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A TQFT of Turaev–Viro Type on Shaped Triangulations

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Abstract. A shaped triangulation is a finite triangulation of an oriented pseudo-three-manifold where each tetrahedron carries dihedral angles of an ideal hyperbolic tetrahedron. To each shaped triangulation, we associate a quantum partition function in the form of an absolutely convergent state integral which is invariant under shaped 3–2 Pachner moves and invariant with respect to shape gauge transformations generated by total dihedral angles around internal edges through the Neumann–Zagier Poisson bracket. Similarly to Turaev–Viro theory, the state variables live on edges of the triangulation but take their values on the whole real axis. The tetrahedral weight functions are composed of three hyperbolic gamma functions in a way that they enjoy a manifest tetrahedral symmetry. We conjecture that for shaped triangulations of closed three-manifolds, our partition function is twice the absolute value squared of the partition function of Techmüller TQFT defined by Andersen and Kashaev. This is similar to the known relationship between the Turaev–Viro and the Witten–Reshetikhin–Turaev invariants of three-manifolds. We also discuss interpretations of our construction in terms of three-dimensional supersymmetric field theories related to triangulated three-dimensional manifolds.

1. Introduction

Topological quantum field theories were discovered and axiomatized by Atiyah [2], Segal [42] and Witten [56]. First examples in $2+1$ dimensions were constructed by Reshetikhin and Turaev [39,40,52] using the combinatorial framework of Kirby calculus, and by Turaev and Viro [53] using the framework of triangulations and Pachner moves. The algebraic ingredients of both constructions come from the finite-dimensional representation category of the quantum group $U_q(sl(2))$ at roots of unity. For example, the basic building

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elements in Turaev–Viro construction are tetrahedral weight functions given by $6j$ symbols. These theories have been the subject of much subsequent investigation in the works of Blanchet, Habegger, Masbaum, Vogel, Barrett, Westbury, Turaev, Virelizier, Balsam, Kirillov and others [6–8,30,54]. A related but somewhat different line of development was initiated by Kashaev in [29] where a state sum invariant of links in three-manifolds was defined using the combinatorics of charged triangulations where the charges are algebraic versions of dihedral angles of ideal hyperbolic tetrahedra in finite cyclic groups. This approach has been subsequently developed by Baseilhac, Benedetti, Geer, Kashaev, Turaev [3,21]. The common feature of all these theories is that the partition functions are always given by finite state sums.

On the other hand, the idea of partition functions of Turaev–Viro type originates from the work of Ponzano and Regge [37] where, based on $SU(2)$ $6j$ symbols, a lattice version of quantum $2+1$ gravity was suggested, but this theory was not complete and remained of restricted use because of problems of convergence of infinite sums. Similar problems of convergence appear when one tries to construct combinatorial versions of quantum Chern–Simons theories with non-compact gauge groups. For example, a connected component of $\mathrm{PSL}(2,\mathbb{R})$ Chern–Simons theory is identified with Teichmüller space, and its quantum theory corresponds to a specific class of unitary mapping class group representations in infinite dimensional Hilbert spaces [9,28]. Based on quantum Teichmüller theory, formal state-integral partition functions of triangulated three-manifolds were defined by Hikami, Dimofte, Gukov, Lenells, Zagier, Dijkgraaf, Fuji, Manabe [10,11,14,24,25], mostly for the purposes of quasi-classical expansions, but the question of convergence remained largely open until a mathematically rigorous version of Teichmüller TQFT was suggested in [1]. The convergence property of Teichmüller TQFT is due to its specific underlying combinatorial setting: it is not just triangulations but shaped triangulations where each tetrahedron carries dihedral angles of an ideal hyperbolic tetrahedron. Moreover, the role of dihedral angles is twofold: they not only provide absolute convergence of state integrals but they also implement the complete symmetry with respect to change of edge orientations. Although, shaped triangulations are similar to charged triangulations of [29], the positivity condition of dihedral angles imposes important restrictions on construction of topologically invariant partition functions.

The purpose of this paper is to suggest yet another TQFT based on combinatorics of shaped triangulations. As its basic building block is defined in terms of Faddeev's quantum dilogarithm [17] and the absolute convergence of partition functions relies on the positivity of dihedral angles, it is similar to the Teichmüller TQFT. As a consequence, we are still restricted in our abilities of constructing topologically invariant partition functions in the sense that the 2-3 shaped Pachner move is not always applicable. On the other hand, unlike the Teichmüller TQFT, our tetrahedral weight functions enjoy manifest tetrahedral symmetry and the partition function is well defined on any shaped triangulation without any extra topological restrictions.

We now describe our construction in precise terms.

1.1. States, State Potentials, and State Gauge Invariance

Let Y be a CW complex. Denote by $\Delta_i(Y)$ the set of i-dimensional cells of Y. A *state* of Y is a map $s\colon \Delta_1(Y) \to \mathbb{R}$. A *state potential* is a map $g\colon \Delta_0(Y) \to \mathbb{R}$. Define a linear *state gauge* map

$$b\colon \mathbb{R}^{\Delta_0(Y)} \to \mathbb{R}^{\Delta_1(Y)}, \quad bg(e) = g(\partial_0 e) + g(\partial_1 e), \tag{1}$$

where $\partial_i e$, $i \in \{0,1\}$, are the two end points of e (they coincide if the edge is a loop). A state is called *pure gauge* if it finds itself in the image of the state gauge map. The pure gauge states constitute a vector subspace of the state space.

Let S be a set. A function $f\colon \mathbb{R}^{\Delta_1(Y)} \to S$ is called *state gauge invariant* at state s if $f(s+bg) = f(s)$ for any state potential g.

A *(state) gauge fixing* at vertex $v \in \Delta_0(Y)$ is a linear form λ on the vector space of states $\mathbb{R}^{\Delta_1(Y)}$ such that

$$\langle \lambda, bg \rangle = g(v), \quad \forall g \in \mathbb{R}^{\Delta_0(Y)}. \tag{2}$$

Note that a gauge fixing at a vertex may not exist if the state gauge map is not injective.

In what follows, a real-valued function defined on only a subset of vertices will always be thought of as a state potential having zero values on the vertices where initially it was not defined.

1.2. Shaped Tetrahedra and Their Boltzmann Weights

Let T be an oriented tetrahedron embedded into $\mathbb{R}^3$ together with its standard CW complex structure. Let $\Box(T)$ be the set of normal quadrilateral types (to be called *quads*) in T which is in bijection with the set of pairs of opposite edges of T. We fix the action of $\mathbb{Z}/3\mathbb{Z} = \{1, \tau, \tau^2\}$ on $\Box(T)$ so that the images of a quad q under the action are $q, q' = \tau(q)$ and $q'' = \tau^2(q)$ corresponding to the clockwise cyclic order of three edges around a vertex (as seen from the outside of the tetrahedron). We say T is *shaped tetrahedron* if it is provided with a *dihedral angle map* $\alpha\colon \Box(T) \to]0,\pi[$, such that $\alpha(q)+\alpha(q')+\alpha(q'') = \pi$. Associated to α, the complex shape variables entering Thurston's hyperbolicity equations are given by a map $z_\alpha\colon \Box(T) \to \mathbb{C}\backslash\{0,1\}$ defined by the formula

$$z_\alpha(q) = \mathrm{e}^{i\alpha(q)} \sin\alpha(q'')/\sin\alpha(q'). \tag{3}$$

Any state $s\colon \Delta_1(T) \to \mathbb{R}$ induces a map $\tilde{s}\colon \Box(T) \to \mathbb{R}$ defined by the formula $\tilde{s}(q) = s(e) + s(e')$, where the e and e' are the opposite edges separated by q. To each pair (T,s) consisting of a shaped tetrahedron T and a state s of T, we associate the following *Boltzmann weight*

$$B(T,s) := \prod_{q\in\Box(T)} \gamma^{(2)}\left(\frac{\omega_1+\omega_2}{\pi}\alpha(q) + \sqrt{-\omega_1\omega_2}(\tilde{s}(q') - \tilde{s}(q''));\omega_1,\omega_2\right) \tag{4}$$

where function $\gamma^{(2)}(z;\omega_1,\omega_2)$ is defined below in (30) with $\omega_1, \omega_2 \in \mathbb{C}$ and $\omega_1/\omega_2 \notin (-\infty, 0]$. It is easily verified that this Boltzmann weight is state gauge invariant at any state.

