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Differential Equations with Infinitely Many Derivatives and the Borel Transform

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Abstract. Differential equations with infinitely many derivatives, sometimes also referred to as "nonlocal" differential equations, appear frequently in branches of modern physics such as string theory, gravitation and cosmology. We properly interpret and solve linear equations in this class with a special focus on a solution method based on the Borel transform. This method is a far-reaching generalization of previous studies of nonlocal equations via Laplace and Fourier transforms, see for instance (Barnaby and Kamran, J High Energy Phys 02:40, 2008; Górka et al., Class Quantum Gravity 29:065017, 2012; Górka et al., Ann Henri Poincaré 14:947–966, 2013). We reconsider "generalized" initial value problems within the present approach and we disprove various conjectures found in modern physics literature. We illustrate various analytic phenomena that can occur with concrete examples, and we also treat efficient implementations of the theory.

1. Introduction

The mathematical study of differential equations with infinitely many derivatives began over a century ago, and it has reappeared intermittently since then, see e.g. [9–11,28]. For instance in [9], the author considers the homogeneous equation

$$Lu = \sum_{n=0}^{\infty} a_n \, D^n u(x,t) = 0, \quad a_n \in \mathbb{C}, \tag{1.1}$$

where D indicates differentiation with respect to x and the coefficients a_n satisfy some adequate conditions. Much more recently Teixeira [31], has used an abstract theory of ordinary differential equations on Banach spaces to study evolution problems of the form $u_t(x,t) = \sum_{n=0}^{\infty} a_n(t,x)\, D^n u(x,t) + \phi(x,t)$,

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thereby generalizing [9], and two of the present authors in collaboration with Górka, see [15,18], have considered nonlinear equations of the form $f(\Delta)u = U(\cdot, u)$ assuming that f satisfies an "ellipticity" condition.

Now, ordinary differential equations with infinitely many derivatives—which may be thought of as ordinary differential equations of infinite order—have been extensively considered in contemporary string theory, quantum gravity and cosmology, see for instance [1–5,7,8,14,26,33,34] and also [35]. For example, a rather simple looking equation appearing in the study of cosmological models (the Friedmann model, see [1, Section 9]) is

$$e^{\tau(-\partial_t^2-3H\partial_t)}u = m^2\, u, \tag{1.2}$$

in which τ, H and m are constant numbers. Also, special classes of ordinary differential equations of infinite order appear in connection with theta functions [29] and complex analysis [22,23]. As observed in [31], these works, in addition to the classical literature [9–11,28], naturally lead us to the problem of understanding ordinary nonlocal equations in spaces of analytic functions.

In actual fact, there are several techniques for defining the left-hand side of equations such as (1.2) and for finding solutions, most of these based on the Fourier and Laplace transforms (see for instance [4,5,16,17]). In this paper, we rely further on complex function theory and we generalize these classical approaches using the Borel transform, which we introduce in Definition 2.2 below. We will see that, in particular, this point of view allows us to prove that nonlocal operators such as $e^{\tau(-\partial_t^2-3H\partial_t)}$ (and more generally, nonlocal operators determined by entire functions) can be understood via power series expansions, see Theorem 3.2.

We now summarize the contents of this work. We consider the linear equation

$$\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) = g(x), \quad x \in \mathbb{R}, \tag{1.3}$$

where the symbol $\phi(z) = \sum_{k=0}^{\infty} a_k z^k$ is an entire function, f is unknown, and g is given. More precisely we assume that f and g belong to the class Exp, that is, they are restrictions to the real line $\mathbb{R}$ of entire functions of exponential type. As we shall see, there are several benefits of working with the class Exp. First of all Exp is a rather large space, containing e.g. all band-limited functions in $L^2(\mathbb{R})$ as well as all exponential functions. We note that when ϕ is a polynomial, i.e. when (1.3) reduces to an ordinary differential equation, then the span of exponential functions of the form $x^m \mathsf{e}^{\lambda x}$ [where $\phi(\lambda) = 0$ and m is less than the multiplicity of λ], equals the set of homogenous solutions to (1.3). A pleasant fact is that these functions are in Exp, which is not the case for $L^2(\mathbb{R})$ or more general classes of Sobolev type spaces. Relying on the Borel transform, we give a simple explicit formula for the solution of (1.3), given $g \in$ Exp, see Eq. (4.4) below, and moreover, we show that the homogenous solutions are obtained by a direct generalization of the situation for ODE's. Summing up, if we know the Borel transform of g and the zeroes of ϕ, all solutions to (1.3) in Exp can be represented explicitly. This is done in Sects. 2–5.

The approach just outlined seems to us much more satisfactory than the one developed in our previous works [16,17]. In those papers also appear explicit representations for solutions of equations such as (1.3), but the representations therein are valid only for functions ϕ and g satisfying some technical assumptions needed for the inversion of the Laplace transform. These restrictions are absent here: as explained above, we simply ask that the right-hand side of (1.3) belongs to Exp.

Yet another good feature of working with Exp is that there can be no ambiguity as to how $\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x)$ should be interpreted: we will define an appropriate functional calculus in Sect. 3 and we shall see that (Theorem 3.2)

$$\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f = \lim_{K\to\infty} \sum_{k=0}^{K} a_k \frac{\mathrm{d}^k f}{\mathrm{d}x^k}, \quad f \in \mathrm{Exp}, \tag{1.4}$$

with respect to uniform convergence on compact sets. We remark that a general convergence theorem of this type does not hold in other natural spaces, e.g., the space C_c^∞ of infinitely differentiable functions with compact support, for we shall prove that the right-hand side diverges in L^1_{loc} for all $f \in C_c^\infty$ and $\phi(z) = \mathsf{e}^{z^2}$ (Example 4). This observation rules out convergence in any weighted L^p-space, and with the same method we can also show that it does not converge in any classical Sobolev space. This is remarkable since $\mathsf{e}^{(\frac{\mathrm{d}}{\mathrm{d}x})^2}$ does have a physically relevant interpretation in e.g. $L^2(\mathbb{R})$ (convolution with the heat kernel) which arises when considering the heat equation on the line. Since a subset of Exp is dense in $L^2(\mathbb{R})$ (we can take, for instance, the set of Hermite functions) this physically relevant interpretation of $\mathsf{e}^{(\frac{\mathrm{d}}{\mathrm{d}x})^2}$ on $L^2(\mathbb{R})$ is easily derived from the general theory of this paper. We refer to Example 10 for more details.

After the general theory has been developed in Sects. 2–5, we consider particular cases in Sects. 6 and 7. In particular, we show that linear equations of convolution type can be reformulated as (1.3), and hence solved via the Borel transform. Section 8 is devoted to showing the power of the present approach in comparison with the other methods mentioned initially. As we shall see, the various possible interpretations of the operator $\phi(\frac{\mathrm{d}}{\mathrm{d}x})$ coincide on common domains of definition.

Section 9 sheds further light on initial value problems, see [16,17]. A recurring issue in the physics literature is how to appropriately impose initial value conditions that uniquely specify the solution, in analogy with the theory of ODE's [2,4,26]. Here, the fact that we are working with real analytic functions will be of great benefit, for it allows us to use deep results from the theory of entire functions of finite order, to (dis)prove some natural conjectures [2,26]. In particular, we shall show that if the exponent of convergence of the zeroes of ϕ is > 1, then any function in $L^2(I)$, in which I is a finite interval, is arbitrarily close in $L^2(I)$ to a solution of the equation $\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) = 0$. See Proposition 9.1 for definitions and a precise statement. The paper ends with Sect. 10 treating the issue of how to efficiently implement stable solvers based on the methods presented in this paper.

2. Functions of Exponential Type

We provide a review of key properties of functions of exponential type. This is an extensively studied subject, and many more details can be found e.g. in [6,25] and in Polya's paper [27].

Definition 2.1. An entire function F is said to be of exponential type if there exists a $C, \tau > 0$ such that

$$|F(z)| \leq C\mathsf{e}^{\tau|z|}.$$

The number τ is called an exponential bound of F, and the infimum of all possible τ's is called the exponential type of F. The set of all entire functions of exponential type will be denoted by $\mathrm{Exp}(\mathbb{C})$.

Examples of functions of exponential type include $\sin(z)$, $\cos(z)$ as well as all polynomials, but not e.g. e^{z^2}. We will denote by $\mathrm{Exp}(\mathbb{R})$ the space of all functions f such that there exists an $F \in \mathrm{Exp}(\mathbb{C})$ with $f = F|_{\mathbb{R}}$ (F restricted to $\mathbb{R}$). Note that $\mathrm{Exp}(\mathbb{R})$ contains e.g. all band-limited functions in $L^2(\mathbb{R})$, i.e. all functions whose Fourier transform has compact support. This is not hard to deduce by applying the Fourier inversion formula, but it also follows from Proposition 2.4 and formula (2.10) below. When there is no risk of confusion, we will make no distinction between f on $\mathbb{R}$ and its extension to $\mathbb{C}$, and simply write $f \in \mathrm{Exp}$.

