



On the Elliptic Genera of Manifolds of Spin(7) Holonomy

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Abstract. Superstring compactification on a manifold of Spin(7) holonomy gives rise to a 2d worldsheet conformal field theory with an extended supersymmetry algebra. The $\mathcal{N} = 1$ superconformal algebra is extended by additional generators of spins 2 and $5/2$, and instead of just superconformal symmetry one has a $c = 12$ realization of the symmetry group $SW(3/2, 2)$. In this paper, we compute the characters of this supergroup and decompose the elliptic genus of a general Spin(7) compactification in terms of these characters. We find suggestive relations to various sporadic groups, which are made more precise in a companion paper.

1. Introduction

Berger's classification of holonomy groups [1] allows for only a few possibilities yielding supersymmetric vacua of the superstring. Beyond the low-dimensional avatars of the sequence of spaces of $SU(n)$ or $Sp(n)$ holonomy (comprising Calabi–Yau threefolds and fourfolds and low-dimensional hyperKähler manifolds), there are two exceptional holonomy groups that are relevant: G_2 and Spin(7). Compact examples of such spaces, of possible interest for string compactification, were first constructed by Joyce—for Spin(7) holonomy good discussions appear in [2, 3]. Our goal in this paper is to compute the elliptic genera of Spin(7) manifolds, and to make some observations about interesting connections between geometry, number theory, and group theory at which they hint. A more precise version of these connections is derived in the companion paper [4].

Our study will proceed as follows. In Sect. 2, we derive on macroscopic grounds (using simple arguments about the allowed space of modular functions) the elliptic genus of a Spin(7) manifold X . This yields a one-parameter

family of modular functions under the congruence subgroup Γ_θ ; the single parameter is determined by the Euler character $\chi(X)$.

We then wish to decompose the elliptic genus into a sum of irreducible characters of the worldsheet superconformal field theory. It has been known since the work of Shatashvili and Vafa [5] that the $\mathcal{N} = 1$ superconformal algebra is extended in this case by the addition of new generators of spins 2 and $5/2$. This yields a $c = 12$ realization of the $\mathcal{SW}(3/2, 2)$ algebra. While many properties of the representations were studied in [6], the full set of characters have not yet been determined (although a conjecture for the massive characters, which we will confirm, appears in [7]). In Sect. 3, we discuss the derivation of the characters of the algebra.

In Sect. 4, we decompose the elliptic genus into irreducible characters. We find a suggestive connection with representation theory of various sporadic simple groups. It is difficult, however, to make this connection precise beyond the level of numerology for reasons we discuss in Sect. 5. A precise connection between a chiral conformal field theory with the same superconformal symmetry group and these groups appears in [4].

Indeed, a good part of our motivation for undertaking this study was the 2010 observation of Eguchi, Ooguri, and Tachikawa (EOT) regarding K3 models [8]. K3 sigma models enjoy $\mathcal{N} = (4, 4)$ superconformal symmetry. EOT observed that in the decomposition of the elliptic genus of K3 in terms of $\mathcal{N} = 4$ characters, the coefficients could be expressed as sums of dimensions of irreducible representations of M_{24} , the largest Mathieu group. Explicitly, the characters of the $\mathcal{N} = 4$ algebra are

$$\text{ch}_{h=\frac{1}{4}, l=0}(\tau, z) = -\frac{iy^{\frac{1}{2}}\theta_1(\tau, z)}{\eta(\tau)^3} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n+1)} y^n}{1 - y q^n} \tag{1.1}$$

$$\text{ch}_{h=n+\frac{1}{4}, l=\frac{1}{2}}(\tau, z) = q^{n-\frac{1}{8}} \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \tag{1.2}$$

in terms of which the K3 elliptic genus may be written

$$\begin{aligned} \mathcal{Z}_{elliptic}^{K3}(\tau, z) &= 8 \left[\left(\frac{\theta_2(\tau, z)}{\theta_2(\tau, 0)} \right)^2 + \left(\frac{\theta_3(\tau, z)}{\theta_3(\tau, 0)} \right)^2 + \left(\frac{\theta_4(\tau, z)}{\theta_4(\tau, 0)} \right)^2 \right] \\ &= 24 \text{ch}_{h=\frac{1}{4}, l=0}(\tau, z) + \sum_{n=0}^{\infty} A_n \text{ch}_{h=n+\frac{1}{4}, l=\frac{1}{2}}(\tau, z). \end{aligned} \tag{1.3}$$

The observation of EOT, applied to the first few coefficients, is

$$\begin{aligned} A_0 &= -2 = -1 - 1, \\ A_1 &= 90 = 45 + \overline{45}, \\ A_2 &= 462 = 231 + \overline{231}, \\ A_3 &= 1540 = 770 + \overline{770}, \\ A_4 &= 4554 = 2277 + \overline{2277}, \\ &\dots \end{aligned} \tag{1.4}$$

The decompositions and twining genera were put on firm footing in [9–12]. It was proven in [13] that all the A_n for $n \geq 1$ are sums of irreducible representations of M_{24} with only positive coefficients. However, a natural construction of the full M_{24} group acting on a module, such as one associated to K3 sigma models, is unknown. Mukai [14] showed that the groups of symplectic automorphisms of any K3—that is, automorphisms that act trivially on the holomorphic 2-form—fall into certain subgroups of M_{23} and hence are insufficient to explain the appearance of M_{24} . Moreover, [15] showed that no K3 sigma model admits the full M_{24} as a symmetry and some sigma models possess symmetries that lie outside M_{24} , yet in Co_1 . Further aspects of this moonshine and its precise relation to K3, as well as various extensions, have been explored in [16–29]. For a recent review, see [30].

While the precise nature of the relation of M_{24} to K3 σ -models remains unclear, it is clearly worthwhile to look for similar connections in other classes of supersymmetric string compactifications. Especially relevant to our work are various precise versions of mock modular moonshine associated with the superconformal field theory on the chiral E_8 lattice [20] (building on the earlier work in [31]; see also the recent related work [32]). Our study of $Spin(7)$ manifolds here suggested an extension of the latter approach, and indeed using the same characters we find here, a precise moonshine conjecture relating M_{24} , a certain $c = 12$ conformal field theory with $SW(3/2, 2)$ symmetry, and various mock modular forms that appear in the closely related companion work [4].

Readers primarily interested in the main results of the paper can skip directly to Sect. 4.

2. Elliptic Genus for $Spin(7)$ manifolds

Let us begin by calculating the elliptic genera for manifolds of $Spin(7)$ holonomy. We will derive the answer using general arguments based on modularity and pole structure, and then check our result using orbifold techniques to directly compute the answer in some of the examples constructed by Joyce [2, 3].

2.1. Genera(lities)

The elliptic genus can be defined for any compact even-dimensional spin manifold¹ [33]. Let us first consider the NS, + elliptic genus, defined as

$$Z_{NS,+}(\tau) = \text{Tr}_{NS,R}(-1)^{F_R} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}. \quad (2.1)$$

Note that this trace acts as a Witten index on the right-movers, so the \bar{q} dependence drops out leaving us with a holomorphic object.

It will also be convenient to define elliptic genera with different boundary conditions and insertions in the trace:

¹ The genus as defined here vanishes for odd-dimensional manifolds.

$$\begin{aligned}
 \chi &= \text{Tr}_{R,R}(-1)^F q^{L_0-c/24} \bar{q}^{\bar{L}_0-c/24} \\
 Z_{R,+}(\tau) &= \text{Tr}_{R,R}(-1)^{F_R} q^{L_0-c/24} \bar{q}^{\bar{L}_0-c/24} \\
 Z_{NS,-}(\tau) &= \text{Tr}_{NS,R}(-1)^F q^{L_0-c/24} \bar{q}^{\bar{L}_0-c/24}
 \end{aligned}
 \tag{2.2}$$

where $(-1)^F$ is the fermion sign operator. χ is the Witten index, and the rest of the genera are functions of τ , which are invariant under a level-2 congruence subgroup of $SL(2, \mathbb{Z})$ that preserves the appropriate spin structure on the torus.

By considering elements of $SL(2, \mathbb{Z})$ that exchange spin structures, we can relate $Z_{R,+}$, $Z_{NS,-}$, and $Z_{NS,+}$ to each other. In particular

$$\begin{aligned}
 Z_{R,+}(\tau) &= Z_{NS,-}(-1/\tau) \\
 Z_{R,+}(\tau) &= -Z_{NS,+}(-1/\tau + 1).
 \end{aligned}
 \tag{2.3}$$

The congruence subgroup that preserves the spin structure of $Z_{NS,+}(\tau)$ (antiperiodic fermions in space and time) is given by

$$\Gamma_\theta = \left\{ \gamma \in SL(2, \mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2} \right\},
 \tag{2.4}$$

which is the subgroup generated by $\tau \rightarrow \tau+2$ and $\tau \rightarrow -1/\tau$. The fundamental domain of Γ_θ has genus 0, and the corresponding normalized (i.e. with constant removed) Hauptmodul is

$$\begin{aligned}
 K(\tau) &= \frac{\Delta(\tau)^2}{\Delta(2\tau)\Delta(\tau/2)} - 24 \\
 &= q^{-1/2} + 276q^{1/2} + 2048q + O(q)^{3/2},
 \end{aligned}
 \tag{2.5}$$

where $\Delta(\tau) = \eta(\tau)^{24}$ and $q = \exp(2\pi i\tau)$. In particular, any function invariant under Γ_θ that is meromorphic on the upper half-plane and at ∞ can be written as a rational function of $K(\tau)$. By looking at the pole structure of a function, we can specify the rational coefficients of $K(\tau)$.

Just as in [34], the function $Z_{NS,+}$ is convergent for $0 < |q| < 1$ due to the $\text{Tr } q^{L_0}$ so there are only potential poles at $\tau = i\infty$ and $\tau = 1$. The NS ground state energy of $-\frac{d}{16}$ gives $Z_{NS,+}(\tau)$, a pole of order $\frac{d}{16}$ at $\tau = i\infty$, where d is the real dimension of the manifold. For Spin(7) manifolds, $d = 8$ and the pole will be of order $\frac{1}{2}$. At $\tau = 1$, the other cusp, the function is regular because the Ramond sector genus $Z_{R,+}$ is regular at the high-temperature cusp (since its ground states have zero energy). Transforming to $Z_{NS,+}$ using (2.3), we see

$$\lim_{\tau \rightarrow i\infty} Z_{R,+}(\tau) = -Z_{NS,+}(1).$$

Thus, we can write the elliptic genus as

$$Z_{NS,+}(\tau) = c_0 + c_1 K(\tau)
 \tag{2.6}$$

for constants c_0 and c_1 . From (2.3), we can get the R_+ elliptic genus as well.

2.2. Specialization to Spin(7)

The only remaining freedom is to fix the two coefficients in (2.6). To do this, we now turn to geometric properties of the elliptic genus. (Alternatively, one could try to match the results to explicit computations in two models, particularly if one knows which topological quantities the constants depend on.)

Witten emphasized the geometric interpretation of the elliptic genus as a character-valued index of a (twisted) Dirac operator on loop space $\mathcal{L}M$, for any spin manifold M with tangent bundle² T_M [35]. Construct the following symmetric and antisymmetric products of bundles:

$$R(T_M) = \bigotimes_{\substack{l \in \mathbb{Z} \\ l > 0}} S_{q^l}(T_M) \otimes \bigotimes_{\substack{l \in \mathbb{Z} + \frac{1}{2} \\ l > 0}} \bigwedge (T_M) \tag{2.7}$$

where $\bigwedge_t T_M = \sum_{i=0}^{dim M} \bigwedge^i T_M t^i$ and $S_t T_M = \sum_{i=0}^{\infty} S^i T_M t^i$. Then, the NS-R elliptic genus for the even-dimensional manifold M with $dim M = d$ can be written as

$$Z_{NS,+} = q^{-d/16} \left\langle \hat{A}(M) \text{ch } R(T_M), M \right\rangle. \tag{2.8}$$

The inner product is often rewritten as an integral over M . Here, $\hat{A}(M)$ is the \hat{A} -class of the manifold. If M has total Pontryagin class $p(T_M) = \prod_i (1 + u_i)$, then we may write $\hat{A}(M) = \prod_i \frac{\sqrt{u_i/2}}{\sinh(\sqrt{u_i}/2)}$, where we have used the splitting principle. In what follows, we will frequently use the relation between Pontryagin and Chern classes: $p_i(T_M) = (-1)^i c_{2i}(T_C)$.

The Fourier development of the elliptic genus is given by

$$Z_{NS,+} = q^{-d/16} \sum_l \text{index}(R_l) q^{l/2} \tag{2.9}$$

where R_l is the representation of $R(T_M)$ multiplying the l th power of q in (2.7). For Spin(7) manifolds, we want to match this to Eq. (2.6); to fix two undetermined constants, we need to determine two of the R_l . We will choose the simplest cases, $R_0 = 1$ and $R_1 = T_M$, and compute their indices below.