1.3. Shaped Triangulations and Their Boltzmann Weights

A *triangulation* is an oriented pseudo-three-manifold obtained from finitely many tetrahedra in $\mathbb{R}^3$ by gluing them along triangular faces through orientation reversing affine CW homeomorphisms. Any triangulation X is naturally a CW complex and its boundary ∂X is the CW subcomplex composed of unglued triangular faces. We will use the following notation:

$$\Delta_i(\mathring{X}) := \Delta_i(X)\backslash\Delta_i(\partial X). \tag{5}$$

A *shaped triangulation* is a triangulation where all tetrahedra are shaped. Similarly to the case of one shaped tetrahedron, to each pair (X, s) consisting of a shaped triangulation X and a state s of X, we associate a Boltzmann weight

$$B(X,s) := \prod_{T\in\Delta_3(X)} B\left(T, s|_{\Delta_1(T)}\right). \tag{6}$$

Again, this Boltzmann weight is state gauge invariant at any state.

1.4. The Partition Function of Shaped Triangulations

A *boundary state* of a triangulation X is a state of its boundary. We have the natural linear restriction map from the vector space of states of X to the vector space of its boundary states

$$\partial\colon \mathbb{R}^{\Delta_1(X)} \to \mathbb{R}^{\Delta_1(\partial X)}, \tag{7}$$

and for any boundary state s, we have a canonical identification of the preimage $\partial^{-1}(s)$ with the vector space $\mathbb{R}^{\Delta_1(\mathring{X})}$ of real-valued functions on the interior edges of X.

A *state gauge fixing* in the interior of a triangulation X is a collection

$$\lambda = \{\lambda_v\}_{v\in\Delta_0(\mathring{X})} \tag{8}$$

of gauge fixings at all interior vertices. Notice that for any triangulation the state gauge map is injective and state gauge fixings exist at any vertex.

To any triple (X, t, λ), where X is a shaped triangulation, t is a boundary state of X, and λ is a state gauge fixing in the interior of X, we associate a *partition function*

$$W_b(X,t,\lambda) := \int_{\partial^{-1}(t)} B(X,s)\delta\left(\langle\lambda, s\rangle\right) \mathrm{d}s, \tag{9}$$

where

$$\delta(\langle\lambda,s\rangle) := \prod_{v\in\Delta_0(\mathring{X})} \delta\left(\langle\lambda_v, s\rangle\right), \quad \mathrm{d}s := \prod_{v\in\Delta_1(\mathring{X})} \mathrm{d}s(e) \tag{10}$$

and $b := \sqrt{\frac{\omega_1}{\omega_2}}$. The main result of this paper is the following theorem where we use the notions of *shaped 3-2 Pachner moves* and *shape gauge transformations* considered in [1].

Theorem 1. *The partition function $W_b(X, t, \lambda)$ is an absolutely convergent integral independent of the choice of the state gauge fixing λ, invariant under shaped 3-2 Pachner moves, and invariant under the shape gauge transformations induced by interior edges.*

Several examples of explicit calculations make us to believe that for shaped triangulations of closed three-manifolds, when the Teichmüller TQFT is defined as well, our partition function is twice the absolute value squared of the partition function of the Techmüller TQFT. This is similar to the known relationship between the Turaev–Viro and the Witten–Reshetikhin–Turaev invariants of three-manifolds.

Conjecture 1. *Let (X, ℓ_X) be an admissible shaped leveled branched triangulation of a closed oriented compact three-manifold in the sense of* [1]. *Then the following equality holds true*

$$2\,|F_\hbar(X, \ell_X)|^2 = W_b(X, \lambda) \tag{11}$$

where $\hbar = (b+b^{-1})^{-2} \in \mathbb{R}_{>0}$ and $F_\hbar(X, \ell_X)$ is the Teichmüller TQFT partition function of (X, ℓ_X).

Following [1,29], we can construct an invariant of pairs (a compact closed oriented three-manifold M, a knot K in M). The corresponding combinatorial framework is that of one-vertex Hamiltonian triangulations (or H-triangulations). These are one vertex shaped Δ-triangulations of M where the knot K is represented by a distinguished edge, the shape structures being degenerate in the sense that the total dihedral angle on the knot is zero and all other edges are balanced. As these are not shape structures in the strict sense of the word, one cannot be sure that the state integrals in the partition function will not diverge. For that reason, one should approach them by true shape structures, provided one deals with triangulations which allow such approach. As in the case of Teichmüller TQFT of [1], after analytic continuation to complex angles, the partition functions can have poles in the total angle ϵ around the knot. That means that it makes sense to consider ratios of partition functions, before taking the limit $\epsilon \to 0$ and balancing all other edges. Let $\omega_X \in \mathbb{R}^{\Delta_1(X)}$ be the assignment of total dihedral angles around the edges of X, and let $\tau \in \mathbb{R}^{\Delta_1(X)}$ assign the zero value to the distinguished edge representing the knot and 2π to any other edge. We define the *renormalized partition function* (*RPF*) by the formula

$$\check{W}_b(X, \lambda) := \frac{W_b(X, \lambda)}{2\,|\Phi_b(c_b - c_b\epsilon/\pi)|^2}, \tag{12}$$

where ϵ is the value of ω_X on the knot (see also Appendix for notation), and the *balanced renormalized partition function* (*BRPF*)

$$\tilde{W}_b(X, \lambda) := \lim_{\omega_X \to \tau} \check{W}_b(X, \lambda). \tag{13}$$

Theorem 1 implies that BRPF is a function on the set of equivalence classes of pairs (M, K) with respect to the equivalence relation generated by shaped

3-2 Pachner moves along the non-distinguished edges. In the case where X is admissible in the sense of [1], Conjecture 1 implies the equality

$$\tilde{W}_b(X,\lambda) = \left|\tilde{F}_\hbar(X,\ell_X)\right|^2, \tag{14}$$

where

$$\tilde{F}_\hbar(X,\ell_X) := \lim_{\omega_X\to\tau} \frac{F_\hbar(X,\ell_X)}{\Phi_b(c_b - c_b\epsilon/\pi)} \tag{15}$$

is the (balanced) renormalized partition function of the Teichmüller TQFT. In the case of $M = S^3$, by explicit calculations we prove equality (14) for the knots 3_1, 4_1, 5_2. We also calculate the BRPF for the knot 6_1.

The rest of this paper is organized as follows. Section 2 contains the proof of the main Theorem 1. In Sect. 3, using the elliptic beta integral, we derive the pentagon identity which underlies the invariance of our partition function with respect to shaped 3-2 Pachner moves. In Sect. 4, we provide examples of concrete calculations and verify Conjecture 1. In particular, we verify the equality (14) between knot invariants. Section 5 is devoted to some considerations from the perspective of 3d supersymmetric field theories. Namely, based on our construction, we get a class of 3d supersymmetric field theories defined on a squashed three-sphere S_b^3 related to triangulated three-manifolds. The latter relation is known as 3d/3d correspondence which is the topic of recent study [12,13,49,50]. In Sect. 6, we discuss a bit more the gauge equivalence relation on the shape structures and the relationship of our theory to representations of three-manifold fundamental groups into $\mathrm{PSL}(2,\mathbb{C})$ and simplicial $\mathrm{PSL}(2,\mathbb{R})$ Chern–Simons theory. Appendices contain some technical information on the special functions used.

2. Proof of Theorem 1

Lemma 1. *Let X be a shaped triangulation, and let s and s' be states of X such that $B(X,s'+ts) = B(X,s')$ for any $t\in\mathbb{R}$. Then the state s is in the image of the state gauge map.*

Proof. By a straightforward verification, the statement of the lemma is true if X is a disjoint union of unglued tetrahedra. Thus, it suffices to prove that if triangulation X is obtained from a triangulation Y by identification of two triangular faces f and f', and the statement of the lemma is true for Y, then it is also true for X.