The class Exp will be the domain of the nonlocal differential operators we consider in this paper, while we will pose no restriction on the class of symbols ϕ appearing in our equations, except that they should be entire functions. We can give a concrete integral representation of the functions in Exp via the Borel transform which we define as follows:

Definition 2.2. Suppose that $f(z) = \sum_{k=0}^{\infty} b_k z^k$ belongs to $\mathrm{Exp}(\mathbb{C})$. The Borel transform of f is defined as

$$\mathcal{B}(f)(z) = \sum_{k=0}^{\infty} \frac{k!\, b_k}{z^{k+1}}. \tag{2.1}$$

If τ is the exponential type of f, it is a classical fact (see e.g. [6, Theorem 5.3.1]) that the series in (2.1) converges for all $|z| > \tau$, whereas this is false for all smaller discs. Even more, if we let S be the *conjugate diagram of* $\mathcal{B}(f)$, i.e. the smallest convex set contained in $\{z\colon |z| \leq \tau\}$ such that $\mathcal{B}(f)$ extends by analyticity outside S, then a classical theorem due to Polya tells us that S can be characterized in terms of the growth of f along rays emanating from $z = 0$. We refer to [6, Chapter 5] for the details, but we remark that this is closely connected with the following alternative definition of $\mathcal{B}(f)$ via the Laplace transform $\mathcal{L}$; given z with $|z| > \tau$ we have

$$\mathcal{B}(f)(re^{i\theta}) = \mathsf{e}^{-i\theta}\int_0^\infty f(t\mathsf{e}^{i\theta})\mathsf{e}^{-rt}\,\mathrm{d}t = \mathsf{e}^{-i\theta}\mathcal{L}\left(f(\cdot\,\mathsf{e}^{i\theta})\right). \tag{2.2}$$

We note that both (2.1) and (2.2) give concrete formulas for calculating the Borel transform of an explicitly given function $f \in \mathrm{Exp}(\mathbb{C})$. If one only knows f on $\mathbb{R}$, neither of the above formulas can be evaluated, but we will see in Sect. 10 how this can be circumvented. The following formula, which says how to recover f from $\mathcal{B}(f)$, is crucial for our theory. Given any $R > \tau$ we have

$$f(z) = \int_{|\zeta|=R} \mathsf{e}^{z\zeta}\mathcal{B}(f)(\zeta)\,\frac{\mathrm{d}\zeta}{2\pi\mathsf{i}}. \tag{2.3}$$

(See [6, Theorem 5.3.5]).

Example 1. Let $f(x) = \mathsf{e}^x$. Then $f(z) = \sum_{k=0}^{\infty}\frac{z^k}{k!}$ and $f(z)$ is obviously of exponential type $\tau = 1$. We have,

$$\mathcal{B}(\mathsf{e}^x) = \sum_{k=0}^{\infty}\frac{1}{\zeta^{k+1}} = \frac{1}{\zeta}\frac{1}{1-1/\zeta} = \frac{1}{\zeta-1},$$

which converges for $|\zeta| > 1$, as predicted by the theory. In this case, we note that the convex set S given by Polya's theorem mentioned above is significantly smaller than $\{\zeta\colon |\zeta| \le 1\}$; obviously $S = \{1\}$. Now, we can invert $\mathcal{B}$: for $R > 1$, (2.3) implies that

$$\mathsf{e}^x = \int_0^{2\pi} \mathsf{e}^{xR\mathsf{e}^{\mathsf{i}\theta}}\,\frac{R\mathsf{e}^{\mathsf{i}\theta}}{R\mathsf{e}^{\mathsf{i}\theta}-1}\,\frac{\mathrm{d}\theta}{2\pi} \tag{2.4}$$

which can be verified directly by expanding $\mathsf{e}^{xR\mathsf{e}^{\mathsf{i}\theta}}$ and $(1-\mathsf{e}^{-\mathsf{i}\theta}/R)^{-1}$ in power series, integrating and collecting non-zero terms.

To work with expressions of the type (2.3), we now introduce an extension of the Fourier–Laplace transform [19,20]. It will turn out convenient (but it is not strictly necessary) to also work with distributions. To define distributions on $\mathbb{C}$ we identify it with $\mathbb{R}^2$ via $\zeta = \xi + \mathsf{i}\eta \leftrightarrow (\xi, \eta)$. We refer to [19, Sects. 2.1, 2.3], for basic results on distributions (of compact support). Recall in particular that any distribution d of compact support K automatically has finite order N, and that then one can find $C > 0$ such that

$$|\langle \mathrm{d}, \tau\rangle| \le C \sum_{n_1+n_2\le N} \|\partial_\xi^{n_1}\partial_\eta^{n_2}\tau(\zeta)\|_{L^\infty(K)} \tag{2.5}$$

for all test functions τ. Moreover, [19, Section 2.3] implies that the following definition is coherent.

Definition 2.3. If d is a distribution with compact support in $\mathbb{C}$, we set

$$\mathcal{P}(\mathrm{d})(z) = \langle \mathsf{e}^{z\zeta}, \mathrm{d}\rangle, \quad z \in \mathbb{C} \tag{2.6}$$

where ζ represents the independent variable on which d acts. If $\mathrm{d} = \mu$ is a measure, Eq. (2.6) clearly reduces to

$$\mathcal{P}(\mu)(z) = \int_{\mathbb{C}} \mathsf{e}^{z\zeta}\,\mathrm{d}\mu(\zeta).$$

The following proposition shows that the image of $\mathcal{P}$ equals Exp.

Proposition 2.4. *If* d *is a distribution with compact support* K *contained in* $\{\zeta\colon |\zeta|<R\}$*, then* $\mathcal{P}(\mathrm{d})\in \mathrm{Exp}$ *and*

$$|\mathcal{P}(\mathrm{d})(z)| \le C\mathsf{e}^{R|z|} \tag{2.7}$$

for some $C>0$*. Conversely, given any* $f\in\mathrm{Exp}$ *we can pick a measure* μ_f *such that* $f=\mathcal{P}(\mu_f)$*. The measure* μ_f *is not unique, on the contrary, if* f *has exponential type* τ *and* $R>\tau$ *is arbitrary,* μ_f *can be chosen to have support on* $\{\zeta\colon |\zeta|=R\}$*.*

Proof. If $\mathrm{d}=\mu$ is a complex measure (of finite variation) the first part follows as

$$|\mathcal{P}(\mu)(z)| \le \int_{|\zeta|\le R} \mathsf{e}^{|z|R}\,\mathrm{d}|\mu|(\zeta) \le \mathsf{e}^{|z|R}\|\mu\|.$$

The corresponding result for distributions is a bit more complicated. We take $R_1<R$ such that $K\subset\{\zeta\colon |\zeta|<R_1\}$, and we let $\chi\in C^\infty(\mathbb{C})$ be a bump function which is identically 1 on the support of d and identically 0 outside the ball $\{\zeta\colon |\zeta|<R_1\}$. By (2.6) it then follows that

$$\begin{aligned}|\mathcal{P}(\mathrm{d})(z)| &= \left|\left\langle \mathrm{d}(\zeta),\mathsf{e}^{z\zeta}\right\rangle\right| = \left|\left\langle \mathrm{d}(\zeta),\chi(\zeta)\mathsf{e}^{z\zeta}\right\rangle\right| \\ &\le C\sum_{n_1+n_2\le N} \|\partial_\xi^{n_1}\partial_\eta^{n_2}\chi(\zeta)\mathsf{e}^{z\zeta}\|_{L^\infty(K)},\end{aligned}$$

where C is a constant, N is the order of d and, as stated in (2.5), the inequality follows from the definition of distributions, see also (2.3.1) in [19]. We easily conclude that

$$|\mathcal{P}(\mathrm{d})(z)| \le C_1(1+|z|^N)\mathsf{e}^{R_1|z|}$$

for some $C_1>0$. Since $R_1<R$, the inequality (2.7) follows. This shows that $Im\mathcal{P}\subset\mathrm{Exp}$. The reverse inclusion follows by realizing that the integral representation (2.3) can be rewritten as $f=\mathcal{P}(\mu_{f,R})$ with the measure $\mu_{f,R}$ defined as

$$\mu_{f,R}(E) = \int_{\{\theta\colon\, R\mathsf{e}^{\mathsf{i}\theta}\in E\}} \mathcal{B}(f)(R\mathsf{e}^{\mathsf{i}\theta})\; R\mathsf{e}^{\mathsf{i}\theta}\,\frac{\mathrm{d}\theta}{2\pi}, \quad E\subset\mathbb{C}. \tag{2.8}$$

□

Note that for any distribution d with compact support, Eq. (2.3) implies that the function $\mathcal{P}(\mathrm{d})$ can be equivalently represented by a measure supported on a circle. Despite this, we will see in Sect. 6 that it can be much easier to work directly with distributions.

Example 2. Continuing Example 1, we have

$$\mathsf{e}^x = \mathcal{P}(\delta_1)(x)$$

where δ_1 is the Dirac measure with support at $\{1\}$. On the other hand, by (2.4) and (2.8) we have that $\mathsf{e}^x=\mathcal{P}(\mu_{\mathsf{e}^x})$ with

$$\mu_{\mathsf{e}^x}(E) = \int_{\{\theta\colon\, R\mathsf{e}^{\mathsf{i}\theta}\in E\}} \frac{R\mathsf{e}^{\mathsf{i}\theta}}{R\mathsf{e}^{\mathsf{i}\theta}-1}\,\frac{\mathrm{d}\theta}{2\pi}, \quad E\subset\mathbb{C}$$

and $R>1$.

Example 3. The function $f(x) = \cos x$ is a function of exponential type 1 since

$$\cos x = \frac{1}{2}\left(\mathsf{e}^{\mathsf{i}x} + \mathsf{e}^{-\mathsf{i}x}\right) = \mathcal{P}\left(\frac{1}{2}(\delta_{\mathsf{i}} + \delta_{-\mathsf{i}})\right),$$

in which $\delta_{\pm\mathsf{i}}$ are Dirac measures with support at $\pm\mathsf{i}$. The Borel transform is easily calculated to $\mathcal{B}(\cos x) = \frac{1}{2}\left(\frac{1}{\zeta-\mathsf{i}} + \frac{1}{\zeta+\mathsf{i}}\right) = \frac{\zeta}{\zeta^2+1}$, and hence for $R > 1$ we have

$$\cos x = \int_0^{2\pi} \mathsf{e}^{xR\mathsf{e}^{\mathsf{i}\theta}} \frac{R\mathsf{e}^{\mathsf{i}\theta}}{R\mathsf{e}^{2\mathsf{i}\theta}+1} \frac{\mathrm{d}\theta}{2\pi}.$$

More generally, letting Pol denote the set of all polynomials, we have that any function of the form

$$f(x) = \sum_{\text{finite}} p_k(x)\mathsf{e}^{\zeta_k x}, \quad p_k \in \text{Pol},\ \zeta_k \in \mathbb{C}$$

is in Exp and can therefore be expressed as (2.3). However, we remark that a more direct representation is given by letting d be the distribution $\sum p_k(-\partial_\xi)\delta_{\zeta_k}$, (where the independent variable is $\zeta = \xi + \mathsf{i}\eta$), and using (2.6). Indeed,

$$\begin{aligned}\mathcal{P}(\mathrm{d})(x) &= \left\langle \mathsf{e}^{x(\xi+\mathsf{i}\eta)}, \sum p_k(-\partial_\xi)\delta_{\zeta_k}\right\rangle \\ &= \sum\langle p_k(\partial_\xi)\mathsf{e}^{x(\xi+\mathsf{i}\eta)}, \delta_{\zeta_k}\rangle = \sum\langle p_k(x)\mathsf{e}^{x(\xi+\mathsf{i}\eta)}, \delta_{\zeta_k}\rangle = f(x)\end{aligned}$$

by standard calculation rules for distributions.

