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Quantum Scattering in a Periodically Pulsed Magnetic Field

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Abstract. In this paper, we study the quantum dynamics of a charged particle in the plane in the presence of a periodically pulsed magnetic field perpendicular to the plane. We show that by controlling the cycle when the magnetic field is switched on and off appropriately, the result of the asymptotic completeness of wave operators can be obtained under the assumption that the potential V satisfies the decaying condition $|V(x)| \leq C(1+|x|)^{-\rho}$ for some $\rho > 0$.

1. Introduction

The purpose of this paper was to study of the quantum dynamics of a charged particle in the plane $\boldsymbol{R}^2$ in the presence of a periodically pulsed magnetic field perpendicular to the plane.

We consider a quantum system of a charged particle moving in the plane $\boldsymbol{R}^2$ in the presence of a periodically pulsed magnetic field $\boldsymbol{B}(t)$ which is perpendicular to the plane. We suppose that positive constants B and T_B are given, and that $\boldsymbol{B}(t) = (0, 0, B(t)) \in \boldsymbol{R}^3$ is given by

$$B(t) = \begin{cases} B, & t \in \bigcup_{n\in\boldsymbol{Z}}[nT, nT+T_B) =: I_B, \\ 0, & t \in \bigcup_{n\in\boldsymbol{Z}}[nT+T_B, (n+1)T) =: I_0, \end{cases} \tag{1.1}$$

for some T with $T > T_B$. T is the period of $\boldsymbol{B}(t)$. We put $T_0 := T - T_B > 0$ for simplicity. Then the free Hamiltonian under consideration is defined by

$$H_0(t) = \frac{1}{2m}(p - qA(t,x))^2 \tag{1.2}$$

acting on $\mathscr{H} := L^2(\boldsymbol{R}^2)$, where $m > 0$, $q \in \boldsymbol{R}\backslash\{0\}$, $x = (x_1, x_2)$ and $p = (p_1, p_2) = (-i\partial_1, -i\partial_2)$ are the mass, the charge, the position, and the momentum of the charged particle, respectively, and

$$A(t,x) = \left(-\frac{B(t)}{2}x_2, \frac{B(t)}{2}x_1\right) = \begin{cases} \left(-\frac{B}{2}x_2, \frac{B}{2}x_1\right) =: A(x), & t \in I_B, \\ (0,0), & t \in I_0, \end{cases} \tag{1.3}$$

is the vector potential in the symmetric gauge. By introducing the operator J defined as

$$(Jx)^{\mathrm{T}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x^{\mathrm{T}}, \tag{1.4}$$

$A(x)$ can be written as

$$A(x) = \frac{B}{2}Jx. \tag{1.5}$$

Here x^{T} denotes the transpose of x. Then $H_0(t)$ is represented as

$$H_0(t) = \begin{cases} H_0^B, & t \in I_B, \\ H_0^0, & t \in I_0, \end{cases} \tag{1.6}$$

where the free Landau Hamiltonian H_0^B and the free Schrödinger operator H_0^0 are given by

$$H_0^B = \frac{1}{2m}D^2, \quad H_0^0 = \frac{1}{2m}p^2. \tag{1.7}$$

D is the momentum of the charged particle in the presence of the constant magnetic field $\boldsymbol{B} = (0,0,B)$, which is given by

$$D = (D_1, D_2) = \left(p_1 + \frac{qB}{2}x_2, p_2 - \frac{qB}{2}x_1\right) = p - qA(x) = p - \frac{qB}{2}Jx. \tag{1.8}$$

Let $U_0(t,s)$ be the propagator generated by $H_0(t)$ (in the sense of Theorem 2 of Huang [7]). By (1.6) and the self-adjointness of H_0^B and H_0^0, $U_0(t,0)$ is represented as

$$U_0(t,0) = \begin{cases} \mathrm{e}^{-i(t-nT)H_0^B} U_0(T,0)^n, & t \in [nT, nT+T_B), \\ \mathrm{e}^{-i(t-(nT+T_B))H_0^0} \mathrm{e}^{-iT_B H_0^B} U_0(T,0)^n, & t \in [nT+T_B, (n+1)T), \end{cases} \tag{1.9}$$

with $n \in \boldsymbol{Z}$, where

$$U_0(T,0) = \mathrm{e}^{-iT_0 H_0^0} \mathrm{e}^{-iT_B H_0^B} \tag{1.10}$$

is the Floquet operator associated with $H_0(t)$, $U_0(T,0)^0 = \mathrm{Id}$, and $U_0(T,0)^n = (U_0(T,0)^*)^{-n}$ when $-n \in \boldsymbol{N}$. Put

$$\omega := \frac{qB}{m}, \quad \bar{\omega} := \frac{\omega}{2}, \quad \bar{\bar{\omega}} := \frac{\bar{\omega}}{2} = \frac{\omega}{4}. \tag{1.11}$$

$|\omega|$ is the Larmor frequency of the charged particle in the presence of the constant magnetic field $\boldsymbol{B}$. As is well known,

$$\sigma(H_0^B) = \sigma_{\mathrm{pp}}(H_0^B) = \left\{ |\omega|\left(n + \frac{1}{2}\right) \;\middle|\; n \in \boldsymbol{N} \cup \{0\} \right\} \tag{1.12}$$

holds. This is one of the most remarkable properties of H_0^B. Each eigenvalue of H_0^B is called a Landau level. Equation (1.12) implies that

$$e^{-i(2\pi/|\omega|)H_0^B} = -\mathrm{Id} \tag{1.13}$$

holds. Taking account of this fact, we always assume $0 < T_B < 2\pi/|\omega|$, that is,

$$0 < |\bar{\omega}|T_B < \pi \tag{1.14}$$

for the sake of simplicity.

In order to capture the distinctive features of this quantum system, we first watch the corresponding classical orbits: We denote the position and the canonical momentum by $x_{\mathrm{cl}}(t) = (x_{\mathrm{cl},1}(t), x_{\mathrm{cl},2}(t))$ and $\xi_{\mathrm{cl}}(t) = (\xi_{\mathrm{cl},1}(t), \xi_{\mathrm{cl},2}(t))$, respectively. Suppose $n \in \boldsymbol{Z}$. In the interval $(nT, nT + T_B)$, $(x_{\mathrm{cl}}(t), \xi_{\mathrm{cl}}(t))$ satisfies Hamilton's equations

$$\frac{\mathrm{d}x_{\mathrm{cl}}}{\mathrm{d}t}(t) = \frac{1}{m}\left(\xi_{\mathrm{cl}}(t) - \frac{qB}{2}Jx_{\mathrm{cl}}(t)\right), \quad \frac{\mathrm{d}\xi_{\mathrm{cl}}}{\mathrm{d}t}(t) = -\frac{qB}{2m}J\left(\xi_{\mathrm{cl}}(t) - \frac{qB}{2}Jx_{\mathrm{cl}}(t)\right) \tag{1.15}$$

because $B(t) \equiv B$ on $[nT, nT + T_B)$. Putting

$$D_{\mathrm{cl}}(t) := \xi_{\mathrm{cl}}(t) - \frac{qB}{2}Jx_{\mathrm{cl}}(t), \quad k_{\mathrm{cl}}(t) := \xi_{\mathrm{cl}}(t) + \frac{qB}{2}Jx_{\mathrm{cl}}(t), \tag{1.16}$$

$(D_{\mathrm{cl}}(t), k_{\mathrm{cl}}(t))$ satisfies

$$\frac{\mathrm{d}D_{\mathrm{cl}}}{\mathrm{d}t}(t) = -\omega J D_{\mathrm{cl}}(t), \quad \frac{\mathrm{d}k_{\mathrm{cl}}}{\mathrm{d}t}(t) = 0. \tag{1.17}$$

Hence we see that

$$D_{\mathrm{cl}}(t) = \hat{R}(-\omega\tilde{t}_n)D_{\mathrm{cl}}(nT), \quad k_{\mathrm{cl}}(t) \equiv k_{\mathrm{cl}}(nT) \tag{1.18}$$

hold in the interval $[nT, nT + T_B]$. Here $\hat{R}(\theta)$ is the rotation operator defined by

$$(\hat{R}(\theta)x)^{\mathrm{T}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} x^{\mathrm{T}}, \quad \theta \in \boldsymbol{R}, \tag{1.19}$$

and we put $\tilde{t}_n := t - nT$ with $t \in \boldsymbol{R}$ for simplicity. We note that J is the generator of $\{\hat{R}(\theta)\}_{\theta \in \boldsymbol{R}}$. Equation (1.18) yields

$$\begin{aligned} x_{\mathrm{cl}}(t) &= \frac{1}{qB}J(D_{\mathrm{cl}}(t) - k_{\mathrm{cl}}(t)) = \frac{1}{qB}\hat{R}(-\omega\tilde{t}_n)JD_{\mathrm{cl}}(nT) - \frac{1}{qB}Jk_{\mathrm{cl}}(nT) \\ &= \frac{1}{2}(\hat{R}(-\omega\tilde{t}_n) + 1)x_{\mathrm{cl}}(nT) + \frac{1}{qB}(\hat{R}(-\omega\tilde{t}_n) - 1)J\xi_{\mathrm{cl}}(nT), \\ \xi_{\mathrm{cl}}(t) &= \frac{1}{2}(D_{\mathrm{cl}}(t) + k_{\mathrm{cl}}(t)) = \frac{1}{2}\hat{R}(-\omega\tilde{t}_n)D_{\mathrm{cl}}(nT) + \frac{1}{2}k_{\mathrm{cl}}(nT) \\ &= \frac{1}{2}(\hat{R}(-\omega\tilde{t}_n) + 1)\xi_{\mathrm{cl}}(nT) - \frac{qB}{4}(\hat{R}(-\omega\tilde{t}_n) - 1)Jx_{\mathrm{cl}}(nT) \end{aligned} \tag{1.20}$$

for $t \in [nT, nT + T_B]$. Here we used $J^2 = -1$, and $[\hat{R}(-\omega t), J] = 0$ for any $t \in \boldsymbol{R}$. By using

$$\hat{R}(-\omega t)+1=2\cos(-\bar{\omega}t)\hat{R}(-\bar{\omega}t),\quad (\hat{R}(-\omega t)-1)J=-2\sin(-\bar{\omega}t)\hat{R}(-\bar{\omega}t),$$
$$\cos(-\bar{\omega}t)=\cos(|\bar{\omega}|t),\quad \sin(-\bar{\omega}t)=-(\operatorname{sgn}\bar{\omega})\sin(|\bar{\omega}|t),$$

(1.20) can be written as

$$\begin{pmatrix} x_{\mathrm{cl}}(t) \\ \xi_{\mathrm{cl}}(t) \end{pmatrix} = L_{|\bar{\omega}|}(\tilde{t}_n)\begin{pmatrix} \hat{R}(-\bar{\omega}\tilde{t}_n)x_{\mathrm{cl}}(nT) \\ \hat{R}(-\bar{\omega}\tilde{t}_n)\xi_{\mathrm{cl}}(nT) \end{pmatrix} = L_{|\bar{\omega}|}(\tilde{t}_n)\hat{\mathcal{R}}(-\bar{\omega}\tilde{t}_n)\begin{pmatrix} x_{\mathrm{cl}}(nT) \\ \xi_{\mathrm{cl}}(nT) \end{pmatrix} \tag{1.21}$$

with

$$\begin{aligned} L_{|\bar{\omega}|}(t) &:= \begin{pmatrix} \cos(|\bar{\omega}|t) & \dfrac{1}{m|\bar{\omega}|}\sin(|\bar{\omega}|t) \\ -m|\bar{\omega}|\sin(|\bar{\omega}|t) & \cos(|\bar{\omega}|t) \end{pmatrix}, \quad t\in\boldsymbol{R}, \\ \hat{\mathcal{R}}(\theta) &:= \begin{pmatrix} \hat{R}(\theta) & 0 \\ 0 & \hat{R}(\theta) \end{pmatrix}, \quad \theta\in\boldsymbol{R}. \end{aligned} \tag{1.22}$$

On the other hand, in the interval $[nT+T_B,(n+1)T]$,

$$\begin{pmatrix} x_{\mathrm{cl}}(t) \\ \xi_{\mathrm{cl}}(t) \end{pmatrix} = L_{+0}(\tilde{t}_n-T_B)\begin{pmatrix} x_{\mathrm{cl}}(nT+T_B) \\ \xi_{\mathrm{cl}}(nT+T_B) \end{pmatrix} \tag{1.23}$$

holds with

$$L_{+0}(t) := \begin{pmatrix} 1 & \dfrac{1}{m}t \\ 0 & 1 \end{pmatrix}, \quad t\in\boldsymbol{R}, \tag{1.24}$$

because $B(t)\equiv 0$ on $[nT+T_B,(n+1)T)$. In particular, it follows from (1.21) and (1.23) that

$$\begin{pmatrix} x_{\mathrm{cl}}((n+1)T) \\ \xi_{\mathrm{cl}}((n+1)T) \end{pmatrix} = L_{+0}(T_0)\begin{pmatrix} x_{\mathrm{cl}}(nT+T_B) \\ \xi_{\mathrm{cl}}(nT+T_B) \end{pmatrix} = L\hat{\mathcal{R}}(-\bar{\omega}T_B)\begin{pmatrix} x_{\mathrm{cl}}(nT) \\ \xi_{\mathrm{cl}}(nT) \end{pmatrix} \tag{1.25}$$

holds for any $n\in\boldsymbol{Z}$. Here L is given by

$$\begin{aligned} L &= \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = L_{+0}(T_0)L_{|\bar{\omega}|}(T_B) \\ &= \begin{pmatrix} \cos(|\bar{\omega}|T_B)-|\bar{\omega}|T_0\sin(|\bar{\omega}|T_B) & \dfrac{1}{m|\bar{\omega}|}(|\bar{\omega}|T_0\cos(|\bar{\omega}|T_B)+\sin(|\bar{\omega}|T_B)) \\ -m|\bar{\omega}|\sin(|\bar{\omega}|T_B) & \cos(|\bar{\omega}|T_B) \end{pmatrix}. \end{aligned} \tag{1.26}$$

$L\hat{\mathcal{R}}(-\bar{\omega}T_B)$ is called the Floquet matrix of Hamilton's equations (1.15). Since $L_{+0}(t)$, $L_{|\bar{\omega}|}(t)\in \mathrm{SL}(2,\boldsymbol{R})$, we see that $L\in\mathrm{SL}(2,\boldsymbol{R})$. Equation (1.25) yields

$$\begin{pmatrix} x_{\mathrm{cl}}(nT) \\ \xi_{\mathrm{cl}}(nT) \end{pmatrix} = L^n\hat{\mathcal{R}}(-n\bar{\omega}T_B)\begin{pmatrix} x_{\mathrm{cl}}(0) \\ \xi_{\mathrm{cl}}(0) \end{pmatrix} \tag{1.27}$$

for any $n\in\boldsymbol{Z}$. Here L^0 is equal to the identity matrix E, and L^n with $-n\in\boldsymbol{N}$ denotes $(L^{-1})^{-n}$. Thus we obtain the solution $(x_{\mathrm{cl}}(t),\xi_{\mathrm{cl}}(t))$ of (1.15) with the initial value $(x_{\mathrm{cl}}(0),\xi_{\mathrm{cl}}(0))$, by virtue of (1.21), (1.23) and (1.27).