Denote by $p\colon Y\to X$ the identification projection, and by $p^*\colon \mathbb{R}^{\Delta_i(X)}\to\mathbb{R}^{\Delta_i(Y)}$ the corresponding pull-back maps. Let s and s' be states of X such that

$$B(X,s'+ts) = B(X,s'),\quad \forall t\in\mathbb{R}. \tag{16}$$

Using the fact that $B(X,r) = B(Y,p^*(r))$ for any state r of X, equation (16) is equivalent to

$$B(Y,p^*(s')+tp^*(s)) = B(Y,p^*(s')),\quad \forall t\in\mathbb{R}. \tag{17}$$

As we assume that the statement of the lemma is true for Y, there exists $g \in \mathbb{R}^{\Delta_0(Y)}$ such that $p^*(s) = bg$. Let us show that there exists $g' \in \mathbb{R}^{\Delta_0(X)}$ such that $g = p^*(g')$. Indeed, let triangles f and f' have respective vertices v_i and v'_i and edges e_i and e'_i for $i \in \{1, 2, 3\}$ such that

$$\partial e_i = \{v_j, v_k\}, \quad \partial e'_i = \{v'_j, v'_k\}, \quad \{i, j, k\} = \{1, 2, 3\}, \tag{18}$$

and

$$p(e_i) = p(e'_i), \quad p(v_i) = p(v'_i), \quad i \in \{1, 2, 3\}. \tag{19}$$

That means that when applied to edges e_i and e'_i, the equality $p^*(s) = bg$ gives

$$g(v_j) + g(v_k) = g(v'_j) + g(v'_k) \Leftrightarrow g(v_i) - g(v'_i) = \xi := \sum_{m=1}^{3} (g(v_m) - g(v'_m)). \tag{20}$$

Taking sum over i in the last equation, we obtain $\xi = 3\xi \Leftrightarrow \xi = 0$ which implies that $g(v_i) = g(v'_i)$ for any $i \in \{1, 2, 3\}$, i.e., $g = p^*(g')$. Thus, we have the equality $p^*(s) = bp^*(g') = p^*(bg')$, and as p^* is injective, we conclude that $s = bg'$. $\square$

Proof of Theorem 1. By injectivity of the state gauge map in the case of triangulations and Lemma 1, the state gauge map image of the group $\mathbb{R}^{\Delta_0(\mathring{X})}$ is the maximal translation subgroup of the state space of X which leaves invariant the boundary state t and the Boltzmann weight $B(X, s)$. On the other hand, the product of delta functions $\delta(\langle \lambda, s \rangle)$ restricts the integral to a hyperplane in the space $\mathbb{R}^{\Delta_1(\mathring{X})} \simeq b^{-1}(t)$ which intersects any orbit of this group action in a unique point, while the Boltzmann weight exponentially decays along any direction in this hyperplane. This implies that the integral in (9) is absolutely convergent.

Independence on the choice of the state gauge fixing λ easily follows through the use of a simplest finite-dimensional version of the Faddeev–Popov trick in path integrals for gauge invariant systems[1] [33]. Indeed, if s is a state of X and λ' a state gauge fixing in the interior of X, then we have the identity

$$1 = \int_{\mathbb{R}^{\Delta_0(\mathring{X})}} \delta\left(\langle \lambda', s + bg \rangle\right) \mathrm{d}g, \tag{21}$$

where

$$\mathrm{d}g := \prod_{v \in \Delta_0(\mathring{X})} \mathrm{d}g(v). \tag{22}$$

Inserting (21) into (9), exchanging the order of integrations, shifting the integration state variables, using the gauge invariance of the Boltzmann weight, again exchanging the order of integrations, and again using identity (21) with λ' replaced by λ, we obtain

[1] Similarly to QED, our system is linear and the Faddeev–Popov determinant is trivial so that no ghosts are needed.

$$\begin{aligned}
W_b(X,t,\lambda) &= \int_{\partial^{-1}(t)} B(X,s)\delta\left(\langle\lambda,s\rangle\right)\mathrm{d}s\\
&= \int_{\partial^{-1}(t)} B(X,s)\delta\left(\langle\lambda,s\rangle\right)\left(\int_{\mathbb{R}^{\Delta_0(\mathring{X})}}\delta\left(\langle\lambda',s+bg\rangle\right)\mathrm{d}g\right)\mathrm{d}s\\
&= \int_{\mathbb{R}^{\Delta_0(\mathring{X})}}\left(\int_{\partial^{-1}(t)} B(X,s)\delta\left(\langle\lambda,s\rangle\right)\delta\left(\langle\lambda',s+bg\rangle\right)\mathrm{d}s\right)\mathrm{d}g\\
&= \int_{\mathbb{R}^{\Delta_0(\mathring{X})}}\left(\int_{\partial^{-1}(t)} B(X,s-bg)\delta\left(\langle\lambda,s-bg\rangle\right)\delta\left(\langle\lambda',s\rangle\right)\mathrm{d}s\right)\mathrm{d}g\\
&= \int_{\mathbb{R}^{\Delta_0(\mathring{X})}}\left(\int_{\partial^{-1}(t)} B(X,s)\delta\left(\langle\lambda,s-bg\rangle\right)\delta\left(\langle\lambda',s\rangle\right)\mathrm{d}s\right)\mathrm{d}g\\
&= \int_{\partial^{-1}(t)} B(X,s)\delta\left(\langle\lambda',s\rangle\right)\left(\int_{\mathbb{R}^{\Delta_0(\mathring{X})}}\delta\left(\langle\lambda,s-bg\rangle\right)\mathrm{d}g\right)\mathrm{d}s\\
&= \int_{\partial^{-1}(t)} B(X,s)\delta\left(\langle\lambda',s\rangle\right)\mathrm{d}s = W_b(X,t,\lambda'). \qquad (23)
\end{aligned}$$

Invariance under 3-2 shaped Pachner moves is a consequence of the shaped pentagon identity for the tetrahedral Boltzmann weights, which in its turn is equivalent to identity (47), provided the relevant integration variable does not enter the product of delta functions $\delta\left(\langle\lambda,s\rangle\right)$. This condition can always be satisfied by appropriate choice of λ.

Finally, the gauge transformation in the space of dihedral angles induced by an edge e, see [1], is equivalent to an imaginary shift of the integration variable $s(e)$, which, using the holomorphicity of the Boltzmann weights, can be compensated by an imaginary shift of the integration path in the complex $s(e)$-plane. □

3. Pentagon Identities from Elliptic Beta Integral

Let p and q be two fixed complex numbers inside the unit disk, i.e., $|p|$, $|q|<1$ which will be called *basis parameters*. The *elliptic gamma function* is a meromorphic function on the complex plane defined by the formula [20]

$$\Gamma(z;p,q) := \prod_{k,l=0}^{\infty}\frac{1-z^{-1}p^{k+1}q^{l+1}}{1-zp^kq^l} \qquad (24)$$

known to satisfy *(Spiridonov's) elliptic beta integral* identity [43,44]

$$\kappa\int_{\mathbb{T}}\frac{\prod_{j=1}^{6}\Gamma(s_jz^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)}\frac{\mathrm{d}z}{2\pi iz} = \prod_{1\le k<l\le 6}\Gamma(s_ks_lp,q), \qquad (25)$$

where

$$\kappa := \frac{(p;p)_\infty(q;q)_\infty}{2}, \quad (x;y)_\infty := \prod_{k=0}^{\infty}(1-xy^k), \qquad (26)$$

the parameters $s_1, \ldots, s_6$ being constrained by the *balancing* condition

$$\prod_{j=1}^{6} s_j = pq, \tag{27}$$

and we use the notation

$$\Gamma(a,b;p,q) := \Gamma(a;p,q)\Gamma(b;p,q), \quad \Gamma(az^{\pm 1};p,q) := \Gamma(az;p,q)\Gamma(az^{-1};p,q). \tag{28}$$

The elliptic gamma function has the following limiting value [41]:

$$\lim_{r\to 0} \exp\left(\pi i \frac{2z - \omega_1 - \omega_2}{12r}\right) \Gamma(\mathrm{e}^{2\pi i r z}; \mathrm{e}^{2\pi i r \omega_1}, \mathrm{e}^{2\pi i r \omega_2}) = \gamma^{(2)}(z;\omega_1,\omega_2), \tag{29}$$

with the *hyperbolic gamma function* defined by

$$\gamma^{(2)}(z;\omega_1,\omega_2) := \frac{\exp\left(-\pi i B_{2,2}(z;\omega_1,\omega_2)/2\right)}{\Phi_{\sqrt{\frac{\omega_1}{\omega_2}}}\left(\frac{\omega_1+\omega_2-2z}{2i\sqrt{\omega_1\omega_2}}\right)} \tag{30}$$

where

$$B_{2,2}(z;\omega_1,\omega_2) := \frac{z^2}{\omega_1\omega_2} - \frac{z}{\omega_1} - \frac{z}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2} \tag{31}$$

is the second-order Bernoulli polynomial and

$$\Phi_b(z) := \exp\left(\int_{\mathbb{R}+i0} \frac{\mathrm{e}^{-2izw}\mathrm{d}w}{4\sinh(wb)\sinh(w/b)w}\right) \tag{32}$$

is Faddeev's quantum dilogarithm. As it follows from (29) and explicitly seen from (30) and (31), $\gamma^{(2)}(z;\omega_1,\omega_2)$ is invariant under simultaneous rescalings of its three arguments so that it depends on only two independent homogeneous combinations, for example, $\frac{z}{\omega_2}$ and $b^2 := \frac{\omega_1}{\omega_2}$. Notice, that b^2 is an element of the set $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$, and, using the symmetry of the elliptic gamma function with respect to exchange of p and q, without loss of generality, we can assume that it has a non-negative imaginary part. Moreover, we choose b with strictly positive real part.