A comment before we get started: the \hat{A} -genus, which is the evaluation of the above \hat{A} -class $\hat{A}(M)$ on the fundamental class³ of the manifold, is itself the index of the Dirac operator.⁴ Joyce proved that for simply-connected Spin(7) manifolds, this quantity is simply equal to 1 [2]. This fact follows from the formula for the \hat{A} -genus, $24\hat{A} = -1 + b^1 - b^2 + b^3 + b_4^+ - 2b_4^-$, and a constraint on the Betti numbers of Spin(7) manifolds: $b_3 + b_4^+ - b_2 - 2b_4^- - 1 = 24$. Note that the formula for the \hat{A} -genus can be rewritten in terms of Pontryagin classes as $5760\hat{A} = 7p_1^2 - 4p_2$. In the previous formula and below, we will suppress the argument of the Pontryagin classes and so write p_i to denote $p_i(T_M)$. We

² One could of course consider more general vector bundles on the manifold, but we will not do this here.

³ In plain language, this is an instruction to take the form of top degree in the expansion of $\hat{A}(M)$ and integrate over M .

⁴ In the Kähler case, this is equivalent to the index of $\bar{\partial}$ acting on $(0, q)$ forms, which becomes the holomorphic Euler characteristic $\chi_0 = \sum_q (-1)^q H^{0,q}(M, \mathbb{Z})$.

will also abuse notation using the same expression before and after integrating over M .

First, we look at the $q^{-1/2}$ coefficient. We have $\text{index}(R_0) = \int_M \hat{A}(M) \text{ch}(1)$. For an 8-manifold, we can expand $\hat{A}(M) = 1 - \frac{p_2}{24} + \frac{7p_1^2 - 4p_2}{5760}$. Integrating over M and using Joyce’s result that the \hat{A} -genus equals 1 we just get a coefficient of 1. (In the Calabi–Yau case, we would get $\hat{A} = 2$ because such manifolds preserve more supersymmetry.) In other words, we have just counted the dimension of the space of harmonic spinors.

We also want to find the q^0 coefficient of the genus, which is given by $\text{index}(T) = \int_M \hat{A}(M) \text{ch}(T)$. For any complex bundle V on an 8-manifold, we can expand the Chern character as

$$\begin{aligned} \text{ch}(V) = & \dim(V) + c_1(V) + \frac{1}{2}(c_1(V)^2 - c_2(V)) + \frac{1}{3!}(c_1(V)^3 \\ & - 3c_2(V)c_1(V) + 3c_3(V)) + \frac{1}{4!}(c_1(V)^4 - 4c_2(V)c_1(V)^2 \\ & + 4c_3(V)c_1(V) + 2c_2(V)^2 - 4c_4(V)). \end{aligned} \tag{2.10}$$

For us, $\text{ch}(T)$ denotes the Chern character of the complexification of the tangent bundle, $T_{\mathbb{C}} = T \oplus iT$, though below we will suppress this argument as well. Setting $c_1 = 0$, multiplying by $\hat{A}(M)$, and integrating we are left with

$$\text{index}(T) = \dim(T) \frac{7p_1^2 - 4p_2}{5760} + \frac{p_1 c_2}{24} + \frac{1}{4!}(2c_2^2 - 4c_4), \tag{2.11}$$

where we can set the first term to $\dim(T)$ because it is multiplied by the \hat{A} -genus, which is just 1. It is convenient to use the relation $4p_2 - p_1^2 = 8\chi$ (see the remark below the proof of 7.3 in [36]). Employing the relationship between Chern and Pontryagin classes above, $c_4 = \chi = p_2$ and $-c_2 = p_1$, the relation from [36] becomes $c_2^2 = -4\chi$.

Then, we have

$$\text{index}(T) = \dim(T) - \frac{c_2^2}{24} + \frac{1}{4!}(2c_2^2 - 4\chi) = -\frac{\chi}{6} + \frac{c_2^2}{24} + \dim(T) = -\frac{\chi}{3} \tag{2.12}$$

In the last line, we have used $\dim(T) = \dim_{\mathbb{C}}(T_{\mathbb{C}}) = \dim_{\mathbb{R}}(T)$. Our final result, then, is that for a $\text{Spin}(7)$ manifold M , one has

$$\begin{aligned} Z_{NS,+}(M) &= K(\tau) + \left(8 - \frac{\chi}{3}\right) \\ &= q^{-1/2} + \left(8 - \frac{\chi}{3}\right) + 276q^{1/2} + 2048q + \dots \end{aligned} \tag{2.13}$$

In the rest of the paper, we will decompose (2.13) in terms of characters of $SW(3/2, 2)$ and observe some interesting structure in the coefficients. We discuss the relevant numerology in Sect. 4.

As a final remark, we notice a similarity between our $\text{Spin}(7)$ elliptic genus and the elliptic genus of a Calabi–Yau fourfold [37]. Since Calabi–Yau manifolds enjoy $\mathcal{N} = 2$ worldsheet supersymmetry, an additional grading by a chemical potential z conjugate to the $U(1)$ R-symmetry generator is possible, and the fourfold elliptic genus is a weak Jacobi form of weight 0 and index

2 [37, 38]; upon specialization to $z = 0$, one can again express the fourfold genus as a simple polynomial in $K(\tau)$, with coefficients that can be fixed by geometric data. On the other hand, it is unlikely that one can lift the Spin(7) elliptic genus to a Jacobi form since its worldsheet algebra is extended by an additional Virasoro generator (see Sect. 3) rather than some other Kac–Moody currents.

2.3. Some Explicit Computations

We now compute a few examples of Spin(7) elliptic genera directly, using orbifold techniques. We will see that they obey the general formula (2.13). In [3], Joyce constructs several examples of manifolds with Spin(7) holonomy. We will focus on his toroidal orbifolds of the form T^8/\mathbb{Z}_2^4 . Our computations follow the techniques used in [39]. For our conventions regarding theta functions, see Appendix A.

Joyce gives three examples of T^8/\mathbb{Z}_2^4 orbifolds with Spin(7) holonomy. Label the torus coordinates by (x_1, \dots, x_8) . The orbifolds are as follows.

1. The first example Joyce gives has the following orbifold action:

$$\begin{aligned}
 \alpha &: (x_1, \dots, x_8) \mapsto (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8) \\
 \beta &: (x_1, \dots, x_8) \mapsto (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8) \\
 \gamma &: (x_1, \dots, x_8) \mapsto (\tfrac{1}{2} - x_1, \tfrac{1}{2} - x_2, x_3, x_4, \tfrac{1}{2} - x_5, \tfrac{1}{2} - x_6, x_7, x_8) \\
 \delta &: (x_1, \dots, x_8) \mapsto (-x_1, x_2, \tfrac{1}{2} - x_3, x_4, -x_5, x_6, \tfrac{1}{2} - x_7, x_8).
 \end{aligned}
 \tag{2.14}$$

We need to sum over the untwisted sector and all 15 twisted sectors, each of which must be projected onto \mathbb{Z}_2^4 -invariant states by inserting $(\frac{1+\alpha}{2}) (\frac{1+\beta}{2}) (\frac{1+\gamma}{2}) (\frac{1+\delta}{2})$ into the trace. In total, we will need to sum over 256 separate boundary conditions. Fortunately, most of the sectors’ contributions to the elliptic genus will vanish.

Any sector with both an untwisted space and an untwisted time boundary condition along any of the eight coordinates will have a right-moving fermion zero-mode and thus its contribution will vanish. Similarly, any sector with both a twisted space and a twisted time boundary condition along any of the eight coordinates will have a left-moving fermion zero-mode and its contribution also vanishes. Finally, any insertions in the trace that permute fixed points will not contribute.

The surviving twists are

$$Z_{NS,+} = 16 \left(1_{\alpha\beta} \square + \alpha\beta \square_1 + \beta \square_\alpha + \alpha \square_\beta \right)
 \tag{2.15}$$

where the 16 includes both the right-moving Ramond ground states and the fixed-point contributions.

Calculating the contributions of each twisted sector to the elliptic genus is an exercise in free-field theory. For example, the $1_{\alpha\beta} \square$ contribution involves a spatial twist for all eight bosons and fermions, and no time twist, making the bosons antiperiodic in space, and the fermions periodic in space (recall we have NS boundary conditions). We insert nothing in

the trace. Finally, we have to take into account the ground state energy of $q^{\frac{1}{2}}$ and ground state degeneracy. The contribution to the elliptic genus is

$$1_{\alpha\beta}^{\square} = 4q^{\frac{1}{2}} \prod_{n=1}^{\infty} \left(\frac{1 + q^n}{1 - q^{n-\frac{1}{2}}} \right)^8 = \frac{\theta_2(\tau, 0)^4}{\theta_4(\tau, 0)^4}. \tag{2.16}$$

The sum over all sectors is

$$\begin{aligned} Z_{NS,+} &= 16 \left(\frac{\theta_2(\tau, 0)^4}{\theta_4(\tau, 0)^4} + \frac{\theta_4(\tau, 0)^4}{\theta_2(\tau, 0)^4} - 2 \right) \\ &= \frac{1}{\sqrt{q}} - 40 + 276\sqrt{q} + 2048q + 11202q^{3/2} + \dots \\ &= K(\tau) - 40. \end{aligned} \tag{2.17}$$

If we do a similar computation for the R+ elliptic genus, we get

$$\begin{aligned} Z_{R,+} &= 16 \left(\frac{\theta_3(\tau, 0)^4}{\theta_4(\tau, 0)^4} + \frac{\theta_4(\tau, 0)^4}{\theta_3(\tau, 0)^4} + 2 \right) \\ &= 64 + 4096q + 98304q^2 + 1228800q^3 + \dots \end{aligned} \tag{2.18}$$

Joyce gives a resolution of this orbifold to a smooth manifold with Euler character $\chi = 144$, which matches (2.13) [2,3].

- 2. The second example is a T^8/\mathbb{Z}_2^4 with the \mathbb{Z}_2 's acting as

$$\begin{aligned} \alpha &: (x_1, \dots, x_8) \mapsto (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8) \\ \beta &: (x_1, \dots, x_8) \mapsto (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8) \\ \gamma &: (x_1, \dots, x_8) \mapsto \left(\frac{1}{2} - x_1, \frac{1}{2} - x_2, x_3, x_4, -x_5, -x_6, x_7, x_8\right) \\ \delta &: (x_1, \dots, x_8) \mapsto \left(-x_1, x_2, -x_3, x_4, \frac{1}{2} - x_5, x_6, \frac{1}{2} - x_7, x_8\right). \end{aligned} \tag{2.19}$$

The contributing twists to the elliptic genus in this example are the same as in the previous, so we again get

$$\begin{aligned} Z_{NS,+} &= 16 \left(\frac{\theta_2(\tau, 0)^4}{\theta_4(\tau, 0)^4} + \frac{\theta_4(\tau, 0)^4}{\theta_2(\tau, 0)^4} - 2 \right) \\ &= \frac{1}{\sqrt{q}} - 40 + 276\sqrt{q} + 2048q + 11202q^{3/2} + \dots \end{aligned} \tag{2.20}$$

and similarly

$$\begin{aligned} Z_{R,+} &= 16 \left(\frac{\theta_3(\tau, 0)^4}{\theta_4(\tau, 0)^4} + \frac{\theta_4(\tau, 0)^4}{\theta_3(\tau, 0)^4} + 2 \right) \\ &= 64 + 4096q + 98304q^2 + 1228800q^3 + \dots \end{aligned} \tag{2.21}$$

Joyce gives several inequivalent resolutions of this orbifold, but all of them have Euler character $\chi = 144$.