The transform $\mathcal{P}$ is an extension of both the Fourier and Laplace transforms, since the former appears if we restrict $\mathcal{P}$ to measures μ supported on $\mathsf{i}\mathbb{R}$ and the latter if we restrict $\mathcal{P}$ to measures supported on $\mathbb{R}^-$. More concretely, if $u\colon \mathbb{R}^+ \to \mathbb{C}$ has compact support and we define the distribution $u(-\xi)\delta_0(\eta)$ via

$$\langle u(-\xi)\delta_0(\eta), \tau(\xi,\eta)\rangle = \int u(-\xi)\tau(\xi,0)\ \mathrm{d}\xi,$$

we get

$$\begin{aligned}\mathcal{P}\big(u(-\xi)\delta_0(\eta)\big)(z) &= \langle u(-\xi)\delta_0(\eta), \mathsf{e}^{(\xi+\mathsf{i}\eta)z}\rangle \\ &= \int \mathsf{e}^{\xi z}u(-\xi)\ \mathrm{d}\xi = \int \mathsf{e}^{-\xi z}u(\xi)\ \mathrm{d}\xi = \mathcal{L}(u)(z), \quad (2.9)\end{aligned}$$

where $\mathcal{L}$ denotes the Laplace transform. Clearly $u(-\xi)\delta_0(\eta)$ can be identified with the measure μ given by

$$\mu(E) = \int_{E\cap\mathbb{R}} u(-\xi)\ \mathrm{d}\xi, \quad E \subset \mathbb{C}.$$

Similarly, let $\mathcal{F}$ be the (unitary) Fourier transform. Given $u\colon \mathbb{R} \to \mathbb{C}$ with compact support we have

$$\mathcal{P}\big(\delta_0(\xi)u(-\eta)\big)(z) = \int \mathsf{e}^{\mathsf{i}\eta z}u(-\eta)\ \mathrm{d}\eta = \sqrt{2\pi}\mathcal{F}(u)(z) \qquad (2.10)$$

and a short argument shows that $\delta_0(\xi)u(-\eta)$ can be identified with the measure

$$\mu(E) = \int_{\mathrm{i}E\,\cap\,\mathbb{R}} u(\eta)\;\mathrm{d}\eta, \quad E \subset \mathbb{C}.$$

One could thus consider $\mathcal{P}$ as a mere extension of the Fourier transform and call it something like "the extended Fourier–Laplace transform" (see [19,20]). However, we avoid this since both the Fourier and Laplace transforms are intimately connected with their respective domains of definition, with associated inverses, etc. The transform $\mathcal{P}$ is different from the Fourier–Laplace transform in that its argument d is allowed to have support on $\mathbb{C}$ as opposed to $\mathbb{R}$, which significantly alters most properties.

3. Defining the Functional Calculus of $\frac{\mathrm{d}}{\mathrm{d}x}$

In this section, we return to the differential equation (1.3) and show that the left-hand side is well defined for any $f \in \mathrm{Exp}$ in accordance with the limit (1.4) (with respect to the topology of uniform convergence on compacts). We recall that if d is a distribution and $\phi \in C^\infty(\mathbb{C})$, we can define a new distribution $\phi\mathrm{d}$ via $\langle \phi\mathrm{d}, \tau\rangle = \langle \mathrm{d}, \phi\tau\rangle$, where τ represents a test function. If $\mathrm{d} = \mu$ is a measure, $\phi\mu$ is also a measure and $\int_{\mathbb{C}} \tau\;\mathrm{d}(\phi\mu) = \int_{\mathbb{C}} \tau\phi\;\mathrm{d}\mu$.

Partially motivated by [16,17], we make the following definition.

Definition 3.1. Given an entire function $\phi(z) = \sum_{k=0}^{\infty} a_k z^k$, $f \in \mathrm{Exp}$ and a distribution d_f such that $f = \mathcal{P}(\mathrm{d}_f)$, we define the operator $\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f$ as

$$\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f = \mathcal{P}(\phi\,\mathrm{d}_f). \tag{3.1}$$

The main theorem of this section states that this definition makes sense *and* that the interpretation of the operator $\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$ as a power series is correct if we take as its domain the space Exp.

Theorem 3.2. *Given an entire function* $\phi(z) = \sum_{k=0}^{\infty} a_k z^k$, $f \in \mathrm{Exp}$ *and a distribution* d_f *such that* $f = \mathcal{P}(\mathrm{d}_f)$, *we have*

$$\lim_{K\to\infty} \sum_{k=0}^{K} a_k \frac{\mathrm{d}^k f}{\mathrm{d}x^k} = \mathcal{P}(\phi\,\mathrm{d}_f) \tag{3.2}$$

uniformly on compacts. In particular the limit (3.2) *exists and equals* $\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f$.

Proof. If $\mathrm{d}_f = \mu_f$ is a measure with compact support, then the identity

$$\frac{\mathrm{d}}{\mathrm{d}z} f = \frac{\mathrm{d}}{\mathrm{d}z}\mathcal{P}(\mathrm{d}_f) = \mathcal{P}(\zeta \mathrm{d}_f) \tag{3.3}$$

follows by the dominated convergence theorem. The formula is true also in the general case, and the proof builds on the same estimate we used in Proposition 2.4. We omit the details. By repeated use of (3.3) we get

$$\sum_{k=0}^{K} a_k \frac{\mathrm{d}^k f}{\mathrm{d}z^k} - \mathcal{P}(\phi\mu_f) = \mathcal{P}\left(\left(\sum_{k=0}^{K} a_k \zeta^k - \phi(\zeta)\right) \mathrm{d}_f\right)$$
$$= \left\langle \mathrm{d}_f, \left(\sum_{k=0}^{K} a_k \zeta^k - \phi(\zeta)\right) \mathsf{e}^{z\zeta} \right\rangle.$$

Since $\sum_{k=0}^{K} a_k \zeta^k$ converges uniformly on compacts to $\phi(\zeta)$, we have that

$$\lim_{K\to\infty} \sum_{k=0}^{K} a_k \frac{\mathrm{d}^k f}{\mathrm{d}z^k}(z) = \lim_{K\to\infty} \sum_{k=0}^{K} a_k \frac{\mathrm{d}^k}{\mathrm{d}z^k} \mathcal{P}(\mu_f)(z)$$
$$= \lim_{K\to\infty} \sum_{k=0}^{K} a_k \frac{\mathrm{d}^k}{\mathrm{d}z^k} \int_{\mathbb{C}} \mathsf{e}^{z\zeta} \mathrm{d}\mu_f(\zeta) = \lim_{K\to\infty} \int_{\mathbb{C}} \sum_{k=0}^{K} a_k \zeta^k \mathsf{e}^{z\zeta} \mathrm{d}\mu_f(\zeta)$$
$$= \int_{\mathbb{C}} \phi(\zeta) \mathsf{e}^{z\zeta} \mathrm{d}\mu_f(\zeta) = \mathcal{P}(\phi\mu_f)(z).$$

It is also easy to see that the limit is uniform on compact sets. If d_f is a distribution, the result follows by a similar modification as in Proposition 2.4. □

For instance, an elementary use of Definition 3.1 gives

$$\mathsf{e}^{y\partial_x} u(x) = \int \mathsf{e}^{x\zeta} \mathsf{e}^{y\zeta} \mathrm{d}\mu_u(\zeta) = \int \mathsf{e}^{(x+y)\zeta} \mathrm{d}\mu_u(\zeta) = u(x+y). \tag{3.4}$$

To further stress the importance of Exp as domain of our operators $\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$, we present an example which gives a particular ϕ such that the convergence of (1.4) is not compatible with functions of compact support. We denote by $L^1_{\mathrm{loc}}(\mathbb{R})$ the set of all functions whose restriction to any compact interval I is integrable.

Example 4. Claim: if $f \in C^\infty(\mathbb{R})$ has compact support, the sequence $f_K = \sum_{k=0}^{K} \frac{\partial_x^{2k} f}{k!}$ does not converge in $L^1_{\mathrm{loc}}(\mathbb{R})$ (hence neither does it converge in any weighted $L^p(\mathbb{R})$-space, by Hölder's inequality). Thus, in $L^1_{\mathrm{loc}}(\mathbb{R})$ we cannot define $\mathsf{e}^{(\partial_x)^2}$ using power series.