Among the properties of the hyperbolic gamma function is the inversion relation

$$\gamma^{(2)}(z,\omega_1+\omega_2-z;\omega_1,\omega_2) = 1 \tag{33}$$

and the asymptotic formulae

$$\begin{aligned} \lim_{|u|\to\infty} \mathrm{e}^{\frac{\pi i}{2}B_{2,2}(u;\omega_1,\omega_2)}\gamma^{(2)}(u;\omega_1,\omega_2) &= 1, \quad \text{if } \arg\omega_1 < \arg u < \arg\omega_2 + \pi, \\ \lim_{|u|\to\infty} \mathrm{e}^{-\frac{\pi i}{2}B_{2,2}(u;\omega_1,\omega_2)}\gamma^{(2)}(u;\omega_1,\omega_2) &= 1, \quad \text{if } \arg\omega_1 - \pi < \arg u < \arg\omega_2. \end{aligned} \tag{34}$$

The inversion relation (33) is equivalent to the relation

$$\Phi_b(z)\Phi_b(-z) = \Phi_b(0)^2\mathrm{e}^{\pi i z^2} \tag{35}$$

which can be proved using (32) and Cauchy's residue theorem. The asymptotic formulae can be easily proved in the case of complex b using the inversion relation and the formula

$$\Phi_b(z) = \frac{\left(-\mathrm{e}^{2\pi bz+\pi ib^2}; \mathrm{e}^{2\pi ib^2}\right)_\infty}{\left(-\mathrm{e}^{2\pi b^{-1}z-\pi ib^{-2}}; \mathrm{e}^{-2\pi ib^{-2}}\right)_\infty}. \tag{36}$$

Some of other properties of the hyperbolic gamma function and Faddeev's quantum dilogarithm are collected in the Appendix.

The hyperbolic gamma function can further be reduced to the usual gamma function in the limit [41]:

$$\lim_{\omega_2\to\infty} \left(\frac{\omega_2}{2\pi\omega_1}\right)^{\frac{z}{\omega_1}-\frac{1}{2}} \gamma^{(2)}(z;\omega_1,\omega_2) = \frac{\Gamma(z/\omega_1)}{\sqrt{2\pi}}. \tag{37}$$

3.1. The First Pentagon Identity

In what follows, we will use the notation

$$\gamma^{(2)}(a,b;\omega_1,\omega_2) := \gamma^{(2)}(a;\omega_1,\omega_2)\gamma^{(2)}(b;\omega_1,\omega_2) \tag{38}$$

and

$$\gamma^{(2)}(a\pm u;\omega_1,\omega_2) := \gamma^{(2)}(a+u;\omega_1,\omega_2)\gamma^{(2)}(a-u;\omega_1,\omega_2). \tag{39}$$

In the elliptic beta integral (25), we introduce a new parameterization

$$z = \mathrm{e}^{2\pi iru}, \quad p = \mathrm{e}^{2\pi ir\omega_1}, \quad q = \mathrm{e}^{2\pi ir\omega_2}, \quad s_j = \mathrm{e}^{2\pi ir\alpha_j}, \quad j\in\{1,\ldots,6\}, \tag{40}$$

and use the limit (29) to get

$$\frac{1}{2}\int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^6 \gamma^{(2)}(\alpha_j\pm u;\omega_1,\omega_2)}{\gamma^{(2)}(\pm 2u;\omega_1,\omega_2)} \frac{\mathrm{d}u}{i\sqrt{\omega_1\omega_2}} = \prod_{1\le k<l\le 6} \gamma^{(2)}(\alpha_k+\alpha_l;\omega_1,\omega_2), \tag{41}$$

where the contour of integration is the imaginary axis directed upwards and in addition to the equality

$$\sum_{j=1}^{6} \alpha_j = \omega_1+\omega_2 \tag{42}$$

coming from the balancing condition (27), the variables α_j should also be restricted by the conditions that the contour of integration separates the zeros and poles of each of the hyperbolic gamma functions entering the integrand.

Now, following [46], we introduce yet another parameterization

$$\alpha_j = \mu + a_j, \quad \alpha_{j+3} = -\mu + b_j, \quad j\in\{1,2,3\}, \tag{43}$$

which transforms the balancing condition (42) into the equality

$$\sum_{j=1}^{3}(a_j+b_j) = \omega_1+\omega_2. \tag{44}$$

In equality (41), shifting the integration variable $u \to u + \mu$ and taking the limit $\mu \to \infty$, we get

$$\int_{-i\infty}^{i\infty} \prod_{j=1}^{3} \gamma^{(2)}(a_j - u, b_j + u; \omega_1, \omega_2) \frac{\mathrm{d}u}{i\sqrt{\omega_1\omega_2}} = \prod_{k,l=1}^{3} \gamma^{(2)}(a_k + b_l; \omega_1, \omega_2). \tag{45}$$

Using the following function

$$\mathcal{B}(x, y) := \frac{\gamma^{(2)}(x, y; \omega_1, \omega_2)}{\gamma^{(2)}(x + y; \omega_1, \omega_2)}, \tag{46}$$

we rewrite (45) in the form of *pentagon* (five term) identity

$$\int_{-i\infty}^{i\infty} \prod_{j=1}^{3} \mathcal{B}(a_j - u, b_j + u) \frac{\mathrm{d}u}{i\sqrt{\omega_1\omega_2}} = \mathcal{B}(a_2 + b_1, a_3 + b_2)\mathcal{B}(a_1 + b_2, a_3 + b_1), \tag{47}$$

where we used the inversion relation for the hyperbolic gamma function. Recall that in (47) the contour of integration is the imaginary axis directed upwards and the parameters a_j and b_j, apart from satisfying (44), are such that the contour of integration separates the zeros and poles of each hyperbolic gamma function entering the integrand. The pentagon relation (47) can be interpreted in geometrical terms as is seen in Fig. 1. Note that by the inversion formula (33), we have

$$\mathcal{B}(x, y) = \gamma^{(2)}(x; \omega_1, \omega_2)\gamma^{(2)}(y; \omega_1, \omega_2)\gamma^{(2)}(\omega_1 + \omega_2 - x - y; \omega_1, \omega_2) \tag{48}$$

so that the right-hand side of (47) corresponds to the union of two tetrahedra while the left-hand side to three tetrahedra. One also has the following orthogonality relations for the $\mathcal{B}$ function:

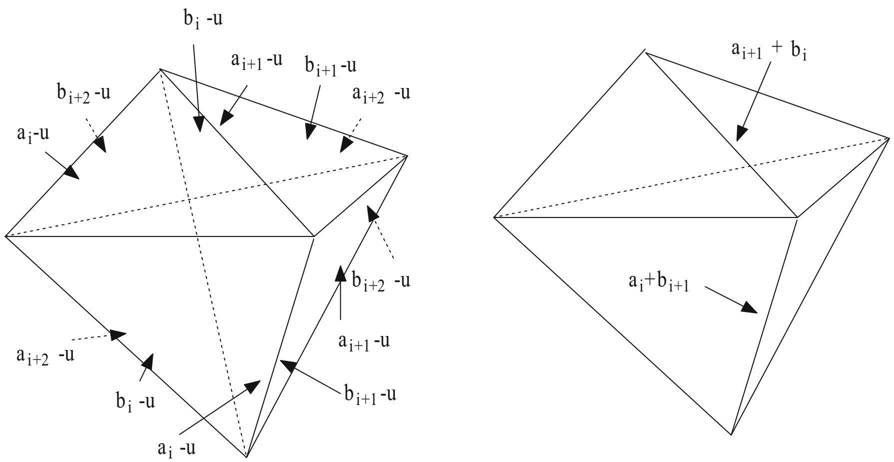

FIGURE 1. 2-3 moves

$$\int_{\mathbb{R}} \mathcal{B}(a-iu,b+iu)\mathcal{B}(-a-iu,-b+iu)\frac{\mathrm{d}u}{\sqrt{\omega_1\omega_2}} = 2i\sqrt{\omega_1\omega_2}\delta(a-b),$$
$$\int_{\mathbb{R}} \mathcal{B}(a-iu,a+iu)\mathcal{B}(-a-i(u+b),-a+i(u+b))\frac{\mathrm{d}u}{\sqrt{\omega_1\omega_2}} = 2\sqrt{\omega_1\omega_2}\delta(b). \tag{49}$$

It would be natural to ask if one could give also a geometrical interpretation to the original elliptic beta integral, but to the best of our knowledge it is not known yet. However, recently in papers [5], it was realized that the elliptic beta integral can be interpreted as a star–triangle relation (see also [46]) with the Boltzmann weight

$$W_\alpha(x,y) = \Gamma(\mathrm{e}^{\alpha}x^{\pm 1}y^{\pm 1};p,q). \tag{50}$$

Taking into account the fact that the elliptic hypergeometric integrals describe specific partition functions of $4d\,\mathcal{N}=1$ SYM theories known as superconformal indices [15,47], one possibly could try to interpret Spiridonov's identity (25) in terms of 3-3 Pachner moves of triangulated four-manifolds, see, for example, [31,32].