- 3. Joyce provides a final example with \mathbb{Z}_2 group actions

$$\begin{aligned} \alpha &: (x_1, \dots, x_8) \mapsto (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8) \\ \beta &: (x_1, \dots, x_8) \mapsto (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8) \end{aligned}$$

$$\begin{aligned} \gamma &: (x_1, \dots, x_8) \mapsto (-x_1, -x_2, x_3, x_4, -x_5, -x_6, x_7, x_8) \\ \delta &: (x_1, \dots, x_8) \mapsto \left(\frac{1}{2} - x_1, x_2, \frac{1}{2} - x_3, x_4, \frac{1}{2} - x_5, x_6, \frac{1}{2} - x_7, x_8\right). \end{aligned} \tag{2.22}$$

Unlike in the previous two examples, this orbifold has three \mathbb{Z}_2 actions that do not involve coordinate shifts whereas the previous examples had only two. Thus, we get more sectors contributing to the elliptic genus. The contributing twists are

$$\begin{aligned} Z_{NS,+} &= 16 \left(1_{\alpha\beta} \square + \alpha\beta 1_{\alpha} \square + \beta 1_{\alpha} \square + \alpha 1_{\beta} \square + \beta\gamma 1_{\alpha\gamma} \square + \alpha\gamma 1_{\beta\gamma} \square + \gamma 1_{\alpha\beta\gamma} \square + \alpha\beta\gamma 1_{\gamma} \square \right) \\ &= 16 \left(\frac{\theta_2(\tau, 0)^4}{\theta_4(\tau, 0)^4} + \frac{\theta_4(\tau, 0)^4}{\theta_2(\tau, 0)^4} - 6 \right) \\ &= \frac{1}{\sqrt{q}} - 104 + 276\sqrt{q} + 2048q + 11202q^{3/2} + \dots \end{aligned} \tag{2.23}$$

The same analysis in the $R+$ elliptic genus gives

$$\begin{aligned} Z_{R,+} &= 16 \left(\frac{\theta_3(\tau, 0)^4}{\theta_4(\tau, 0)^4} + \frac{\theta_4(\tau, 0)^4}{\theta_3(\tau, 0)^4} + 6 \right) \\ &= 128 + 4096q + 98304q^2 + 1228800q^3 + \dots \end{aligned} \tag{2.24}$$

Joyce gives several inequivalent resolutions of this orbifold, and it has Euler character $\chi = 336 - 48k - 48l + 12kl$, with $k, l = 0, 1, \dots, 8$. The orbifold with these choices of phases (as opposed to possible other choices of discrete torsion) matches onto the results Joyce finds for resolutions with $k = l = 0$ or $k = l = 8$.

4. We can take the same toroidal orbifold as above and put in some choice of discrete torsion: in each twisted sector Hilbert space \mathcal{H}_h , introduce a phase $\epsilon(g, h)$, as done in [40]. In fact, at each fixed point, we can introduce a different choice of torsion, $\epsilon_f(g, h)$, as done in [41]. However, there are consistency conditions that must be satisfied. In particular, for the action to be a representation of \mathbb{Z}_2^4 , we require

$$\epsilon_f(g, h)\epsilon_f(g', h) = \epsilon_f(gg', h). \tag{2.25}$$

Moreover, modular invariance requires

$$\begin{aligned} g \square_h(\tau + 1) &= gh \square_h(\tau) \\ g \square_h(-1/\tau) &= h^{-1} \square_g(\tau). \end{aligned} \tag{2.26}$$

Note that for \mathbb{Z}_2^4 , (2.25) and (2.26) imply $\epsilon_f(g, 1) = \epsilon_f(1, g) = 1$ for all g , and that all the phases $\epsilon_f(g, h)$ are ± 1 since $\epsilon_f(g, h)^2 = \epsilon_f(1, h) = 1$.

We can rewrite the first line of (2.23) in full generality as

$$\begin{aligned} Z_{NS,+} &= 16 \left(1_{\alpha\beta} \square + \alpha\beta 1_{\alpha} \square \right) \\ &\quad + \sum_f \left(\epsilon_f(\beta, \alpha)\beta 1_{\alpha} \square \right) + \sum_f \left(\epsilon_f(\alpha, \beta)\alpha 1_{\beta} \square \right) \end{aligned}$$

$$\begin{aligned}
 & + \sum_f \left(\epsilon_f(\beta\gamma, \alpha\gamma)\beta\gamma \square_{\alpha\gamma} \right) + \sum_f \left(\epsilon_f(\alpha\gamma, \beta\gamma)\alpha\gamma \square_{\beta\gamma} \right) \\
 & + \sum_f \left(\epsilon_f(\gamma, \alpha\beta\gamma)\gamma \square_{\alpha\beta\gamma} \right) + \sum_f \left(\epsilon_f(\alpha\beta\gamma, \gamma)\alpha\beta\gamma \square_{\gamma} \right). \quad (2.27)
 \end{aligned}$$

Using (2.26) repeatedly, we can rewrite every term in (2.27) in terms of the $\alpha\beta$ -twisted sector: $\epsilon_f(g, \alpha\beta)$ for some g . Then, we need to make a consistent choice of torsion for each of the fixed points in the $\alpha\beta$ -twisted sector to solve for the elliptic genus. The choice

$$\begin{aligned}
 \epsilon_f(\alpha, \alpha\beta) &= -1 \\
 \epsilon_f(\gamma, \alpha\beta) &= -1
 \end{aligned} \quad (2.28)$$

for all 128 fixed points f is a consistent choice which gives

$$\begin{aligned}
 Z_{NS,+} &= 16 \left(\frac{\theta_2(\tau, 0)^4}{\theta_4(\tau, 0)^4} + \frac{\theta_2(\tau, 0)^4}{\theta_4(\tau, 0)^4} + 2 \right) \\
 &= \frac{1}{\sqrt{q}} + 24 + 276\sqrt{q} + 2048q + 11202q^{3/2} + \dots \quad (2.29)
 \end{aligned}$$

which corresponds to a manifold with $\chi = -48$. Joyce’s resolutions with $k = 0, l = 8$ or $k = 8, l = 0$ give such a χ .

The same choice of torsion in the $R+$ elliptic genus gives

$$\begin{aligned}
 Z_{R,+} &= 16 \left(\frac{\theta_3(\tau, 0)^4}{\theta_4(\tau, 0)^4} + \frac{\theta_4(\tau, 0)^4}{\theta_3(\tau, 0)^4} - 2 \right) \\
 &= 4096q + 98304q^2 + 1228800q^3 + \dots \quad (2.30)
 \end{aligned}$$

It might be interesting to match possible choices of torsion in this class of models with all possible geometric resolutions; similar results in the context of manifolds of G_2 holonomy were obtained in [41].

3. Toward Characters of the “Spin(7) Algebra”

In this section, we review the structure of the chiral algebra for a sigma-model with target a manifold of Spin(7) holonomy. The algebra was first studied in the context of Spin(7) compactifications in [5]. We discuss our conventions and present explicit details in Appendix B.

The reduced holonomy of a Spin(7) manifold M implies the existence of a nowhere-vanishing self-dual 4-form, Ω . Ω can be written in closed form in terms of a local vielbein $\sum_{i=1}^8 e_i \otimes e_i$ (see equations 2.1–2.3 of [5]), where the e_i are in the fundamental of $O(8)$. We may choose an embedding $\text{Spin}(7) \subset O(8)$ such that the 8-dimensional spinor representation of $O(8)$ decomposes as $\mathbf{7} \oplus \mathbf{1}$. Then viewing the 8-dimensional vector representation of $O(8)$ as a spinor of Spin(7), the fourfold antisymmetric product of this spinor includes the singlet Ω . Moreover, a Spin(7) manifold is characterized by three independent Betti numbers that we can express as b_2, b_3, b_4^\pm subject to the constraint $b_3 + b_4^+ - b_2 - 2b_4^- - 1 = 24$. The dimension of the Spin(7) moduli space is $b_4^- + 1$.

To find a geometry-inspired construction of the algebra, we start with the $\mathcal{N} = 1$ superconformal algebra (SCA) and extend it by new generators. In the case of a Calabi–Yau sigma model, one must add an $U(1)$ current and impose closure of the algebra. In the case of a $\text{Spin}(7)$ manifold, something more exotic happens. The analog of the $U(1) = U(n)/SU(n)$ is in this case a sector isomorphic to $SO(8)/\text{Spin}(7)$. Computing the central charge of this coset, $4 - 7/2 = 1/2$, we can see that the result is the Ising model. Thus, the Ising sector will also produce an analog of the spectral flow isomorphism enjoyed by Calabi–Yau sigma models.

As worked out in detail by Shatashvili and Vafa using a free-field representation, the algebra is generated by the operators:

$$L_n, G_n, X_n, M_n \quad (3.1)$$

where L_n, G_n are the usual Virasoro generators and their superpartners. X_n are the Fourier modes of a new spin-2 operator produced by replacing the e_i in Ω with target space fermions. Checking the OPEs and enforcing closure of the algebra results in a spin-5/2 superpartner for X_n that we call M_n . Note that G_n, M_n are fermionic and will both be half-integrally graded in the NS sector. One can check that X_n is related to an Ising model stress-energy tensor T_{I} by $T_{\text{I}} = X/8$; see Appendix B for the full algebra.

Finally, we note in passing that even in the absence of a $U(1)$ current there is a version of “spectral flow” enabling one to transform between the NS and R sectors. Consider the R sector ground states, which have $h = c/24 = 1/2$. Label each state by $|h_{\text{R}}, h_{\text{I}}\rangle$ where h_{I} is the eigenvalue of T_{I} and h_{R} is the eigenvalue of T_{R} such that $T = T_{\text{R}} + T_{\text{I}}$. For a unitary theory, we have only three classes of such states, because there are only three admissible weights in the Ising sector: $0, \frac{1}{16}, \frac{1}{2}$. Therefore, one of these ground states has weight in the Ising sector only: $h_{\text{tot}} = h_{\text{I}} = 1/2$. Shatashvili and Vafa utilize the fusion rules for this operator, which is isomorphic to the Ising energy operator ϵ , to map Ramond ground states to certain NS sector highest weight states and show it generates the spectral flow-like isomorphism. We will use this isomorphism to map our NS sector characters to the R sector and thus obtain the complete set of characters.

3.1. Unitary Highest Weight States

Next, we would like to study the representation theory of this algebra and locate the unitary highest weight states in the NS sector, to compute their characters. Here, we will briefly sketch some results of Gepner and Noyvert [6], who first extensively studied the representation theory of the algebra and computed the Kac determinant.⁵ The Kac determinant is a useful first step in finding the irreducible representations of the algebra and for many algebras, like the Virasoro algebra, it is also sufficient.

We will see, however, that the Kac determinant fails to provide complete information about the structure of the maximal proper submodule generated by null states in several important ways.

⁵ The operators in [5] and [6] are related by $X(z) = 8A(z)$, $M(z) = \frac{\sqrt{23}}{3}U(z) + \frac{1}{6}\partial G(z)$.

Gepner and Noyvert evaluated the Kac determinant for the $SW(3/2, 2)$ highest weight modules using the Coulomb gas formalism. Highest weight vectors are written as $|h, x\rangle$ and satisfy

$$\begin{aligned} X_0|h, x\rangle &= x|h, x\rangle \\ L_0|h, x\rangle &= h|h, x\rangle \\ \mathcal{O}_n|h, x\rangle &= 0, \forall n > 0. \end{aligned} \tag{3.2}$$

The Kac determinant of such a module has a closed-form expression

$$\begin{aligned} \det M_N^{NS}(h, a) &= \prod_{1 \leq mn \leq 2N} (f_{m,n})^{P_{NS}(N-mn/2)} \\ &\times \prod_{1 \leq jk \leq N} (g_{j,k})^{P_{NS}^l(N-jk)} \prod_{1 \leq l \leq 2N} (d_l)^{\bar{P}_{NS}(N-l/2)}. \end{aligned} \tag{3.3}$$

The $f, g,$ and h curves are given by⁶

$$f_{m,n} = \frac{1}{192} (4 + 24x - (5m - 3n)^2) \tag{3.4}$$

$$g_{j,k} = \frac{1}{8} (49 - 120h + 24x - (10j + 3k)^2) \tag{3.5}$$

$$d_l = \frac{9}{64} - \frac{3h}{4} + h^2 - \frac{53l^2}{288} + \frac{5hl^2}{12} + \frac{25l^4}{576} - \frac{l^2x}{6}. \tag{3.6}$$

The generating functions P_{NS} describe the number of states in the free-field theory. The free-field representation comprises two bosons and their superpartners, so we have

$$\begin{aligned} \sum_n P_{NS}(n)q^n &= \prod_{k=1}^{\infty} \left(\frac{1 + q^{k-1/2}}{1 - q^k} \right)^2 = \mathcal{P}(\tau) \\ \sum_n \bar{P}_{NS}^l(n)q^n &= \frac{1}{(1 + q^{l/2})} \prod_{k=1}^{\infty} \left(\frac{1 + q^{k-1/2}}{1 - q^k} \right)^2 = \bar{\mathcal{P}}^l(\tau). \end{aligned} \tag{3.7}$$

$\bar{\mathcal{P}}^l(\tau)$ comes when we have a fermionic level- l operator that annihilates the highest weight state: $O_{-l}|h\rangle = 0$. This is probably most familiar from the $\mathcal{N} = 2$ algebra, which is discussed in more detail in Appendix B of [42]. There exists a null vector at level-1/2 which is annihilated by $G_{-1/2}$. Therefore, at each level we must remove basis elements containing $G_{-1/2}$ and divide out the corresponding factor, $(1 + x^{1/2})$, from the partition function.

Something similar happens for the Spin(7) algebra. For example, $(\frac{5}{2}G_{-1/2} + \frac{1}{2}M_{-1/2})|0, 0\rangle$ is a singular highest weight state annihilated by $\frac{1}{2}G_{-1/2} - M_{-1/2}$. Similarly, $(\frac{1}{2}G_{-1/2} - M_{-1/2})|\frac{1}{2}, \frac{1}{2}\rangle$ is a singular highest weight state annihilated by $\frac{7}{2}G_{-1/2} - M_{-1/2}$. Both of these singular states produce Verma modules generated by $\bar{\mathcal{P}}^1(\tau) = \frac{\mathcal{P}(\tau)}{1+x^{\frac{1}{2}}}$.