Now we will compute L^n: since the characteristic equation of L is

$$\lambda^2 - (L_{11} + L_{22})\lambda + 1 = 0$$

by $\det L = 1$, the eigenvalues $\lambda_\pm$ of L are given as

$$\lambda_\pm := \lambda_0 \pm \sqrt{\lambda_0^2 - 1}, \quad \lambda_0 := \cos(|\bar{\omega}|T_B) - \frac{1}{2}|\bar{\omega}|T_0 \sin(|\bar{\omega}|T_B) \in \boldsymbol{R}, \tag{1.28}$$

and satisfy

$$\lambda_+ + \lambda_- = 2\lambda_0 = L_{11} + L_{22}, \quad \lambda_+\lambda_- = 1. \tag{1.29}$$

By (1.14), $\sin(|\bar{\omega}|T_B) > 0$ and $-1 < \cos(|\bar{\omega}|T_B) < 1$ hold. Hence we have $L_{21} < 0$ and $\lambda_0 < \cos(|\bar{\omega}|T_B) < 1$. In particular, $\lambda_0 - 1 < 0$ holds. We first consider the case where $\lambda_0^2 - 1 \neq 0$, which is equivalent to $\lambda_0 + 1 \neq 0$. By using the fact

$$(L - \lambda_\pm E)\begin{pmatrix} L_{22} - \lambda_\pm \\ -L_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1.30}$$

L can be diagonalized as

$$P^{-1}LP = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}, \quad P = \begin{pmatrix} L_{22} - \lambda_+ & L_{22} - \lambda_- \\ -L_{21} & -L_{21} \end{pmatrix}.$$

Hence, by straightforward calculation, we obtain

$$L^n = \begin{pmatrix} L_{11}\mu_n - \mu_{n-1} & L_{12}\mu_n \\ L_{21}\mu_n & L_{22}\mu_n - \mu_{n-1} \end{pmatrix} \tag{1.31}$$

for $n \in \boldsymbol{Z}$, where

$$\mu_n := \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-} = \frac{\lambda_+^n - \lambda_-^n}{2\sqrt{\lambda_0^2 - 1}}. \tag{1.32}$$

Here we used $L_{22} - \lambda_\pm = -(L_{11} - \lambda_\mp)$ and $(L_{22} - \lambda_+)(L_{22} - \lambda_-) = -(L_{22} - \lambda_+)(L_{11} - \lambda_+) = -L_{12}L_{21}$. In particular, we have

$$\begin{aligned} x_{\mathrm{cl}}(nT) &= \hat{R}(-n\bar{\omega}T_B)\{(L_{11}\mu_n - \mu_{n-1})x_{\mathrm{cl}}(0) + L_{12}\mu_n\xi_{\mathrm{cl}}(0)\}. \\ &= \frac{\lambda_+^n}{2\sqrt{\lambda_0^2 - 1}}\hat{R}(-n\bar{\omega}T_B)\{(L_{11} - \lambda_-)x_{\mathrm{cl}}(0) + L_{12}\xi_{\mathrm{cl}}(0)\} \\ &\quad - \frac{\lambda_-^n}{2\sqrt{\lambda_0^2 - 1}}\hat{R}(-n\bar{\omega}T_B)\{(L_{11} - \lambda_+)x_{\mathrm{cl}}(0) + L_{12}\xi_{\mathrm{cl}}(0)\} \\ &=: x_{\mathrm{cl}}^+(nT) + x_{\mathrm{cl}}^-(nT). \end{aligned} \tag{1.33}$$

Now we consider the subcase where $\lambda_0 + 1 < 0$, which can be written as

$$T_0 > \frac{2(\cos(|\bar{\omega}|T_B) + 1)}{|\bar{\omega}|\sin(|\bar{\omega}|T_B)} = \frac{\cos(|\bar{\bar{\omega}}|T_B)}{|\bar{\bar{\omega}}|\sin(|\bar{\bar{\omega}}|T_B)} =: T_{0,\mathrm{cr}} > 0. \tag{1.34}$$

Since $\lambda_0^2 - 1 = (\lambda_0 + 1)(\lambda_0 - 1) > 0$, we see that

$$\lambda_- < \lambda_0 < -1 < \lambda_+ = \lambda_-^{-1} < 0. \tag{1.35}$$

Since $|\lambda_-| > 1 > |\lambda_+|$, we have

$$\lim_{n\to\pm\infty} |\lambda_\pm|^n = 0, \quad \lim_{n\to\pm\infty} |\lambda_\mp|^n = +\infty, \tag{1.36}$$

which yields

$$\lim_{n\to\pm\infty} (x_{\mathrm{cl}}(nT) - x_{\mathrm{cl}}^{\mp}(nT)) = \lim_{n\to\pm\infty} x_{\mathrm{cl}}^{\pm}(nT) = 0, \tag{1.37}$$

and

$$\lim_{n\to\pm\infty} |x_{\mathrm{cl}}(nT)| = \infty \tag{1.38}$$

if $(L_{11} - \lambda_\pm)x_{\mathrm{cl}}(0) + L_{12}\xi_{\mathrm{cl}}(0) \neq 0$. Since $x_{\mathrm{cl}}^{\mp}(nT)$ can be written as

$$\frac{\mp|\lambda_\mp|^n}{2\sqrt{\lambda_0^2-1}}\hat{R}(n(\operatorname{sgn}\bar{\omega})(\pi - |\bar{\omega}|T_B))\{(L_{11} - \lambda_\pm)x_{\mathrm{cl}}(0) + L_{12}\xi_{\mathrm{cl}}(0)\},$$

$x_{\mathrm{cl}}^{\mp}(nT)$ is located on the logarithmic spiral

$$\frac{\mp \mathrm{e}^{t\log|\lambda_\mp|/T}}{2\sqrt{\lambda_0^2-1}}\hat{R}\left(\frac{(\operatorname{sgn}\bar{\omega})(\pi - |\bar{\omega}|T_B)}{T}t\right)\{(L_{11} - \lambda_\pm)x_{\mathrm{cl}}(0) + L_{12}\xi_{\mathrm{cl}}(0)\}.$$

If we overlook the behavior of $x_{\mathrm{cl}}(t)$ in each interval $(nT, (n+1)T)$, then this makes us expect that classical orbits behave asymptotically like logarithmic spirals. Taking account of $L_{21} < 0$, $L_{11} - \lambda_+ = -|\bar{\omega}|T_0 \sin(|\bar{\omega}|T_B)/2 - \sqrt{\lambda_0^2-1} < 0$ and $(L_{11} - \lambda_+)(L_{11} - \lambda_-) = -L_{12}L_{21}$, we see that $L_{11} - \lambda_- = 0$ if and only if $L_{12} = 0$. In the case where $L_{11} - \lambda_- = L_{12} = 0$, we have

$$L^n = \begin{pmatrix} \lambda_-^n & 0 \\ L_{21}\mu_n & \lambda_+^n \end{pmatrix}, \quad \lambda_+ = \cos(|\bar{\omega}|T_B), \quad \lambda_- = \frac{1}{\cos(|\bar{\omega}|T_B)}, \tag{1.39}$$

for $n \in \boldsymbol{Z}$, which yields

$$\lim_{n\to-\infty} x_{\mathrm{cl}}(nT) = 0, \tag{1.40}$$

even if $x_{\mathrm{cl}}(0) \neq 0$. When $|\bar{\omega}|T_B = \pi/2$, we have $L_{12} = 1/(m|\bar{\omega}|) \neq 0$; while, when $|\bar{\omega}|T_B \neq \pi/2$, $L_{12} \neq 0$ is equivalent to

$$T_0 \neq -\frac{\sin(|\bar{\omega}|T_B)}{|\bar{\omega}|\cos(|\bar{\omega}|T_B)} = \frac{\sin(|\bar{\bar{\omega}}|T_B)\cos(|\bar{\bar{\omega}}|T_B)}{|\bar{\bar{\omega}}|(2\sin^2(|\bar{\bar{\omega}}|T_B) - 1)} =: T_{0,\mathrm{res}}. \tag{1.41}$$

When $0 < |\bar{\omega}|T_B < \pi/2$, (1.41) is satisfied automatically because of $T_{0,\mathrm{res}} < 0$; while, when $\pi/2 < |\bar{\omega}|T_B < \pi$, we have to assume (1.41) additionally to guarantee $L_{12} \neq 0$, because $T_{0,\mathrm{res}} > T_{0,\mathrm{cr}}$.

Next we consider the subcase where $\lambda_0 + 1 > 0$, which is equivalent to $T_0 < T_{0,\mathrm{cr}}$. Then $\lambda_0^2 - 1 = (\lambda_0+1)(\lambda_0-1) < 0$ holds. Since $\overline{\lambda_+} = \lambda_-$ and $\lambda_+\lambda_- = 1$, there exists a unique $\vartheta \in (0, \pi)$ such that $\lambda_\pm$ can be represented as $\lambda_\pm = \mathrm{e}^{\pm i\vartheta}$. By using this ϑ, μ_n can be represented as $\sin(n\vartheta)/\sqrt{1-\lambda_0^2}$. Hence $x_{\mathrm{cl}}(t)$ is bounded in t.

Next we consider the case where $\lambda_0^2 - 1 = 0$, which is equivalent to $\lambda_0 + 1 = 0$ by $\lambda_0 - 1 < 0$. Then $T_0 = T_{0,\mathrm{cr}}$ holds. By using the fact

$$(L-\lambda_0 E)\begin{pmatrix} L_{22}-\lambda_0 \\ -L_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (L-\lambda_0 E)\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} L_{22}-\lambda_0 \\ -L_{21} \end{pmatrix} \tag{1.42}$$

because of $-(L_{11}-\lambda_0) = L_{22}-\lambda_0$, we see that L is equivalent to a Jordan matrix:

$$P^{-1}LP = \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix}, \quad P = \begin{pmatrix} L_{22}-\lambda_0 & -1 \\ -L_{21} & 0 \end{pmatrix}.$$

Hence, by straightforward calculation, we obtain

$$L^n = \begin{pmatrix} L_{11}n\lambda_0^{n-1} - (n-1)\lambda_0^{n-2} & L_{12}n\lambda_0^{n-1} \\ L_{21}n\lambda_0^{n-1} & L_{22}n\lambda_0^{n-1} - (n-1)\lambda_0^{n-2} \end{pmatrix} \tag{1.43}$$

for $n \in \boldsymbol{Z}$. Here we used $L_{22}-\lambda_0 = -(L_{11}-\lambda_0)$, $(L_{22}-\lambda_0)^2 = -(L_{22}-\lambda_0)(L_{11}-\lambda_0) = -L_{12}L_{21}$, and $\lambda_0^2 = 1$. In the same way as above, this makes us expect that classical orbits behave asymptotically like Archimedes' spirals.

Now we return to the quantum system under consideration. We first watch the spectral properties of the Floquet operator $U_0(T,0)$. To this end, we study the behavior of the time-dependent observables

$$x(t) = U_0(t,0)^* x U_0(t,0), \quad p(t) = U_0(t,0)^* p U_0(t,0). \tag{1.44}$$

As will be seen in Sect. 2, $(x(t), p(t))$ satisfies formally Hamilton's equations (1.15) by replacement $(x_{\mathrm{cl}}(t), \xi_{\mathrm{cl}}(t))$ with $(x(t), p(t))$. Therefore, the following theorem can be obtained easily by the above argument:

Theorem 1.1. *Suppose* (1.14).
1. *When* $T_0 > T_{0,\mathrm{cr}}$,

$$\begin{aligned} \lim_{n\to\pm\infty} |\lambda_\mp|^{-n} \|xU_0(nT,0)\varphi\|_{\mathscr{H}^2} &= \frac{1}{2\sqrt{\lambda_0^2-1}} \|\{(L_{11}-\lambda_\pm)x + L_{12}p\}\varphi\|_{\mathscr{H}^2}, \\ \lim_{n\to\pm\infty} |\lambda_\mp|^{-n} \|pU_0(nT,0)\varphi\|_{\mathscr{H}^2} &= \frac{1}{2\sqrt{\lambda_0^2-1}} \|\{L_{21}x + (L_{22}-\lambda_\pm)p\}\varphi\|_{\mathscr{H}^2} \end{aligned} \tag{1.45}$$

hold for $\varphi \in \mathscr{S}(\boldsymbol{R}^2)$.
2. *When* $T_0 = T_{0,\mathrm{cr}}$,

$$\begin{aligned} \lim_{n\to\pm\infty} |n|^{-1} \|xU_0(nT,0)\varphi\|_{\mathscr{H}^2} &= \|\{(L_{11}-\lambda_0)x + L_{12}p\}\varphi\|_{\mathscr{H}^2}, \\ \lim_{n\to\pm\infty} |n|^{-1} \|pU_0(nT,0)\varphi\|_{\mathscr{H}^2} &= \|\{L_{21}x + (L_{22}-\lambda_0)p\}\varphi\|_{\mathscr{H}^2} \end{aligned} \tag{1.46}$$

hold for $\varphi \in \mathscr{S}(\boldsymbol{R}^2)$. *Here* $\lambda_0 = -1$.
3. *When* $T_0 \geq T_{0,\mathrm{cr}}$, *one has*

$$\sup_{t\in\boldsymbol{R}} \|xU_0(t,0)\varphi\|_{\mathscr{H}^2} = \infty, \quad \sup_{t\in\boldsymbol{R}} \|pU_0(t,0)\varphi\|_{\mathscr{H}^2} = \infty \tag{1.47}$$

for $0 \neq \varphi \in \mathscr{S}(\boldsymbol{R}^2)$; *while, when* $0 < T_0 < T_{0,\mathrm{cr}}$, *one has*

$$\sup_{t\in\boldsymbol{R}} \|xU_0(t,0)\varphi\|_{\mathscr{H}^2} < \infty, \quad \sup_{t\in\boldsymbol{R}} \|pU_0(t,0)\varphi\|_{\mathscr{H}^2} < \infty. \tag{1.48}$$

Corollary 1.2. *Suppose* (1.14). *When* $0 < T_0 < T_{0,\mathrm{cr}}$, $U_0(T,0)$ *has a pure point spectrum, that is,*

$$\mathscr{H} = \mathscr{H}^2_{\mathrm{pp}}(U_0(T,0)), \tag{1.49}$$

where $\mathscr{H}^2_{\mathrm{pp}}(U_0(T,0))$ *is the pure point spectral subspace associated with* $U_0(T,0)$.

In order to obtain this corollary of Theorem 1.1, we have only to use Theorem 1 of [7].