3.2. The Second Pentagon Identity

Let us rewrite (47) in the form

$$\int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{3}\gamma^{(2)}(a_j-u;\omega_1,\omega_2)\prod_{k=1}^{2}\gamma^{(2)}(b_k+u;\omega_1,\omega_2)}{\gamma^{(2)}(\sum_{l=1}^{3}a_l+b_1+b_2-u;\omega_1,\omega_2)}\frac{\mathrm{d}u}{i\sqrt{\omega_1\omega_2}}$$
$$= \prod_{k,l=1}^{3}\gamma^{(2)}(a_k+b_l;\omega_1,\omega_2), \tag{51}$$

Applying the limit $\omega_2\to\infty$ to (51) and using (37) we get

$$\int_{-i\infty}^{i\infty} B(a_1+u,b_1-u)B(a_2+u,b_2-u)B(a_3+u,a_1+a_2+b_1+b_2)\frac{\mathrm{d}u}{2\pi i}$$
$$= B(a_2+b_1,a_3+b_2)B(a_1+b_2,a_3+b_1), \tag{52}$$

where $B(x,y)$ is the usual beta function

$$B(x,y) \;=\; \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{53}$$

It is interesting to note that written in terms of fifteen gamma functions, identity (52) is known as Barnes' second lemma [4], see also [20, formula (4.1.3)], but to the best of our knowledge, it was not noticed that it can be written as a pentagon relation for the beta function.

Taking the limit $\omega_2\to\infty$ in (49) one also gets analogous orthogonality relations for the beta function.

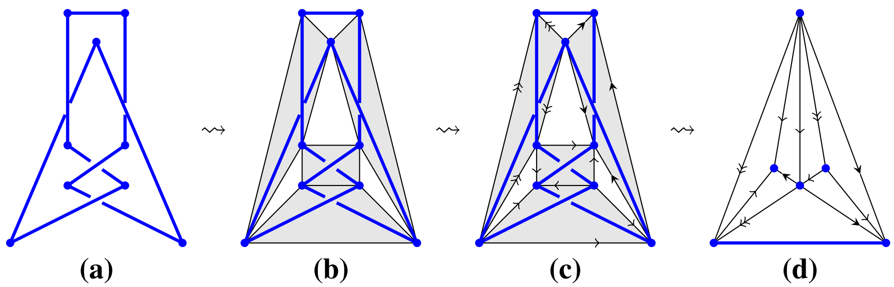

Figure 2. Construction of a one-vertex cell decomposition of the pair $(S^3, 4_1)$: **a** a knot diagram with a cellular structure; **b** the induced cellular decomposition of S^3 with crossings contained in shaded tetrahedral cells; **c** each shaded tetrahedron is collapsed to a segment represented by four oriented edges of the tetrahedron; **d** the result of gluing of the cells $c_\pm$ along the 2-cell corresponding to the outer region of the knot diagram

4. Examples of Calculations

4.1. One-Vertex *H*-Triangulations

Let (M, K) be a pair consisting of a closed oriented three-manifold M and a knot $K \subset M$. A *one-vertex H-triangulation* of (M, K) is, by definition, a delta triangulation of M with one vertex and a distinguished edge representing the knot K. A one-vertex H-triangulation of a knot in S^3 can be constructed starting from a non-trivial knot diagram D as follows.[2]

First, take S^3 as the standard one point compactification of $\mathbb{R}^3$ and fix an embedding $S^2 \subset S^3$ as the closure of the standard embedding $\mathbb{R}^2 \subset \mathbb{R}^3$, $(x, y) \mapsto (x, y, 0)$. Next, we choose D as the image of a polygonal knot $K \subset \mathbb{R}^2 \times [-\epsilon, \epsilon] \subset \mathbb{R}^3$ under the orthogonal projection of $\mathbb{R}^3 \to \mathbb{R}^2$, with a small positive real ϵ, so that the vertices of K project to mid-points of the edges of D, except for one distinguished edge, to which we assume two vertices of K are projected. We consider K together with its natural cellular decomposition. In the case of the figure-eight knot, the diagram in Fig. 2a satisfies all these conditions where the images of the vertices of the cellular decomposition of K are represented by small filled circles and the distinguished edge of D is the one containing the uppermost horizontal segment.

The cellular decomposition of K extends to that of S^3 by keeping the same vertex set and adding new edges so that each crossing point of D is surrounded by images of four new edges as in Fig. 2b where the higher dimensional cells are given by tetrahedral cells contained in $\mathbb{R}^2 \times [-\epsilon, \epsilon]$, together with their own natural cellular structure and which are projected to shaded quadrilaterals containing the crossings of D, and also by two 3-cells $\tilde{c}_\pm$ given by intersections

[2] We will think of a knot diagram as a four valent planar graph with vertices at the crossing points.

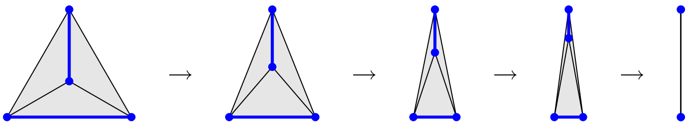

FIGURE 3. An isotopy which collapses a tetrahedron to a segment

of the complements of the tetrahedral cells in S^3 with two balls B_+ and B_- obtained as closures in S^3 of the upper and lower half spaces respectively, i.e., $B_\pm = \overline{\{(x, y, z) \mid \pm z \geq 0\}}$ so that $S^3 = B_+ \cup B_-$ and $S^2 = B_+ \cap B_-$.

From the constructed cellular decomposition of S^3, we produce a new cellular complex by an isotopy which starts from the identity map and ends with a projection to a topological quotient space (still homeomorphic to S^3) with respect to an equivalence relation under which all points of K, except the segment which projects to the distinguished edge of D, are equivalent to each other, and each tetrahedral cell is collapsed to a single edge as in Fig. 3. The resulting cellular complex is composed of two 3-cells $c_\pm$, the images of $\tilde{c}_\pm$, while the 2-skeleton is given by the complementary regions of the (shaded) quadrilaterals containing the crossings of the diagram D as in Fig. 2c where the non-collapsed part of K is the uppermost horizontal segment, and the 2-cells are given by non-shaded regions, namely four triangular cells and one quadrilateral cell corresponding to the outer region. The orientations on the edges with different types of arrows allow to keep track the information on their identifications, namely, one type of arrow corresponds to one and the same (geometrical) edge, the image of the corresponding tetrahedra.

By gluing two 3-cells $c_\pm$ together along the 2-cell corresponding to the outer region in the knot diagram, we obtain a cellular complex given by one 3-cell whose boundary is composed of the remaining 2-cells, each 2-cell appearing twice with opposite orientations as in Fig. 2d where the boundary 2-sphere of the 3-cell is identified with the coordinate plane compactified to a 2-sphere by adding a point at infinity. The obtained complex is (non-canonically) transformed into a Δ-triangulation by cutting the 3-cell into tetrahedra. In our example, this is achieved easily by cutting along two new triangular 2-cells, see Fig. 4b, where a linear order of vertices of each tetrahedron is induced by the directions of arrows on the edges.

In the rest of this section, we present few examples of calculations of partition functions for one-vertex H-triangulations of knots in S^3.