⁶ We correct a small typo on p. 15, Eq. (7.10) of [6]. The second line of the expression for $g_{j,k}^{NS}$ should read $\frac{1}{18} \left((3+c)(1-j) + 9\frac{1-k}{2} \right) \left((3+c)(1+j) + 9\frac{1+k}{2} \right)$.

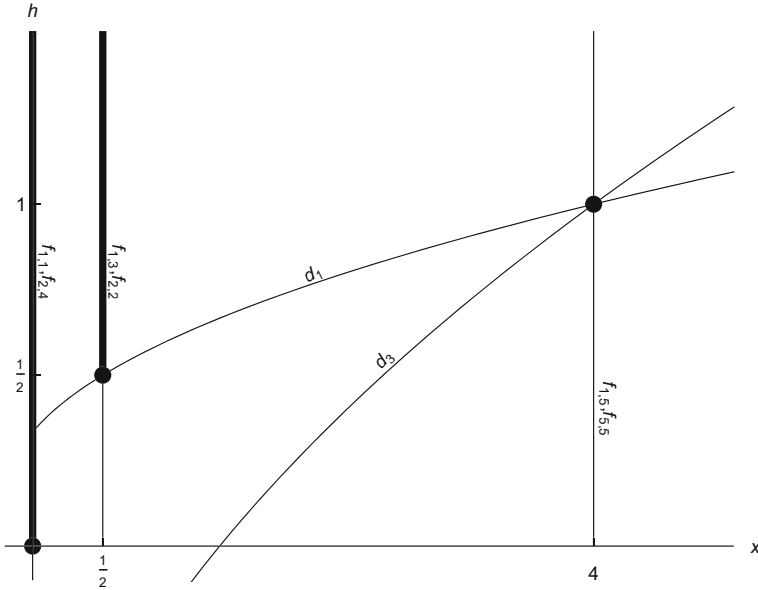


FIGURE 1. The vanishing curves of the f , d , and g curves in the Kac determinant in terms of the eigenvalues of the highest weight state, taken from [6]. The *black dots* indicate the location of massless unitary modules and the two *vertical black lines* are continua of massive unitary modules

To find the unitary representations, we need the vanishing curves of the f , g , and d curves. The first few curves are plotted in Fig. 1 in terms of the h and x , the L_0 and X_0 eigenvalues of the highest weight space, respectively. Gepner and Noyvert specified which of the modules are unitary and the results are also in the figure.

Gepner and Noyvert write down two continuous series of unitary highest weight representations that we call massive representations and three discrete representations that we call massless representations [6]. The massive representations are labeled by highest weight states $|h, 0\rangle$ with $h > 0$ and $|h, \frac{1}{2}\rangle$ with $h > \frac{1}{2}$. The massless characters are labeled by highest weight states $|0, 0\rangle$ (the vacuum), $|\frac{1}{2}, \frac{1}{2}\rangle$, and $|1, 4\rangle$.

3.2. The Content of Our Characters

In this section, we construct (conjectural) characters for the massive and massless unitary highest weight representations of the Spin(7) algebra in the NS sector.

3.2.1. Massive Characters. In the case of the massive characters, it is sufficient to look for solutions of the Kac determinant to find the location of singular vectors, then to treat these singular vectors as new highest weight states and look

for new singular vectors, and so on. The character of the unitary representation is then found by subtracting out all singular vectors via inclusion/exclusion.⁷

As an example, consider a Verma module generated by the highest weight state $|h\rangle$. Suppose this state has two singular descendants, $|n_1\rangle, |n_2\rangle$, whose contributions we wish to subtract from the character. If we denote the modules generated by a highest weight vector λ by $M(\lambda)$, then the expression for the character becomes $\text{ch } M(h) - \text{ch } M(n_1) - \text{ch } M(n_2)$. However, if $|n_1\rangle$ and $|n_2\rangle$ share a singular descendant, $|n_3\rangle$, then in subtracting $M(n_{1,2})$ we have doubly subtracted $M(n_3)$ and therefore must add a term to our character to compensate:

$$\chi = \text{ch } M(h) - \text{ch } M(n_1) - \text{ch } M(n_2) + \text{ch } M(n_3).$$

Let us first compute the character for $|h, 0\rangle$ for some $h > 0$. The other massive character will be treated identically; all that differs is finding the particular quantum numbers.

The first thing to note is that the only vanishing curve “intersecting” the two massive states is the $f_{m,n}$ curve in Fig. 1. While it is possible in general for the descendant singular vectors to admit descendants of their own that satisfy the $g_{j,k}$ or d_l equations (and this will happen in the massless case), it does not happen here.

The characters will in general have the form

$$\chi_{h,0} = q^{h-\frac{1}{2}} \mathcal{P}(\tau)(1 - f_{\text{singular}}), \tag{3.8}$$

where $\mathcal{P}(\tau)$ is defined in (3.7) and f_{singular} is the contribution of all the singular vectors.

We plug $x = 0, h > 0$ into $f_{m,n}$ and search for solutions that satisfy $m, n \in \mathbb{Z}, m + n \in 2\mathbb{Z}$. We find two sets of solutions:

$$\begin{aligned} n_{1,k} &= (3k + 2), \quad m_{1,k} = (5k + 4), \quad k = 0, 1, \dots \\ n_{2,k} &= (3k + 1), \quad m_{2,k} = (5k + 1), \quad k = 0, 1, \dots \end{aligned} \tag{3.9}$$

Because these new singular vectors are all by themselves highest weight, they generate their own Verma module from which we can find new singular vectors, by finding solutions to $f_{m,n} = 0$ and plugging in $h + \frac{n_{i,k}m_{i,k}}{2}$, $i = 1, 2$. For example, using as our highest weight vector the singular vector at level $\frac{1}{2}$ from the first set of solutions ($i = 2$ and $k = 0$) we find:

$$\begin{aligned} n_{1,j} &= (3j + 2), \quad m_{1,j} = (5j + 6), \quad j = 0, 1, \dots \\ n_{2,j} &= (3j + 4), \quad m_{2,j} = (5j + 4), \quad j = 0, 1, \dots \end{aligned} \tag{3.10}$$

After iterating this procedure, we exhaust the singular vectors and organize the result in the embedding diagram in Fig. 2. Nodes in this diagram are singular vectors with the indicated set of quantum numbers under L_0 and X_0 .

⁷ We do not expect exotica like subsingular vectors to appear in the massive characters, which have, for example only a single Kac determinant vanishing curve, much like the conventional Virasoro case. This expectation is borne out by explicit numerical checks and the complementary computations of [7]. For a more precise explanation of the unusual features of \mathcal{W} -algebra representation theory see, e.g., [43,44].

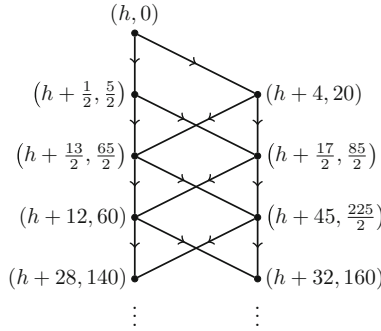


FIGURE 2. The embedding diagram that determines the character $\chi_{h,0}$

Arrows indicate descendants; there is an arrow from some singular vector v to w if w is in the submodule generated by v .

The embedding structure is necessary to determine how to subtract all singular vectors, to prevent overcounting. The answer for $\chi_{h,0}$ is

$$\begin{aligned} \chi_{h,0} = q^{h-\frac{1}{2}} \mathcal{P}(\tau) & \left(1 - \sum_{k=0}^{\infty} q^{\frac{(3k+1)(5k+1)}{2}} - \sum_{k=0}^{\infty} q^{\frac{(3k+2)(5k+4)}{2}} \right. \\ & \left. + \sum_{k=0}^{\infty} q^{\frac{(3k+2)(5k+6)+1}{2}} + \sum_{k=0}^{\infty} q^{\frac{(3k+4)(5k+4)+1}{2}} \right). \end{aligned} \tag{3.11}$$

After repeating the same procedure for the other massive tower, we find

$$\begin{aligned} \chi_{h,\frac{1}{2}} = q^{h-\frac{1}{2}} \mathcal{P}(\tau) & \left(1 - \sum_{k=0}^{\infty} q^{\frac{(3k+1)(5k+3)}{2}} - \sum_{k=0}^{\infty} q^{\frac{(3k+2)(5k+2)}{2}} \right. \\ & \left. + \sum_{k=0}^{\infty} q^{\frac{(3k+1)(5k+7)+4}{2}} + \sum_{k=0}^{\infty} q^{\frac{(3k+5)(5k+3)+4}{2}} \right), \end{aligned} \tag{3.12}$$

whose embedding diagram is shown in Fig. 3.

These answers agree with those derived in [7], who employed a coset construction of the algebra.

3.2.2. Massless Characters. We now turn to the three discrete massless unitary highest weight representations. For these computations, the Kac determinant proves to be insufficient to obtain the correct character formulae. To assist us, we employ numeric methods that compute the characters to finite order by explicitly constructing the algebra. The Mathematica code and the details of the algorithm are reported in [45].

The Kac determinant can fail to provide complete information about the characters in the following four ways [46, 47]:

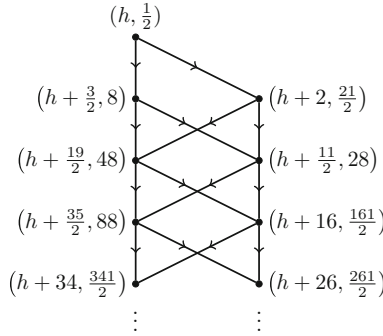


FIGURE 3. The embedding diagram that determines the character $\chi_{h, \frac{1}{2}}$

1. The Kac determinant may propose states that evaluate to zero.
2. The Kac determinant may fail to identify the complete embedding structure among Verma modules by failing to find arrows in the embedding diagram.
3. The Kac determinant will not provide information about multiplicity of states with identical eigenvalues.
4. If there exist null states that are descendants of a unitary highest weight state that are neither highest weight themselves (singular) nor descendants of singular vectors, the Kac determinant will fail to find them. These states may become singular in the quotient module constructed by modding out the original highest weight module by all singular vectors, and are sometimes called subsingular vectors.⁸

The fourth item is frequently assumed not to occur in computations of characters (and in many of the most physically interesting algebras, it does not). Unhappily, these states seem to appear in two of our three massless characters and are generally quite complicated. We present an explicit subsingular vector in Appendix C. Though we believe we have found all such vectors, it would be desirable to have a proof of this and, more generally, a systematic analytical way for finding all subsingular vectors, including those at very high levels inaccessible to numerics.⁹

The first three subtleties may be dealt with systematically by explicitly constructing the offending states. This has been done to great effect in, for example, [46, 47]. We proceed numerically in a similar spirit, supplementing this approach with more standard computations from the Kac determinant as

⁸ This is closely related to the mathematical notion of a primitive vector, a vector v along with a submodule U of the entire module V such that $v \notin U$, but $v + U$ is singular in V/U . A subsingular vector is a primitive vector where U is the space generated by singular vectors in V .

⁹ We are grateful to Daniel Bump and Valentin Buciumas for preliminary discussions on this point.

described above. In the course of these computations, we find the following BPS-like relations are satisfied by the characters:

$$\begin{aligned} \tilde{\chi}_{0,0} + \tilde{\chi}_{\frac{1}{2},\frac{1}{2}} &= q^{-n} \chi_{n,0} \\ \tilde{\chi}_{\frac{1}{2},\frac{1}{2}} + \tilde{\chi}_{1,4} &= q^{-n} \chi_{n+\frac{1}{2},\frac{1}{2}} \end{aligned} \tag{3.13}$$

where $\tilde{\chi}_{h,x}$ denotes the massless characters with internal Ising weight $\frac{x}{8}$. We will derive these equations in Sect. 3.4, but for now let us motivate them heuristically. Very roughly, one can see this at the level of the Kac determinant and embedding diagrams by noticing the following. The massive towers only possess singular vectors generating full Verma modules via $\mathcal{P}(\tau)$, coming from $f_{m,n}$ solutions while the massless towers possess additional d_l solutions which generate “truncated” modules (i.e. modules with partition function $\bar{\mathcal{P}}^l(\tau)$). However, two d_l Verma modules can sum up to a contribution equivalent to a single $f_{m,n}$ Verma module. One can track this elaborate series of splittings in the embedding diagrams of all the characters and convince oneself that, at least up to a certain order in q , these relations are satisfied. The analog of this in the $\mathcal{N} = 2$ case is that the short Verma module comes from BPS states satisfying $G_{-1/2}|\text{state}\rangle = 0$ which then have truncated modules as described earlier. However, as is commonly known, one long multiplet can split into two short multiplets at the $h \rightarrow q/2$ threshold.