By using $(x_{\mathrm{cl}}(t), \xi_{\mathrm{cl}}(t))$, we will introduce $(\tilde{x}_{\mathrm{cl}}(t), \tilde{\xi}_{\mathrm{cl}}(t))$ as follows: for $t \in [nT, (n+1)T)$ with $n \in \boldsymbol{Z}$, we put

$$\begin{aligned} \tilde{x}_{\mathrm{cl}}(t) &:= \begin{cases} \hat{R}(\bar{\omega}\tilde{t}_n + n\bar{\omega}T_B)x_{\mathrm{cl}}(t), & t \in [nT, nT+T_B), \\ \hat{R}((n+1)\bar{\omega}T_B)x_{\mathrm{cl}}(t), & t \in [nT+T_B, (n+1)T), \end{cases} \\ \tilde{\xi}_{\mathrm{cl}}(t) &:= \begin{cases} \hat{R}(\bar{\omega}\tilde{t}_n + n\bar{\omega}T_B)\xi_{\mathrm{cl}}(t), & t \in [nT, nT+T_B), \\ \hat{R}((n+1)\bar{\omega}T_B)\xi_{\mathrm{cl}}(t), & t \in [nT+T_B, (n+1)T), \end{cases} \end{aligned} \tag{1.50}$$

where $\tilde{t}_n = t - nT$. We note that $\bar{\omega}\tilde{t}_n + n\bar{\omega}T_B$ can be written as $\bar{\omega}(t - nT_0)$. By simple computation, we see that $(\tilde{x}_{\mathrm{cl}}(t), \tilde{\xi}_{\mathrm{cl}}(t))$ satisfies Hamilton's equations

$$\frac{\mathrm{d}\tilde{x}_{\mathrm{cl}}}{\mathrm{d}t}(t) = \frac{1}{m}\tilde{\xi}_{\mathrm{cl}}(t), \quad \frac{\mathrm{d}\tilde{\xi}_{\mathrm{cl}}}{\mathrm{d}t}(t) = -m\bar{\omega}(t)^2\tilde{x}_{\mathrm{cl}}(t) \tag{1.51}$$

with $\bar{\omega}(t) := qB(t)/(2m)$. What we emphasize here is that the Floquet matrix of (1.51) is given by the matrix L in (1.26) and that the corresponding quantum system is governed by the T-periodic two-dimensional quantum harmonic oscillator Hamiltonian

$$H_{0,|\bar{\omega}|}(t) = \frac{1}{2m}p^2 + \frac{m}{2}|\bar{\omega}(t)|^2 x^2. \tag{1.52}$$

By virtue of the above argument, especially when $T_0 > T_{0,\mathrm{cr}}$ and $T_0 \neq T_{0,\mathrm{res}}$, one can see the exponential amplification property of the propagator $U_{0,|\bar{\omega}|}(t,s)$ generated by $H_{0,|\bar{\omega}|}(t)$ (see Hagedorn–Loss–Slawny [4] for related results about general T-periodic quantum harmonic oscillator Hamiltonians).

From now on, we will focus on the case where $T_0 > T_{0,\mathrm{cr}}$ and $T_0 \neq T_{0,\mathrm{res}}$. In this case, it can be expected that the charged particle moves away from the origin along orbits which behave asymptotically like some logarithmic spirals as $t \to \pm\infty$, as we have seen above. Thus one can consider some scattering problems for this situation. In this paper, we treat the problem of the asymptotic completeness.

Now we will state the assumption on the time-independent potential V:
$(V)_\rho$ V is a real-valued continuous function on $\boldsymbol{R}^2$ satisfying the decaying condition

$$|V(x)| \leq C\langle x\rangle^{-\rho} \tag{1.53}$$

with $\rho > 0$, where $\langle x\rangle = \sqrt{1+x^2}$.
Here we introduce the time-periodic Hamiltonian $H(t)$ given by

$$H(t) := H_0(t) + V, \tag{1.54}$$

and the propagator $U(t,s)$ generated by $H(t)$. $H(t)$ is represented as

$$H(t) = \begin{cases} H^B := H_0^B + V, & t \in I_B, \\ H^0 := H_0^0 + V, & t \in I_0. \end{cases} \tag{1.55}$$

We note that under the condition $(V)_\rho$ for some $\rho > 0$, H^B and H^0 are self-adjoint on $\mathscr{D}(H_0^B)$ and $\mathscr{D}(H_0^0)$, respectively. $U(t,0)$ is represented as

$$U(t,0) = \begin{cases} \mathrm{e}^{-i(t-nT)H^B} U(T,0)^n, & t \in [nT, nT+T_B), \\ \mathrm{e}^{-i(t-(nT+T_B))H^0} \mathrm{e}^{-iT_B H^B} U(T,0)^n, & t \in [nT+T_B, (n+1)T), \end{cases} \tag{1.56}$$

with $n \in \boldsymbol{Z}$, where

$$U(T,0) = \mathrm{e}^{-iT_0 H^0} \mathrm{e}^{-iT_B H^B} \tag{1.57}$$

is the Floquet operator associated with $H(t)$. The main result of this paper is as follows:

Theorem 1.3. *Suppose* (1.14), *and that* T_0 *satisfies* (1.34). *When* $\pi/2 < |\bar{\omega}|T_B < \pi$, *assume that* T_0 *satisfies* (1.41) *additionally. Assume that* V *satisfies the condition* $(V)_\rho$ *for some* $\rho > 0$. *Then the wave operators*

$$W^\pm = \operatorname*{s-lim}_{t \to \pm\infty} U(t,0)^* U_0(t,0) \tag{1.58}$$

exist and are asymptotically complete:

$$\mathrm{Ran}(W^\pm) = \mathscr{H}_{\mathrm{ac}}(U(T,0)). \tag{1.59}$$

Here $\mathscr{H}_{\mathrm{ac}}(U(T,0))$ *is the absolutely continuous spectral subspace associated with* $U(T,0)$.

As far as the authors know, there are very few results on quantum scattering in a time-periodic magnetic field. In Korotyaev [10], the free Hamiltonian

$$h_0(t) = -\frac{1}{2}\Delta - b(t)\tilde{L} + p(t)x_1^2/2 \tag{1.60}$$

on $L^2(\boldsymbol{R}^{m_1} \times \boldsymbol{R}^{m_2})$ was considered, where $x = (x_1, x_2) \in \boldsymbol{R}^{m_1} \times \boldsymbol{R}^{m_2}$, $x_1 = (x_{11}, \ldots, x_{1m_1})$, $\tilde{L} = -i(x_{11}\partial_{x_{12}} - x_{12}\partial_{x_{11}})$ for $m_1 \geq 2$; $\tilde{L} = 0$ for $m_1 = 1$, $b(t)$ and $p(t)$ are T-periodic real-valued continuous functions on $\boldsymbol{R}$. Under the implicit assumption on $p(t)$ that the Hill equation $y''(t) + p(t)y(t) = 0$ has solutions $y_1(t) = \mathrm{e}^{\lambda t}\chi_1(t)$ and $y_2(t) = \mathrm{e}^{-\lambda t}\chi_2(t)$, where $\lambda > 0$, and $\chi_1(t)$ and $\chi_2(t)$ are periodic in t, Korotyaev obtained the result corresponding to Theorem 1.3. Our result can be recognized as an extension of the result of [10] with $p(t) = b(t)^2$, $m_1 = 2$ and $m_2 = 0$ to the case where $\boldsymbol{B}(t)$ is a periodically pulsed magnetic field. What we would like to emphasize here is that the implicit assumption on $p(t)$ mentioned above can be replaced by the explicit conditions $T_0 > T_{0,\mathrm{cr}}$ and $T_0 \neq T_{0,\mathrm{res}}$ for our model. In [10], Korotyaev also treated the case where the Hill equation $y''(t) + p(t)y(t) = 0$ has solutions $y_1(t) = ty_2(t) + \chi_1(t)$ and $y_2(t) = \chi_2(t)$. Here $\chi_1(t)$ and $\chi_2(t)$ are periodic and antiperiodic in t, respectively. This is corresponding to the case where $T_0 = T_{0,\mathrm{cr}}$ for our model.

In the proof of Theorem 1.3, the limiting absorption principle for the Floquet Hamiltonian K_0 associated with $H_0(t)$ plays an important role: put $\mathscr{K} := L^2(\boldsymbol{T};\mathscr{H})$, where $\boldsymbol{T} = \boldsymbol{R}/(T\boldsymbol{Z})$, and introduce a family of unitary operators $\{\mathscr{U}_0(\sigma)\}_{\sigma\in\boldsymbol{R}}$ on $\mathscr{K}$ as

$$(\mathscr{U}_0(\sigma)f)(t) = U_0(t,t-\sigma)f(t-\sigma), \quad f \in \mathscr{K}. \tag{1.61}$$

Then $\{\mathscr{U}_0(\sigma)\}_{\sigma\in\boldsymbol{R}}$ forms a strongly continuous one-parameter unitary group on $\mathscr{K}$. By virtue of the Stone theorem, one can write $\mathscr{U}_0(\sigma) = \mathrm{e}^{-i\sigma K_0}$ with a certain self-adjoint operator K_0 on $\mathscr{K}$. K_0 is called the Floquet Hamiltonian associated with $H_0(t)$. As will be seen in Sect. 5, when $T_0 > T_{0,\mathrm{cr}}$ and $T_0 \neq T_{0,\mathrm{res}}$, we have obtained the limiting absorption principle for K_0; while, when $T_0 = T_{0,\mathrm{cr}}$, we have not obtained it yet, although it can be shown by the results of [4] that the Floquet operator $U_0(T,0)$ has a pure absolutely continuous spectrum, which implies that the Floquet Hamiltonian K_0 also has a pure absolutely continuous spectrum. When $T_0 > T_{0,\mathrm{cr}}$ and $T_0 \neq T_{0,\mathrm{res}}$, we introduce the Floquet Hamiltonian K associated with $H(t)$ similarly and obtain the result of the existence and the asymptotic completeness of

$$\mathscr{W}^{\pm} = \operatorname*{s-lim}_{\sigma\to\pm\infty} \mathrm{e}^{i\sigma K}\mathrm{e}^{-i\sigma K_0} \tag{1.62}$$

(see Theorem 5.6), by utilizing the abstract stationary scattering theory. Then, by virtue of the Howland–Yajima method (see, e.g. [5,6,13]), one can also obtain the asymptotic completeness of $W^{\pm}$ by showing the existence of $W^{\pm}$. This is an outline of the proof of Theorem 1.3.

The plan of this paper is as follows: in Sect. 2, we prove Theorem 1.1. In Sect. 3, we derive the integral kernel of $U_0(t,0)$. In Sect. 4, using the integral kernel of $U_0(t,0)$, we show the local compactness property of K_0. In Sect. 5, when $T_0 > T_{0,\mathrm{cr}}$ and $T_0 \neq T_{0,\mathrm{res}}$, we obtain the limiting absorption principles for K_0 and K, which yield the existence and the asymptotic completeness of $\mathscr{W}^{\pm}$. In Sect. 6, when $T_0 > T_{0,\mathrm{cr}}$ and $T_0 \neq T_{0,\mathrm{res}}$, we prove the existence and the asymptotic completeness of $W^{\pm}$ under the assumptions in Theorem 1.3.

2. Proof of Theorem 1.1

We first introduce the pseudomomentum k of the charged particle by

$$k = (k_1,k_2) = (p_1 - qBx_2/2, p_2 + qBx_1/2) = p + qA(x). \tag{2.1}$$

Here we note that the commutation relations

$$i[D_1,D_2] = -qB, \quad i[k_1,k_2] = qB, \quad i[D_{j_1},k_{j_2}] = 0 \quad (j_1,j_2 \in \{1,2\}) \tag{2.2}$$

hold (see, e.g. Avron–Herbst–Simon [1,2] and Gérard–Łaba [3]). Now we put

$$D(t) = U_0(t,0)^* D U_0(t,0), \quad k(t) = U_0(t,0)^* k U_0(t,0) \tag{2.3}$$

for the sake of brevity. Suppose $n \in \boldsymbol{Z}$. It follows from (2.2) that $(D(t),k(t))$ satisfies

$$\frac{\mathrm{d}D}{\mathrm{d}t}(t) = -\omega J D(t), \quad \frac{\mathrm{d}k}{\mathrm{d}t}(t) = 0 \tag{2.4}$$

for $t \in (nT, nT + T_B)$ (cf. (1.17)). Hence we have for $t \in [nT, nT + T_B]$,

$$D(t) = \hat{R}(-\omega\tilde{t}_n)D(nT), \quad k(t) \equiv k(nT) \tag{2.5}$$

(cf. (1.18)). Since $(x(t), p(t))$ in (1.44) can be written as

$$x(t) = \frac{1}{qB}J(D(t) - k(t)), \quad p(t) = \frac{1}{2}(D(t) + k(t)) \tag{2.6}$$

(cf. (1.20)), one can obtain

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = L_{|\bar{\omega}|}(\tilde{t}_n)\hat{\mathcal{R}}(-\bar{\omega}\tilde{t}_n)\begin{pmatrix} x(nT) \\ p(nT) \end{pmatrix} \tag{2.7}$$

for $t \in [nT, nT + T_B]$ in the same way as in Sect. 1 (see (1.21)). On the other hand, for $t \in [nT + T_B, (n+1)T]$, one can obtain

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = L_{+0}(\tilde{t}_n - T_B)\begin{pmatrix} x(nT + T_B) \\ p(nT + T_B) \end{pmatrix} \tag{2.8}$$

(cf. (1.23)). By virtue of (2.7) and (2.8), one can see easily that if

$$\bar{x}(\varphi) := \sup_{n\in\boldsymbol{Z}} \|x(nT)\varphi\|_{\mathscr{H}^2} < \infty \quad \text{and} \quad \bar{p}(\varphi) := \sup_{n\in\boldsymbol{Z}} \|p(nT)\varphi\|_{\mathscr{H}^2} < \infty,$$

then

$$\bar{\bar{x}}(\varphi) := \sup_{t\in\boldsymbol{R}} \|x(t)\varphi\|_{\mathscr{H}^2} < \infty \quad \text{and} \quad \bar{\bar{p}}(\varphi) := \sup_{t\in\boldsymbol{R}} \|p(t)\varphi\|_{\mathscr{H}^2} < \infty;$$

while, if $\bar{x}(\varphi) = \infty$ or $\bar{p}(\varphi) = \infty$ holds, then both $\bar{\bar{x}}(\varphi) = \infty$ and $\bar{\bar{p}}(\varphi) = \infty$ hold, by virtue of (2.7). Hence we have only to study about the finiteness of $\bar{x}(\varphi)$ and $\bar{p}(\varphi)$. By (2.7) and (2.8), one can obtain

$$\begin{pmatrix} x((n+1)T) \\ p((n+1)T) \end{pmatrix} = L\hat{\mathcal{R}}(-\bar{\omega}T_B)\begin{pmatrix} x(nT) \\ p(nT) \end{pmatrix} \tag{2.9}$$

in the same way as in Sect. 1 (see (1.25)). Here L is given by (1.26). Equation (2.9) yields

$$\begin{pmatrix} x(nT) \\ p(nT) \end{pmatrix} = L^n\hat{\mathcal{R}}(-n\bar{\omega}T_B)\begin{pmatrix} x \\ p \end{pmatrix} \tag{2.10}$$

for any $n \in \boldsymbol{Z}$ (cf. (1.27)). Thus we have only to use the explicit form of L^n, which was already obtained in Sect. 1, to study $\bar{x}(\varphi)$ and $\bar{p}(\varphi)$:

Case I. $0 < T_0 < T_{0,\mathrm{cr}}$:

Since $|\mu_n| \le 1/\sqrt{1-\lambda_0^2}$ by $\mu_n = \sin(n\vartheta)/\sqrt{1-\lambda_0^2}$ as mentioned in Sect. 1, one can see easily that $\bar{x}(\varphi) < \infty$ and $\bar{p}(\varphi) < \infty$.

