(a_{2j+1} a_{2j}^* a_{2j-1} A),$$
$$0 = \varphi_{\mathrm{occup}}(A a_{2j+1} a_{2j}^* a_{2j-1}) = \varphi_{\mathrm{occup}}(a_{2j+1} a_{2j}^* a_{2j-1} A),$$

since by the Cauchy–Schwarz inequality

$$\begin{aligned}
&\left|\varphi_{\mathrm{emp}}(A a_{2j+1} a_{2j}^* a_{2j-1})\right|^2 \\
&\quad \le \varphi_{\mathrm{emp}}\left(\left(A a_{2j+1} a_{2j}^*\right)\left(A a_{2j+1} a_{2j}^*\right)^*\right) \cdot \varphi_{\mathrm{emp}}\left(a_{2j-1}^* a_{2j-1}\right) = 0, \\
&\qquad \left|\varphi_{\mathrm{emp}}(a_{2j+1} a_{2j}^* a_{2j-1} A)\right|^2 = \left|\varphi_{\mathrm{emp}}(-a_{2j}^* a_{2j+1} a_{2j-1} A)\right|^2 \\
&\quad = \left|\varphi_{\mathrm{emp}}(a_{2j}^* a_{2j+1} a_{2j-1} A)\right|^2 \\
&\quad \le \varphi_{\mathrm{emp}}\left(a_{2j}^* a_{2j}\right) \cdot \varphi_{\mathrm{emp}}\left(\left(a_{2j+1} a_{2j-1} A\right)^* \left(a_{2j+1} a_{2j-1} A\right)\right) = 0,
\end{aligned}$$

and similar results hold for φ_{occup}. Thus we obtain for any $A \in \mathcal{A}_\circ$ and $\mathrm{X} \Subset \mathbb{Z}$

$$0 = \varphi_{\mathrm{emp}}\left(A \Psi_{\mathrm{Nic}}(\mathrm{X})\right) = \varphi_{\mathrm{emp}}\left(\Psi_{\mathrm{Nic}}(\mathrm{X}) A\right),$$
$$0 = \varphi_{\mathrm{occup}}\left(A \Psi_{\mathrm{Nic}}(\mathrm{X})\right) = \varphi_{\mathrm{occup}}\left(\Psi_{\mathrm{Nic}}(\mathrm{X}) A\right).$$

These yield for any $A \in \mathcal{A}_\circ$

$$\varphi_{\mathrm{emp}}\left(\delta_{\Psi_{\mathrm{Nic}}}(A)\right) = \sum_{\mathrm{X} \Subset \mathbb{Z}} \varphi_{\mathrm{emp}}\left([\Psi_{\mathrm{Nic}}(\mathrm{X}),\ A]_\gamma\right) = 0, \tag{5.4}$$

and

$$\varphi_{\mathrm{occup}}\left(\delta_{\Psi_{\mathrm{Nic}}}(A)\right) = 0. \tag{5.5}$$

Namely both φ_{emp} and φ_{occup} are invariant under $\delta_{\Psi_{\mathrm{Nic}}}$. We have shown $\Psi_{\mathrm{Nic}} \in \mathscr{C}^\sharp_{unbroken}$. By Theorem 4.21 $\Psi_{\mathrm{Nic}} \in \mathscr{C}^\sharp_{unbroken}$ generates supersymmetric C^*-dynamics on the CAR algebra.

In the following we provide other two examples. Let

$$\Psi_{\mathrm{Fen}}\left(\{i-1,\, i,\, i+1\}\right) := a_i P_{<i>} \quad \text{for each } \{i-1,\, i,\, i+1\} \quad (i \in \mathbb{Z}),$$
$$\Psi_{\mathrm{Fen}}\left(\mathrm{I}\right) := 0 \quad \text{for any other } \mathrm{I} \Subset \mathbb{Z}, \tag{5.6}$$

where

$$P_{<i>} := (1 - a_{i-1}^* a_{i-1})(1 - a_{i+1}^* a_{i+1}) \quad \text{for each} \quad i \in \mathbb{Z}. \tag{5.7}$$

This model is due to Fendly et al. [8]. We easily verify that $\Psi_{\mathrm{Fen}} \in \mathscr{C}^\sharp$ by using Proposition 4.15. We also see that the state φ_{occup} given in (5.3) is invariant under the superderivation $\delta_{\Psi_{\mathrm{Fen}}}$ associated with Ψ_{Fen}. Hence $\Psi_{\mathrm{Fen}} \in \mathscr{C}^\sharp_{unbroken}$.