Below, we present the embedding diagrams (to finite level) obtained by explicitly constructing null states, and in the case of all singular vectors, determining their parentage (i.e. from which highest weight vector they descended). These appear in Figs. 4, 5, and 6.

Figures 5 and 6 use additional formalism from the massive case. Given a module V , the diagram is separated into two subdiagrams. The subdiagram on the right is the space U of singular vectors as in the massless case. The circle on the left represents a subsingular vector in V , that is, a singular vector in V/U . The tower on the left is the diagram for the Verma module W generated by a highest weight vector with the same quantum numbers as the subsingular vector. The \times s on the left are the singular vectors in W that are in the kernel of the induced homomorphism $W \rightarrow U/V$.

We conjecture closed-form expressions for the character formulae by extrapolating the structure of the diagrams to higher level, imposing satisfaction of (3.13), and when possible, finding singular vector locations of the Kac determinant in closed form. Our confidence in our character formulae is bolstered by the agreement of our massive characters with those in [7].

We conjecture the massless characters $\tilde{\chi}_{h,x}$ to be

$$\begin{aligned} \tilde{\chi}_{0,0} = q^{-\frac{1}{2}} \mathcal{P}(\tau) & \left(1 - \sum_{k=0}^{\infty} q^{\frac{15}{2}k^2 + 4k + \frac{1}{2}} + \frac{q^{\frac{15}{2}k^2 + 2k + \frac{1}{2}}}{1 + q^{\frac{6k+1}{2}}} - \frac{q^{\frac{15}{2}k^2 + 7k + 2}}{1 + q^{\frac{6k+3}{2}}} \right. \\ & \left. - q^{\frac{15}{2}k^2 + 14k + \frac{13}{2}} + \frac{q^{\frac{15}{2}k^2 + 14k + \frac{11}{2}}}{1 + q^{\frac{6k+3}{2}}} - \frac{q^{\frac{15}{2}k^2 + 19k + 11}}{1 + q^{\frac{6k+5}{2}}} \right) \end{aligned}$$

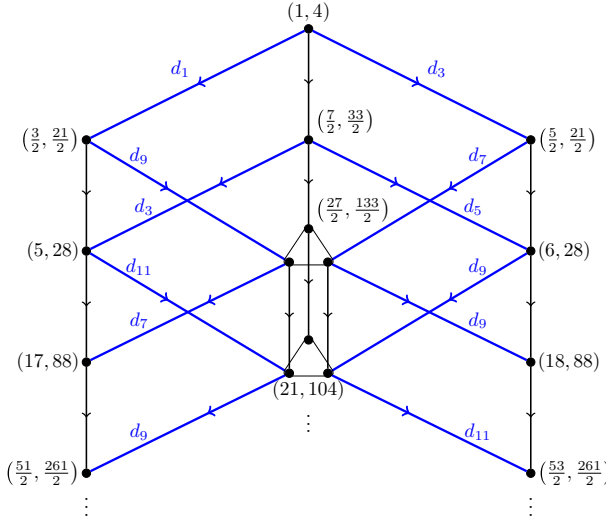


FIGURE 4. The embedding diagram for the massless character $\tilde{\chi}_{1,4}$. Blue arrows represent d-curve descendants. The triangles are written in the notation of Dörrzapf [48]: each triangle and the three dots represent a two-dimensional space of singular vectors with the same quantum numbers. The three dots each represent a singular vector that generates the one-dimensional intersection of the singular space with the descendants of the source of the arrow. These dots are pairwise linearly independent (color figure online)

$$\begin{aligned}
 \tilde{\chi}_{\frac{1}{2}, \frac{1}{2}} &= \mathcal{P}(\tau) \left(1 - \sum_{k=0}^{\infty} q^{\frac{15}{2}k^2 + 7k + \frac{3}{2}} - \frac{q^{\frac{15}{2}k^2 + 16k + 8}}{1 + q^{\frac{6k+5}{2}}} + \frac{q^{\frac{15}{2}k^2 + 5k + \frac{1}{2}}}{1 + q^{\frac{6k+1}{2}}} \right. \\
 &\quad \left. - q^{\frac{15}{2}k^2 + 17k + \frac{19}{2}} + \frac{q^{\frac{15}{2}k^2 + 11k + \frac{7}{2}}}{1 + q^{\frac{6k+3}{2}}} - \frac{q^{\frac{15}{2}k^2 + 10k + 3}}{1 + q^{\frac{6k+3}{2}}} \right) \\
 \tilde{\chi}_{1,4} &= q^{\frac{1}{2}} \mathcal{P}(\tau) \left(1 - \sum_{k=0}^{\infty} q^{\frac{15}{2}k^2 + 10k + \frac{5}{2}} - q^{\frac{15}{2}k^2 + 20k + \frac{25}{2}} + \frac{q^{\frac{15}{2}k^2 + 8k + \frac{1}{2}}}{1 + q^{\frac{6k+1}{2}}} \right. \\
 &\quad \left. - \frac{q^{\frac{15}{2}k^2 + 13k + 4}}{1 + q^{\frac{6k+3}{2}}} + \frac{q^{\frac{15}{2}k^2 + 8k + \frac{3}{2}}}{1 + q^{\frac{6k+3}{2}}} - \frac{q^{\frac{15}{2}k^2 + 13k + 5}}{1 + q^{\frac{6k+5}{2}}} \right). \tag{3.14}
 \end{aligned}$$

3.3. What Kind of Mockery is This?

We would like to understand the modular properties of our conjectural NS sector characters, and we will do so by relating them to standard modular and mock modular forms of a single variable. In Sect. 3.3.1, we will discuss the mock modular properties of the massless characters, and in Sect. 3.3.2, we will

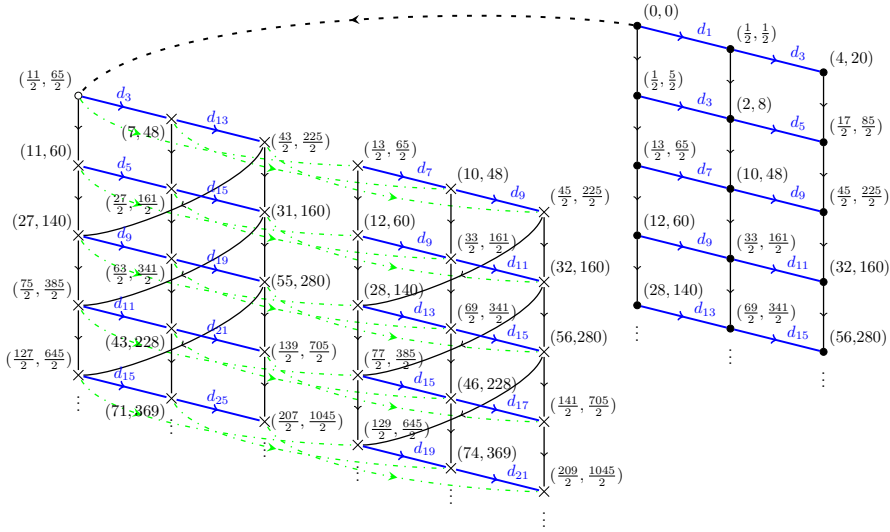


FIGURE 5. The embedding diagram which determines the massless character $\tilde{\chi}_{0,0}$. The *green lines* represent solutions to $g_{1,1} = 0$ and $g_{1,3} = 0$ in the Kac determinant (color figure online)

show how the massive characters transform as a two-component vector-valued modular form under Γ_θ .¹⁰

3.3.1. Massless Characters. First we define some useful functions,

$$\theta_{m,r}(\tau, z) = \sum_{k=r \pmod{2m}} q^{\frac{k^2}{4m}} y^{2mk} \tag{3.15}$$

and

$$f_u^{(m)}(\tau, z) = \sum_{k \in \mathbb{Z}} \frac{q^{mk^2} y^{2mk}}{1 - yq^k e^{-2\pi i u}}. \tag{3.16}$$

Note that $f_u^{(m)}$ has the so-called elliptic transformation property for $2m \in \mathbb{Z}$:

$$f_u^{(m)}(\tau, z) = f_u^{(m)}(\tau, z + 1) = q^m y^{2m} f_u^{(m)}(\tau, z + \tau),$$

and $\theta_{m,r}(\tau, z) = \theta_{m,-r}(\tau, -z)$. We will also make use of a single variable version of this function

$$\theta_{m,r}(\tau) = \theta_{m,r}(\tau, 0), \tag{3.17}$$

which is what we mean whenever z is suppressed, satisfying $\theta_{m,r}(\tau) = \theta_{m,-r}(\tau) = \theta_{m,r+2m}(\tau)$, and

$$\tilde{\theta}_{m,r}(\tau) = \theta_{m,r}(\tau) + \theta_{m,r-m}(\tau) \tag{3.18}$$

¹⁰ Some of the following (mock) modular identities are also discussed in [4], in which the characters are labeled by their Ising weights a , rather than $x = 8a$. We will label quantities by x throughout.

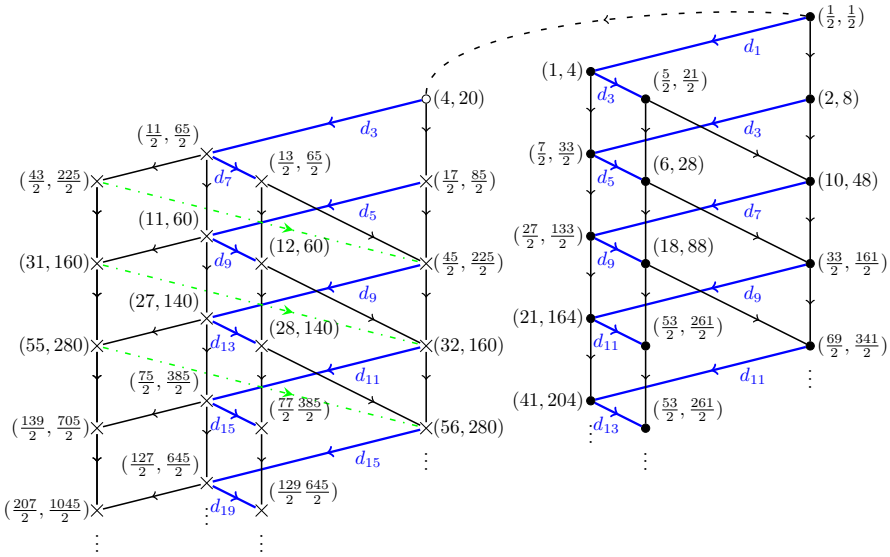


FIGURE 6. The embedding diagram which determines the massless character $\tilde{\chi}_{\frac{1}{2}, \frac{1}{2}}$. The green arrows are g -curves. The rightward arrows in the subsingular diagram are actually g -curves in the Kac determinant, but since they follow a d -curve, they induce d -type descendants in the target singular vector as though they were the d -curve indicated in the diagram (color figure online)

satisfying $\tilde{\theta}_{m,r} = \tilde{\theta}_{m,-r} = \tilde{\theta}_{m,r+m}$. In [49], it was shown that one can define a (non-holomorphic) completion of $f_u^{(m)}(\tau, z)$

$$\hat{f}_u^{(m)}(\tau, \bar{\tau}, z) = f_u^{(m)}(\tau, z) - \frac{1}{2} \sum_{r \bmod 2m} R_{m,r}(\tau, u) \theta_{m,r}(\tau, z) \tag{3.19}$$

which transforms as a Jacobi form of weight 1 and index m . Here, we have defined

$$R_{m,r}(\tau, u) = \sum_{k=r \bmod 2m} \left(\operatorname{sgn} \left(k + \frac{1}{2} \right) - E \left(\left(k + 2m \frac{\Im u}{\Im \tau} \right) \sqrt{\frac{\Im}{\tau} m} \right) \right) q^{-\frac{k^2}{4m}} e^{-2\pi i k u} \tag{3.20}$$

where

$$E(z) = \operatorname{sgn}(z) \left(1 - \int_{z^2}^{\infty} dt t^{-\frac{1}{2}} e^{-\pi t} \right). \tag{3.21}$$