To prove this claim, we first recall the Paley–Wiener theorem (see e.g. [24, p. 132] or [19]), which says that for any $u \in L^2(\mathbb{R})$ with compact support the number

$$b_u = \limsup_{y\to\infty} \log |\mathcal{F}(u)(\mathrm{i}y)|/y \tag{3.5}$$

is finite and equals the right endpoint of the support of u. Similarly, the left endpoint a_u of $\mathsf{supp}\ u$ is given by $a_u = -\limsup_{y\to\infty} \log |\mathcal{F}(u)(-\mathrm{i}y)|/y$. Now, suppose the limit exists in $L^1_{\mathrm{loc}}(\mathbb{R})$ and denote it by g. Clearly $\mathsf{supp}\ f_K \subset \mathsf{supp}\ f = [a_f, b_f]$ and so

$$a_f \leq \mathsf{supp}\ g \leq b_f \tag{3.6}$$

and $(f_K)_{K=1}^\infty$ converges to g in $L^1(\mathbb{R})$. Thus

$$\lim_{K\to\infty} \mathcal{F}(f_K)(z) = \lim_{K\to\infty} \int_a^b f_K(t) \mathsf{e}^{-\mathrm{i}zt} \frac{\mathrm{d}t}{\sqrt{2\pi}} = \int_a^b g(t) \mathsf{e}^{-\mathrm{i}zt} \frac{\mathrm{d}t}{\sqrt{2\pi}} = \mathcal{F}(g)(z).$$

On the other hand, by partial integration we obtain

$$\begin{aligned}\mathcal{F}(f_K)(z) &= \int_a^b \sum_{k=0}^{K} \frac{\partial_t^{2k} f(t)}{k!}\, \mathsf{e}^{-\mathsf{i}zt} \frac{\mathrm{d}t}{\sqrt{2\pi}} \\ &= \sum_{k=0}^{K} \frac{(-z)^{2k}}{k!} \int_a^b f(t) \mathsf{e}^{-\mathsf{i}zt} \frac{\mathrm{d}t}{\sqrt{2\pi}} = \sum_{k=0}^{K} \frac{(-z)^{2k}}{k!} \mathcal{F}(f)(z),\end{aligned}$$

and therefore

$$\mathcal{F}(g)(z) = \mathsf{e}^{-z^2} \mathcal{F}(f)(z).$$

Combining (3.5) and (3.6) we have

$$\begin{aligned}b_f \geq b_g &= \limsup_{y\to\infty} \frac{\log|\mathcal{F}(g)(\mathsf{i}y)|}{y} = \limsup_{y\to\infty} \frac{\log|\mathsf{e}^{y^2}\mathcal{F}(f)(\mathsf{i}y)|}{y} \\ &= \limsup_{y\to\infty}\ y + \frac{\log|\mathcal{F}(f)(\mathsf{i}y)|}{y} = \infty + b_f = \infty.\end{aligned}$$

This contradiction proves the claim.

In the above example, we worked with the concrete function $\phi(z) = \mathsf{e}^{z^2}$, which is an entire function of order 2. Based on deep results on entire functions, (see [25]), it is possible to show that the same contradiction arises for any ϕ of order > 1 with regular growth. On the other hand, it is well known that $\mathsf{e}^{-\partial_x^2} f(x)$ appears in the solution of the heat equation on the line. In this case, $\mathsf{e}^{-\partial_x^2} f(x)$ has the physically correct interpretation $\frac{1}{\sqrt{4\pi}} f * \mathsf{e}^{-x^2/4}$, which at first seems to be a contradiction. This is not the case, as we further discuss in Sects. 6–8. The connection with the heat equation is worked out in Example 10. In the coming two sections, we develop the general theory for solving (1.3).

4. Particular Solutions

Using the machinery developed in the previous section, the existence of a solution to (1.3) is immediate. Given an entire function ϕ we let $\mathcal{Z}(\phi)$ denote the set of zeroes of ϕ and $|\mathcal{Z}(\phi)|$ the set of their respective modulii.

Theorem 4.1. *Let ϕ be an entire function. The equation*

$$\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) = g(x), \quad g \in \mathrm{Exp}, \tag{4.1}$$

always has at least one solution. More precisely, if $g = \mathcal{P}(\mathrm{d}_g)$ and $\operatorname{supp}\, \mathrm{d}_g \cap \mathcal{Z}(\phi) = \emptyset$*, and then a solution is given by*

$$f = \mathcal{P}\left(\frac{\mathrm{d}_g}{\phi}\right). \tag{4.2}$$

Proof. Because of Theorem 3.2 (see Eqs. (3.1), (3.2)), Eq. (4.1) is equivalent to

$$\mathcal{P}(\phi\, \mathrm{d}_f) = g, \tag{4.3}$$

in which the "unknown" is the distribution d_f. We set $\mathrm{d}_f = (1/\phi)\mathrm{d}_g$. That d_f solves (4.3) is immediate (since multiplication of functions and distributions is associative, see [19]). Equation (3.1) then tells us that f given by (4.2) solves (4.1). □

Note that by Proposition 2.4 we can always pick $\mathrm{d}_g = \mu_g$ to be a measure supported on a circle of radius $R \notin |\mathcal{Z}(\phi)|$, so that the condition $\mathsf{supp}\ \mathrm{d}_g \cap \mathcal{Z}(\phi) = \emptyset$ is met. In particular, for every $R \notin |\mathcal{Z}(\phi)|$ greater than an exponential bound τ of g (recall Definition 2.1), Eq. (4.2) can be explicitly evaluated as

$$f(x) = \int_{|\zeta|=R} \mathsf{e}^{x\zeta} \frac{\mathcal{B}(g)(\zeta)}{\phi(\zeta)} \frac{\mathrm{d}\zeta}{2\pi\mathsf{i}}, \tag{4.4}$$

that is, we take μ_g as in (2.8). By Cauchy's theorem and the remarks following Definition 2.2, this argument can be taken one step further which we state as a separate corollary.

Corollary 4.2. *Let γ be any simply connected closed curve avoiding $\mathcal{Z}(\phi)$ with the conjugate diagram S in its interior. Then*

$$f(x) = \int_{\gamma} \mathsf{e}^{x\zeta} \frac{\mathcal{B}(g)(\zeta)}{\phi(\zeta)} \frac{\mathrm{d}\zeta}{2\pi\mathsf{i}}$$

solves (4.1).

Sometimes it is, however, better to work with distributions than with the above integrals, as the example below illustrates.

Example 5. In the previous section we noted that

$$\sum p_k(x)\mathsf{e}^{\zeta_k x} = \mathcal{P}\left(\sum p_k(-\partial_\xi)\delta_{\zeta_k}\right)$$

for finite sums. If the right-hand side g has this form and $\mathrm{d}_g = \sum p_k(-\partial_\xi)\delta_{\zeta_k}$, then (4.2) evaluates to

$$\begin{aligned} f(x) = \mathcal{P}\left(\frac{\sum p_k(-\partial_\xi)\delta_{\zeta_k}}{\phi}\right) &= \left\langle \mathsf{e}^{x(\xi+\mathsf{i}\eta)}, \frac{\sum p_k(-\partial_\xi)\delta_{\zeta_k}}{\phi} \right\rangle \\ &= \sum \left\langle p_k(\partial_\xi)\frac{\mathsf{e}^{x(\xi+\mathsf{i}\eta)}}{\phi(\xi+\mathsf{i}\eta)}, \delta_{\zeta_k} \right\rangle, \end{aligned}$$

assuming that $\mathcal{Z}(\phi)\cap\{\zeta_k\} = \emptyset$. If this is not the case, we are forced to work with the more cumbersome expression (4.4), or resort to some perturbation analysis. Let us assume that $\mathcal{Z}(\phi) \cap \{\zeta_k\} = \emptyset$. For concrete values of p_k and ϕ, the last expression above can clearly be evaluated explicitly. We see immediately that the solution f will be of the form $f(x) = \sum q_k(x)\mathsf{e}^{\zeta_k x}$ as well, for some $q_k \in \mathrm{Pol}$.

Theorem 4.1 certainly includes (using appropriate limiting processes) classical results on equations with infinitely many derivatives appearing in [11], see also [4,16,17] and Sects. 5 and 8 below. In particular, let us consider Eq. (4.1) and the measure

$$\mathrm{d}\mu_g(\zeta) = \chi_{Re(\zeta)=c}(\zeta)\mathcal{L}(g)(\zeta)\mathrm{d}\zeta.$$

Then,

$$\mathcal{P}(\mu_g) = \int_{\mathbb{C}} \mathsf{e}^{z\zeta} \mathrm{d}\mu_g(\zeta) = \int_{Re(\zeta)=c} \mathsf{e}^{z\zeta} \mathcal{L}(g)(\zeta) \mathrm{d}\zeta = g(z),$$

and we can solve (4.1) using (4.2). We obtain

$$f(z) = \mathcal{P}((1/\phi)\mu_g)(z) = \int_{Re(\zeta)=c} \mathsf{e}^{z\zeta} \frac{\mathcal{L}(g)(\zeta)}{\phi(\zeta)} \mathrm{d}\zeta = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(g)(\zeta)}{\phi(\zeta)}\right),$$

which is Equation (17) in [17]. We note that, as anticipated in Sect. 1, this formula for f demands that ϕ and g satisfy conditions so that $\mathcal{L}(g)/\phi$ is in the image of the Laplace transform. This requirement is superfluous if we use the transform $\mathcal{P}$ and compute f via (4.2) for a suitable measure d_g.

We end this section with the comment that the reader may also find in [11] an early use of the Borel transform in the context of equations with infinitely many derivatives.

5. Homogeneous Solutions

An immediate question in connection with Theorem 4.1 is whether different d_g's give rise to the same solution f. This is not the case, which is to be expected since if ϕ is a polynomial (so that (4.1) is simply a constant coefficient ordinary differential equation) then the homogenous solutions are linear combinations of exponential functions $\mathsf{e}^{\lambda x}$ where $\phi(\lambda) = 0$. This points out another pleasant fact when working with Exp, as opposed to traditional spaces like $L^2(\mathbb{R})$ or Sobolev spaces, namely that homogenous solutions are inside of the space.

Theorem 5.1. *Let $f \in \mathrm{Exp}$ be a solution to $\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) = 0$ of exponential type τ. Let $\{\zeta_k\}_{k=0}^{\infty} = \mathcal{Z}(\phi)$ be an enumeration of $\mathcal{Z}(\phi)$ and let m_k denote the multiplicity of ζ_k. Then there are polynomials p_k of degree $< m_k$ such that*

$$f(x) = \sum_{|\zeta_k| \le \tau} p_k(x) \mathsf{e}^{\zeta_k x}. \tag{5.1}$$

Note that the sum in (5.1) is necessarily finite, since zeroes of analytic functions are discrete. Also note that if the zeroes of ϕ have multiplicity 1 then (5.1) reduces to

$$f(x) = \sum_{|\zeta_k| \le \tau} c_k \mathsf{e}^{\zeta_k x}, \quad c_k \in \mathbb{C}.$$

The proof of the theorem is given at the end of this section. We first note the following immediate corollaries to Theorems 4.1 and 5.1.