4.2. Notation and Some Useful Formulae

For a tetrahedron T, we will denote ij_T the edge ij of T for $\{i, j\} \subset \{0, 1, 2, 3\}$. If $\alpha \in \mathbb{R}^{\square(T)}$ is an arrangement of dihedral angles, then we denote $\alpha_i := \alpha(q_{T,i})$ with $q_{T,i} \in \square(T)$ being the quad separating the opposite edge pair $(0i, jk)$ of T. For a shaped Δ-triangulation X, we will let denote $\omega_X \in \mathbb{R}^{\Delta_1(X)}$ the *weight* function which assigns the total dihedral angles around the edges of X. We will work only with H-triangulations where the distinguished edge belongs to only one tetrahedron which will always be denoted H, and we choose the orientation

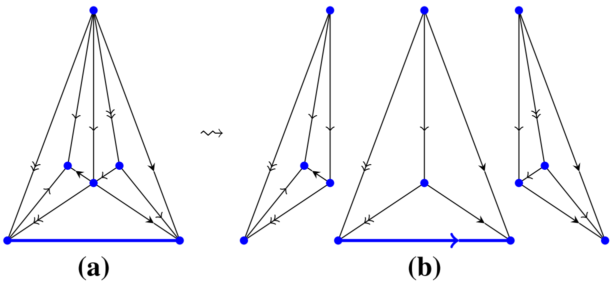

FIGURE 4. A one-vertex H-triangulation of the pair $(S^3, 4_1)$ with three tetrahedra

on the distinguished edge so that H will always be a positive tetrahedron (provided this is possible).

We will use the following notation

$$\Delta := (\omega_1 + \omega_2)/\pi, \quad \nabla := \sqrt{\omega_1 \omega_2}, \tag{54}$$

and also

$$w(x) := c_b \left(1 - \frac{x}{\pi}\right) \tag{55}$$

with the property

$$w(x) - w(y) = w(x - y) - c_b = -c_b \frac{x - y}{\pi}. \tag{56}$$

Further, we will use the ψ function defined as

$$\psi(x, y) := \Psi(x, -x, y) = \int_{\mathbb{R}} \frac{\Phi_b(t + x)}{\Phi_b(t - x)} \mathrm{e}^{2\pi i y t} \mathrm{d}t, \tag{57}$$

see also (147). We have the equality

$$\mathcal{B}\left(\Delta\mu + i\nabla x, \Delta\nu + i\nabla y\right) = \psi\left(w(\mu) + \frac{x}{2}, y - 2c_b \frac{\nu}{\pi}\right) \tag{58}$$

which is equivalent to (146). We can also write

$$\mathcal{B}\left(\Delta\mu + i\nabla x, \Delta\nu + i\nabla y\right) = \int_{\mathbb{R}^2} \mathcal{K}(x, y, s, t) \mathcal{M}_{\nu, \pi-\mu-\nu}(s, t) \mathrm{d}s \mathrm{d}t \tag{59}$$

with a tempered distribution

$$\mathcal{K}(x, y, s, t) := \mathrm{e}^{\pi i (s+t) y} \delta(x - s + t) \in \mathcal{S}'(\mathbb{R}^4), \tag{60}$$

and a test function

$$\mathcal{M}_{\mu,\nu}(s, t) := \bar{\varphi}_{\mu,\nu}(s) \varphi_{\mu,\nu}(t) \tag{61}$$

where

$$\varphi_{\mu,\nu}(z) := \frac{\mathrm{e}^{-2ic_b \mu z}}{\Phi_b\left(z - c_b \frac{\mu+\nu}{\pi}\right)} \in \mathcal{S}(\mathbb{R}) \tag{62}$$

accumulates all the dependence of the model on the quantum parameter b and the shape structure given by the dihedral angle variables μ and ν. Notice that equality (59) easily follows from (57), while the function $\varphi_{\mu,\nu}(z)$, up to a phase independent of z, is the function $\psi_{\frac{\mu}{2\pi},\frac{\nu}{2\pi}}(z)$ of [1, Section 4]. In particular, it has the following important symmetry properties: one with respect to the complex conjugation

$$\bar{\varphi}_{\mu,\nu}(z) \simeq \mathrm{e}^{\pi i z^2} \varphi_{\nu,\mu}(-z) \tag{63}$$

and another with respect to the Fourier transformation

$$\tilde{\varphi}_{\mu,\nu}(z) := \int_{\mathbb{R}} \mathrm{e}^{-2\pi i y z} \varphi_{\mu,\nu}(y) \mathrm{d}y \simeq \mathrm{e}^{\pi i z^2} \varphi_{\nu,\pi-\mu-\nu}(z), \tag{64}$$

where $\simeq$ means an equality up to a phase factor independent of z.

Formula (59) allows us to separate the state and shape variables in the sense that the Boltzmann weight (6) of a shaped Δ-triangulation X in state $s \in \mathbb{R}^{\Delta_1(X)}$ can be written in the form

$$B(X, s) = \int_{\mathbb{R}^{2\Delta_3(X)}} K(X, s, u) M(X, u) \mathrm{d}u \tag{65}$$

where $K(X, s, u)$ is a tempered distribution in $\mathcal{S}'\left(\mathbb{R}^{\Delta_1(X) \sqcup 2\Delta_3(X)}\right)$ independent of the shape structure of X. Substituting (65) in (9), one can exchange the order of integrations and get the formula

$$W_b(X, t, \lambda) = \int_{\mathbb{R}^{2\Delta_3(X)}} L(X, t, u, \lambda) M(X, u) \mathrm{d}u \tag{66}$$

where

$$L(X, t, u, \lambda) = \int_{\partial^{-1}(t)} K(X, s, u) \delta\left(\langle \lambda, s \rangle\right) \mathrm{d}s \tag{67}$$

is another tempered distribution in $\mathcal{S}'\left(\mathbb{R}^{\Delta_1(\partial X) \sqcup 2\Delta_3(X)}\right)$ still independent of the shape structure of X. This way of calculating the partition function is very convenient in practice as we will see below.

4.3. One-Vertex H-Triangulation of $(S^3, 3_1)$

We construct a one-vertex one-tetrahedron H-triangulation X of the pair $(S^3, 3_1)$ according to Fig. 5. Namely, as explained in Sect. 4.1, the diagram (a) of the trefoil knot induces a cellular decomposition (b) of S^3 which, upon removing the 2-cell corresponding to the outer region of (b), immediately gives rise to H-triangulation (c). Thus, we have one tetrahedron H with the face identifications

$$\partial_i H \sim \partial_{3-i} H, \quad i \in \{0, 1\}. \tag{68}$$

The quotient space X is a triangulation of S^3 with only one vertex v and two edges: e_1, the edge $\bullet\!\!\longrightarrow\!\!\bullet$ in Fig. 5c, which is knotted like trefoil and has as pre-image the only edge 03 and e_2, the edge $\bullet\!\!\longrightarrow\!\!\bullet$ in Fig. 5c, having as pre-images all other five edges of H. The Boltzmann weight reads

$$B(X, s) = \mathcal{B}\left(\Delta\alpha_3, \Delta\alpha_1 + i\nabla(s_2 - s_1)\right) \tag{69}$$

where $\alpha_i := \alpha(q_i)$ with $q_i \in \square(H)$ being the quad separating the opposite edge pair $(0i, jk)$ of H, and $s_i := s(e_i)$. Formula (59) implies a decomposition of the form (65) with

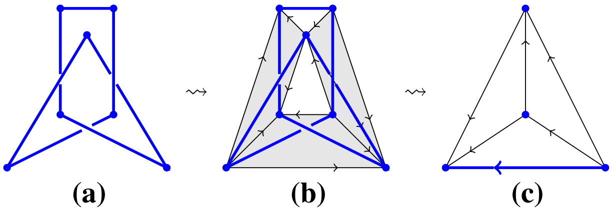

FIGURE 5. Construction of a one-vertex H-triangulation of the pair $(S^3, 3_1)$

$$K(X,s,u) = \mathcal{K}(0, s_2 - s_1, u_1, u_2) = \mathrm{e}^{2\pi i u_1(s_2-s_1)}\delta(u_2-u_1) \tag{70}$$

and

$$M(X,u) = \mathcal{M}_{\alpha_1,\alpha_2}(u_1,u_2) \tag{71}$$

where $(u_1,u_2) := u(H)$.

Choosing the gauge fixing map λ so that $\langle\lambda_v, s\rangle = s_1/2$, we first calculate the integral in (67):

$$\begin{aligned} L(X,u,\lambda) &= \int_{\mathbb{R}^2} K(X,s,u)\delta(s_1/2)\mathrm{d}s_1\mathrm{d}s_2 \\ &= 2\delta(u_2-u_1)\int_{\mathbb{R}^2} \mathrm{e}^{2\pi i u_1 s_2} ds_2 = 2\delta(u_1)\delta(u_2) \end{aligned} \tag{72}$$

which, upon substitution into (66), immediately gives the following result for the partition function:

$$W_b(X,\lambda) = 2\mathcal{M}_{\alpha_1,\alpha_2}(0,0) = 2|\varphi_{\alpha_1,\alpha_2}(0)|^2 = 2\left|\Phi_b(w(\alpha_3))\right|^2. \tag{73}$$

Using the definitions of the RPF and BRPF in (12) and (13), respectively, we conclude that $\tilde{W}_b(X,\lambda) = \check{W}_b(X,\lambda) = 1$.