Case II. $T_0 > T_{0,\mathrm{cr}}$:

By using (1.31) and (1.36), we have

$$\begin{aligned}\lim_{n\to\pm\infty} |\lambda_\mp|^{-n}\|x(nT)\varphi\|_{\mathscr{H}^2} &= \frac{1}{2\sqrt{\lambda_0^2-1}}\|\{(L_{11}-\lambda_\pm)x + L_{12}p\}\varphi\|_{\mathscr{H}^2} \\ &= \frac{1}{2\sqrt{\lambda_0^2-1}}|L_{11}-\lambda_\pm|\|(x + \tilde{L}_{12,\pm}p)\varphi\|_{\mathscr{H}^2},\end{aligned}$$

$$\lim_{n\to\pm\infty} |\lambda_\mp|^{-n}\|p(nT)\varphi\|_{\mathscr{H}^2} = \frac{1}{2\sqrt{\lambda_0^2-1}}\|\{L_{21}x + (L_{22}-\lambda_\pm)p\}\varphi\|_{\mathscr{H}^2}$$
$$= \frac{1}{2\sqrt{\lambda_0^2-1}}|L_{21}|\|(x+\tilde{L}_{12,\pm}p)\varphi\|_{\mathscr{H}^2},$$

if $L_{11}-\lambda_- \neq 0$. Here $\tilde{L}_{12,\pm} := (L_{22}-\lambda_\pm)/L_{21}$. But it can be seen easily that the above result is valid also when $L_{11}-\lambda_- = 0$, since $L_{11}-\lambda_- = L_{12} = 0$. These imply $\bar{x}(\varphi) = \infty$ and $\bar{p}(\varphi) = \infty$ for $0 \neq \varphi \in \mathscr{S}(\boldsymbol{R}^2)$.

Case III. $T_0 = T_{0,\mathrm{cr}}$:

We first note that $|\lambda_0^n| = 1$ holds for any $n \in \boldsymbol{Z}$, because $\lambda_0 = -1$. Taking account of $L_{11}-\lambda_0 = -|\bar{\omega}|T_0\sin(|\bar{\omega}|T_B)/2 < 0$ by (1.14), we have

$$\lim_{n\to\pm\infty} |n|^{-1}\|x(nT)\varphi\|_{\mathscr{H}^2} = \|\{(L_{11}-\lambda_0)x + L_{12}p\}\varphi\|_{\mathscr{H}^2}$$
$$= |L_{11}-\lambda_0|\|(x+\tilde{L}_{12,0}p)\varphi\|_{\mathscr{H}^2},$$
$$\lim_{n\to\pm\infty} |n|^{-1}\|p(nT)\varphi\|_{\mathscr{H}^2} = \|\{L_{21}x + (L_{22}-\lambda_0)p\}\varphi\|_{\mathscr{H}^2}$$
$$= |L_{21}|\|(x+\tilde{L}_{12,0}p)\varphi\|_{\mathscr{H}^2}.$$

Here $\tilde{L}_{12,0} := (L_{22}-\lambda_0)/L_{21}$. These imply $\bar{x}(\varphi) = \infty$ and $\bar{p}(\varphi) = \infty$ for $0 \neq \varphi \in \mathscr{S}(\boldsymbol{R}^2)$.

Thus the proof of Theorem 1.1 is completed.

3. Integral Kernel of the Free Propagator

In this section, we would like to find the integral kernel of the free propagator $U_0(t,s)$ generated by $H_0(t)$. From now on, we denote by $S_0(t,s;x,y)$ the integral kernel of $U_0(t,s)$. And, we put $\tilde{S}_0(t;x,y) := S_0(t,0;x,y)$, which is the integral kernel of $U_0(t,0)$ obviously.

(I) $\tilde{S}_0(t;x,y)$ for $t \in [0,T]$:

We first consider the case where $t \in [0,T_B]$. $\tilde{S}_0(t;x,y)$ is just the integral kernel $S_0^B(t;x,y)$ of $\mathrm{e}^{-itH_0^B}$. Now we introduce

$$H_{0,|\bar{\omega}|} := \frac{1}{2m}p^2 + \frac{m}{2}|\bar{\omega}|^2x^2, \quad \tilde{L} := x_1p_2 - x_2p_1. \tag{3.1}$$

$\tilde{L}$ is called the angular momentum. It is well known that $\mathrm{e}^{-itH_0^B}$ can be represented as $\mathrm{e}^{i\bar{\omega}t\tilde{L}}\mathrm{e}^{-itH_{0,|\bar{\omega}|}}$, because $H_0^B = H_{0,|\bar{\omega}|} - \bar{\omega}\tilde{L}$ and $[H_{0,|\bar{\omega}|},\tilde{L}] = 0$. Here we note that

$$(\mathrm{e}^{i\bar{\omega}t\tilde{L}}\psi)(x) = \psi(\hat{R}(\bar{\omega}t)x), \quad \psi \in L^2(\boldsymbol{R}^2),$$

holds. On the other hand, by virtue of Mehler's formula, the integral kernel $S_{0,|\bar{\omega}|}(t;x,y)$ of $\mathrm{e}^{-itH_{0,|\bar{\omega}|}}$ is given by

$$S_{0,|\bar{\omega}|}(t;x,y) = \frac{m|\bar{\omega}|}{2\pi i\sin(|\bar{\omega}|t)}\mathrm{e}^{im|\bar{\omega}|(\cos(|\bar{\omega}|t)(x^2+y^2)-2x\cdot y)/(2\sin(|\bar{\omega}|t))} \tag{3.2}$$

when $\sin(|\bar{\omega}|t) \neq 0$, which is satisfied in the interval $(0, T_B]$ by (1.14). Then $\tilde{S}_0(t; x, y)$ with $t \in (0, T_B]$ can be represented as

$$\begin{aligned}\tilde{S}_0(t; x, y) &= S_0^B(t; x, y) = S_{0,|\bar{\omega}|}(t; \hat{R}(\bar{\omega}t)x, y) \\ &= \frac{m|\bar{\omega}|}{2\pi i \sin(|\bar{\omega}|t)} \mathrm{e}^{im|\bar{\omega}| \cot(|\bar{\omega}|t)x^2/2} \\ &\quad \times \mathrm{e}^{-im|\bar{\omega}|(\hat{R}(\bar{\omega}t)x)\cdot y/\sin(|\bar{\omega}|t)} \mathrm{e}^{im|\bar{\omega}|\cot(|\bar{\omega}|t)y^2/2} \end{aligned} \tag{3.3}$$

(see also Avron–Herbst–Simon [1]). Of course, $\tilde{S}_0(0; x, y)$ is equal to $\delta(x - y)$ as a distributional kernel, where δ is the Dirac delta function. Putting

$$T_{B,\mathrm{exc}} := \frac{\pi}{2|\bar{\omega}|}, \tag{3.4}$$

we have

$$\tilde{S}_0(T_{B,\mathrm{exc}}; x, y) = \frac{m|\bar{\omega}|}{2\pi i} \mathrm{e}^{-im|\bar{\omega}|(\hat{R}(\bar{\omega}T_{B,\mathrm{exc}})x)\cdot y}, \tag{3.5}$$

if $T_{B,\mathrm{exc}} \leq T_B$, since $\cos(|\bar{\omega}|T_{B,\mathrm{exc}}) = 0$ and $\sin(|\bar{\omega}|T_{B,\mathrm{exc}}) = 1$. Now we focus on the case where $t \neq T_{B,\mathrm{exc}}$. For the sake of simplicity, we will write (3.3) as the formula

$$\begin{aligned}\tilde{S}_0(t; x, y) &= \frac{1}{2\pi i c_0(t)\theta_0(t)} \mathrm{e}^{ix^2/(2\theta_0(t))} \\ &\quad \times \mathrm{e}^{-i(\hat{R}(\phi_0(t))x)\cdot y/(c_0(t)\theta_0(t))} \mathrm{e}^{i\sigma_0(t)y^2/(2\theta_0(t))} \end{aligned} \tag{3.6}$$

with $\theta_0(t) := \tan(|\bar{\omega}|t)/(m|\bar{\omega}|)$, $c_0(t) := \cos(|\bar{\omega}|t)$, $\phi_0(t) := \bar{\omega}t$ and $\sigma_0(t) := 1$ for $t \in [0, T_B] \backslash \{T_{B,\mathrm{exc}}\}$. Here we introduce

$$S_0^0(t; x) = \begin{cases} \delta(x), & t = 0, \\ \dfrac{1}{2\pi i t} \mathrm{e}^{ix^2/(2t)}, & t \neq 0. \end{cases} \tag{3.7}$$

It is well known that $S_0^0(t; x - y)$ is the integral kernel of $\mathrm{e}^{-itp^2/2}$. Obviously, the integral kernel of $\mathrm{e}^{-itH_0^0}$ can be represented as $S_0^0(t/m; x - y)$. By using this S_0^0, we will give a representation of $\tilde{S}_0(t; x, y)$: since

$$\begin{aligned} & x^2 - 2\frac{\hat{R}(\phi_0(t))x \cdot y}{c_0(t)} + \sigma_0(t)y^2 \\ &= \sigma_0(t)\left(\frac{\hat{R}(\phi_0(t))x}{c_0(t)\sigma_0(t)} - y\right)^2 + \left(1 - \frac{1}{c_0(t)^2\sigma_0(t)}\right)x^2 \\ &= \left(x - \frac{\hat{R}(-\phi_0(t))y}{c_0(t)}\right)^2 + \left(\sigma_0(t) - \frac{1}{c_0(t)^2}\right)y^2, \end{aligned}$$

$\tilde{S}_0(t; x, y)$ can be written as

$$\tilde{S}_0(t;x,y) = \frac{1}{c_0(t)\sigma_0(t)} \mathrm{e}^{i\{1-1/(c_0(t)^2\sigma_0(t))\}x^2/(2\theta_0(t))} S_0^0\left(\frac{\theta_0(t)}{\sigma_0(t)}; \frac{\hat{R}(\phi_0(t))x}{c_0(t)\sigma_0(t)} - y\right)$$
$$= \frac{1}{c_0(t)} \mathrm{e}^{i\{\sigma_0(t)-1/c_0(t)^2\}y^2/(2\theta_0(t))} S_0^0\left(\theta_0(t); x - \frac{\hat{R}(-\phi_0(t))y}{c_0(t)}\right) \tag{3.8}$$

for $t \in (0,T_B]\backslash\{T_{B,\mathrm{exc}}\}$. Since

$$\sigma_0(s) - \frac{1}{c_0(s)^2} = -\tan^2(|\bar{\omega}|s) = -m|\bar{\omega}|\tan(|\bar{\omega}|s)\theta_0(s)$$

for $s \in (0,T_B]\backslash\{T_{B,\mathrm{exc}}\}$, one can see that $\tilde{S}_0(s;x,y)$ converges to $\delta(x-y)$ as $s \to +0$. Hence, (3.8) with $t=0$ is also valid.

We next consider the case where $t \in (T_B,T]$. Then $\tilde{S}_0(t;x,y)$ is just the integral kernel of $\mathrm{e}^{-i(t-T_B)H_0^0}\mathrm{e}^{-iT_BH_0^B}$. When $T_B = T_{B,\mathrm{exc}}$, one can obtain

$$\tilde{S}_0(t;x,y) = \frac{m|\bar{\omega}|}{2\pi i} \mathrm{e}^{-im|\bar{\omega}|(\hat{R}(\bar{\omega}T_B)x)\cdot y} \mathrm{e}^{-im|\bar{\omega}|^2(t-T_B)y^2/2} \tag{3.9}$$

by calculating the Fourier transform of $S_0^0((t-T_B)/m;\cdot)$, since (3.5) holds. On the other hand, when $T_B \neq T_{B,\mathrm{exc}}$, one can obtain

$$\tilde{S}_0(t;x,y) = \frac{1}{c_0(T_B)} \mathrm{e}^{i\{\sigma_0(T_B)-1/c_0(T_B)^2\}y^2/(2\theta_0(T_B))} \times S_0^0\left(\theta_0(t); x - \frac{\hat{R}(-\phi_0(T_B))y}{c_0(T_B)}\right) \tag{3.10}$$

with $\theta_0(t) := (t-T_B)/m + \theta_0(T_B)$ for $t \in (T_B,T]$, if $\theta_0(t) \neq 0$, by taking account of that $S_0^0(t+s;x-y)$ is the integral kernel of $\mathrm{e}^{-i(t+s)p^2/2} = \mathrm{e}^{-itp^2/2}\mathrm{e}^{-isp^2/2}$. Then (3.10) can be represented as (3.6) by putting $c_0(t) := c_0(T_B)$, $\phi_0(t) := \phi_0(T_B)$ and

$$\sigma_0(t) := \left(\sigma_0(T_B) - \frac{1}{c_0(T_B)^2}\right)\frac{\theta_0(t)}{\theta_0(T_B)} + \frac{1}{c_0(T_B)^2}$$

for $t \in (T_B,T]$. In particular,

$$\left(\sigma_0(t) - \frac{1}{c_0(t)^2}\right)\frac{1}{\theta_0(t)} \equiv \left(\sigma_0(T_B) - \frac{1}{c_0(T_B)^2}\right)\frac{1}{\theta_0(T_B)} = -m|\bar{\omega}|\tan(|\bar{\omega}|T_B)$$

holds on $(T_B,T]$, which yields $\sigma_0(t) = 1/c_0(t)^2 - m|\bar{\omega}|\tan(|\bar{\omega}|T_B)\theta_0(t) = 1 - |\bar{\omega}|(t-T_B)\tan(|\bar{\omega}|T_B)$. By using this and (3.8), we have

$$\tilde{S}_0(t;x,y) = \frac{1}{c_0(t)\sigma_0(t)} \mathrm{e}^{-im|\bar{\omega}|\tan(|\bar{\omega}|T_B)x^2/(2\sigma_0(t))} S_0^0\left(\frac{\theta_0(t)}{\sigma_0(t)}; \frac{\hat{R}(\phi_0(t))x}{c_0(t)\sigma_0(t)} - y\right).$$

By this formula, $\tilde{S}_0(t; x, y)$ with $\theta_0(t) = 0$ can be also given as a distributional kernel. Consequently, $\tilde{S}_0(t; x, y)$ for $t \in [0, T]$ is represented as (3.6) with

$$\begin{aligned}
\theta_0(t) &= \begin{cases} \dfrac{\tan(|\bar{\omega}|t)}{m|\bar{\omega}|} & (t \in [0, T_B]) \\ \dfrac{1}{m}\left(t - T_B + \dfrac{\tan(|\bar{\omega}|T_B)}{|\bar{\omega}|}\right) & (t \in (T_B, T]), \end{cases} \\
c_0(t) &= \begin{cases} \cos(|\bar{\omega}|t) & (t \in [0, T_B]) \\ \cos(|\bar{\omega}|T_B) & (t \in (T_B, T]), \end{cases} \\
\phi_0(t) &= \begin{cases} \bar{\omega}t & (t \in [0, T_B]) \\ \bar{\omega}T_B & (t \in (T_B, T]), \end{cases} \\
\sigma_0(t) &= \begin{cases} 1 & (t \in [0, T_B]) \\ 1 - |\bar{\omega}|(t - T_B)\tan(|\bar{\omega}|T_B) & (t \in (T_B, T]). \end{cases}
\end{aligned} \tag{3.11}$$

Here, in the case where $T_B = T_{B,\mathrm{exc}}$ and $t \in [T_B, T]$, one has only to recognize $1/\theta_0(t)$, $1/(c_0(t)\theta_0(t))$ and $\sigma_0(t)/\theta_0(t)$ as 0, $m|\bar{\omega}|$ and $-m|\bar{\omega}|^2(t - T_B)$, respectively, by taking account of $\cos(|\bar{\omega}|T_B) = 0$ and $\sin(|\bar{\omega}|T_B) = 1$.