Corollary 5.2. $\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$ *is invertible in Exp if and only if the symbol ϕ has no zeroes. In this case,*

$$\left(\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\right)^{-1} = \frac{1}{\phi}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

By Proposition 2.4 we also easily get

Corollary 5.3. *Let $f_1 = \mathcal{P}(\mu_1)$ and $f_2 = \mathcal{P}(\mu_2)$ be solutions to the non-homogeneous Eq.* (4.1)*, where the measures* (*or distributions*) μ_1, μ_2 *have compact support. Set* $R = \sup\{|z|\colon z \in \mathsf{supp}\ \mu_1 \cup \mu_2\}$*. Then $f_1 - f_2$ is given by* (5.1) *with $\tau = R$.*

To prove Theorem 5.1, we first need a basic lemma. Fix $R > 0$ and set $B_R = \{\zeta\colon |\zeta| \leq R\}$. Let $A(B_R)$ denote the set of continuous functions that are analytic inside B_R, endowed with the supremum norm. Also let $E_z \in A(B_R)$ denote the function $E_z(\zeta) = \mathsf{e}^{\zeta z}$, $\zeta \in B_R$. Given an entire function ϕ we set

$$\mathcal{M}_{\phi,R} = \mathsf{cl}\left(\mathsf{Span}\ \{E_z\, \phi\colon z \in \mathbb{C}\}\right),$$

where cl denotes the closure in $A(B_R)$.

Lemma 5.4. *Let ϕ be an entire function and let $R \notin |\mathcal{Z}(\phi)|$. Let $\{\zeta_k\}_{k=1}^K$ be an enumeration of $\mathcal{Z}(\phi) \cap B_R$ and let m_k denote their corresponding multiplicities. Then*

$$\mathcal{M}_{\phi,R} = \{\psi \in A(B_R)\colon \psi \textit{ is zero at } \zeta_k \textit{ with multiplicity} \geq m_k, \quad 1 \leq k \leq K\}.$$

Proof. Since point evaluations are continuous functionals on $A(B_R)$, it follows that $\mathcal{M}_{\phi,R}$ is included in the set to the right. Conversely, let ψ be in this set. Then $\psi/\phi \in A(B_R)$ and it is well known that Pol is dense in $A(B_R)$. (To see this, first show that a given function $\tau \in A(B_R)$ can be approximated arbitrarily well by a dilation $\tau_r = \tau(r\cdot)$, $r < 1$, and then use that the Taylor series of an analytic function converges uniformly on compacts). Thus, given any $\epsilon > 0$ there is a $p \in \mathrm{Pol}$ with $\|\psi/\phi - p\|_{A(B_R)} < \epsilon$ and it follows that $\psi \in \mathsf{cl}\,(\phi\mathrm{Pol})$. Moreover,

$$\mathrm{Pol} \subset \mathsf{cl}\left(\mathrm{span}\,\{E_z\colon z \in \mathbb{C}\}\right).$$

To see this, note that the right-hand side is an algebra which contains 1 and ζ, which is easily seen by considering the limit of $(\mathsf{e}^{z\zeta} - 1)/z$ as $z \to 0$. Thus

$$\psi \in \mathsf{cl}\,(\phi\,\mathrm{Pol}) \subset \mathsf{cl}\left(\mathrm{span}\,\{E_z\,\phi\colon z \in \mathbb{C}\}\right),$$

as desired. □

Proof of Theorem 5.1*.* Since $|\mathcal{Z}(\phi)|$ is discrete we can find an $R > \tau$ such that there are no zeroes in $\tau < |\zeta| \leq R$. By Proposition 2.4 we can write $f = \mathcal{P}(\mu)$ where μ is a measure supported on $\{\zeta\colon |\zeta| = R\}$. By Theorem 3.2 we have $\mathcal{P}(\phi\mu) \equiv 0$. This implies that μ defines a continuous linear functional on $A(B_R)$ which annihilates $\mathcal{M}_{\phi,R}$. By Lemma 5.4 it follows that $A(B_R)/\mathcal{M}_{\phi,R}$ is finite dimensional and that an equivalence class is uniquely specified by the values

$$\left\{\frac{\mathrm{d}^j}{\mathrm{d}\zeta^j}\psi(\zeta_k)\colon 1 \leq k \leq K, \quad 0 \leq j < m_k\right\}.$$

It thus follows by linear algebra that for any linear functional l on the space $A(B_R)/\mathcal{M}_{\phi,R}$ there are $p_k \in \mathrm{Pol}$ of degree $< m_k$ such that

$$l(\psi) = \sum_{k=1}^{K} p_k\left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)\psi\Big|_{\zeta_k}.$$

In particular this is true for the functional induced by μ and thus

$$f(x) = \mathcal{P}(\mu)(x) = \int \mathsf{e}^{x\zeta}\ \mathrm{d}\mu(\zeta) = \sum_{k=1}^{K} p_k\left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)\mathsf{e}^{x\zeta}\Big|_{\zeta_k} = \sum_{k=1}^{K} p_k(x)\mathsf{e}^{x\zeta_k},$$

which is what we wanted to prove. □

Note that if $\phi(z) = \sum_{k=0}^{\infty} a_k z^k$ has infinitely many zeroes $\{\zeta_k\}_{k=1}^{\infty}$, then it is very likely that we can choose non-zero coefficients c_k such that

$$f(x) = \sum_{k=1}^{\infty} c_k \mathsf{e}^{x\zeta_k}$$

defines a solution to $\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) = 0$ in some sense. However, the above function will not be in Exp, and hence this case falls outside the analysis presented here. One way to work with larger spaces than Exp in a general framework is to consider $\mathcal{P}(\mu)$ for measures μ that do not have compact support, but that decay fast enough that

$$\mathcal{P}(\phi\mu)(z) = \int \mathsf{e}^{z\zeta}\phi(\zeta)\ \mathrm{d}\mu(\zeta)$$

is convergent for all $z \in \mathbb{C}$. Such approach has been anticipated e.g. by Carmichael (see [11, Sect. 4], references therein) and will be considered elsewhere.

We will devote the remainder of this paper to examples (Sects. 6, 7), a comparison with other methods (Sect. 8), a discussion of initial value problems (Sect. 9) and the issue of how to practically implement the theory developed above (Sect. 10).

6. Translation-Differential Equations

We consider some special classes of symbols ϕ's in this section and the next, beginning with functions of the form

$$\phi(z) = \sum_{k=1}^{n} p_k(z)\mathsf{e}^{\zeta_k z}, \quad p_k \in \mathrm{Pol},\ \zeta_k \in \mathbb{C}. \tag{6.1}$$

We call the corresponding equation $\phi(\frac{\mathrm{d}}{\mathrm{d}x})f = g$ a translation-differential equation, since it is easily seen (recall (3.4)) that the equation reduces to

$$\sum p_k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x+\zeta_k) = g(x), \tag{6.2}$$

and is thus a combination of classical ordinary differential equations and translation equations.

Example 6. Set $\phi(z) = 2z\cosh(z) = z(\mathsf{e}^z + \mathsf{e}^{-z})$. Then (6.2) reduces to

$$f'(x+1) + f'(x-1) = g(x). \tag{6.3}$$

Returning to the discussion at the end of Sect. 3, this example shows that for *particular choices of* ϕ, Exp may be unnecessarily restrictive as domain of the operator $\phi(\frac{\mathrm{d}}{\mathrm{d}x})$. For instance, $C^1(\mathbb{R})$ would be a natural environment to work with in the present case.

We now consider an even simpler example. $L^2_{\mathrm{loc}}(\mathbb{R})$ will denote the set of functions which are in $L^2(I)$ when restricted to any compact interval $I \subset \mathbb{R}$.

Example 7. Put $\phi(z) = \mathsf{e}^z - 1$. The corresponding Eq. (6.2) with $g \equiv 0$ is

$$f(x+1) - f(x) = 0.$$

This equation makes perfect sense e.g. in L^2_{loc}, and clearly the solutions in this space are all functions with period 1. Since $\mathcal{Z}(\phi) = \{\mathsf{i}2\pi k\}_{k\in\mathbb{Z}}$, Theorem 5.3 shows that any solution in Exp is of the form

$$\sum_{\text{finite}} c_k \mathsf{e}^{\mathsf{i}2\pi k x}, \quad c_k \in \mathbb{C},$$

which indeed is a function of period 1. Now, if we allow infinite sums above, then, by standard Fourier series on an interval, we can reach any 1-periodic function in $L^2_{\mathrm{loc}}(\mathbb{R})$. Returning to the discussion at the end of Sect. 5, we see that Theorem 5.1 can be used to find solutions to $\phi(\frac{\mathrm{d}}{\mathrm{d}x})f = 0$ outside of Exp given that $\phi(\frac{\mathrm{d}}{\mathrm{d}x})$ has a proper interpretation in the larger space in question. We will leave a formal study of when this is possible for a separate work.

7. Convolution Equations

As a second example of a particular class of symbols ϕ's, fix $u \in L^1(\mathbb{R})$ with compact support and let us consider $\phi(z) = \int_{\mathbb{R}} \mathsf{e}^{-zx} u(x)\ \mathrm{d}x$. Given $f \in$ Exp we then have

$$\begin{aligned}
\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) &= \mathcal{P}(\phi\mu_f)(x) = \int_{\mathbb{C}} \mathsf{e}^{x\zeta}\phi(\zeta)\ \mathrm{d}\mu_f(\zeta) \\
&= \int_{\mathbb{C}} \mathsf{e}^{x\zeta} \int_{\mathbb{R}} \mathsf{e}^{-\zeta y} u(y)\ \mathrm{d}y\ \mathrm{d}\mu_f(\zeta) = \int_{\mathbb{R}} u(y) \int_{\mathbb{C}} \mathsf{e}^{\zeta(x-y)}\ \mathrm{d}\mu_f(\zeta)\ \mathrm{d}y \\
&= \int_{\mathbb{R}} u(y) f(x-y)\ \mathrm{d}y = (u * f)(x), \qquad (7.1)
\end{aligned}$$

where the use of Fubini's theorem is allowed due to the compact support of both u and μ_f. In other words, *all convolution equations where the convolver has compact support fit in the general theory developed in the earlier sections.* This example clearly demonstrates the nonlocal nature of $\phi(\frac{\mathrm{d}}{\mathrm{d}x})$, even when ϕ is a function of exponential type. For ϕ as above, one natural space in which to consider the equation

$$\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f = 0 \qquad (7.2)$$

would be $C(\mathbb{R})$, the space of all continuous functions with the topology of uniform convergence on compacts. It is well known that its dual is the set of distributions with compact support. By Theorem 5.1 we easily get that the closure in $C(\mathbb{R})$ of $\mathsf{Span}\ \{\mathsf{e}^{\zeta x}\}_{\mathcal{Z}(\phi)}$ satisfies $u * f = 0$. That no other functions in $C(\mathbb{R})$ satisfy this is true but difficult to show. It was proved by Schwartz, see [30].