One may consider the following much simpler model.

$$\Psi_{\mathrm{trivial}}\left(\{i\}\right) := a_i \quad \text{for each } i \in \mathbb{Z},$$
$$\Psi_{\mathrm{trivial}}\left(\mathrm{I}\right) := 0 \quad \text{for any other } \mathrm{I} \Subset \mathbb{Z}. \tag{5.8}$$

This is given by deleting the projections $P_{<i>}$ in the form (5.6) of Ψ_{Fen}. We see that $\Psi_{\mathrm{trivial}} \in \mathscr{C}^{\sharp}$ by Proposition 4.15. The corresponding time generator $d_{\Psi_{\mathrm{trivial}}\,0} \equiv \delta_{\Psi_{\mathrm{trivial}}}{}^{*} \cdot \delta_{\Psi_{\mathrm{trivial}}} + \delta_{\Psi_{\mathrm{trivial}}} \cdot \delta_{\Psi_{\mathrm{trivial}}}{}^{*}$ is a zero map on $\mathcal{A}_{\circ}$. So the time evolution $\{\alpha_t^{\Psi_{\mathrm{trivial}}}\,;\ t \in \mathbb{R}\}$ on the total system $\mathcal{F}$ is trivial. The supersymmetry for Ψ_{trivial} is spontaneously broken, since for each $i \in \mathbb{Z}$

$$\delta_{\Psi_{\mathrm{trivial}}}(a_i^*) = \{a_i,\ a_i^*\} = 1,$$

and then for any state ω on $\mathcal{F}$

$$\omega\left(\delta_{\Psi_{\mathrm{trivial}}}(a_i^*)\right) = \omega(1) = 1 \neq 0.$$

Remark 5.1. The fully occupied state φ_{occup} may be unphysical for the Fendly's model, see [8]. However, there exist (many) "physical" supersymmetric states on finite regions as noted in [9, 23]. Any cluster point of such physical states in the infinite-volume limit gives a supersymmetric state by Proposition 2.11. We can use any such supersymmetric state in the place of φ_{occup}.

Remark 5.2. We have seen that Ψ_{trivial} gives a supersymmetry breaking model. Hence Proposition 4.18 cannot be applied to Ψ_{trivial}. Nevertheless, the commutativity relations between its (trivial) global time evolution and its associated superderivations as in Proposition 4.19 are obviously saturated.

6. Abstraction

One encounters various difficulties to formulate supersymmetry in C^*-algebra. It seems, however, that those are mixed up in the literature. We intend to find an appropriate solution to each of them. Buchholz–Grundling have succeeded in formulating a simple supersymmetry model by introducing a new C^*-algebra called the resolvent algebra in [6]. In this work [6] the crucial difficulty lies at unboundedness of boson fields. On the other hand, we have focused on fermion lattice systems in this paper. We have discussed C^*-algebraic formulation of hidden supersymmetry in fermion lattice systems where no boson field exists. As the fermion system is defined on the CAR algebra, we can make use of several techniques of C^*-algebra theory. We have seriously considered some mathematical problems related to the infinite volume limit.

In this final section, we shall provide a general scheme of supersymmetric C^*-dynamical systems based on our C^*-algebraic formulation of supersymmetric fermion lattice systems. By this abstraction we can see the general structure more clearly.