We can rewrite the massless characters in terms of specializations of the function (3.16) to particular values of y and u , as

$$\begin{aligned} \tilde{\chi}_{0,0} = q^{-\frac{3}{8}}\mathcal{P}(\tau) & \left(f_{-\frac{7\tau}{10}+\frac{1}{2}}^{(5)}\left(6\tau, -\frac{\tau}{5}\right) + q^{\frac{13}{2}}f_{-\frac{7\tau}{10}+\frac{1}{2}}^{(5)}\left(6\tau, \frac{14\tau}{5}\right) \right. \\ & \left. - q^{\frac{1}{2}}f_{-\frac{7\tau}{10}+\frac{1}{2}}^{(5)}\left(6\tau, \frac{4\tau}{5}\right) - q^4f_{-\frac{7\tau}{10}+\frac{1}{2}}^{(5)}\left(6\tau, -\frac{11\tau}{5}\right) \right), \end{aligned} \tag{3.22}$$

$$\begin{aligned} \tilde{\chi}_{\frac{1}{2},\frac{1}{2}} = q^{\frac{1}{8}}\mathcal{P}(\tau) & \left(f_{-\frac{\tau}{10}+\frac{1}{2}}^{(5)}\left(6\tau, \frac{2\tau}{5}\right) + q^{\frac{11}{2}}f_{-\frac{\tau}{10}+\frac{1}{2}}^{(5)}\left(6\tau, -\frac{13\tau}{5}\right) \right. \\ & \left. - f_{\frac{\tau}{10}+\frac{1}{2}}^{(5)}\left(6\tau, -\frac{4\tau}{5}\right) - q^{\frac{7}{2}}f_{\frac{\tau}{10}+\frac{1}{2}}^{(5)}\left(6\tau, \frac{11\tau}{5}\right) \right), \end{aligned} \tag{3.23}$$

$$\begin{aligned} \tilde{\chi}_{1,4} = \mathcal{P}(\tau) & \left(q^{\frac{5}{8}}\left(f_{\frac{\tau}{2}+\frac{1}{2}}^{(5)}(6\tau, \tau) - f_{\frac{\tau}{2}+\frac{1}{2}}^{(5)}(6\tau, -\tau)\right) \right. \\ & \left. + q^{\frac{25}{8}}\left(f_{\frac{\tau}{2}+\frac{1}{2}}^{(5)}(6\tau, -2\tau) - f_{\frac{\tau}{2}+\frac{1}{2}}^{(5)}(6\tau, 2\tau)\right) \right), \end{aligned} \tag{3.24}$$

where we recall that

$$\mathcal{P}(\tau) = \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})^2\eta(2\tau)^2}.$$

Note that $\mathcal{P}(\tau)$ transforms as a weight -1 modular form under the subgroup Γ_θ . Each of these characters is composed of holomorphic two-component vector-valued mock modular forms which can be completed into non-holomorphic (two-component, vector-valued) modular forms via Eq. (3.19) with some specialization of u and y . For example, for the case of $\tilde{\chi}_{1,4}$, defining

$$\begin{aligned} \bar{\mu}^{NS} = q^{\frac{5}{8}} & \left(f_{\frac{\tau}{2}+\frac{1}{2}}^{(5)}(6\tau, \tau) - f_{\frac{\tau}{2}+\frac{1}{2}}^{(5)}(6\tau, -\tau) \right) \\ & + q^{\frac{25}{8}}\left(f_{\frac{\tau}{2}+\frac{1}{2}}^{(5)}(6\tau, -2\tau) - f_{\frac{\tau}{2}+\frac{1}{2}}^{(5)}(6\tau, 2\tau)\right), \end{aligned} \tag{3.25}$$

we see that we can define a completion, $\hat{\mu}^{NS}(\tau, \bar{\tau})$,

$$\hat{\mu}^{NS}(\tau, \bar{\tau}) = \bar{\mu}^{NS}(\tau) - \frac{1}{2} \frac{1}{\sqrt{60i}} \int_{-\bar{\tau}}^{i\infty} d\tau' (\tau' + \tau)^{-\frac{1}{2}} \theta_{NS}(\tau) \cdot S(\tau') \tag{3.26}$$

which transforms as a weight 1 modular form under Γ_θ ,¹¹ where we have defined

$$\begin{aligned} \theta_{NS}(\tau) & = \begin{pmatrix} \Theta_{1/2}^{NS}(\tau) \\ \Theta_0^{NS}(\tau) \end{pmatrix}, \\ \Theta_0^{NS}(\tau) & \equiv \tilde{\theta}_{30,2}(\tau) - \tilde{\theta}_{30,8}(\tau) = \sum_{k \in \mathbb{Z}} \epsilon_0^{NS}(k) q^{\frac{k^2}{120}} \end{aligned} \tag{3.27}$$

$$\Theta_{1/2}^{NS}(\tau) \equiv \tilde{\theta}_{30,4}(\tau) - \tilde{\theta}_{30,14}(\tau) = \sum_{k \in \mathbb{Z}} \epsilon_{1/2}^{NS}(k) q^{\frac{k^2}{120}}, \tag{3.28}$$

¹¹ We thank Miranda Cheng for finding an error in an earlier version of this section.

with

$$\begin{aligned} \epsilon_0^{NS}(x) &= \begin{cases} 1 & k = 2, 28 \pmod{60} \\ -1 & k = -8, -22 \pmod{60} \\ 0 & \text{otherwise} \end{cases} \\ \epsilon_{1/2}^{NS}(k) &= \begin{cases} 1 & k = 4, 26 \pmod{60} \\ -1 & k = -14, -16 \pmod{60} \\ 0 & \text{otherwise} \end{cases}, \\ S(\tau) &= \begin{pmatrix} S_1 \\ S_7 \end{pmatrix}, \end{aligned}$$

and finally,

$$S_\alpha(\tau) = \sum_{k \in \mathbb{Z}} k \epsilon_\alpha^R(k) q^{\frac{k^2}{120}}, \quad \alpha = 1, 7 \tag{3.29}$$

satisfying

$$\begin{aligned} \epsilon_1^R(x) &= \begin{cases} 1 & k = 1, 29 \pmod{60} \\ -1 & k = -11, -19 \pmod{60} \\ 0 & \text{otherwise} \end{cases} \\ \epsilon_7^R(k) &= \begin{cases} 1 & k = -7, -23 \pmod{60} \\ -1 & k = 17, 13 \pmod{60} \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Though seemingly unwieldy, these definitions prove useful when deriving transformations under the modular group. Including the factor of $\mathcal{P}(\tau)$, the character $\tilde{\chi}_{1,4} = \mathcal{P}(\tau) \overline{\mu}^{NS}$ as a whole transforms as a weight 0 mock modular form under Γ_θ .

3.3.2. Massive Characters. Now we would like to discuss the modular properties of the massive characters. After some mathematical manipulation, the characters for the massive states in the NS sector given in equations (3.11) and (3.12) can then be written as

$$\chi_{h,0} = q^{h - \frac{49}{120}} \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})^2 \eta(2\tau)^2} \left(\tilde{\theta}_{30,2}(\tau) - \tilde{\theta}_{30,8}(\tau) \right) \equiv q^{h - \frac{49}{120}} \mathcal{P}(\tau) \Theta_0^{NS} \tag{3.30}$$

and

$$\chi_{h,\frac{1}{2}} = q^{h - \frac{61}{120}} \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})^2 \eta(2\tau)^2} \left(\tilde{\theta}_{30,4}(\tau) - \tilde{\theta}_{30,14}(\tau) \right) \equiv q^{h - \frac{61}{120}} \mathcal{P}(\tau) \Theta_{1/2}^{NS} \tag{3.31}$$

We will show that the theta functions appearing in these characters transform as a two-component vector under Γ_θ .

First, consider how $\theta_{m,r}(\tau)$ for a fixed m and r transforms under $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -1/\tau$. We have

$$\theta_{m,r}(\tau + 1) = e \left(\frac{r^2}{4m} \right) \theta_{m,r}(\tau) \tag{3.32}$$

for the T transformation where we use the shorthand $e(x) = e^{2\pi ix}$. For the S transformation, we have

$$\begin{aligned} \theta_{m,r}\left(-\frac{1}{\tau}\right) &= \sum_{k \in \mathbb{Z}} e^{-\frac{2\pi i}{\tau}\left(mk^2 + rk + \frac{r^2}{4m}\right)} \\ &= e\left(-\frac{r^2}{4m\tau}\right) \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} dx e^{2\pi i x k} e^{-\frac{2\pi i}{\tau}(mx^2 + rx)} \\ &= \sqrt{\frac{-i\tau}{2m}} \sum_{k \in \mathbb{Z}} e^{\frac{2\pi i \tau k^2}{4m}} e\left(-\frac{rk}{2m}\right) \end{aligned} \tag{3.33}$$

where in the second line we have used the Poisson transformation formula, and in the third we used

$$\int_{-\infty}^{\infty} dx e^{2\pi i x y} e^{-tx^2} = \frac{e^{-\pi y^2/t}}{\sqrt{t}}. \tag{3.34}$$

Now let us try to write Eq. (3.33) in terms of theta functions of the same m but different r . Note that

$$\sum_{k \in \mathbb{Z}} e^{\frac{2\pi i \tau k^2}{4m}} e\left(-\frac{rk}{2m}\right) = \sum_{r'=-m+1}^m \sum_{k \in \mathbb{Z}} e^{2\pi i \tau\left(mk^2 + r'k + \frac{r'^2}{4m}\right)} e\left(-\frac{rr'}{2m}\right), \tag{3.35}$$

so finally¹²

$$\theta_{m,r}\left(-\frac{1}{\tau}\right) = \sqrt{\frac{-i\tau}{2m}} \sum_{r'=-m+1}^m e\left(-\frac{rr'}{2m}\right) \theta_{m,r'}(\tau). \tag{3.36}$$

Using formulas (3.32) and (3.36), the vector $\underline{\theta}_{NS}(\tau)$ defined in the previous section transforms in the following way under the generators of Γ_θ , T^2 and S :

$$\underline{\theta}_{NS}(\tau + 2) = \rho(T^2) \cdot \underline{\theta}_{NS}(\tau), \quad \underline{\theta}_{NS}\left(-\frac{1}{\tau}\right) = \sqrt{\tau} \rho(S) \cdot \underline{\theta}_{NS}(\tau) \tag{3.37}$$

where

$$\rho(T^2) = \begin{pmatrix} e\left(\frac{32}{120}\right) & 0 \\ 0 & e\left(\frac{8}{120}\right) \end{pmatrix}$$

and

$$\rho(S) = e\left(-\frac{1}{8}\right) \begin{pmatrix} -\sqrt{\frac{2}{5+\sqrt{5}}} & \sqrt{\frac{2}{5-\sqrt{5}}} \\ \sqrt{\frac{2}{5-\sqrt{5}}} & \sqrt{\frac{2}{5+\sqrt{5}}} \end{pmatrix}.$$

¹² This S-transformation formula is often repackaged in the literature by defining $S_{rr'}^{(\theta)} \equiv \frac{1}{\sqrt{2m}} e^{\frac{i\pi rr'}{m}}$ so that $\theta_{m,r}\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} S^{(\theta)} \theta_m(\tau, z)$. Similarly, one often defines $T_{rr'}^\theta \equiv e^{\frac{\pi i r^2}{2m}} \delta_{r,r'}$ so $\theta_m(\tau + 1, z) = T^{(\theta)} \theta_m(\tau, z)$.

3.4. Characters in the Ramond sector

In the Ramond sector, the massive states have two components (see [6]) and are labeled by 8 times the Ising dimensions $(x_1, x_2) = (8a_1, 8a_2)$, and total dimension h . The isomorphism between NS and Ramond sector states is given explicitly in Table 1, where states are labeled by $|8a, h\rangle$ and the isomorphism identifies the states in each row. Recall that this “spectral flow” isomorphism is generated by the internal Ising sector. Specifically, the Ramond ground state with its total weight equal to its Ising weight, $\frac{1}{2}$, is isomorphic to the Ising energy operator $[\epsilon]$ and fusion of this operator with other states generates a flow to the NS sector. Crucially, fusion of this operator with itself gives the identity. For convenience, we reproduce the Ising fusion rules in terms of the dimension $\frac{1}{16}$ operator σ , the identity 0 , and the dimension $\frac{1}{2}$ operator ϵ :

$$\begin{aligned} [\epsilon][\epsilon] &= [0] \\ [\epsilon][\sigma] &= [\sigma] \\ [\sigma][\sigma] &= [0] + [\epsilon]. \end{aligned} \tag{3.38}$$

Following the same steps as described above for the derivation of the characters in the NS sector, we use the Ramond sector Kac determinant derived in [6] to conjecture the following characters in the Ramond sector:

$$\begin{aligned} \chi_{(0, \frac{1}{2}), h + \frac{1}{2}}^R &= 2q^{h - \frac{1}{120}} \frac{\eta(2\tau)^2}{\eta(\tau)^4} \left(\tilde{\theta}_{30,1}(\tau) - \tilde{\theta}_{30,11}(\tau) \right) \\ &\equiv 2q^{h - \frac{1}{120}} \frac{\eta(2\tau)^2}{\eta(\tau)^4} \Theta_{(0,1/2)}^R \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} \chi_{(\frac{1}{2}, 4), h + \frac{1}{2}}^R &= 2q^{h - \frac{49}{120}} \frac{\eta(2\tau)^2}{\eta(\tau)^4} \left(\tilde{\theta}_{30,7}(\tau) - \tilde{\theta}_{30,17}(\tau) \right) \\ &\equiv 2q^{h - \frac{49}{120}} \frac{\eta(2\tau)^2}{\eta(\tau)^4} \Theta_{(1/2,4)}^R. \end{aligned} \tag{3.40}$$

TABLE 1. The unitary irreducible highest weight representations of the Spin(7) algebra

| NS | | R |
|--|--------|---|
| $ 0, 0\rangle$ | \iff | $ 4, \frac{1}{2}\rangle$ |
| $ \frac{1}{2}, \frac{1}{2}\rangle$ | \iff | $ \frac{1}{2}, \frac{1}{2}\rangle$ |
| $ 4, 1\rangle$ | \iff | $ 0, \frac{1}{2}\rangle$ |
| $ 0, x\rangle$ | \iff | $ (\frac{1}{2}, 4), \frac{1}{2} + x\rangle$ |
| $ \frac{1}{2}, \frac{1}{2} + x\rangle$ | \iff | $ (0, \frac{1}{2}), \frac{1}{2} + x\rangle$ |

States in the same row are isomorphic. Recall that we are labeling the states by their eigenvalues $|x_0, h\rangle = |8a_0, h\rangle$

We can rewrite $\Theta^R(\tau)$ for convenience as

$$\Theta_{(0,1/2)}^R(\tau) = \sum_{k \in \mathbb{Z}} \epsilon_1^R(k) q^{\frac{k^2}{120}} \tag{3.41}$$

$$\Theta_{(1/2,4)}^R(\tau) = \sum_{k \in \mathbb{Z}} \epsilon_7^R(k) q^{\frac{k^2}{120}}, \tag{3.42}$$

with $\epsilon_{1,7}^R(k)$ as in 3.30.