Example 8. We start with an easy example, namely $u = \chi_{[-1,0]}$, and suppose we wish to find all solutions to $u * f = 0$ in L^2_{loc}, not just in Exp. Then $\phi(z) = \frac{\mathsf{e}^z - 1}{z}$ with zeroes $\{2\pi \mathsf{i} k\}_{k \in \mathbb{Z}\setminus\{0\}}$, so the solutions in Exp are $\{\mathsf{e}^{2\pi \mathsf{i} k x}\}_{k \in \mathbb{Z}\setminus\{0\}}$. Upon taking the closure in L^2_{loc} we get all functions with period 1 that are orthogonal to $\chi_{[0,1]}$. That this set is precisely the set of all solutions to the original equation $\chi_{[-1,0]} * f = 0$ is a consequence of Beurling's theorem (see [24]) and a short argument, we omit the details.

Example 9. Consider the convolution equation

$$f = \Delta u * f + g \tag{7.3}$$

on the real line (i.e. $\Delta = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$), where f is the unknown function. If u has compact support and $f \in \mathrm{Exp}$ then

$$\begin{aligned}\Delta u * f &= \Delta(u * f) = \Delta_x \int_{\mathbb{C}} \mathsf{e}^{x\zeta} \int_{\mathbb{R}} \mathsf{e}^{-\zeta y} u(y)\ \mathrm{d}y\ \mathrm{d}\mu_f(\zeta)\\ &= \int_{\mathbb{C}} \zeta^2 \mathsf{e}^{x\zeta} \int_{\mathbb{R}} \mathsf{e}^{-\zeta y} u(y)\ \mathrm{d}y\ \mathrm{d}\mu_f(\zeta) = \mathcal{P}\left(\zeta^2 \int_{\mathbb{R}} \mathsf{e}^{-\zeta y} u(y)\ \mathrm{d}y\ \mu_f(\zeta)\right).\end{aligned}$$

Hence, setting $\phi(z) = z^2 \int_{\mathbb{R}} \mathsf{e}^{-zx} u(x)\ \mathrm{d}x$, we have by Theorem 3.2 that

$$\Delta u * f = \mathcal{P}(\phi \mu_f) = \phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f.$$

Setting $\psi = 1 - \phi$, we get that (7.3) is equivalent with $\psi(\frac{\mathrm{d}}{\mathrm{d}x}) f = g$, which thus can be solved by Theorem 4.1, given that $g \in \mathrm{Exp}$. Moreover, Theorem 5.1 tells us that the zeroes of ψ determine the homogenous solutions in Exp and, as before, we can retrieve more solutions by taking limits in some appropriate topology.

8. Review of Other Methods and Examples

We now compare the methods of this paper with other classical approaches to the development of a functional calculus for the operator $\frac{\mathrm{d}}{\mathrm{d}x}$. The standard approach is to notice that $-\mathsf{i}\frac{\mathrm{d}}{\mathrm{d}x}$ is an unbounded self-adjoint operator on e.g. $L^2(\mathbb{R})$ and, therefore, the spectral theorem provides us with a functional calculus. The corresponding spectral projection operator turns out to be the Fourier transform and hence, given any $\psi \in L^\infty(\mathbb{R})$ we get (by definition)

$$\psi\left(-\mathsf{i}\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) = \mathcal{F}^{-1}(\psi \mathcal{F}(f)) = \frac{1}{\sqrt{2\pi}} \int \psi(\xi) \widehat{f}(\xi) \mathsf{e}^{\mathsf{i}x\xi}\ \mathrm{d}\xi. \tag{8.1}$$

If we allow ψ to be unbounded, the same definition works but we get convergence issues in the integral (8.1). There are various ways of dealing with this. The operator-theoretical standpoint is that $\psi\left(-\mathsf{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)$ is a densely defined operator, and hence the domain of definition is restricted to those f's such that $\psi(\xi)\widehat{f}(\xi) \in L^2(\mathbb{R})$. Certainly, this domain is smaller than Exp. The second possibility is to assume that ψ is a Schwartz distribution, $\psi \in \mathcal{S}'$, and $f \in \mathcal{S}$ (or vice versa). In this case, the integral in (8.1) is symbolic but the theory is on solid ground. Moving one step further we can let ψ depend on x as well, and we arrive at the theory of pseudo-differential operators, [21], of which (8.1) is a particular case. Besides the fact that this approach restricts the symbols we can use, it does not allow us to consider the case of equations on half intervals, see for instance [4,16,17]. A third option is to work with functions whose Fourier transforms are functions or distributions with compact support, see for instance [13,32]. This alternative is clearly the closest to the approach developed in the present paper, the main difference being that the corresponding functions spaces are much smaller than Exp, since they contain no functions with exponential growth.

If $f \in \mathrm{Exp} \cap L^2(\mathbb{R})$ and if $\psi \in \mathcal{S}'$ is the restriction to $\mathbb{R}$ of an entire function, the approaches sketched above coincide with the one presented in this paper. More precisely, by the Paley–Wiener theorem we have that functions in $\mathrm{Exp} \cap L^2(\mathbb{R})$ are band limited, i.e. $\mathcal{F}^{-1}(f) = \check{f}$ has compact support. By (2.10) we then have

$$f(x) = \mathcal{F}(\check{f})(x) = \frac{1}{\sqrt{2\pi}} \mathcal{P}(\delta_0(\xi)\check{f}(-\eta))(x) = \frac{1}{\sqrt{2\pi}} \mathcal{P}(\delta_0(\xi)\hat{f}(\eta))(x).$$

With $\phi(z) = \psi(-\mathrm{i}z)$ Theorem 3.2 yields

$$\begin{aligned} \phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) &= \frac{1}{\sqrt{2\pi}} \mathcal{P}(\phi(\zeta)\delta_0(\xi)\hat{f}(\eta))(x) = \frac{1}{\sqrt{2\pi}} \langle \mathrm{e}^{\zeta x}\phi(\zeta), \delta_0(\xi)\hat{f}(\eta)\rangle \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\eta x} \phi(\mathrm{i}\eta)\hat{f}(\eta)\, \mathrm{d}\eta = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\eta x} \psi(\eta)\widehat{f}(\eta)\, \mathrm{d}\eta, \end{aligned} \quad (8.2)$$

which is precisely (8.1). In particular, we note that the limit (3.2) indeed exists. It is immediate that a solution to $\phi(\frac{\mathrm{d}}{\mathrm{d}x})f = g$ is given by the classical formula

$$f = \mathcal{F}^{-1}\left(\frac{\widehat{g}}{\psi}\right), \quad (8.3)$$

given that the right-hand side is well defined. This clearly runs into problems e.g. if ψ has zeroes on $\mathbb{R}$, an obstacle which is completely absent in Theorem 4.1 due to the flexibility in choosing μ_g.

To further highlight some differences, recall Example 6. We first remark that with the “Fourier transform approach” (8.1), the corresponding symbol is $\psi(x) = 2\mathrm{i}x\cos(x)$. However, for solving (4.1) the formula (8.3) will not work due to the fact that $\cos(x)$ has zeroes on $\mathbb{R}$. Despite this, there exists a multitude of solutions given by the theory in Sects. 4 and 5. Of course, the issue of the zeroes may be overcome by using tricks, but we wish to demonstrate the simplicity and clarity of the approach developed here. Moreover, we remark that if we depart from real exponents ζ_k in (6.2), the Fourier transform approach *is* out of the picture.

We now take a look at a more intricate example which highlight both differences and similarities.

Example 10. Consider the heat equation on $\mathbb{R}$,

$$\begin{cases} (\partial_t - \partial_x^2) f(x,t) = 0, & t > 0 \\ f(x,0) = g(x) \end{cases}$$

where $g(x)$ is the initial heat distribution along $\mathbb{R}$. From a naive point of view, treating ∂_x^2 as a number and recalling the theory of ordinary differential equations, the solution should be

$$f(x,t) = \mathsf{e}^{t\partial_x^2} g(x). \tag{8.4}$$

Let $g(x) = \chi_{\mathbb{R}^+}$, i.e. the characteristic function of the positive real axis. Using the distribution theory interpretation of (8.4), there is no problem defining $\widehat{g}$, and then (8.1) gives

$$f(x,t) = \frac{1}{\sqrt{2\pi}} \int \mathsf{e}^{-t\xi^2} \widehat{g}(\xi) \mathsf{e}^{\mathrm{i}x\xi} \, \mathrm{d}\xi \tag{8.5}$$

which is well defined since $\mathsf{e}^{-t\xi^2} \in \mathcal{S}$. Recall that $\mathcal{F}^{-1}(\mathsf{e}^{-t\xi^2}) = \frac{1}{\sqrt{2t}} \mathsf{e}^{-x^2/4t}$ and that the formula $\mathcal{F}^{-1}(\phi\psi) = \frac{1}{\sqrt{2\pi}} \check{\psi} * \check{\phi}$ also holds if $\psi \in \mathcal{S}$ and $\phi \in \mathcal{S}'$. Hence, (8.5) can be recast as

$$f(x,t) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}(\mathsf{e}^{-t\xi^2}) * g = \frac{\mathsf{e}^{-x^2/4t}}{2\sqrt{\pi t}} * \chi_{\mathbb{R}^+},$$

which is the well-known correct solution of the problem; $\frac{\mathsf{e}^{x^2/4t}}{2\sqrt{\pi t}}$ is an approximate identity as $t \to 0$ which satisfies the heat equation for every $t > 0$, as is easily verified. Moreover, the above solution corresponds with our physical intuition; if we have an isolated infinite bar with temperature 1 on $\mathbb{R}^+$ and 0 on $\mathbb{R}^-$, then the temperature should eventually even out to 1/2 over the entire bar, which is precisely what happens above.