Let $\mathcal{F}$ denote a general graded C^*-algebra and let γ denote its grading automorphism γ. Let δ be a superderivation of $\mathcal{F}$ whose domain $\mathcal{A}_{\circ}$ is a globally γ-invariant $*$-subalgebra. Let $\{\alpha_t\,;\ t \in \mathbb{R}\}$ denote a one-parameter group of $*$-automorphisms of $\mathcal{F}$. The superderivation δ and its conjugate superderivation δ^* generate supersymmetry transformation on $\mathcal{F}$, and $\{\alpha_t\,;\ t \in \mathbb{R}\}$ denotes a global time evolution on $\mathcal{F}$. The pair $\{\delta,\ \alpha_t\}$ is the basic building kit. We shall list assumptions on $\{\delta,\ \alpha_t\}$ in the following.

The first assumption is obvious.

• Time evolution preserves the grading:

$$\alpha_t \cdot \gamma = \gamma \cdot \alpha_t \quad \text{for each} \quad t \in \mathbb{R}. \tag{6.1}$$

We need some continuity for the time evolution with respect to time.

• Pointwise norm continuity:

$$\lim_{t \to 0} \|\alpha_t(F) - F\| \longrightarrow 0 \quad \text{for every} \quad F \in \mathcal{F}. \tag{6.2}$$

This condition on time evolution is satisfied by quantum spin lattice models and fermion lattice models for short-range interactions [4]. However, this cannot be expected for general boson systems [4]. If the time evolution $\{\alpha_t\,;\ t \in \mathbb{R}\}$ satisfies this continuity condition, then there exists its infinitesimal generator defined by

$$\begin{aligned} \mathcal{D}_{d_0} &:= \left\{X \in \mathcal{F}\,;\ \ \lim_{t \to 0} \frac{1}{t}\big(\alpha_t(X) - X\big) \text{ exists in norm in } \mathcal{F}\right\}, \\ d_0(X) &:= -i \frac{d}{dt}\alpha_t(X)\Big|_{t=0} \in \mathcal{F} \ \text{ for } X \in \mathcal{D}_{d_0}. \end{aligned} \tag{6.3}$$

Of course, the Leibniz rule is satisfied by this:

$$d_0(XY) = d_0(X)Y + Xd_0(Y) \quad \text{for} \quad X, Y \in \mathcal{D}_{d_0}.$$

It has been known that $\mathcal{D}_{d_0}$ is a norm-dense $*$-subalgebra of $\mathcal{F}$, see [3, Proposition 3.1.6]. From (6.1) it follows that

$$\gamma(\mathcal{D}_{d_0}) = \mathcal{D}_{d_0}, \quad d_0 \cdot \gamma = \gamma \cdot d_0 \quad \text{on} \quad \mathcal{D}_{d_0}.$$

Remark 6.1. As defined in (6.3) the derivation d_0 is closed, whereas the superderivation δ on $\mathcal{A}_\circ$ is not assumed to be closed. Later δ will be assumed to be *closable*.

To encode essential information of dynamics on the domain of the superderivation, we postulate the following.

• The domain of the superderivation is "large":

$$\mathcal{A}_\circ \text{ is norm-dense in } \ \mathcal{F}. \tag{6.4}$$

To relate the superderivation with the time evolution the following assumption will be convenient.

• The domain of the superderivation is included in that of the time generator:

$$\mathcal{A}_\circ \subset \mathcal{D}_{d_0}. \tag{6.5}$$

We assume differentiability of the superderivation in the following sense. This is crucial.

• Differentiability of the superderivation:

$$\delta(\mathcal{A}_\circ) \subset \mathcal{A}_\circ. \tag{6.6}$$

It is immediate to see that the associated superderivations δ^*, $\delta_{\mathrm{s},1}$ and $\delta_{\mathrm{s},2}$ satisfy this differentiability as well. Actually from (2.8), (6.6) and the γ-invariance of $\mathcal{A}_\circ$, it follows that

$$\delta^*(\mathcal{A}_\circ) \subset \mathcal{A}_\circ. \tag{6.7}$$

From (2.11), (6.6), (6.7) it follows that

$$\delta_{\mathrm{s},1}(\mathcal{A}_\circ) \subset \mathcal{A}_\circ, \quad \delta_{\mathrm{s},2}(\mathcal{A}_\circ) \subset \mathcal{A}_\circ. \tag{6.8}$$

By (6.6), (6.7), (6.8) any composition of δ, δ^*, $\delta_{\mathrm{s},1}$, $\delta_{\mathrm{s},2}$ can be defined on $\mathcal{A}_\circ$.