Spectral flow relates $\chi_{(0, \frac{1}{2}), h}^R \iff \chi_{\frac{1}{2}, h}^{NS}$ and $\chi_{(\frac{1}{2}, 4), h}^R \iff \chi_{0, h}^{NS}$. Let us pause here to derive this correspondence, as it will be a prerequisite to confirming (3.13). In the massive sector, as in the massless sector, we must identify the primary state that does not change the external (non-Ising) dimension of a state with which it fuses, and which produces the identity upon fusion with itself. This is perhaps clearer when writing out the external dimensions of each component, which then sum with each Ising dimension to equal the total h . The unique candidate is then $|(\frac{1}{2}, 4), \frac{1}{2} + x\rangle$ which is necessarily mapped to $|0, x\rangle$ in the NS sector. Note that the vector of external dimensions for this state is $(\frac{7}{16} + x, x)$, so the second component of the state has the properties we require. Similarly, fusion of this state with the other massive state, $|(0, \frac{1}{2}), \frac{1}{2} + x\rangle$, produces $|\frac{1}{2}, \frac{1}{2} + x\rangle$ in the NS sector.

The important massless character will be $|0, \frac{1}{2}\rangle$ which we denote as $\tilde{\chi}_0^R$. This is given by

$$\begin{aligned} \tilde{\chi}_0^R = 2 \frac{\eta(2\tau)^2}{\eta(\tau)^4} & \left(f_{\frac{\tau}{2} + \frac{1}{2}}^{(5)} \left(6\tau, \frac{\tau}{2} \right) - f_{\frac{\tau}{2} + \frac{1}{2}}^{(5)} \left(6\tau, -\frac{\tau}{2} \right) \right. \\ & \left. + q^5 f_{\frac{\tau}{2} + \frac{1}{2}}^{(5)} \left(6\tau, -\frac{5\tau}{2} \right) - q^5 f_{\frac{\tau}{2} + \frac{1}{2}}^{(5)} \left(6\tau, \frac{5\tau}{2} \right) \right). \end{aligned} \tag{3.43}$$

Spectral flow relates this character to $\tilde{\chi}_{1,4}$. The Ramond sector BPS relations

$$\tilde{\chi}_0^R + \tilde{\chi}_{\frac{1}{2}}^R = q^{-n} \chi_{(0, \frac{1}{2}), \frac{1}{2} + n}^R \tag{3.44}$$

and

$$\tilde{\chi}_{\frac{1}{2}}^R + \tilde{\chi}_4^R = q^{-n} \chi_{(\frac{1}{2}, 4), \frac{1}{2} + n}^R. \tag{3.45}$$

can give us the remaining two massless characters. In this sector, the BPS relations are essentially forced upon us as we approach the threshold weight, since we demand that the unitary massive state decomposes into (unitary) representations of the internal Ising subalgebra. If we then apply the spectral flow operator (i.e. Ising fusion rules) to the resulting Ramond ground states on the left-hand side of (3.44) and write the result at the level of characters, we precisely reproduce the left-hand side of (3.13) in the NS sector. Thus, since we have demonstrated that Ising fusion maps both the left- and right-hand sides of (3.13) to those of (3.44), and moreover that the relations (3.44) are correct, we may feel reassured that employing (3.13) and (3.44) in deriving our characters is justified.

One can do a similar analysis to that of the previous section to understand the modular properties of our conjectural Ramond sector characters. In the case of $\tilde{\chi}_0^R$, defining

$$\begin{aligned} \bar{\mu}^R &= f_{\frac{\tau}{2}+\frac{1}{2}}^{(5)}\left(6\tau, \frac{\tau}{2}\right) - f_{\frac{\tau}{2}+\frac{1}{2}}^{(5)}\left(6\tau, -\frac{\tau}{2}\right) \\ &\quad + q^5 f_{\frac{\tau}{2}+\frac{1}{2}}^{(5)}\left(6\tau, -\frac{5\tau}{2}\right) - q^5 f_{\frac{\tau}{2}+\frac{1}{2}}^{(5)}\left(6\tau, \frac{5\tau}{2}\right), \end{aligned} \tag{3.46}$$

this can be completed to a non-holomorphic weight 1 modular form which transforms under $\Gamma_0(2)$,

$$\hat{\mu}^R(\tau, \bar{\tau}) = \bar{\mu}^R(\tau) - \frac{1}{2} \frac{1}{\sqrt{60i}} \int_{-\bar{\tau}}^{i\infty} d\tau' (\tau' + \tau)^{-\frac{1}{2}} \underline{\theta}_R(\tau) \cdot \underline{S}(\tau') \tag{3.47}$$

where now

$$\underline{\theta}_R(\tau) = \begin{pmatrix} \Theta_{(0,1/2)}^R \\ \Theta_{(1/2,4)}^R \end{pmatrix},$$

and $\underline{S}(\tau)$ is the same as in the previous section. The factor $\frac{\eta(2\tau)^2}{\eta(\tau)^4}$ is a weight -1 modular form under $\Gamma_0(2)$, and thus $\tilde{\chi}_0^R$ as a whole transforms as a weight 0 mock modular form under $\Gamma_0(2)$. $Z_{R,+}$ is a trace with periodic boundary conditions on the fermions on the space-like cycle of the torus and antiperiodic boundary conditions on the time-like cycle. It is straightforward to check that $\Gamma_0(2)$ is the congruence subgroup of $SL(2, \mathbb{Z})$ that preserves this spin structure. In particular, the spin structure is invariant under $\tau \mapsto \tau + 1$ but not $\tau \mapsto -\frac{1}{\tau}$.

It is easy to see that under T the Θ^R transform as

$$\Theta_{(0,1/2)}^R(\tau + 1) = e\left(\frac{1}{120}\right) \Theta_{(0,1/2)}^R(\tau), \tag{3.48}$$

$$\Theta_{(1/2,4)}^R(\tau + 1) = e\left(\frac{49}{120}\right) \Theta_{(1/2,4)}^R(\tau). \tag{3.49}$$

The second generator of $\Gamma_0(2)$ can be written as $ST^{-2}ST^{-1}$.

We can write the two generators of $\Gamma_0(2)$ as matrices which act on this vector in the following way (stripping off the overall factor of τ by convention):

$$\rho(T) = \begin{pmatrix} e\left(\frac{1}{120}\right) & 0 \\ 0 & e\left(\frac{49}{120}\right) \end{pmatrix} \tag{3.50}$$

and

$$\rho(\tilde{S}) = i \begin{pmatrix} e\left(-\frac{15}{120}\right) \sqrt{\frac{2}{(5-\sqrt{5})}} & e\left(-\frac{39}{120}\right) \sqrt{\frac{2}{(5+\sqrt{5})}} \\ e\left(\frac{9}{120}\right) \sqrt{\frac{2}{(5+\sqrt{5})}} & e\left(\frac{45}{120}\right) \sqrt{\frac{2}{(5-\sqrt{5})}} \end{pmatrix} \tag{3.51}$$

where $\tilde{S} = ST^{-2}ST^{-1}$.

4. Decomposition of the Elliptic Genus into Characters

In this section, we explicitly decompose the elliptic genera into our Spin(7) characters. We stress that although our closed-form expressions for the characters are conjectural, the decompositions are performed to finite order in q , where our numerics explicitly confirm the character formulae.

Recall our BPS equations, (3.13). We will use these relations to write the decomposition of the elliptic genus in a suggestive way, completely analogous to the decompositions presented in [20].

Recall the NS sector elliptic genus for a Spin(7) manifold, X is given by the trace:

$$Z_{NS,+}(\tau) = \text{Tr}_{NS,R}(-1)^{F_R} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}. \tag{4.1}$$

As discussed in Sect. 2, up to a constant term determined by the Euler character of X (which vanishes if $\chi(X) = 24$), this has the q -expansion

$$Z_{NS,+} = \frac{1}{\sqrt{q}} + 276\sqrt{q} + 2048q + 11202q^{3/2} + \dots \tag{4.2}$$

In what follows we simply set the constant term to zero. We discuss the possible meanings of the decompositions in this section in the conclusion; and in the companion [4] we describe some precise realizations of super-modules with vanishing constant which realize the various symmetries we discuss below.

First, we note that these coefficients decompose nicely into dimensions of irreducible representations of Co_1 . For example, 276 is itself a dimension of an irrep of Co_1 . Additionally, $2048 = \mathbf{1} + \mathbf{276} + \mathbf{1771}$, $11202 = \mathbf{1} + \mathbf{276} + \mathbf{299} + \mathbf{1771} + \mathbf{8855}$, etc. In fact, the decomposition into $\mathcal{N} = 1$ super-Virasoro characters would suggest a relation to the Conway module described in [31] and recently reviewed in [20].

Instead, we would like to decompose into Spin(7) characters. The most general decomposition into Ramond sector characters of the Spin(7) algebra has the form

$$Z_{R,+} = A_0 \tilde{\chi}_0^R + A_{\frac{1}{2}} \tilde{\chi}_{\frac{1}{2}}^R + A_4 \tilde{\chi}_4^R + \sum_{n=1}^{\infty} B_n \chi_{(0, \frac{1}{2}), \frac{1}{2}+n}^R + \sum_{n=1}^{\infty} C_n \chi_{(\frac{1}{2}, 4), \frac{1}{2}+n}^R \tag{4.3}$$

where now $A_0, A_{\frac{1}{2}}, A_4, B_n$, and C_n are all (positive integer) constants. The NS, + elliptic genus has a similar expansion in terms of the NS characters, with coefficients $a_0, a_{1/2}, a_4, b_n, c_n$. The isomorphism between Ramond and NS sectors gives us the following relations between expansion coefficients: $a_0 = A_4$, $a_{\frac{1}{2}} = A_{\frac{1}{2}}$, $a_4 = A_0$, $b_n = C_n$, and $c_n = B_n$.

Now we have enough information to fix all of the constants in the decomposition. Using that $a_0 = A_4 = 1$ and $a_{\frac{1}{2}} = A_{\frac{1}{2}} = 0$, at $O(1)$ in the decomposition of $Z_{R,+}$ we get that $A_0 = a_4 = 23$. With this coefficient fixed, the entire decomposition is now fixed.

Once these coefficients are fixed, we get the following for the first few massive coefficients,

$$b_1 = 253, \quad b_2 = 7359, \quad b_3 = 95128, \dots \tag{4.4}$$

and

$$c_1 = 1771, \quad c_2 = 35650, \quad c_3 = 374141, \dots \tag{4.5}$$

Using the above relations, we can repackage the decomposition in the following form, which is entirely analogous to the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ decompositions studied in [20]. First define

$$\begin{aligned} f_1(\tau) &= q^{-\frac{1}{120}}(-1 + c_1q + c_2q^2 + c_3q^3 + \dots) \\ &= q^{-\frac{1}{120}}(-1 + 1771q + 35650q^2 + 374141q^3 + \dots) \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} f_7(\tau) &= q^{-\frac{49}{120}}(1 + b_1q + b_2q^2 + b_3q^3 + \dots), \\ &= q^{-\frac{49}{120}}(1 + 253q + 7359q^2 + 95128q^3 + \dots). \end{aligned} \tag{4.7}$$

Then, we can rewrite

$$Z_{R,+} = 2 \frac{\eta(2\tau)^2}{\eta(\tau)^4} \left(24\bar{\mu}^R + f_1(\tau)\Theta_{(0,1/2)}^R(\tau) + f_7(\tau)\Theta_{(1/2,4)}^R(\tau) \right) \tag{4.8}$$

or, in the NS sector,

$$Z_{NS,+} = \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})^2 \eta(2\tau)^2} \left(24\bar{\mu}^{NS} + f_1(\tau)\Theta_{1/2}^{NS}(\tau) + f_7(\tau)\Theta_0^{NS}(\tau) \right), \tag{4.9}$$

where $f_1(\tau)$ and $f_7(\tau)$ are as defined above, and $\bar{\mu}^{NS}$ and $\bar{\mu}^R$ are as defined in the previous section. Using the transformation properties of these functions derived in the previous section, we can predict how the functions $f_1(\tau)$ and $f_7(\tau)$ transform under $SL(2, \mathbb{Z})$.