(II) $\tilde{S}_0(nT; x, y)$ for $n \in \boldsymbol{N}$:

For the sake of simplicity, we focus on the general case, that is, the case where $T_B \neq T_{B,\mathrm{exc}}$, $\theta_0(T) \neq 0$, and $\sigma_0(T) \neq 0$. By virtue of (3.6), the integral kernel $\tilde{S}_0(T; x, y)$ of the Floquet operator $U_0(T, 0)$ is given by

$$\tilde{S}_0(T; x, y) = \frac{1}{2\pi i c_1\theta_1} \mathrm{e}^{ix^2/(2\theta_1)} \mathrm{e}^{-i(\hat{R}(\phi_1)x)\cdot y/(c_1\theta_1)} \mathrm{e}^{i\sigma_1 y^2/(2\theta_1)} \tag{3.12}$$

with

$$\begin{aligned}
\theta_1 &= \theta_0(T) = \frac{L_{12}}{L_{22}}, \quad c_1 = c_0(T) = L_{22}, \\
\phi_1 &= \phi_0(T) = \bar{\omega}T_B, \quad \sigma_1 = \sigma_0(T) = \frac{L_{11}}{L_{22}}.
\end{aligned} \tag{3.13}$$

Then one can find the following representation of the integral kernel $\tilde{S}_0(nT; x, y)$ of $U_0(nT, 0)$ with $n \in \boldsymbol{N}$:

$$\tilde{S}_0(nT; x, y) = \frac{1}{2\pi i c_n\theta_n} \mathrm{e}^{ix^2/(2\theta_n)} \mathrm{e}^{-i(\hat{R}(\phi_n)x)\cdot y/(c_n\theta_n)} \mathrm{e}^{i\sigma_n y^2/(2\theta_n)}. \tag{3.14}$$

In fact, in the same way as above, by using

$$\begin{aligned}
\tilde{S}_0(T; x, y) &= \frac{1}{c_1\sigma_1} \mathrm{e}^{i\{1 - 1/(c_1^2\sigma_1)\}x^2/(2\theta_1)} S_0^0\left(\frac{\theta_1}{\sigma_1}; \frac{\hat{R}(\phi_1)x}{c_1\sigma_1} - y\right), \\
\tilde{S}_0(nT; x, y) &= \frac{1}{c_n} \mathrm{e}^{i\{\sigma_n - 1/c_n^2\}y^2/(2\theta_n)} S_0^0\left(\theta_n; x - \frac{\hat{R}(-\phi_n)y}{c_n}\right)
\end{aligned} \tag{3.15}$$

we have

$$\tilde{S}_0((n+1)T;x,y) = \frac{1}{c_1\sigma_1 c_n} \mathrm{e}^{i\{1-1/(c_1^2\sigma_1)\}x^2/(2\theta_1)} \mathrm{e}^{i\{\sigma_n - 1/c_n^2\}y^2/(2\theta_n)} \times S_0^0\left(\frac{\theta_1}{\sigma_1}+\theta_n; \frac{\hat{R}(\phi_1)x}{c_1\sigma_1} - \frac{\hat{R}(-\phi_n)y}{c_n}\right) \tag{3.16}$$

in the same way as above. By equating the coefficients of x^2, $(\hat{R}(\phi)x)\cdot y$ and y^2 in the exponents of the exponential functions in (3.16), the recurrence relations

$$\frac{1}{\theta_{n+1}} = \left(1 - \frac{1}{c_1^2\sigma_1}\right)\frac{1}{\theta_1} + \frac{1}{(c_1\sigma_1)^2(\theta_1/\sigma_1+\theta_n)}, \tag{3.17}$$

$$\frac{1}{c_{n+1}\theta_{n+1}} = \frac{1}{c_1\sigma_1 c_n(\theta_1/\sigma_1+\theta_n)}, \tag{3.18}$$

$$\frac{\sigma_{n+1}}{\theta_{n+1}} = \left(\sigma_n - \frac{1}{c_n^2}\right)\frac{1}{\theta_n} + \frac{1}{c_n^2(\theta_1/\sigma_1+\theta_n)}, \tag{3.19}$$

$$\phi_{n+1} = \phi_1 + \phi_n, \tag{3.20}$$

can be obtained. Obviously, we have

$$\phi_n = n\phi_1 = n\bar{\omega}T_B \tag{3.21}$$

by (3.20). Since (3.17) can be written as

$$\begin{aligned} \frac{1}{\theta_{n+1}} &= \left(1 - \frac{1}{L_{11}L_{22}}\right)\frac{L_{22}}{L_{12}} + \frac{1}{L_{11}^2(L_{12}/L_{11}+\theta_n)} \\ &= \frac{L_{21}}{L_{11}} + \frac{1}{L_{11}(L_{11}\theta_n+L_{12})} = \frac{L_{21}(L_{11}\theta_n+L_{12})+1}{L_{11}(L_{11}\theta_n+L_{12})} = \frac{L_{21}\theta_n+L_{22}}{L_{11}\theta_n+L_{12}} \end{aligned}$$

by $c_1^2\sigma_1 = L_{11}L_{22}$, $c_1\sigma_1 = L_{11}$ and $1 = L_{11}L_{22} - L_{12}L_{21}$, we obtain

$$\theta_{n+1} = \frac{L_{11}\theta_n + L_{12}}{L_{21}\theta_n + L_{22}}. \tag{3.22}$$

We will solve these recurrence relations:

Case 1. $T_0 \neq T_{0,\mathrm{cr}}$:

Taking account of (1.30), we put

$$\alpha_\pm := \frac{L_{22}-\lambda_\pm}{-L_{21}} = \frac{L_{11}-\lambda_\mp}{L_{21}}. \tag{3.23}$$

Then we see that $\alpha_\pm$ are the roots of the equation

$$\alpha = \frac{L_{11}\alpha + L_{12}}{L_{21}\alpha + L_{22}}. \tag{3.24}$$

Since

$$\theta_{n+1} - \alpha_\pm = \frac{(L_{11}L_{22}-L_{12}L_{21})(\theta_n-\alpha_\pm)}{(L_{21}\alpha_\pm+L_{22})(L_{21}\theta_n+L_{22})} = \frac{\theta_n - \alpha_\pm}{\lambda_\pm(L_{21}\theta_n+L_{22})}$$

by $L_{11}L_{22} - L_{12}L_{21} = 1$ and $L_{21}\alpha_\pm + L_{22} = \lambda_\pm$, we have

$$\frac{\theta_{n+1}-\alpha_+}{\theta_{n+1}-\alpha_-} = \frac{\lambda_-}{\lambda_+} \times \frac{\theta_n - \alpha_+}{\theta_n - \alpha_-},$$

which yields

$$\theta_n = \frac{\alpha_+(\theta_1 - \alpha_-)\lambda_+^{n-1} - \alpha_-(\theta_1 - \alpha_+)\lambda_-^{n-1}}{(\theta_1 - \alpha_-)\lambda_+^{n-1} - (\theta_1 - \alpha_+)\lambda_-^{n-1}} = \frac{L_{12}\mu_n}{L_{22}\mu_n - \mu_{n-1}}. \quad (3.25)$$

Here we used

$$\theta_1 - \alpha_\pm = \frac{L_{12}}{L_{22}} - \frac{L_{11} - \lambda_\mp}{L_{21}} = \frac{L_{22}\lambda_\mp - 1}{L_{21}L_{22}} = \frac{\lambda_\mp(L_{22} - \lambda_\pm)}{L_{21}L_{22}},$$
$$\alpha_\mp(\theta_1 - \alpha_\pm) = \frac{\lambda_\mp(L_{11} - \lambda_\pm)(L_{22} - \lambda_\pm)}{L_{21}^2 L_{22}} = \frac{L_{12}\lambda_\mp}{L_{21}L_{22}}$$

by $L_{11}L_{22} - L_{12}L_{21} = 1$ and $(L_{11} - \lambda_\pm)(L_{22} - \lambda_\pm) = L_{12}L_{21}$. Then, by (3.17), (3.18) and (3.25), we have

$$\frac{c_n}{c_{n+1}} = L_{11}\left(\frac{1}{\theta_{n+1}} - \frac{L_{21}}{L_{11}}\right)\theta_{n+1} = L_{11} - L_{21}\theta_{n+1} = L_{11} - \frac{L_{12}L_{21}\mu_{n+1}}{L_{22}\mu_{n+1} - \mu_n}$$
$$= \frac{\mu_{n+1} - L_{11}\mu_n}{L_{22}\mu_{n+1} - \mu_n} = \frac{L_{22}\mu_n - \mu_{n-1}}{L_{22}\mu_{n+1} - \mu_n}.$$

Here we used $L_{11}L_{22} - L_{12}L_{21} = 1$ and $\mu_{n+1} = (\lambda_+ + \lambda_-)\mu_n - \mu_{n-1} = (L_{11} + L_{22})\mu_n - \mu_{n-1}$. This yields

$$c_n = c_1 \prod_{k=1}^{n-1} \frac{c_{k+1}}{c_k} = L_{22}\mu_n - \mu_{n-1}$$

by $c_1 = L_{22}$, $\mu_1 = 1$ and $\mu_0 = 0$. We finally consider (3.19). Here we note that

$$\frac{\sigma_{n+1}}{\theta_{n+1}} - \frac{\sigma_n}{\theta_n} = \frac{1}{c_n^2}\left(\frac{1}{L_{12}/L_{11} + \theta_n} - \frac{1}{\theta_n}\right) = -\frac{L_{12}/L_{11}}{c_n^2\theta_n(L_{12}/L_{11} + \theta_n)}$$
$$= -\frac{L_{12}}{c_n\theta_n(L_{12}c_n + L_{11}c_n\theta_n)} = -\frac{1}{L_{12}\mu_{n+1}\mu_n}$$
$$= -\frac{1}{L_{12}}\left(\frac{\mu_n}{\mu_{n+1}} - \frac{\mu_{n-1}}{\mu_n}\right)$$

holds by $c_n\theta_n = L_{12}\mu_n$, $L_{12}c_n + L_{11}c_n\theta_n = L_{12}(L_{22}\mu_n - \mu_{n-1} + L_{11}\mu_n) = L_{12}\mu_{n+1}$ and $\mu_n^2 - \mu_{n+1}\mu_{n-1} = (\lambda_+ - \lambda_-)^2/(\lambda_+ - \lambda_-)^2 = 1$. Thus we have

$$\frac{\sigma_n}{\theta_n} = \frac{1}{L_{12}}\left(-\frac{\mu_{n-1}}{\mu_n} + L_{11}\right) = \frac{L_{11}\mu_n - \mu_{n-1}}{L_{12}\mu_n},$$

which yields

$$\sigma_n = \frac{L_{11}\mu_n - \mu_{n-1}}{L_{22}\mu_n - \mu_{n-1}}.$$

Therefore, the solutions of (3.17), (3.18) and (3.19) are given by

$$\theta_n = \frac{L_{12}\mu_n}{L_{22}\mu_n - \mu_{n-1}}, \quad c_n = L_{22}\mu_n - \mu_{n-1}, \quad \sigma_n = \frac{L_{11}\mu_n - \mu_{n-1}}{L_{22}\mu_n - \mu_{n-1}}. \quad (3.26)$$

Case 2. $T_0 = T_{0,\mathrm{cr}}$:

Taking account of (1.42), we put

$$\alpha_0 = \frac{L_{22} - \lambda_0}{-L_{21}}. \tag{3.27}$$

Then one can obtain

$$\theta_{n+1} - \alpha_0 = \frac{\theta_n - \alpha_0}{\lambda_0(L_{21}\theta_n + L_{22})}$$

in the same way as in case 1, which yields

$$\frac{1}{\theta_{n+1} - \alpha_0} = \frac{\lambda_0(L_{21}\alpha_0 + L_{22})}{\theta_n - \alpha_0} + \lambda_0 L_{21} = \frac{1}{\theta_n - \alpha_0} + \lambda_0 L_{21},$$

where we used $\lambda_0^2 = 1$. Hence we obtain

$$\theta_n = \alpha_0 + \frac{\theta_1 - \alpha_0}{1 + (n-1)\lambda_0 L_{21}(\theta_1 - \alpha_0)} = \frac{\theta_1 + (n-1)\lambda_0 L_{21}\alpha_0(\theta_1 - \alpha_0)}{1 + (n-1)\lambda_0 L_{21}(\theta_1 - \alpha_0)},$$

which can be written as

$$\theta_n = \frac{nL_{12}}{nL_{22} - (n-1)\lambda_0}.$$

Here we used $\lambda_0^2 = 1$ and

$$\theta_1 = \frac{L_{12}}{L_{22}}, \quad \theta_1 - \alpha_0 = \frac{L_{22}\lambda_0 - 1}{L_{21}L_{22}}, \quad \alpha_0(\theta_1 - \alpha_0) = \frac{L_{12}\lambda_0}{L_{21}L_{22}}.$$

Then, in the same way as in case 1, one can solve (3.18) and (3.19). Therefore, the solutions of (3.17), (3.18) and (3.19) are given by

$$\begin{aligned} \theta_n &= \frac{nL_{12}}{nL_{22} - (n-1)\lambda_0}, \quad c_n = \lambda_0^{n-1}\{nL_{22} - (n-1)\lambda_0\}, \\ \sigma_n &= \frac{nL_{11} - (n-1)\lambda_0}{nL_{22} - (n-1)\lambda_0}. \end{aligned} \tag{3.28}$$

Now we will mention exceptional cases: when $T_B = T_{B,\mathrm{exc}}$, $L_{22} = 0$ holds. Then we will recognize $1/\theta_1$, $1/(c_1\theta_1)$ and σ_1/θ_1 as 0, $m|\bar{\omega}|$ and $-m|\bar{\omega}|^2 T_0$, respectively, as mentioned in the end of (I). By using these, we see that θ_n, c_n and σ_n with $n \geq 2$ can be given by (3.26) or (3.28). Also in the case where $\sigma_1 = \sigma_0(T) = 0$, that is, $L_{11} = 0$, θ_n, c_n and σ_n can be given by (3.26) or (3.28). In the case where $\theta_1 = \theta_0(T) = 0$, that is, $L_{12} = 0$, one can obtain

$$\begin{aligned} \tilde{S}_0(nT; x, y) &= \lambda_+^n \mathrm{e}^{-im|\bar{\omega}|\cos(|\bar{\omega}|T_B)(1-\lambda_+^{2n})x^2/(2\sin(|\bar{\omega}|T_B))} \\ &\quad \times \delta(\lambda_+^n \hat{R}(\phi(nT))x - y) \\ &= \lambda_-^n \mathrm{e}^{-im|\bar{\omega}|\cos(|\bar{\omega}|T_B)(1-\lambda_-^{2n})y^2/(2\sin(|\bar{\omega}|T_B))} \\ &\quad \times \delta(x - \lambda_-^n \hat{R}(-\phi(nT))y) \end{aligned} \tag{3.29}$$

with $\lambda_+ = \lambda_-^{-1} = \cos(|\bar{\omega}|T_B)$ (cf. (1.39)), by using the argument in (I). Such nT's should be called *resonant times*. Here we used the facts that

$$\left(\sigma_n - \frac{1}{c_n^2}\right)\frac{1}{\theta_n} = \left(\sigma_1 - \frac{1}{c_1^2}\right)\frac{1}{\theta_1} \times \sum_{k=1}^{n} \frac{1}{c_{k-1}^2}$$

holds in the case where $L_{12} \neq 0$, where $c_0 = 1$ and that $c_k = \lambda_+^k$ holds in the case where $L_{12} = 0$.