On the other hand, if we try to interpret (8.4) via the limit (1.4), we obviously get nonsense, for we then have $\phi(z) = \sum_{k=0}^{\infty} \frac{t^k}{k!} z^{2k}$ but

$$\frac{\mathsf{e}^{x^2/4t}}{2\sqrt{\pi t}} * \chi_{\mathbb{R}^+} \neq \lim_{K\to\infty} \sum_{k=0}^{K} \frac{t^k}{k!} \frac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}} \chi_{\mathbb{R}^+}$$

since the right-hand side for each fixed K equals $\chi_{\mathbb{R}^+}$ plus a distribution with support at 0, which in no sensible topology can converge to 1/2.

At first sight, this seems to be in contrast with the equivalence of (8.1) and (8.2). However, the complications arise from the fact that $\chi_{\mathbb{R}^+}$ does not lie in Exp. For $g \in \mathrm{Exp}$, the calculations in Sect. 7 show that (8.4) reduces to

$$f(x,t) = \frac{\mathsf{e}^{-x^2/4t}}{2\sqrt{\pi t}} * g \tag{8.6}$$

also with the functional calculus developed in this paper. More precisely, $\mathsf{e}^{tz^2} = \int \mathsf{e}^{-zx} \frac{\mathsf{e}^{-x^2/4t}}{2\sqrt{\pi t}}$ so (8.4) follows by the calculation (7.1).

9. On Initial Value Problems

The search for an appropriate substitute of initial value conditions (i.e. conditions on a solution f to $\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) = g(x)$ given at $x = 0$ which uniquely determine f), is an important topic in various branches of theoretical physics, see for instance [1,2,4,14,26]. We argue in this section that no such substitute exists that will work in general, although it is clearly possible for specific choices of ϕ. For example, if ϕ has finitely many zeroes, say N counted with multiplicity, then it clearly follows from Theorems 4.1 and 5.1 that

$$\begin{cases} \phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f = g, & g \in \mathrm{Exp} \\ \frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}} f(0) = c_j, & j = 1, \ldots, N \end{cases} \tag{9.1}$$

has a unique solution f in Exp for all fixed g and $c_1, \ldots, c_N \in \mathbb{C}$. A framework via Laplace transform which includes this case has been developed in [16, 17]. However, when ϕ has infinitely many zeroes we see that to specify f we will need an infinite amount of conditions at $x = 0$. It has been observed several times before [14,26] that if f is analytic this is the same as uniquely determining f before solving the equation (since the Taylor series of an entire function defines it uniquely). Another option, as is argued e.g. in [2], is to consider the "initial value problem"

$$\begin{cases} \phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f = g \\ f|_I = F \end{cases} \tag{9.2}$$

(where $f|_I$ means the restriction of f to I). Again we run into trouble if we ask that the solution $f \in \mathrm{Exp}$, as F then uniquely determines f (since we can get the Taylor series at a point in I from F) and hence the equation $\phi(\frac{\mathrm{d}}{\mathrm{d}x})f = g$ again becomes redundant.

Thus, for the question of appropriate IVP's to make sense we have to consider non-entire functions. Let us thus suppose that ϕ is such that $\phi(\frac{\mathrm{d}}{\mathrm{d}x})f = g$ has a physically sensible interpretation in some space $\mathcal{H}$ which does not contain only analytic functions, as in Examples 6–10 and [16,17]. For instance, in Example 7, the particular equation considered there has a natural interpretation outside of Exp, for example in L^2_{loc}, and initial data on the interval $I = [0, 1)$ uniquely specifies f [by setting $f(x + n) = F(x)$ for all $x \in [0, 1)$ and $n \in \mathbb{Z}$]. Thus, supposing that f is given on $[0, r]$ with $r < 1$, the equation is underdetermined and for $r > 1$ overdetermined. However, the fact that in Example 7 there exists an interval of a precise length such that the values of f there uniquely determine f on $\mathbb{R}$, is clearly related to the very simple structure of the symbol $\phi(z) = \mathsf{e}^z - 1$, and it is bound to fail in general. For example, dividing $e^z - 1$ by z, we obtain Example 8 (we can keep $\mathcal{H} = L^2_{\mathrm{loc}}$). Then the appropriate length of I is clearly still 1, but now the equation

$$\begin{cases} \phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f = 0 \\ f|_I = 1 \end{cases}$$

has no solution, as was shown in Example 8. As one takes into account ϕ's with less structure, we get more erratic behavior. We now take a look at an example where the symbol ϕ is such that

$$\begin{cases} \phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f = 0 \\ f|_I = F \end{cases} \tag{9.3}$$

has a solution *for all* I *and* $F \in L^2(I)$, but where no interval I is sufficient to *uniquely* determine a solution.

Example 11. Let $2m > r_m > m$, $m \in \mathbb{N}$, be numbers such that $\frac{r_m}{r_{m'}} \notin \mathbb{Q}$ for all pairs $m \neq m'$. Consider the set

$$Z = \{r_m n \colon m \in \mathbb{N}, \quad n \in \mathbb{Z}\backslash\{0\}\}.$$

Given any $\rho > 1$ we have

$$\sum_{\zeta \in Z} \frac{1}{|\zeta|^\rho} = \sum_{n \in \mathbb{Z}\backslash\{0\}} \frac{1}{n^\rho} \sum_{m \in \mathbb{N}} \frac{1}{r_m^\rho} \leq 2 \sum_{n \in \mathbb{N}} \frac{1}{n^\rho} \sum_{m \in \mathbb{N}} \frac{1}{m^\rho} < \infty.$$

By the theory of Hadamard decompositions of analytic functions, there exists an entire function ϕ of order 1 with zeroes Z (each with multiplicity 1). By Theorem 5.1 the solutions to the equation

$$\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f = 0 \tag{9.4}$$

are given by

$$\mathsf{Span}\ \cup_{\zeta \in Z} \{\mathsf{e}^{\zeta x}\}, \tag{9.5}$$

where Span means taking all possible finite linear combinations. However, we will again assume that the nature of the problem (9.4) is such that we can take limits in L^2_{loc} and still get relevant solutions. We then claim that given any interval I and any $F \in L^2(I)$, the equation system (9.3) always has infinitely many solutions.

To prove this, we need to recall the classical theorem of Levinson on completeness of bases of exponential functions. Given a discrete set $\tilde{Z} \subset \mathbb{C}$ that does not contain 0, we define $n_{\tilde{Z}}(r) = \#\{\zeta \in \tilde{Z} \colon |\zeta| < r\}$, (# denotes the number of points in a set), and set

$$N_{\tilde{Z}}(r) = \int_0^r \frac{n_{\tilde{Z}}(t)}{t}\, \mathrm{d}t.$$

Given an interval I, Levinson's theorem says that

$$\mathsf{Span}\ \{\mathsf{e}^{\zeta x}\}_{\zeta \in \tilde{Z}}$$

is dense in $L^2(I)$ if

$$\limsup_{r \to \infty} N_{\tilde{Z}}(r) - \frac{|I| r}{\pi} + \frac{\ln r}{2} > -\infty, \tag{9.6}$$

where $|I|$ denotes the length of I (see e.g. [6] or [36]). We will show that (9.6) is satisfied for $\tilde{Z} = Z$ and any I. Assuming this for the moment, our example is complete; for given F on I we can pick a larger interval J and extend F in different ways to J. Since all these extensions then lie in the $L^2(J)$-closure of $\mathsf{Span}\ \{\mathsf{e}^{\zeta x}\}_{\zeta \in Z}$, they are all solutions to (9.4) in the weaker sense considered here.

Now, Eq. (9.6) follows if we show that given any $C > 0$ there exists an R with

$$N_Z(r) \geq Cr, \quad r > R. \tag{9.7}$$

Set $Z_m = \{r_m n \colon n \in \mathbb{Z}\backslash\{0\}\}$ so that $Z = \cup_{m \in \mathbb{N}} Z_m$. Then

$$n_{Z_m}(r) \geq \frac{2r}{r_m} - 2$$

so

$$N_{Z_m}(r) = \int_{r_m}^{r} \frac{n_{Z_m}(x)}{x}\,\mathrm{d}x \geq \int_{r_m}^{r} \frac{2}{r_m} - \frac{2}{x}\,\mathrm{d}x = 2\left(\frac{r}{r_m} - 1 - \ln\left(\frac{r}{r_m}\right)\right)$$

for $r > r_m$. It is easy to show that $\ln(x) \leq x/\mathsf{e}$ so we can continue to get

$$N_{Z_m}(r) \geq \left(2 - \frac{2}{\mathsf{e}}\right)\frac{r}{r_m} - 2 \geq \left(2 - \frac{2}{\mathsf{e}}\right)\frac{r}{2m} - 2.$$

Hence, given any $M \in N$ and r large enough, we have

$$N_Z(r) \geq \sum_{m=1}^{M} N_{Z_m}(r) \geq \left(2 - \frac{2}{\mathsf{e}}\right)\left(\sum_{m=1}^{M} \frac{1}{m}\right) r - 2M$$

and (9.7) immediately follows since $\sum_{m=1}^{M} \frac{1}{m} = \infty$.