In particular, the modular group $SL(2, \mathbb{Z})$ is generated by the matrices $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The multiplier system ρ of \underline{f} on the entirety of $SL(2, \mathbb{Z})$ is generated by

$$\rho(T) = \begin{pmatrix} e\left(-\frac{1}{120}\right) & 0 \\ 0 & e\left(-\frac{49}{120}\right) \end{pmatrix} \tag{4.10}$$

$$\rho(S) = e\left(\frac{1}{8}\right) \begin{pmatrix} -\sqrt{\frac{2}{5+\sqrt{5}}} & \sqrt{\frac{2}{5-\sqrt{5}}} \\ \sqrt{\frac{2}{5-\sqrt{5}}} & \sqrt{\frac{2}{5+\sqrt{5}}} \end{pmatrix}. \tag{4.11}$$

Mock modularity and Rademacher summability have been proposed as a hallmark of moonshine [50]. Using standard techniques for computing vector-valued Rademacher sums[51], it can be seen that the weight- $\frac{1}{2}$ Rademacher sum generated by the polar part $(q^{-\frac{1}{120}}, q^{-\frac{49}{120}})$ and above multiplier system over $SL(2, \mathbb{Z})$ gives \underline{f} exactly. The Rademacher summability of the \underline{f} serves as a weak independent verification of the decomposition.

It will strike the attentive reader in the post-EOT era that the integer coefficients appearing in the q -series (4.6) and (4.7) are related in a simple way

to irreducible representations of sporadic simple groups. For instance, in f_7 , one can decompose into dimensions of irreps of M_{24} as

$$\begin{aligned} 253 &= \mathbf{253}, \quad 7359 = \mathbf{23} \oplus \mathbf{252} \oplus \mathbf{770} \oplus \overline{\mathbf{770}} \oplus \mathbf{5544} \\ 95128 &= \mathbf{253} \oplus \mathbf{990} \oplus \overline{\mathbf{990}} \oplus 2 \times \mathbf{1265} \oplus \mathbf{1771} \oplus \mathbf{2024} \oplus \mathbf{2277} \oplus \mathbf{3312} \\ &\quad \oplus 2 \times \mathbf{3520} \oplus 2 \times \mathbf{5313} \oplus \mathbf{5544} \oplus \mathbf{5796} \oplus 5 \times \mathbf{10395}, \dots \end{aligned}$$

while in f_1 , one sees

$$1771 = \mathbf{1771}, \quad 35650 = \mathbf{252} \oplus \mathbf{253} \oplus \mathbf{1771} \oplus 2 \times \mathbf{3520} \oplus \mathbf{5544} \oplus 2 \times \mathbf{10395}, \dots$$

The coefficients grow quickly enough that one needs some confidence in the actual existence of an M_{24} symmetry, and a systematic means of determining compositions, to go further. In a companion paper [4], we describe how in a precise chiral CFT with the same chiral algebra and giving rise to the same (chiral) partition function, we can realize an M_{24} symmetry with explicit decompositions (and checks of twining for all M_{24} conjugacy classes) at all orders. This verifies (in a specific example) that the numerology above hides deeper meaning in some specific instance.

Similar decompositions into representations of the sporadic groups Co_2 and Co_3 exist and will be elaborated upon in [4].

5. Conclusions

In this paper, we studied the elliptic genera of manifolds of exceptional holonomy $Spin(7)$. These constitute one of the few classes of possible supersymmetric compactification manifolds for superstrings.

To look for hidden connections between the geometry of these spaces and number theory and representation theory, following the example of [8], it is natural to decompose the elliptic genus into characters of the relevant chiral algebra ($SW(3/2, 2)$ at $c = 12$), and try to find representation-theoretic interpretations for the coefficients. While we find suggestive results in Sect. 4, we note that a precise relationship of any particular $Spin(7)$ manifold with the group M_{24} (or the groups Co_2 and Co_3 , which also appear to play a special role in the q -expansions of the functions $f_1(\tau)$, $f_7(\tau)$) is not indicated by our results. For a given explicit manifold X , for generic $\chi(X)$ there would be an unwanted constant term in the elliptic genus. For those spaces which have $\chi(X) = 24$ and vanishing constant term, one would expect the space to at most admit discrete symmetries whose twinings could agree with M_{24} for a handful of conjugacy classes (of the same orders). Even in the case of the K3 conformal field theory, where (obviously) a unique candidate topology exists realizing the suggestive elliptic genus, a precise statement of Mathieu moonshine has been elusive.

However, there is a simple explicit chiral conformal field theory with the same chiral algebra we studied here, which manifestly realizes the M_{24} moonshine hinted at in Sect. 4. That is, the subject of the companion paper [4]. One could hope that this chiral CFT is a sort of ‘‘Platonic model,’’ indicating the largest symmetries that might be realized at (perhaps non-geometric) points in the moduli space of $Spin(7)$ compactifications. Broadly similar remarks

apply to the constructions based on the $\mathcal{N} = 2, 4$ superconformal algebras in the recent paper [20], and compactifications on Calabi–Yau fourfold or 8-dimensional hyperKähler spaces.

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Appendix A. Jacobi Theta Functions and the Dedekind Eta Function

The Jacobi theta functions are a collection of weight $\frac{1}{2}$, index 1 Jacobi forms. Write $q = \exp(2\pi i\tau)$ and $y = \exp(2\pi iz)$. Then, the Jacobi theta functions can be written as

$$\theta_1(\tau, z) = -iq^{\frac{1}{8}}y^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^n)(1 - y^{-1}q^{n-1}) \tag{A.1}$$

$$= i \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{(n-\frac{1}{2})^2}{2}} y^{n-\frac{1}{2}}$$

$$\theta_2(\tau, z) = q^{\frac{1}{8}}y^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^n)(1 + y^{-1}q^{n-1}) \tag{A.2}$$

$$= \sum_{n=-\infty}^{\infty} q^{\frac{(n-\frac{1}{2})^2}{2}} y^{n-\frac{1}{2}}$$

$$\theta_3(\tau, z) = \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^{n-\frac{1}{2}})(1 + y^{-1}q^{n-\frac{1}{2}}) \tag{A.3}$$

$$= \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} y^n$$

$$\theta_4(\tau, z) = \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^{n-\frac{1}{2}})(1 - y^{-1}q^{n-\frac{1}{2}}) \tag{A.4}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n^2}{2}} y^n. \tag{A.5}$$

The Dedekind Eta function is a modular function of weight $\frac{1}{2}$ which is defined to be

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = q^{\frac{1}{24}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2-n}{2}}. \tag{A.6}$$

Appendix B. $\mathcal{SW}(3/2, 2)$ Algebra

Here, we set our notation for the algebra $\mathcal{SW}(3/2, 2)$ at $c = 12$ and write down the commutation relations. The $\mathcal{SW}(3/2, 2)$ algebra has a basis that consists of operators L_n, X_n , for $n \in \mathbf{Z}$ and M_m, G_m for $m \in \mathbf{Z} + \frac{1}{2}$. We define L and X to be bosonic operators, and M and G to be fermionic.

The normal-ordered product is given by

$$:XY:{}_n = \sum_{k \leq -\Delta_X} X_k Y_{n-k} + (-1)^F \sum_{k > -\Delta_X} Y_{n-k} X_k. \tag{B.1}$$

where $(-1)^F$ is -1 if both X and Y are fermionic and 1 otherwise. The Δ s are the conformal weights of the operators. Recall that $\Delta_L = \Delta_X = 2$, $\Delta_G = \frac{3}{2}$ and $\Delta_M = \frac{5}{2}$.

We are interested in constructing unitary irreducible highest weight representations, which are generated by the action of the algebra on a highest weight state $|h, x\rangle$, where h and x are the eigenvalues of L_0 and X_0 respectively.

We can construct a Hermitian conjugate on this space. The operators in the algebra are also given Hermitian conjugates: $L_n^\dagger = L_{-n}$, $X_n^\dagger = X_{-n}$, and $G_n^\dagger = G_{-n}$. The M operator possesses a non-standard conjugate: $M_n^\dagger = -M_{-n} - \frac{1}{2}G_{-n}$.

The commutation relations of the algebra are given below¹³:

$$\begin{aligned} [L_n, L_m] &= (n - m)L_{n+m} + (n^3 - n)\delta_{n+m} \\ [X_n, X_m] &= \frac{8}{3}(n^3 - n)\delta_{n+m} + 8(n - m)X_{n+m} \\ [L_n, X_m] &= \frac{1}{3}(n^3 - n)\delta_{n+m} + (n - m)X_{n+m} \\ [G_n, X_m] &= \frac{1}{2} \left(n + \frac{1}{2} \right) G_{n+m} + M_{n+m} \\ [X_n, M_m] &= \left(\frac{15}{4} (n + 1) \left(m + \frac{3}{2} \right) \right. \\ &\quad \left. - \frac{5}{4} \left(n + m + \frac{3}{2} \right) \left(n + m + \frac{5}{2} \right) \right) G_{n+m} \\ &\quad - \left(-8(n + 1) + \frac{11}{2} \left(n + m + \frac{5}{2} \right) \right) M_{n+m} \\ &\quad - 6:GX:_{n+m} \\ [L_n, G_m] &= \left(\frac{1}{2}n - m \right) G_{n+m} \\ [L_n, M_m] &= \frac{1}{4}n(n + 1)G_{n+m} + \left(\frac{3}{2}n - m \right) M_{n+m} \\ \{G_n, M_m\} &= \frac{2}{3} \left(n^2 - \frac{1}{4} \right) \left(n - \frac{3}{2} \right) \delta_{n+m} \\ &\quad - \left(n + \frac{1}{2} \right) L_{n+m} + (3n - m)X_{n+m} \end{aligned}$$

¹³ We correct some small typos on p. 35 of [5].

$$\begin{aligned}
\{M_n, M_m\} &= -\frac{8}{3} \left(n^2 - \frac{9}{4}\right) \left(n^2 - \frac{1}{4}\right) \delta_{n+m} + \left(\frac{15}{2} \left(m + \frac{3}{2}\right) \left(n + \frac{3}{2}\right)\right. \\
&\quad \left. - \frac{5}{2} (n+m+2)(n+m+3)\right) L_{n+m} \\
&\quad + \left(16 \left(m + \frac{3}{2}\right) \left(n + \frac{3}{2}\right)\right. \\
&\quad \left. - \frac{5}{2} (n+m+2)(n+m+3)\right) X_{n+m} \\
&\quad - 12:LX:_{n+m} + 6:GM:_{n+m} \\
\{G_n, G_m\} &= (4n^2 - 1)\delta_{n+m} + 2L_{n+m}. \tag{B.2}
\end{aligned}$$

Appendix C. Subsingular Vectors

For the sake of completeness and to assist in replicating our results, we provide a construction of a subsingular vector from the module generated by $|\frac{1}{2}, \frac{1}{2}\rangle$. The Verma module possess a subsingular vector with L_0 eigenvalue of 4 and X_0 eigenvalue of 20. Up to addition of an element from the subspace generated by singular vectors, the subsingular vector is at

$$\begin{aligned}
&(G_{-7/2} + \frac{7}{3}G_{-5/2}X_{-1} - \frac{2}{3}M_{-5/2}X_{-1} + 2X_{-3}G_{-1/2} + \frac{10}{9}X_{-2}G_{-3/2} \\
&\quad + \frac{8}{9}X_{-2}M_{-3/2} + \frac{1}{3}G_{-3/2}(X_{-1})^2 + \frac{5}{3}X_{-2}X_{-1}G_{-1/2} \\
&\quad + \frac{1}{6}X_{-2}X_{-1}M_{-1/2} + \frac{23}{21}L_{-1}(X_{-1})^2M_{-1/2} - \frac{11}{192}(X_{-1})^3G_{-1/2} \\
&\quad - \frac{33}{224}(X_{-1})^3M_{-1/2})|\frac{1}{2}, \frac{1}{2}\rangle.
\end{aligned}$$

The $|h = 0, x = 0\rangle$ Verma module possesses a subsingular vector with L_0 eigenvalue of 11/2 and X_0 eigenvalue of 65/2, but the subsingular vector is too complex to usefully write down.

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