(III) $\tilde{S}_0(nT; x, y)$ for $n \in \boldsymbol{Z}\backslash\{0\}$:

Even if $-n \in \boldsymbol{N}$, $\tilde{S}_0(nT; x, y)$ can be written as (3.14) with (3.21) and (3.26) when $T_0 \neq T_{0,\mathrm{cr}}$; while, with (3.21) and (3.28) when $T_0 = T_{0,\mathrm{cr}}$: We first note that $\tilde{S}_0(nT; x, y)$ is equal to $\overline{\tilde{S}_0(-nT; y, x)}$. By equating the coefficients of x^2, $(\hat{R}(\phi)x) \cdot y$ and y^2 in the exponents of the exponential functions in $\tilde{S}_0(nT; x, y)$ and $\overline{\tilde{S}_0(-nT; y, x)}$, we have

$$\frac{1}{\theta_n} = -\frac{\sigma_{-n}}{\theta_{-n}}, \quad \frac{\hat{R}(\phi_n)}{c_n\theta_n} = -\frac{\hat{R}(-\phi_{-n})}{c_{-n}\theta_{-n}}, \quad \frac{\sigma_n}{\theta_n} = -\frac{1}{\theta_{-n}}, \tag{3.30}$$

Then one can see easily that ϕ_n for $-n \in \boldsymbol{N}$ is given by (3.21) and that θ_n, c_n and σ_n for $-n \in \boldsymbol{N}$ are given by (3.26) when $T_0 \neq T_{0,\mathrm{cr}}$, while, by (3.28) when $T_0 = T_{0,\mathrm{cr}}$, respectively. Here we used $\mu_{-n} = -\mu_n$ when $T_0 \neq T_{0,\mathrm{cr}}$.

(IV) $\tilde{S}_0(t + nT; x, y)$ for $n \in \boldsymbol{Z}\backslash\{0\}$ and $t \in [0, T)$:

By using (3.8) and (3.15), one can obtain

$$\begin{aligned}\tilde{S}_0(t + nT; x, y) = {} & \frac{1}{2\pi i c_n(t)\theta_n(t)} \mathrm{e}^{ix^2/(2\theta_n(t))} \\ & \times \mathrm{e}^{-i(\hat{R}(\phi_n(t))x)\cdot y/(c_n(t)\theta_n(t))} \mathrm{e}^{i\sigma_n(t)y^2/(2\theta_n(t))},\end{aligned} \tag{3.31}$$

where

$$\frac{1}{\theta_n(t)} = \left(1 - \frac{1}{c_0(t)^2\sigma_0(t)}\right)\frac{1}{\theta_0(t)} + \frac{1}{(c_0(t)\sigma_0(t))^2(\theta_0(t)/\sigma_0(t) + \theta_n)}, \tag{3.32}$$

$$\frac{1}{c_n(t)\theta_n(t)} = \frac{1}{c_0(t)\sigma_0(t)c_n(\theta_0(t)/\sigma_0(t) + \theta_n)}, \tag{3.33}$$

$$\frac{\sigma_n(t)}{\theta_n(t)} = \left(\sigma_n - \frac{1}{c_n^2}\right)\frac{1}{\theta_n} + \frac{1}{c_n^2(\theta_0(t)/\sigma_0(t) + \theta_n)}, \tag{3.34}$$

$$\phi_n(t) = \phi_0(t) + \phi_n \tag{3.35}$$

in the same way as above.

(V) $S_0(t + nT, s; x, y)$ for $n \in \boldsymbol{Z}$ and t, $s \in [0, T)$:

Since the integral kernel of $U_0(s, 0)^*$ is given by

$$\begin{aligned}\overline{\tilde{S}_0(s; y, x)} = {} & \frac{1}{c_0(s)\sigma_0(s)} \mathrm{e}^{i\{1-1/(c_0(s)^2\sigma_0(s))\}y^2/(-2\theta_0(s))} \\ & \times S_0^0\left(-\frac{\theta_0(s)}{\sigma_0(s)}; x - \frac{\hat{R}(\phi_0(s))y}{c_0(s)\sigma_0(s)}\right),\end{aligned} \tag{3.36}$$

one can obtain the integral kernel $S_0(t + nT, s; x, y)$ of $U_0(t + nT, s)$ as follows:

$$\begin{aligned}& S_0(t + nT, s; x, y) \\ & \quad = \frac{1}{c_n(t)\sigma_n(t)} \mathrm{e}^{i\{1-1/(c_n(t)^2\sigma_n(t))\}x^2/(2\theta_n(t))}\end{aligned}$$

$$\times \frac{1}{c_0(s)\sigma_0(s)} \mathrm{e}^{i\{1-1/(c_0(s)^2\sigma_0(s))\}y^2/(-2\theta_0(s))}$$
$$\times S_0^0 \left(\frac{\theta_n(t)}{\sigma_n(t)} - \frac{\theta_0(s)}{\sigma_0(s)}; \frac{\hat{R}(\phi_n(t))x}{c_n(t)\sigma_n(t)} - \frac{\hat{R}(\phi_0(s))y}{c_0(s)\sigma_0(s)} \right). \tag{3.37}$$

4. Local Compactness Property of the Free Floquet Hamiltonian

Let $\tilde{A}$ be the multiplication by $\tilde{A} \in L^p(\boldsymbol{R}^2)$ with some $p \in [2, \infty]$. Then we will consider the operator $\tilde{A}(K_0 - \zeta)^{-1}\tilde{A}$ with $\zeta \in \boldsymbol{C}\backslash\boldsymbol{R} = \mathbf{H}^+ \cup \mathbf{H}^-$, where $\mathbf{H}^\pm := \{\zeta \in \boldsymbol{C} \mid \pm\mathrm{Im}\,\zeta > 0\}$. It is well known that for $\zeta \in \mathbf{H}^+$ and $f \in \mathscr{K}$, $(\tilde{A}(K_0 - \zeta)^{-1}\tilde{A}f)(t)$ can be represented as

$$\begin{aligned}
(\tilde{A}(K_0 - \zeta)^{-1}\tilde{A})f(t) &= i\int_0^\infty \mathrm{e}^{is\zeta}\tilde{A}(\mathrm{e}^{-isK_0}\tilde{A}f)(t)\,\mathrm{d}s \\
&= i\left\{\sum_{n=1}^\infty \int_0^T \mathrm{e}^{i(t+nT-s)\zeta}\tilde{A}U_0(t+nT,s)(\tilde{A}f)(s)\,\mathrm{d}s \right.\\
&\quad \left. + \int_0^t \mathrm{e}^{i(t-s)\zeta}\tilde{A}U_0(t,s)(\tilde{A}f)(s)\,\mathrm{d}s \right\}
\end{aligned}$$

(see, e.g. Yajima [13]). Also for $\zeta \in \mathbf{H}^-$, a quite similar formula holds.

In order to derive some useful properties of $\tilde{A}(K_0 - \zeta)^{-1}\tilde{A}$, we use the integral kernel $S_0(t+nT, s; x, y)$ of $U_0(t+nT, s)$ for $n \in \boldsymbol{Z}$ and t, $s \in [0, T)$, which was found in Sect. 3. By using the method of Kato [8], one can obtain the estimate

$$\|\tilde{A}U_0(t+nT,s)\tilde{A}f(s)\|_{\mathscr{H}} \le \left(\frac{1}{2\pi|\tilde{d}_n(t,s)|}\right)^{2/p} \|\tilde{A}\|^2_{L^p(\boldsymbol{R}^2)}\|f(s)\|_{\mathscr{H}}, \tag{4.1}$$

where

$$\tilde{d}_n(t,s) = c_n(t)\sigma_n(t)c_0(s)\sigma_0(s)\left(\frac{\theta_n(t)}{\sigma_n(t)} - \frac{\theta_0(s)}{\sigma_0(s)}\right). \tag{4.2}$$

When $T_0 \ne T_{0,\mathrm{cr}}$, $\tilde{d}_n(t,s)$ can be represented by

$$\begin{aligned}
\tilde{d}_n(t,s) &= \frac{1}{m|\bar{\omega}|}\left\{-(L_{11}\mu_n - \mu_{n-1})\Sigma_0(t)\Theta_0(s) + m|\bar{\omega}|L_{12}\mu_n\Sigma_0(t)\Sigma_0(s)\right.\\
&\quad \left. - \frac{L_{21}\mu_n}{m|\bar{\omega}|}\Theta_0(t)\Theta_0(s) + (L_{22}\mu_n - \mu_{n-1})\Theta_0(t)\Sigma_0(s)\right\}\\
&= \frac{-1}{L_{21}(\lambda_+ - \lambda_-)}\{\lambda_+^n\Omega_+(t)\Omega_-(s) - \lambda_-^n\Omega_-(t)\Omega_+(s)\},
\end{aligned} \tag{4.3}$$

while, when $T_0 = T_{0,\mathrm{cr}}$, $\tilde{d}_n(t,s)$ can be represented by

$$\begin{aligned}\tilde{d}_n(t,s) = &\ \frac{\lambda_0^{n-1}}{m|\bar{\omega}|}\Big\{-(nL_{11}-(n-1)\lambda_0)\Sigma_0(t)\Theta_0(s) + m|\bar{\omega}|nL_{12}\Sigma_0(t)\Sigma_0(s) \\ &- \frac{nL_{21}}{m|\bar{\omega}|}\Theta_0(t)\Theta_0(s) + (nL_{22}-(n-1)\lambda_0)\Theta_0(t)\Sigma_0(s)\Big\} \\ = &\ \lambda_0^n\left[\frac{1}{L_{21}}n\Omega_0(t)\Omega_0(s) + \frac{1}{m|\bar{\omega}|}\{\Theta_0(t)\Sigma_0(s) - \Sigma_0(t)\Theta_0(s)\}\right], \end{aligned} \tag{4.4}$$

where

$$\begin{aligned}\Theta_0(t) &:= m|\bar{\omega}|c_0(t)\theta_0(t) \\ &= \begin{cases}\sin(|\bar{\omega}|t) & (t\in[0,T_B]) \\ |\bar{\omega}|(t-T_B)\cos(|\bar{\omega}|T_B) + \sin(|\bar{\omega}|T_B) & (t\in(T_B,T]),\end{cases} \\ \Sigma_0(t) &:= c_0(t)\sigma_0(t) \\ &= \begin{cases}\cos(|\bar{\omega}|t) & (t\in[0,T_B]) \\ \cos(|\bar{\omega}|T_B) - |\bar{\omega}|(t-T_B)\sin(|\bar{\omega}|T_B) & (t\in(T_B,T]),\end{cases} \\ \Omega_\kappa(t) &:= \frac{L_{21}}{m|\bar{\omega}|}\Theta_0(t) - (L_{22}-\lambda_\kappa)\Sigma_0(t) \quad (\kappa\in\{+,-,0\}).\end{aligned} \tag{4.5}$$

Now we introduce the zero set of $\tilde{d}_n(t,\cdot)$

$$Z_n(t) := \left\{s\in[0,T) \;\middle|\; \tilde{d}_n(t,s)=0\right\} \tag{4.6}$$

for $t\in[0,T)$. Since $\tilde{d}(t,s)$ with a fixed t is represented as $C_1\sin(|\bar{\omega}|s) + C_2\cos(|\bar{\omega}|s)$ in the interval $[0,T_B]$, it follows from (1.14) that $\#(Z_n(t)\cap[0,T_B]) \le 1$, while, since $\tilde{d}(t,s)$ with a fixed t is represented as $C_3 s + C_4$ in the interval (T_B,T), $\#(Z_n(t)\cap(T_B,T)) \le 1$ holds. Hence we have $\#(Z_n(t)) \le 2$. Here we denote the cardinal number of the set S by $\#(S)$. If $T_0 \ge T_{0,\mathrm{cr}}$ and $T_0 \ne T_{0,\mathrm{res}}$, then one can show easily that $\#(Z_n(t)) = 1$ for sufficiently large $|n|$, because $\tilde{d}_n(t,0)\tilde{d}_n(t,T) < 0$ holds for sufficiently large $|n|$.