Remark 6.2. The differentiability of superderivations is due to the differential graded algebra (DGA), see [18]. It is one of the desiderata of 'the quantum algebra' by Jaffe et al. [13,14].

We will list more involved assumptions.

- The domain of the superderivation is a core for the time generator:

$$\overline{d_0|_{\mathcal{A}_\circ}} = d_0, \tag{6.9}$$

where the bar on $d_0|_{\mathcal{A}_\circ}$ denotes the norm closure.

We assume the following topological property on the superderivations.

- Norm-closability of the superderivations:

$$\delta : \mathcal{A}_\circ \mapsto \mathcal{F} \ \text{ is norm-closable,} \tag{6.10}$$

and

$$\delta_{\mathrm{s},1} : \mathcal{A}_\circ \mapsto \mathcal{F} \ \text{ is norm-closable,} \quad \delta_{\mathrm{s},2} : \mathcal{A}_\circ \mapsto \mathcal{F} \ \text{ is norm-closable.} \tag{6.11}$$

It has been noted in Proposition 2.9 that (6.10) implies that

$$\delta^* : \mathcal{A}_\circ \mapsto \mathcal{F} \ \text{ is norm-closable.} \tag{6.12}$$

We denote the norm closure of δ by $\overline{\delta}$ and its extended domain by $\mathcal{D}_{\overline{\delta}}$.

Remark 6.3. For the supersymmetric fermion lattice systems presented in Sect. 4, we have guaranteed all of (6.10)–(6.12) in Propositions 4.18.

When the superderivations are norm-closable, we can assume the following relations.

- Time evolution preserves supersymmetry:

$$\begin{aligned}
&\alpha_t(\mathcal{A}_\circ) \subset \mathcal{D}_{\overline{\delta}} \ \text{ and } \ \overline{\delta} \cdot \alpha_t = \alpha_t \cdot \delta \ \text{ on } \ \mathcal{A}_\circ \ \text{ for every } t \in \mathbb{R},\\
&\alpha_t(\mathcal{A}_\circ) \subset \mathcal{D}_{\overline{\delta^*}} \ \text{ and } \ \overline{\delta^*} \cdot \alpha_t = \alpha_t \cdot \delta^* \ \text{ on } \ \mathcal{A}_\circ \ \text{ for every } t \in \mathbb{R},\\
&\alpha_t(\mathcal{A}_\circ) \subset \mathcal{D}_{\overline{\delta_{\mathrm{s},1}}} \ \text{ and } \ \overline{\delta_{\mathrm{s},1}} \cdot \alpha_t = \alpha_t \cdot \delta_{\mathrm{s},1} \ \text{ on } \ \mathcal{A}_\circ \ \text{ for every } t \in \mathbb{R},\\
&\alpha_t(\mathcal{A}_\circ) \subset \mathcal{D}_{\overline{\delta_{\mathrm{s},2}}} \ \text{ and } \ \overline{\delta_{\mathrm{s},2}} \cdot \alpha_t = \alpha_t \cdot \delta_{\mathrm{s},2} \ \text{ on } \ \mathcal{A}_\circ \ \text{ for every } t \in \mathbb{R}.
\end{aligned} \tag{6.13}$$

Remark 6.4. The assumption (6.13) seems indispensable for any natural supersymmetry theory, since the commutativity between the supercharges and the supersymmetric Hamiltonian follows directly from the supersymmetry algebra as noted in Sect. 3. On the other hand, if the domain $\mathcal{A}_\circ$ of the superderivation is not invariant under the global time evolution, then subtle care is required. We have considered this non-trivial problem in Proposition 4.19 in Sect. 4 for the case of fermion lattice systems.

With all the assumptions given so far, we shall propose supersymmetric C^*-dynamics.