4.4. One-Vertex H-Triangulation of $(S^3, 4_1)$

Let H-triangulation X be given by Fig. 4. It consists of two positive tetrahedra, H and R (the central and the right tetrahedra in Fig. 4b), and one negative tetrahedron L (the left tetrahedron in Fig. 4b). One has the following identification of the faces

$$\partial_0 H \sim \partial_1 H,\ \partial_2 H \sim \partial_1 R,\ \partial_3 H \sim \partial_3 L,\ \partial_0 R \sim \partial_2 L,\ \partial_2 R \sim \partial_0 L,\ \partial_3 R \sim \partial_1 L. \tag{74}$$

For $s \in \mathbb{R}^{\Delta_1(X)}$, we get four edge variables

$$\begin{aligned} t &:= s(\bullet\!\!\longrightarrow\!\!\!-\!\!\bullet) = s(23_H), \\ x &:= s(\bullet\!\!\longrightarrow\!\!\!-\!\!\bullet) = s(13_H) = s(03_H) = s(23_R) = s(03_R) = s(13_L), \\ y &:= s(\bullet\!\!\twoheadrightarrow\!\!\!-\!\!\bullet) = s(12_H) = s(02_H) = s(01_R) = s(02_L) = s(12_L), \\ z &:= s(\bullet\!\!\longrightarrow\!\!\!-\!\!\bullet) = s(01_H) = s(02_R) = s(12_R) = s(13_R) = s(01_L) \\ &= s(03_L) = s(23_L). \end{aligned} \tag{75}$$

Denoting by $\alpha \in \mathbb{R}^{\square(H)}$, $\beta \in \mathbb{R}^{\square(R)}$, $\gamma \in \mathbb{R}^{\square(L)}$ the dihedral angle assignments, we have the following weights (total angles) on the edges of X:

$$\begin{aligned}
\epsilon :=&\omega_X(\bullet\!\!\longrightarrow\!\!\bullet) = \alpha_1,\\
\epsilon_1 :=&\omega_X(\bullet\!\!\longrightarrow\!\!\bullet) = 2\pi - \alpha_1 - \beta_2 + \gamma_2,\\
\epsilon_2 :=&\omega_X(\bullet\!\!\twoheadrightarrow\!\!\bullet) = 2\pi - \alpha_1 + \beta_1 - \gamma_1,\\
\epsilon_3 :=&\omega_X(\bullet\!\!\longrightarrow\!\!\bullet) = 6\pi - \epsilon - \epsilon_1 - \epsilon_2.
\end{aligned} \tag{76}$$

The Boltzmann weight function reads as follows:

$$\begin{aligned}
B(X,s) = &\mathcal{B}\left(\Delta\alpha_1, \Delta\alpha_2 + i\nabla(x+y-z-t)\right)\\
&\times \mathcal{B}\left(\Delta\beta_1 + i\nabla(z-x), \Delta\beta_2 + i\nabla(z-y)\right)\\
&\times \mathcal{B}\left(\Delta\gamma_1 + i\nabla(z-x), \Delta\gamma_2 + i\nabla(z-y)\right)
\end{aligned} \tag{77}$$

so that using (59) we obtain decomposition (65) with

$$K(X,s,u) = \mathcal{K}(0, x+y-z-t, h_1, h_2)\mathcal{K}(z-x, z-y, r_1, r_2)\mathcal{K}(z-x, z-y, l_1, l_2) \tag{78}$$

and

$$M(X,u) = \mathcal{M}_{\alpha_2,\alpha_3}(h_1,h_2)\mathcal{M}_{\beta_2,\beta_3}(r_1,r_2)\mathcal{M}_{\gamma_2,\gamma_3}(l_1,l_2) \tag{79}$$

where $(h_1,h_2) := u(H)$, $(r_1,r_2) := u(R)$, $(l_1,l_2) := u(L)$. Choosing the gauge fixing map λ so that $\langle \lambda_v, s\rangle = z/2$, we calculate

$$L(X,u,\lambda) = \int_{\mathbb{R}^4} K(X,s,u)\delta(z/2)\mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z = 2\delta(h_1)\delta(h_2)\check{L}(X,u,\lambda) \tag{80}$$

with

$$\begin{aligned}
\check{L}(X,u,\lambda) &= \int_{\mathbb{R}^2} \mathcal{K}(x,y,r_1,r_2)\mathcal{K}(x,y,l_1,l_2)\mathrm{d}x\mathrm{d}x\\
&= \int_{\mathbb{R}^2} \mathrm{e}^{\pi i(r_1+r_2+l_1+l_2)y}\delta(x-r_1+r_2)\delta(z-l_1+l_2)\mathrm{d}x\mathrm{d}y\\
&= \delta(r_1+l_2)\delta(l_1+r_2).
\end{aligned} \tag{81}$$

Corresponding to the factorization in the second equality of (80), the partition function also factorizes

$$W_b(X,\lambda) = 2\left|\Phi_b(w(\alpha_1))\right|^2 \check{W}_b(X,\lambda) \tag{82}$$

with the RPF

$$\begin{aligned}
\check{W}_b(X,\lambda) &= \int_{\mathbb{R}^2} \mathcal{M}_{\beta_2,\beta_3}(x,y)\mathcal{M}_{\gamma_2,\gamma_3}(-y,-x)\mathrm{d}x\mathrm{d}y\\
&= \left|\int_{\mathbb{R}} \bar{\varphi}_{\beta_2,\beta_3}(-x)\varphi_{\gamma_2,\gamma_3}(x)\mathrm{d}x\right|^2
\end{aligned}$$

$$
\begin{aligned}
&= \left| \int_{\mathbb{R}} \mathrm{e}^{2ic_b x(\beta_2-\gamma_2)} \frac{\Phi_b\left(-x+w(\beta_1)\right)}{\Phi_b\left(x-w(\gamma_1)\right)} \mathrm{d}x \right|^2 \\
&= \left| \int_{\mathbb{R}-i0} \mathrm{e}^{2ic_b z(2\pi-\epsilon-\epsilon_1)} \frac{\Phi_b\left(-z\right)}{\Phi_b\left(z-c_b\frac{\epsilon+\epsilon_2-2\pi}{\pi}\right)} \mathrm{d}z \right|^2
\end{aligned} \tag{83}
$$

where in the last equality, we used the formulae (76) for the weights on the edges of X. In the limit $\epsilon \to 0$ and $\epsilon_i \to 2\pi$, $i \in \{1, 2\}$, the totally balanced renormalized case corresponds to the equality $\beta = \gamma$ with the BRPF

$$
\tilde{W}_b(X, \lambda) = \left| \int_{\mathbb{R}-i0} \frac{\Phi_b\left(-z\right)}{\Phi_b\left(z\right)} \mathrm{d}z \right|^2 \tag{84}
$$

which corresponds to the figure-eight state integral in the Teichmüller TQFT, see [1, formula (38) and subsection 11.5] thus proving formula (14) for this example.

4.5. One-Vertex H-Triangulation of $(S^3, 5_2)$

Following the procedure described in Sect. 4.1, we construct a one-vertex H-triangulation X of the pair $(S^3, 5_2)$ according to Fig. 6, where in (d) one can easily identify four tetrahedra piled up from bottom to top. All tetrahedra are positive. Apart from the distinguished tetrahedron H (the bottom one containing the distinguished edge), we denote the others as T_1, T_2, T_3 in the order from bottom to top. We have the following face identifications:

$$
\begin{aligned}
&\partial_2 H \sim \partial_3 H,\ \partial_0 H \sim \partial_3 T_2,\ \partial_1 H \sim \partial_0 T_1,\ \partial_1 T_1 \sim \partial_0 T_2, \\
&\partial_2 T_1 \sim \partial_1 T_3,\ \partial_3 T_1 \sim \partial_0 T_3,\ \partial_1 T_2 \sim \partial_2 T_3,\ \partial_2 T_2 \sim \partial_3 T_3.
\end{aligned} \tag{85}
$$