Now we consider the case where $\tilde{A} = F(|x|\le R)$ with $R>0$. Here $F(|x|\le R)$ stands for the characteristic function of the set $\left\{x\in\boldsymbol{R}^2 \;\middle|\; |x|\le R\right\}$. We note $\tilde{A}\in L^p(\boldsymbol{R}^2)$ for any $p\in[1,\infty]$. Let $\zeta\in\mathbf{H}^+$ and $\varepsilon>0$. Take an $N_\varepsilon\in\boldsymbol{N}$ such that

$$\sum_{n=N_\varepsilon}^{\infty} \mathrm{e}^{-(n-1)T\,\mathrm{Im}\,\zeta} \le \frac{\varepsilon}{2},$$

and define the operator $J_{\varepsilon,1}$ on $\mathscr{K}$ by

$$(J_{\varepsilon,1}f)(t) = \sum_{n=N_\varepsilon}^{\infty} i\int_0^T \mathrm{e}^{i(t+nT-s)\zeta}\tilde{A}U_0(t+nT,s)(\tilde{A}f)(s)\,\mathrm{d}s.$$

Here we note that

$$\int_0^T \|\tilde{A}U_0(t+nT,s)\tilde{A}f(s)\|_{\mathscr{H}}\,\mathrm{d}s \le \sqrt{T}\|f\|_{\mathscr{K}} \tag{4.7}$$

holds for any $n \in \boldsymbol{N}$, by $\|\tilde{A}\|_{L^\infty(\boldsymbol{R}^2)} = 1$. By using this, we have

$$\|J_{\varepsilon,1} f\|_{\mathscr{K}} \le \frac{\varepsilon}{2} T \|f\|_{\mathscr{K}}.$$

For each $n \in \{0, \ldots, N_\varepsilon - 1\}$, there exists a neighborhood $V_n^\varepsilon(t) \subset [0,T]$ of $Z_n(t)$ such that

$$\int_{V_n^\varepsilon(t)} \|\tilde{A} U_0(t+nT,s) \tilde{A} f(s)\|_{\mathscr{H}} \, \mathrm{d}s \le \frac{\varepsilon}{2N_\varepsilon} \sqrt{T} \|f\|_{\mathscr{K}}. \tag{4.8}$$

By defining the operator $J_{\varepsilon,2}$ on $\mathscr{K}$ by

$$\begin{aligned}(J_{\varepsilon,2} f)(t) = \sum_{n=1}^{N_\varepsilon - 1} i \int_{V_n^\varepsilon(t)} \mathrm{e}^{i(t+nT-s)\zeta} \tilde{A} U_0(t+nT,s)(\tilde{A} f)(s) \, \mathrm{d}s \\ + i \int_{V_0^\varepsilon(t) \cap [0,t]} \mathrm{e}^{i(t-s)\zeta} \tilde{A} U_0(t,s)(\tilde{A} f)(s) \, \mathrm{d}s,\end{aligned}$$

we also have

$$\|J_{\varepsilon,2} f\|_{\mathscr{K}} \le \frac{\varepsilon}{2} T \|f\|_{\mathscr{K}}. \tag{4.9}$$

Putting $J_{\varepsilon,0} := \tilde{A}(K_0 - \zeta)^{-1} \tilde{A} - (J_{\varepsilon,1} + J_{\varepsilon,2})$, in the same way as in Møller [11], one can prove that $J_{\varepsilon,0}$ is compact on $\mathscr{K}$, by virtue of the fact that $\tilde{A} U_0(t + nT, s) \tilde{A}$ is a Hilbert-Schmidt operator on $\mathscr{H}$. This yields the compactness of $\tilde{A}(K_0 - \zeta)^{-1} \tilde{A}$:

Proposition 4.1. *Suppose* (1.14)*. Let $\zeta \in \boldsymbol{C} \backslash \boldsymbol{R}$ and $\tilde{A} = F(|x| \le R)$ with $R > 0$. Then $\tilde{A}(K_0 - \zeta)^{-1} \tilde{A}$ is compact on $\mathscr{K}$.*

By virtue of the first resolvent formula, the local compactness property of K_0 is a direct consequence of Proposition 4.1:

Corollary 4.2. (Local compactness property of K_0) *Suppose* (1.14)*. Let $\zeta \in \boldsymbol{C} \backslash \boldsymbol{R}$ and $\tilde{A} = F(|x| \le R)$ with $R > 0$. Then $\tilde{A}(K_0 - \zeta)^{-1}$ is compact on $\mathscr{K}$.*

5. Limiting Absorption Principles for Floquet Hamiltonians

In this and the next sections, we always assume $T_0 > T_{0,\mathrm{cr}}$, that is, (1.34). And, when $\pi/2 < |\bar{\omega}| T_B < \pi$, we assume additionally $T_0 \ne T_{0,\mathrm{res}}$, that is, (1.41).

We first show the limiting absorption principle for K_0:

Theorem 5.1. (Limiting absorption principle for K_0) *Suppose* (1.14)*, and that T_0 satisfies the conditions stated in Theorem* 1.3*. Let $\kappa > 0$. Then*

$$\sup_{\Lambda \in \boldsymbol{R},\, 0 < \epsilon < 1} \|\langle x \rangle^{-\kappa} (K_0 - (\Lambda \pm i\epsilon))^{-1} \langle x \rangle^{-\kappa}\|_{\mathscr{B}(\mathscr{K})} < \infty \tag{5.1}$$

holds.

Proof. We will watch $\langle x\rangle^{-\kappa}(K_0-\zeta)^{-1}\langle x\rangle^{-\kappa}$ with $\zeta\in\mathbf{H}^+$. Putting $\tilde{A}(x)=\langle x\rangle^{-\kappa}$, we see that $\tilde{A}\in L^p(\boldsymbol{R}^2)$ for $p\in(2/\kappa,\infty]$. Let $n\in\boldsymbol{N}$. Suppose p satisfies $\max\{4,2/\kappa\}<p<\infty$. Putting

$$\begin{aligned}\gamma_{n,1}(s)&:=\frac{1}{m|\bar{\omega}|}\{-(L_{11}\mu_n-\mu_{n-1})\Theta_0(s)+m|\bar{\omega}|L_{12}\mu_n\Sigma_0(s)\},\\ \gamma_{n,2}(s)&:=\frac{1}{m|\bar{\omega}|}\left\{-\frac{L_{21}\mu_n}{m|\bar{\omega}|}\Theta_0(s)+(L_{22}\mu_n-\mu_{n-1})\Sigma_0(s)\right\},\end{aligned}\tag{5.2}$$

$\tilde{d}_n(t,s)$ is written as $\tilde{d}_n(t,s)=\gamma_{n,1}(s)\Sigma_0(t)+\gamma_{n,2}(s)\Theta_0(t)$. When $t\in[0,T_B)$, $\tilde{d}_n(t,s)$ can be represented as

$$\begin{aligned}\tilde{d}_n(t,s)&=\gamma_{n,1}(s)\cos(|\bar{\omega}|t)+\gamma_{n,2}(s)\sin(|\bar{\omega}|t)\\&=(\gamma_{n,1}(s)^2+\gamma_{n,2}(s)^2)^{1/2}\sin(|\bar{\omega}|t+\eta_n(s))\end{aligned}\tag{5.3}$$

with some $\eta_n(s)\in[0,2\pi)$. Then one can obtain

$$\begin{aligned}&\int_0^{T_B}\|\tilde{A}U_0(t+nT,s)\tilde{A}f(s)\|^2_{\mathscr{H}}\,\mathrm{d}t\\&\quad\le C_1(\gamma_{n,1}(s)^2+\gamma_{n,2}(s)^2)^{-2/p}\|\tilde{A}\|^4_{L^p(\boldsymbol{R}^2)}\|f(s)\|^2_{\mathscr{H}},\end{aligned}\tag{5.4}$$

by (4.1) and $-4/p>-1$, even if $\tilde{d}_n(\cdot,s)$ has a zero in the interval $[0,T_B)$. Here C_1 is a positive constant which is independent of n and s. When $t\in[T_B,T)$, $\tilde{d}_n(t,s)$ can be represented as

$$\begin{aligned}\tilde{d}_n(t,s)=&(-\sin(|\bar{\omega}|T_B)\gamma_{n,1}(s)+\cos(|\bar{\omega}|T_B)\gamma_{n,2}(s))|\bar{\omega}|(t-T_B)\\&+\cos(|\bar{\omega}|T_B)\gamma_{n,1}(s)+\sin(|\bar{\omega}|T_B)\gamma_{n,2}(s).\end{aligned}\tag{5.5}$$

Then, in the same way as above, one can also obtain

$$\begin{aligned}&\int_{T_B}^{T}\|\tilde{A}U_0(t+nT,s)\tilde{A}f(s)\|^2_{\mathscr{H}}\,\mathrm{d}t\\&\quad\le C_2|-\sin(|\bar{\omega}|T_B)\gamma_{n,1}(s)+\cos(|\bar{\omega}|T_B)\gamma_{n,2}(s)|^{-4/p}\|\tilde{A}\|^4_{L^p(\boldsymbol{R}^2)}\|f(s)\|^2_{\mathscr{H}},\end{aligned}\tag{5.6}$$

where C_2 is a positive constant which is independent of n and s. From (5.4) and (5.6), we have

$$\int_0^T\left(\int_0^T\|\tilde{A}U_0(t+nT,s)\tilde{A}f(s)\|^2_{\mathscr{H}}\,\mathrm{d}t\right)^{1/2}\mathrm{d}s\le C_3|\lambda_-|^{-2n/p}\|\tilde{A}\|^2_{L^p(\boldsymbol{R}^2)}\|f\|_{\mathscr{K}},\tag{5.7}$$

as well as

$$\int_0^T\left(\int_s^T\|\tilde{A}U_0(t,s)\tilde{A}f(s)\|^2_{\mathscr{H}}\,\mathrm{d}t\right)^{1/2}\mathrm{d}s\le C_3\|\tilde{A}\|^2_{L^p(\boldsymbol{R}^2)}\|f\|_{\mathscr{K}},\tag{5.8}$$

by using the Schwarz inequality. Here C_3 is a positive constant which is independent of n. Then, using Minkowski's integral inequality, $\|(\tilde{A}(K_0-\zeta)^{-1}\tilde{A})$

$f\|_{\mathscr{K}}$ can be estimated as

$$\|(\tilde{A}(K_0-\zeta)^{-1}\tilde{A})f\|_{\mathscr{K}} \leq C_3 \sum_{n=0}^{\infty} |\lambda_-|^{-2n/p} \|\tilde{A}\|^2_{L^p(\boldsymbol{R}^2)} \|f\|_{\mathscr{K}}$$

by $|e^{i(t+nT-s)\zeta}| \leq 1$. Since $\sum_{n=0}^{\infty} |\lambda_-|^{-2n/p} < \infty$ by $|\lambda_-| > 1$, this yields

$$\|\tilde{A}(K_0-\zeta)^{-1}\tilde{A}\|_{\mathscr{B}(\mathscr{K})} \leq C_4 \|\tilde{A}\|^2_{L^p(\boldsymbol{R}^2)}. \tag{5.9}$$

The case where $\zeta \in \mathbf{H}^-$ can treated with similarly. This completes the proof. □

Since $\sum_{n=1}^{\infty} n^{-2/p} = \infty$ for any $p \in (2,\infty)$, our proof of Theorem 5.1 does not work well in the case where $T_0 = T_{0,\mathrm{cr}}$. If the space dimension is equal to 3, one may use $\sum_{n=1}^{\infty} n^{-3/p} < \infty$ for $p \in (2,3)$ even in the case where $T_0 = T_{0,\mathrm{cr}}$.

Corollary 5.2. *Suppose* (1.14), *and that* T_0 *satisfies the conditions stated in Theorem* 1.3. *Then* K_0 *has a pure absolutely continuous spectrum*, *that is*,

$$\mathscr{K}_{\mathrm{ac}}(K_0) = \mathscr{K}, \tag{5.10}$$

where $\mathscr{K}_{\mathrm{ac}}(K_0)$ *is the absolutely continuous spectral subspace associated with* K_0.

Let $V(x)$ be the potential satisfying the condition $(V)_\rho$ for some $\rho > 0$. Let $\tilde{A}$ and $\tilde{B}$ be the multiplication by $\tilde{A}(x) = |V(x)|^{1/2}$ and $\tilde{B}(x) = (\mathrm{sgn} V(x))|V(x)|^{1/2}$, respectively. Then, by virtue of Corollary 4.2 and the proof of Theorem 5.1, the following results can be obtained in the same way as in Yajima [13], so we omit the proofs:

Lemma 5.3. *Suppose* (1.14), *and that* T_0 *satisfies the conditions stated in Theorem* 1.3. *Assume that* V *satisfies the condition* $(V)_\rho$ *for some* $\rho > 0$. *Put* $Q_0(\zeta) := \tilde{B}(K_0-\zeta)^{-1}\tilde{A}$ *with* $\zeta \in \boldsymbol{C}\backslash\boldsymbol{R} = \mathbf{H}^+ \cup \mathbf{H}^-$. *Then* $Q_0(\zeta)$ *has the following properties*:

1. $Q_0(\zeta)$ *is a* $\mathscr{B}(\mathscr{K})$*-valued analytic function on* $\mathbf{H}^\pm$.
2. *For each* $\zeta \in \boldsymbol{C}\backslash\boldsymbol{R}$, $Q_0(\zeta)$ *is compact on* $\mathscr{K}$.
3. $Q_0(\Lambda \pm i\epsilon)$ *have boundary values in* $\mathscr{B}(\mathscr{K})$ *as* $\epsilon \to +0$, *whose convergence is uniformly in* $\Lambda \in \boldsymbol{R}$.

Lemma 5.4. *Suppose* (1.14), *and that* T_0 *satisfies the conditions stated in Theorem* 1.3. *Assume that* V *satisfies the condition* $(V)_\rho$ *for some* $\rho > 0$. *Then, for each* $\zeta \in \boldsymbol{C}\backslash\boldsymbol{R}$,

$$(K-\zeta)^{-1} = (K_0-\zeta)^{-1} - (K_0-\zeta)^{-1}\tilde{A}(1+Q_0(\zeta))^{-1}\tilde{B}(K_0-\zeta)^{-1} \tag{5.11}$$

holds.

Theorem 5.5. (Limiting absorption principle for K) *Suppose* (1.14), *and that* T_0 *satisfies the conditions stated in Theorem* 1.3. *Assume that* V *satisfies the* *condition* $(V)_\rho$ *for some* $\rho > 0$. *Let* $\kappa > 0$. *Then there exists a closed null set* $\Gamma_0 \subset \boldsymbol{R}$ *such that*

$$\sup_{\Lambda \in \boldsymbol{R} \setminus \Gamma_0,\, 0<\epsilon<1} \|\langle x\rangle^{-\kappa}(K-(\Lambda \pm i\epsilon))^{-1}\langle x\rangle^{-\kappa}\|_{\mathscr{B}(\mathscr{K})} < \infty \tag{5.12}$$

holds.

Then the following theorem can be obtained as a direct consequence of the abstract stationary scattering theory (see, e.g. Kato–Kuroda [9]):

Theorem 5.6. *Suppose* (1.14), *and that* T_0 *satisfies the conditions stated in Theorem* 1.3. *Assume that* V *satisfies the condition* $(V)_\rho$ *for some* $\rho > 0$. *Then the strong limits*

$$\mathscr{W}^{\pm} := \operatorname*{s-lim}_{\sigma \to \pm\infty} \mathrm{e}^{i\sigma K}\mathrm{e}^{-i\sigma K_0} \tag{5.13}$$

exist and are asymptotically complete:

$$\mathrm{Ran}(\mathscr{W}^{\pm}) = \mathscr{K}_{\mathrm{ac}}(K). \tag{5.14}$$

6. Existence and Asymptotic Completeness of Physical Wave Operators

At first, we will show the existence of W^+. The existence of W^- can be shown similarly.