Definition 6.5. Let δ denote a superderivation of a graded C^*-algebra $\mathcal{F}$ and let $\mathcal{A}_\circ$ denote its domain subalgebra. Let $\{\alpha_t\,;\ t \in \mathbb{R}\}$ denote a strongly continuous one-parameter group of $*$-automorphisms of $\mathcal{F}$ and let d_0 denote its generator. All the assumptions stated in this section are satisfied. If δ is nilpotent, i.e.:

$$\delta \cdot \delta = \mathbf{0} \quad \text{on} \ \ \mathcal{A}_\circ, \tag{6.14}$$

and the following relation holds

$$d_0 = \delta^* \cdot \delta + \delta \cdot \delta^* \quad \text{on} \ \ \mathcal{A}_\circ, \tag{6.15}$$

then it is said that $\{\delta,\ \alpha_t\}$ generates a supersymmetric C^*-dynamics.

As we have seen in Lemma 4.6 in Sect. 4, the superderivation δ is nilpotent if and only if δ^* is nilpotent:

$$\delta^* \cdot \delta^* = \mathbf{0} \quad \text{on} \ \ \mathcal{A}_\circ. \tag{6.16}$$

We note the following general remark on superderivations.

Proposition 6.6. *Suppose that a superderivation δ is nilpotent and hermite. Then the supersymmetric dynamics induced by $\{\delta,\ \alpha_t\}$ is trivial.*

Proof. By (6.14)–(6.16) together with the hermite property $\delta^* = \delta$, we obtain $d_0 = \mathbf{0}$ on $\mathcal{A}_\circ$. By this and (6.9) the strongly continuous time evolution $\{\alpha_t\,;\ t \in \mathbb{R}\}$ is trivial, i.e., α_t is an identity map for each $t \in \mathbb{R}$. □

We can rewrite the supersymmetric C^*-dynamics given above in terms of the hermite superderivations.

Proposition 6.7. *The set of relations* (6.14) *and* (6.15) *in Definition* 6.5 *is equivalent to the combination of*

$$\delta_{\mathrm{s},1} \cdot \delta_{\mathrm{s},2} + \delta_{\mathrm{s},2} \cdot \delta_{\mathrm{s},1} = \mathbf{0} \quad \text{on } \mathcal{A}_\circ, \tag{6.17}$$

and

$$d_0 = {\delta_{\mathrm{s},1}}^2 = {\delta_{\mathrm{s},2}}^2 \quad \text{on } \mathcal{A}_\circ, \tag{6.18}$$

where the hermite superderivations $\delta_{\mathrm{s},1}$ and $\delta_{\mathrm{s},2}$ are given in (2.11)*.*

Proof. The proof has been essentially done in Proposition 4.9 in Sect. 4. □

Remark 6.8. Another C^*-algebraic framework of supersymmetry is proposed by Jaffe et al. in [14, Sect. 2]. Here the superderivation δ of a graded C^*-algebra $\mathcal{F}$ is defined on the analytic algebra $\mathcal{F}_\alpha$ that consists of the *whole entire analytic elements* for the given strongly continuous time automorphism group α_t, and the supersymmetry relation is defined on $\mathcal{F}_\alpha$. Its justification is not given there. Theorem 4.21 in Sect. 4 gives a concrete realization of C^*-algebraic supersymmetry on the fermion lattice system taking the local algebra for the domain of superderivations. This norm-dense subalgebra is *strictly* contained in $\mathcal{F}_\alpha$.

Finally, we shall propose a naive question. How should the listed assumptions be changed to deal with supersymmetry between fermions and bosons? By comparing this work with [6,19] we shall give some comments. First the norm topology used in (6.2), (6.4) should be replaced by some weaker one. Second, the differentiability of the superderivation (6.6) is likely not to be satisfied. So it should be removed or altered. Third, the assumption (6.13) may be trivial for quantum field theory if the domain of superderivations is made by local subsystems only. Note that general time evolution in fermion (quantum spin) lattice systems is non-local, whereas it is local in relativistic systems due to the finite velocity of its propagation.

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