The conclusion of Example 11 seems to imply that *all* functions are solutions to $\phi(\frac{\mathrm{d}}{\mathrm{d}x})f = 0$ as long as we restrict attention to a finite interval of arbitrary length. This actualizes the question of whether it makes any sense to study the equation $\phi(\frac{\mathrm{d}}{\mathrm{d}x})f = g$ for general entire functions ϕ. We end this section with some comments on this. The discussion requires some acquaintance with the theory of entire functions, for which we refer to [6]. It is easily seen that the symbols ϕ in Sects. 6 and 7 are all of order 1 and *finite type*. The ϕ in Example 11 still has order 1 but will have *infinite type*. Clearly, the situation in Example 11 is related to ϕ having "too many zeroes". The proposition below shows that the phenomenon in Example 11 occurs for all symbols ϕ whose exponent of convergence is greater than 1. We note that this includes the majority of all entire functions of order > 1, since the order and the exponent of convergence coincide if the *canonical factor* is dominant in the Hadamard decomposition, (see [6, Theorem 2.6.5 and Sect. 2.7]). A notable exception to this is the symbol $\phi(z) = \mathsf{e}^{z^2}$ considered earlier, which has order two and no zeroes. We remind the reader that the exponent of convergence is > 1 if there exists a $\kappa > 1$ such that

$$\sum_{\zeta \in \mathcal{Z}(\phi)} \frac{1}{|\zeta|^{\kappa}} = \infty. \tag{9.8}$$

Proposition 9.1. *Let ϕ be an entire function such that the exponent of convergence of the zeroes of ϕ is > 1. Then the solutions to $\phi(\frac{\mathrm{d}}{\mathrm{d}x})f = 0$ in $\mathrm{Exp}(\mathbb{R})$ are dense in $L^2(I)$ for any bounded interval $I \subset \mathbb{R}$.*

Proof. There is no restriction to assume that $0 \notin \mathcal{Z}(\phi)$. It suffices to show that

$$\limsup \frac{N_{\mathcal{Z}(\phi)}(r)}{r} = \infty, \tag{9.9}$$

for then (9.6) holds and hence Levinson's theorem does the job just as in Example 11. Suppose that (9.9) is false. Then $N_{\mathcal{Z}(\phi)}(r) \leq Cr$ for some $C > 0$. Moreover,

$$\begin{aligned} n_{\mathcal{Z}(\phi)}(r) &= n_{\mathcal{Z}(\phi)}(r) \ln \mathsf{e} = n_{\mathcal{Z}(\phi)}(r) \int_r^{\mathsf{e}r} \frac{1}{x}\, \mathrm{d}x \\ &\leq \int_r^{\mathsf{e}r} \frac{n_{\mathcal{Z}(\phi)}(x)}{x}\, \mathrm{d}x \leq N_{\mathcal{Z}(\phi)}(\mathsf{e}r) \leq C\mathsf{e}r. \end{aligned} \tag{9.10}$$

However, with $\kappa > 1$ in (9.8), Lemma 2.5.5 of [6] implies that

$$\infty = \int_0^\infty \frac{n_{\mathcal{Z}(\phi)}(r)}{r^{\kappa+1}}\, \mathrm{d}r$$

which contradicts (9.10), since $\int_1^\infty \frac{C\mathsf{e}r}{r^{k+1}}\, \mathrm{d}r < \infty$. □

10. Implementation

In this section, we consider the problem of computing numerically solutions to equations of the form (4.1). Certainly, the formula (4.4) for solving (4.1) is complete from a theoretical perspective and, if g is explicitly known, then it is perfectly possible to implement it in a computer. However, if g stems from measured data, Eq. (4.4) is completely useless. For, calculating derivatives is a highly unstable operation on a measured signal: calculating $g^{(10)}$ with some acceptable accuracy is usually out of the picture, whereas to get a good approximation of $\mathcal{B}(g)$ via (2.1) we clearly need more than ten Taylor coefficients. The obvious alternative is to try to get $\mathcal{B}(g)$ using (2.2), i.e.

$$\mathcal{B}(g)(\zeta) = \overline{\vartheta} \int_0^\infty g(t\vartheta) \mathsf{e}^{-\zeta\overline{\vartheta}t}\, \mathrm{d}t, \tag{10.1}$$

in which $\zeta = re^{i\theta} = r\vartheta$, but here the obstacle is that (10.1) can only be evaluated for $\vartheta = 1$ or $\vartheta = -1$, since our measurements likely take place in the real world!

With

$$X_+ = \inf\{\xi\colon g(t)\mathsf{e}^{-\xi t} \in L^1(\mathbb{R}^+)\} \tag{10.2}$$

and

$$X_- = \sup\{\xi\colon g(-t)\mathsf{e}^{\xi t} \in L^1(\mathbb{R}^+)\} \tag{10.3}$$

this will give $\mathcal{B}(g)(\zeta)$ on the domain

$$\{\xi + \mathsf{i}\eta\colon \xi > X_+ \text{ or } \xi < X_-\},$$

and hence we find that (4.4) cannot be used since we lack information on the strip

$$\{\xi + \mathsf{i}\eta\colon X_- \leq \xi \leq X_+\}.$$

The following theorem basically says that if ϕ is bounded below on this strip, then (4.1) can be solved using the Fourier transform. For clarity we indicate the independent variable upon which the Fourier transform acts with a subscript.

Theorem 10.1. *Suppose that* ϕ *satisfies*

$$\inf \{ |\phi(\xi + \mathsf{i}\eta)| \colon \xi \in [\xi_-, \xi_+],\ |\eta| > Y_0 \} = \epsilon > 0$$

for some $\xi_+ > X_+$, $\xi_- < X_-$ *and* $\epsilon, Y_0 > 0$*. Then, given* $g \in \mathrm{Exp}$*, the expression*

$$f(x) = \mathsf{e}^{x\xi_+} \mathcal{F}_s^{-1} \left(\frac{\mathcal{F}_t \left(\chi_{\mathbb{R}^+} \mathsf{e}^{-\xi_+ t} g \right)(s)}{\phi(\xi_+ + \mathsf{i}s)} \right)(x) + \mathsf{e}^{x\xi_-} \mathcal{F}_s^{-1} \left(\frac{\mathcal{F}_t \left(\chi_{\mathbb{R}^-} \mathsf{e}^{-\xi_- t} g \right)(s)}{\phi(\xi_- + \mathsf{i}s)} \right)(x), \quad x \in \mathbb{R}, \tag{10.4}$$

provides a solution to $\phi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) = g(x)$ *in* Exp*.*

We remark that, given a sampling of the function g, the corresponding discrete version of (10.4) can be efficiently implemented using the fast Fourier transform [12].

Proof. By Corollary 4.2 we have that

$$f(x) = \int_{\square_Y} \mathsf{e}^{x\zeta} \frac{\mathcal{B}(g)(\zeta)}{\phi(\zeta)} \frac{\mathrm{d}\zeta}{2\pi\mathsf{i}}, \tag{10.5}$$

gives a solution, where $\square_Y$ represents the rectangle with corners $\{\xi_+ \pm \mathsf{i}Y, \xi_- \pm \mathsf{i}Y\}$ and $Y > Y_0$ is a parameter sufficiently large so that the sides are outside the conjugate diagram S of $\mathcal{B}(f)$, and such that $\square_Y \cap \mathcal{Z}(\phi) = \emptyset$. By the assumption on ϕ and Cauchy's theorem, (10.5) does not depend on Y, and hence we can take a limit as $Y \to \infty$. By the equivalence of (10.1), it is easy to see that the two vertical integrals evaluate to

$$\mathsf{e}^{x\xi_+} \int_{-Y}^{Y} \mathsf{e}^{\mathsf{i}\eta x} \frac{\mathcal{F}_t \left(\chi_{\mathbb{R}^+} \mathsf{e}^{-\xi_+ t} g \right)(\eta)}{\phi(\xi_+ + \mathsf{i}\eta)} \frac{\mathrm{d}\eta}{\sqrt{2\pi}} + \mathsf{e}^{x\xi_-} \int_{-Y}^{Y} \mathsf{e}^{\mathsf{i}\eta x} \frac{\mathcal{F}_t \left(\chi_{\mathbb{R}^-} \mathsf{e}^{-\xi_- t} g \right)(\eta)}{\phi(\xi_- + \mathsf{i}\eta)} \frac{\mathrm{d}\eta}{\sqrt{2\pi}},$$

whose limit as $Y \to \infty$ equals (10.4). Hence (10.4) follows if we prove that the horizontal integrals converge to 0 as $Y \to \infty$. We consider the upper one. For each fixed $x \in \mathbb{R}$ we have

$$\left| - \int_{\xi_-}^{\xi_+} \mathsf{e}^{x(\xi + \mathsf{i}Y)} \frac{\mathcal{B}(g)(\xi + \mathsf{i}Y)}{\phi(\xi + \mathsf{i}Y)} \frac{\mathrm{d}\xi}{2\pi\mathsf{i}} \right| \leq \frac{1}{2\pi\epsilon} \sup_{\xi_- < \xi < \xi_+} \left| \mathsf{e}^{x\xi} \mathcal{B}(g)(\xi + \mathsf{i}Y) \right| \leq \mathrm{const} \cdot \sup_{\xi_- < \xi < \xi_+} |\mathcal{B}(g)(\xi + \mathsf{i}Y)|.$$

Since $\mathcal{B}(g)$ is defined by a Laurent series starting at ζ^{-1}, see (2.1), it easily follows that

$$|\mathcal{B}(g)(\zeta)| \leq \text{const}/|\zeta|$$

for large $|\zeta|$, and hence $\lim_{Y\to\infty} \sup_{\xi_- < \xi < \xi_+} |\mathcal{B}(g)(\xi + \mathrm{i}Y)| = 0$, as desired. □

An interesting note is that although the above theorem was derived under the assumption that $g \in \text{Exp}$, the formula (10.4) works for any function g such that X_+ and X_-, as defined in (10.2) and (10.3), are finite. For g in Exp, f given by (10.4) is a solution in the sense of (1.4). However, as was observed in Example 10, this interpretation of our Eq. (1.4) may be too restrictive for the underlying physical reality, and hence it is certainly worthwhile to consider formula (10.4) even for examples where (1.4) is too restrictive.

Theorem 10.1 typically applies to e.g. the translation-differential equations considered in Sect. 6, but it never applies to the convolution equations considered in Sect. 7. However, if g e.g. is a band limited L^2 function (i.e. a function in the Paley–Wiener space), then we have $X_+ = X_- = 0$ and hence $\mathcal{B}(g)$ can be evaluated at all points of $\Box_Y$ if Y is large enough. Thus, (10.5) provides a computable solution as is, given that Y is taken large enough.

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