Let $M(\tau)$, $D(\tau)$ and $\mathscr{F}$ be unitary operators on $\mathscr{H}$ given by

$$\begin{aligned} &(M(\tau)\varphi)(x) = \mathrm{e}^{ix^2/(2\tau)}\varphi(x), \quad (D(\tau)\varphi)(x) = \frac{1}{i\tau}\varphi\left(\frac{x}{\tau}\right), \\ &\mathscr{F}[\varphi](\xi) = \frac{1}{2\pi}\int_{\boldsymbol{R}^2} \mathrm{e}^{-ix\xi}\varphi(x)\,dx, \end{aligned} \tag{6.1}$$

respectively. As is well known, $\mathrm{e}^{-itp^2/2}$ is represented as

$$\mathrm{e}^{-itp^2/2} = M(t)D(t)\mathscr{F}M(t). \tag{6.2}$$

Now we see that $U_0(t+nT,0)$ for $n \in \boldsymbol{Z}$ and $t \in [0,T)$ can be also represented as

$$U_0(t+nT,0) = \mathrm{e}^{i\phi_n(t)\tilde{L}}M(\theta_n(t))D(c_n(t)\theta_n(t))\mathscr{F}M\left(\frac{\theta_n(t)}{\sigma_n(t)}\right) \tag{6.3}$$

by virtue of (3.31). Here we used $(\hat{R}(\phi_n(t))x)^2 = x^2$. Since

$$\lim_{n\to\infty}\frac{\theta_n(t)}{\sigma_n(t)} = \frac{L_{12}}{L_{11}-\lambda_+} =: \hat{\tau}_\infty \tag{6.4}$$

by (3.19) and (3.34), we obtain the following lemma in the same way as in the proof of Theorem IX.31 of Reed–Simon [12]:

Lemma 6.1. *Suppose* (1.14), *and that* T_0 *satisfies the conditions stated in Theorem* 1.3. *Put*

$$\hat{U}_0(t+nT) = \mathrm{e}^{i\phi_n(t)\tilde{L}}M(\theta_n(t))D(c_n(t)\theta_n(t))\mathscr{F}M(\hat{\tau}_\infty) \tag{6.5}$$

for $n \in \boldsymbol{N} \cup \{0\}$ *and* $t \in [0,T)$. *Let* $\varphi \in \mathscr{S}(\boldsymbol{R}^2)$. *Then there exists a constant* $C_\varphi > 0$ *such that*

$$\|U_0(t+nT,0)\varphi - \hat{U}_0(t+nT)\varphi\|_{\mathscr{H}} \le C_\varphi |\lambda_-|^{-2n} \tag{6.6}$$

holds for $n \in \boldsymbol{N} \cup \{0\}$.

Proof. We have only to estimate $\|M(\theta_n(t)/\sigma_n(t))\varphi - M(\hat{\tau}_\infty)\varphi\|_{\mathscr{H}}$, by virtue of the unitarity of $\mathrm{e}^{i\phi_n(t)\tilde{L}} M(\theta_n(t)) D(c_n(t)\theta_n(t))\mathscr{F}$. Noting

$$\left\| M\left(\frac{\theta_n(t)}{\sigma_n(t)}\right)\varphi - M(\hat{\tau}_\infty)\varphi \right\|_{\mathscr{H}} \le \frac{1}{2}\left|\frac{\sigma_n(t)}{\theta_n(t)} - \frac{1}{\hat{\tau}_\infty}\right| \|x^2\varphi\|_{\mathscr{H}},$$
$$\left|\frac{\sigma_n(t)}{\theta_n(t)} - \frac{\sigma_n}{\theta_n}\right| = \left|\frac{-\theta_0(t)/\sigma_0(t)}{c_n^2\theta_n(\theta_0(t)/\sigma_0(t)+\theta_n)}\right| \le C_1|\lambda_-|^{-2n},$$
$$\left|\frac{\sigma_n}{\theta_n} - \frac{1}{\hat{\tau}_\infty}\right| = \left|\frac{(L_{11}\mu_n - \mu_{n-1}) - (L_{11}\mu_n - \lambda_+\mu_n)}{L_{12}\mu_n}\right| \le C_2|\lambda_-|^{-2n}$$

by (3.19) and (3.34), we have

$$\left\| M\left(\frac{\theta_n(t)}{\sigma_n(t)}\right)\varphi - M(\hat{\tau}_\infty)\varphi \right\|_{\mathscr{H}} \le \frac{1}{2}(C_1+C_2)|\lambda_-|^{-2n}\|x^2\varphi\|_{\mathscr{H}},$$

which yields the lemma. □

Lemma 6.2. *Suppose* (1.14), *and that* T_0 *satisfies the conditions stated in Theorem* 1.3. *Assume that* V *satisfies the condition* $(V)_\rho$ *for some* $\rho > 0$. *Let* $\varphi \in \mathscr{S}(\boldsymbol{R}^2)$ *be such that* $\mathscr{F}[M(\hat{\tau}_\infty)\varphi] \in C_0^\infty(\boldsymbol{R}^2\backslash\{0\})$. *Then*

$$\int_0^\infty \|V\hat{U}_0(s)\varphi\|_{\mathscr{H}}\,\mathrm{d}s < \infty \tag{6.7}$$

holds.

Proof. By assumption, one can obtain the estimate

$$\|V\hat{U}_0(t+nT)\varphi\|_{\mathscr{H}} \le C_3(1+|c_n(t)\theta_n(t)|)^{-\rho} \tag{6.8}$$

for $n \in \boldsymbol{N}\cup\{0\}$ and $t \in [0,T)$. By (4.3) and (4.5), we have

$$c_n(t)\theta_n(t) = \tilde{d}_n(t,0) = \frac{\lambda_+ - L_{11}}{L_{21}(\lambda_+ - \lambda_-)}\lambda_+^n\Omega_+(t) - \frac{\lambda_- - L_{11}}{L_{21}(\lambda_+ - \lambda_-)}\lambda_-^n\Omega_-(t).$$

Here we used $\theta_0(0) = 0$, $c_0(0) = \sigma_0(0) = 1$ and $\Omega_\pm(0) = -(L_{22} - \lambda_\pm) = -(\lambda_\mp - L_{11})$. By straightforward calculation, we have

$$\Omega_-(T) = L_{12}L_{21} - L_{11}(L_{22} - \lambda_-) = \lambda_- L_{11} - 1 = \lambda_-\Omega_-(0),$$

which implies $\Omega_-(0)\Omega_-(T) < 0$ by $\lambda_- < 0$. Hence, we see that for sufficiently large n, there exists a unique zero τ_n of $c_n(t)\theta_n(t)$ in the interval $[0,T)$. Put $\tilde{\lambda}_- := (|\lambda_-|+1)/2 > 1$ and $\hat{\lambda}_- := |\lambda_-|/\tilde{\lambda}_- > 1$. Since τ_n's are near the unique zero T_- of $\Omega_-(t)$ in $[0,T)$, there exists an $N_0 \in \boldsymbol{N}$ such that for $n \ge N_0$, the $\tilde{\lambda}_-^{-n}$-neighborhood of τ_n is included in $[0,T)$. In $[0,T)\backslash(\tau_n - \tilde{\lambda}_-^{-n}, \tau_n + \tilde{\lambda}_-^{-n})$,

$$\|V\hat{U}_0(t+nT)\varphi\|_{\mathscr{H}} \le C_4\hat{\lambda}_-^{-n\rho} \tag{6.9}$$

holds. Thus we have

$$\int_0^T \|V\hat{U}_0(t+nT)\varphi\|_{\mathscr{H}}\,\mathrm{d}t \le 2C_3\tilde{\lambda}_-^{-n} + C_4 T\hat{\lambda}_-^{-n\rho} \tag{6.10}$$

for $n \ge N_0$, which yields

$$\int_0^\infty \|V\hat{U}_0(s)\varphi\|_{\mathscr{H}}\,\mathrm{d}s \le C_3 N_0 T + \sum_{n=N_0}^{\infty} (2C_3\tilde{\lambda}_-^{-n} + C_4 T\hat{\lambda}_-^{-n\rho}) < \infty, \tag{6.11}$$

by $\tilde{\lambda}_- > 1$ and $\hat{\lambda}_-^\rho > 1$. This completes the proof. □

By virtue of these two lemmas, one can obtain the following corollary:

Corollary 6.3. *Suppose* (1.14), *and that* T_0 *satisfies the conditions stated in Theorem* 1.3. *Assume that* V *satisfies the condition* $(V)_\rho$ *for some* $\rho > 0$. *Let* $\varphi \in \mathscr{S}(\boldsymbol{R}^2)$ *be such that* $\mathscr{F}[M(\hat{\tau}_\infty)\varphi] \in C_0^\infty(\boldsymbol{R}^2\backslash\{0\})$. *Then*

$$\int_0^\infty \|VU_0(s,0)\varphi\|_{\mathscr{H}}\,\mathrm{d}s < \infty \tag{6.12}$$

holds.

By virtue of this corollary, one can show easily the existence of $W^+\varphi$ for $\varphi \in \mathscr{S}(\boldsymbol{R}^2)$ be such that $\mathscr{F}[M(\hat{\tau}_\infty)\varphi] \in C_0^\infty(\boldsymbol{R}^2\backslash\{0\})$, by using the Cook-Kuroda method. Since $\{\varphi \in \mathscr{S}(\boldsymbol{R}^2) \mid \mathscr{F}[M(\hat{\tau}_\infty)\varphi] \in C_0^\infty(\boldsymbol{R}^2\backslash\{0\})\}$ is dense in $L^2(\boldsymbol{R}^2)$, the proof of the existence of W^+ is completed by the density argument.

Now, by following the argument of Yajima [13], which is the most important part of the so-called Howland–Yajima method, one can prove the asymptotic completeness of $W^\pm$: let $\mathscr{V}$ and $\mathscr{V}_0$ be unitary operators on $\mathscr{K}$ defined by

$$(\mathscr{V}f)(t) = U(t,0)f(t), \quad (\mathscr{V}_0 f)(t) = U_0(t,0)f(t)$$

for $f \in \mathscr{K}$. Let $\hat{\mathscr{U}}$ and $\hat{\mathscr{W}}^\pm$ be the multiplication operators by the Floquet operator $U(T,0)$ and the wave operators $W^\pm$ on $\mathscr{K}$, respectively. Then we have

$$\mathrm{e}^{-iTK} = \mathscr{V}\hat{\mathscr{U}}\mathscr{V}^*, \quad \mathscr{W}^\pm = \mathscr{V}\hat{\mathscr{W}}^\pm\mathscr{V}_0^*,$$

which yield

$$\begin{aligned}
&\mathscr{K}_{\mathrm{ac}}(K) = \mathscr{K}_{\mathrm{ac}}(\mathrm{e}^{-iTK}) = \mathscr{V}\mathscr{K}_{\mathrm{ac}}(\hat{\mathscr{U}}) = \mathscr{V}L^2(\boldsymbol{T};\mathscr{H}_{\mathrm{ac}}(U(T,0))),\\
&\mathrm{Ran}(\mathscr{W}^\pm) = \mathscr{V}\mathrm{Ran}(\hat{\mathscr{W}}^\pm) = \mathscr{V}L^2(\boldsymbol{T};\mathrm{Ran}(W^\pm)).
\end{aligned}$$

By virtue of Theorem 5.6, we obtain (1.59), which is just the asymptotic completeness of $W^\pm$. This completes the proof of Theorem 1.3.

7. Remarks

Suppose that $\boldsymbol{B}(t) = (0, 0, B(t))$ is given by a general T-periodic $B(t) \in C(\boldsymbol{R}; \boldsymbol{R})$. In Korotyaev [10], the following factorization of $U_0(t,0)$ was derived:

$$U_0(t,0) = \mathrm{e}^{i\int_0^t \bar{\omega}(s)\,\mathrm{d}s\tilde{L}} M\left(\frac{-1}{ma_2(t)}\right) iD\left(\frac{1}{\sqrt{ma_1(t)}}\right) \mathrm{e}^{-i\int_0^t a_1(s)\,\mathrm{d}s(p^2+x^2)/2}, \tag{7.1}$$

where $\bar{\omega}(s) = qB(s)/(2m)$, $a_1(t)$ and $a_2(t)$ are the solutions of

$$\begin{cases} a_1'(t) = 2a_1(t)a_2(t), \\ a_2'(t) = a_2(t)^2 - a_1(t)^2 + \bar{\omega}(t)^2 \end{cases} \tag{7.2}$$

with $a_1(0) = 1/m$ and $a_2(0) = 0$. These equations yield Riccati equations

$$(a_2(t) \pm ia_1(t))' = (a_2(t) \pm ia_1(t))^2 + \bar{\omega}(t)^2. \tag{7.3}$$

By putting

$$-(a_2(t) \pm ia_1(t)) = \frac{y_\pm'(t)}{y_\pm(t)},$$

we obtain the Hill equation

$$y_\pm''(t) + \bar{\omega}(t)^2 y_\pm(t) = 0. \tag{7.4}$$

By using real linearly independent solutions $y_1(t)$ and $y_2(t)$ of the Hill equation (7.4), $a_1(t)$ and $a_2(t)$ are represented as

$$\begin{aligned} a_1(t) &= \frac{y_1'(t)y_2(t) - y_1(t)y_2'(t)}{y_1(t)^2 + y_2(t)^2}, \\ a_2(t) &= -\frac{y_1(t)y_1'(t) + y_2(t)y_2'(t)}{y_1(t)^2 + y_2(t)^2}. \end{aligned} \tag{7.5}$$

As mentioned in Sect. 1, under the assumption on $\boldsymbol{B}(t)$ that the Hill equation has solutions $y_1(t) = \mathrm{e}^{\lambda t}\chi_1(t)$ and $y_2(t) = \mathrm{e}^{-\lambda t}\chi_2(t)$ with $\lambda > 0$, and time-periodic functions $\chi_1(t)$ and $\chi_2(t)$, Korotyaev [10] showed the asymptotic completeness of wave operators for potentials V satisfying $|V(x)| \leq C\langle x\rangle^{-\rho}$ with some $\rho > 0$, by using the estimate

$$\frac{1}{\sqrt{ma_1(t)}} = O(\mathrm{e}^{\lambda|t|}).$$

Since $\mathrm{e}^{-it(p^2+x^2)/2}$ is represented as

$$\mathrm{e}^{-it(p^2+x^2)/2} = M(\tan t)D(\sin t)\mathscr{F}M(\tan t), \tag{7.6}$$

$U_0(t,0)$ is also represented as

$$\begin{aligned} U_0(t,0) = \mathrm{e}^{i\int_0^t \bar{\omega}(s)\,\mathrm{d}s\tilde{L}} M\left(\frac{-1}{ma_2(t)}\right) D\left(\frac{1}{\sqrt{ma_1(t)}}\right) M\left(\tan\left(\int_0^t a_1(s)\,\mathrm{d}s\right)\right) \\ \times D\left(\sin\left(\int_0^t a_1(s)\,\mathrm{d}s\right)\right) \mathscr{F} M\left(\tan\left(\int_0^t a_1(s)\,\mathrm{d}s\right)\right). \end{aligned} \tag{7.7}$$

This is quite similar to our factorization of $U_0(t,0)$ which is given as follows:

$$U_0(t,0) = \mathrm{e}^{i\phi(t)\tilde{L}} M(\theta(t)) D(c(t)\theta(t)) \mathscr{F} M\left(\frac{\theta(t)}{\sigma(t)}\right), \tag{7.8}$$

where $\theta(t)$, $c(t)\theta(t)$, $\theta(t)/\sigma(t)$ and $\phi(t)$ should satisfy

$$\begin{cases} \theta'(t) = \dfrac{1}{m}(1 + m^2\bar{\omega}(t)^2\theta(t)^2), \\ \dfrac{(c(t)\theta(t))'}{c(t)\theta(t)} = \dfrac{1}{m}\dfrac{1}{\theta(t)}, \\ \dfrac{(\theta(t)/\sigma(t))'}{(\theta(t)/\sigma(t))^2} = \dfrac{1}{m}\dfrac{1}{(c(t)\theta(t))^2}, \\ \phi'(t) = \bar{\omega}(t). \end{cases} \tag{7.9}$$

The first and second equations yield the Hill equation

$$(c(t)\theta(t))'' + \bar{\omega}(t)^2(c(t)\theta(t)) = 0 \tag{7.10}$$

(cf. (7.4)). Thus, also in our analysis, solutions of the Hill equation (7.10) play an important role, as has been seen in the previous sections.

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