For a state $s \in \mathbb{R}^{\Delta_1(X)}$, denote $s^{(i)}_{kl} \equiv s(kl_{T_i})$ and $s_{kl} \equiv s(kl_H)$, i.e., the variable $s^{(i)}_{kl}$ is the value of s on the edge kl of the tetrahedron T_i while the variable s_{kl} is the value on the edge kl of the tetrahedron H. We have five edge variables

$$
\begin{aligned}
x_0 &:= s(\bullet\!\!\Longrightarrow\!\!\bullet) = s_{01}, \\
x_1 &:= s(\bullet\!\!\longrightarrow\!\!\bullet) = s_{12} = s_{13} = s^{(2)}_{01} = s^{(2)}_{02} = s^{(3)}_{01}, \\
x_2 &:= s(\bullet\!\!\rightarrow\!\!\bullet) = s_{02} = s_{03} = s^{(1)}_{12} = s^{(1)}_{13} = s^{(3)}_{23}, \\
x_3 &:= s(\bullet\!\!\blacktriangleright\!\!\bullet) = s_{23} = s^{(1)}_{02} = s^{(1)}_{23} = s^{(2)}_{12} = s^{(2)}_{23} = s^{(3)}_{13}, \\
x_4 &:= s(\bullet\!\!\twoheadrightarrow\!\!\bullet) = s^{(1)}_{01} = s^{(1)}_{03} = s^{(2)}_{03} = s^{(2)}_{13} = s^{(3)}_{02} = s^{(3)}_{03} = s^{(3)}_{12},
\end{aligned} \tag{86}
$$

Denoting by $\alpha \in \mathbb{R}^{\square(H)}$, $\beta_i \in \mathbb{R}^{\square(T_i)}$, $i \in \{1, 2, 3\}$, the dihedral angle arrangements, we have the following weights on the edges of X:

$$
\begin{aligned}
\epsilon &:= \omega_X(\bullet\!\!\Longrightarrow\!\!\bullet) = \alpha_1, \\
\epsilon_1 &:= \omega_X(\bullet\!\!\longrightarrow\!\!\bullet) = 2\pi - \alpha_1 - \beta_{23} + \beta_{31}, \\
\epsilon_2 &:= \omega_X(\bullet\!\!\rightarrow\!\!\bullet) = 2\pi - \alpha_1 - \beta_{11} + \beta_{31}, \\
\epsilon_3 &:= \omega_X(\bullet\!\!\blacktriangleright\!\!\bullet) = 2\pi + \alpha_1 - \beta_{13} - \beta_{22} + \beta_{32}, \\
\epsilon_4 &:= \omega_X(\bullet\!\!\twoheadrightarrow\!\!\bullet) = 8\pi - \epsilon - \epsilon_1 - \epsilon_2 - \epsilon_3,
\end{aligned} \tag{87}
$$

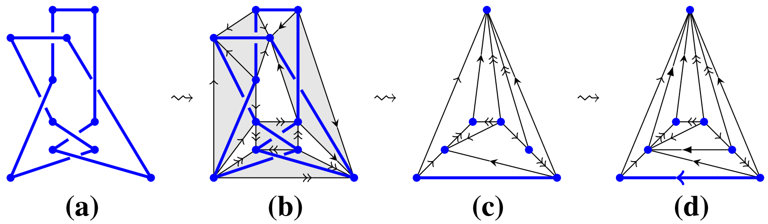

FIGURE 6. Construction of a one-vertex H-triangulation of the pair $(S^3, 5_2)$

where $\alpha_j = \alpha(q_{H,j})$, $\beta_{ij} := (\beta_i)_j := \beta_i(q_{T_i,j})$. The Boltzmann weight function reads as follows:

$$\begin{aligned} B(X,s) = {} & \mathcal{B}\left(\Delta\alpha_1, \Delta\alpha_2 + i\nabla(x_1 + x_2 - x_3 - x_0)\right) \\ & \times \mathcal{B}\left(\Delta\beta_{11} + i\nabla(x_3 - x_4), \Delta\beta_{13} + i\nabla(x_4 - x_2)\right) \\ & \times \mathcal{B}\left(\Delta\beta_{23} + i\nabla(x_3 - x_4), \Delta\beta_{22} + i\nabla(x_4 - x_1)\right) \\ & \times \mathcal{B}\left(\Delta\beta_{31} + i\nabla(x_3 - x_4), \Delta\beta_{32} + i\nabla(2x_4 - x_1 - x_2)\right) \end{aligned} \tag{88}$$

so that we have a decomposition (65) with

$$\begin{aligned} K(X,s,u) = {} & \mathcal{K}(0, x_1 + x_2 - x_3 - x_0, h_1, h_2)\mathcal{K}(x_3 - x_4, x_4 - x_2, u_{11}, u_{12}) \\ & \times \mathcal{K}(x_3 - x_4, x_4 - x_1, u_{21}, u_{22})\mathcal{K}(x_3 - x_4, 2x_4 - x_1 - x_2, u_{31}, u_{32}) \end{aligned} \tag{89}$$

and

$$\begin{aligned} & M(X,u) \\ & \quad = \mathcal{M}_{\alpha_2,\alpha_3}(h_1,h_2)\mathcal{M}_{\beta_{13},\beta_{12}}(u_{11},u_{12})\mathcal{M}_{\beta_{22},\beta_{21}}(u_{21},u_{22})\mathcal{M}_{\beta_{32},\beta_{33}}(u_{31},u_{32}) \end{aligned} \tag{90}$$

where $(h_1, h_2) := u(H)$, $(u_{i1}, u_{i2}) := u(T_i)$. Choosing the gauge fixing map λ so that $\langle \lambda_v, s\rangle = x_4/2$, we calculate

$$L(X,u,\lambda) = \int_{\mathbb{R}^5} K(X,s,u)\delta(x_4/2)\mathrm{d}x_0 \ldots \mathrm{d}x_4 = 2\delta(h_1)\delta(h_2)\check{L}(X,u,\lambda) \tag{91}$$

with

$$\begin{aligned} \check{L}(X,u,\lambda) = {} & \int_{\mathbb{R}^3} \mathcal{K}(x_3, -x_2, u_{11}, u_{12})\mathcal{K}(x_3, -x_1, u_{21}, u_{22}) \\ & \times \mathcal{K}(x_3, -x_1 - x_2, u_{31}, u_{32})\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\ = {} & \delta(u_{11} - u_{21})\delta(u_{11} + u_{32})\delta(u_{12} - u_{22})\delta(u_{12} + u_{31}). \end{aligned} \tag{92}$$

Now, corresponding to the factorization in the second equality of (91), the partition function also straightforwardly factorizes as in (82) with the RPF of the form

$$
\begin{aligned}
\check{W}_b(X,\lambda) &= \left| \int_{\mathbb{R}} \varphi_{\beta_{13},\beta_{12}}(x)\varphi_{\beta_{22},\beta_{21}}(x)\bar{\varphi}_{\beta_{32},\beta_{33}}(-x)\mathrm{d}x \right|^2 \\
&= \left| \int_{\mathbb{R}} \frac{\mathrm{e}^{2ic_b x(\beta_{32}-\beta_{22}-\beta_{13})}\Phi_b\left(w(\beta_{31})-x\right)}{\Phi_b\left(x-w(\beta_{11})\right)\Phi_b\left(x-w(\beta_{23})\right)}\mathrm{d}x \right|^2 \\
&= \left| \int_{\mathbb{R}-i0} \frac{\mathrm{e}^{2ic_b z(\epsilon_3-\epsilon-2\pi)}\Phi_b\left(-z\right)}{\Phi_b\left(z-c_b\frac{\epsilon_2+\epsilon-2\pi}{\pi}\right)\Phi_b\left(z-c_b\frac{\epsilon_1+\epsilon-2\pi}{\pi}\right)}\mathrm{d}z \right|^2. \qquad (93)
\end{aligned}
$$

Thus, in the fully balanced limit $\epsilon \to 0$ and $\epsilon_i \to 2\pi$, $i \in \{1,2,3\}$, we obtain the BRPF

$$
\tilde{W}_b(X,\lambda) = \left| \int_{\mathbb{R}-i0} \frac{\Phi_b\left(-z\right)}{\Phi_b\left(z\right)^2}\mathrm{d}z \right|^2 \qquad (94)
$$

which is consistent with the equality (14), see for the Teichmüller TQFT result in [1, formula (39) and subsection 11.7].

4.6. One-Vertex H-Triangulation of $(S^3, 6_1)$

Starting from a knot diagram, we construct an H-triangulation X of $(S^3, 6_1)$ following the procedure described in Sect. 4.1. The result is given in Fig. 7d where one can easily identify five tetrahedra piled up from bottom to top. Apart from the distinguished tetrahedron H (the bottom one containing the distinguished edge), we denote the others as T_1, T_2, T_3 and T_4 in the order from bottom to top. We have the following face identifications:

$$
\begin{aligned}
&\partial_2 H \sim \partial_3 H,\ \partial_0 H \sim \partial_0 T_1,\ \partial_1 H \sim \partial_3 T_3,\ \partial_1 T_1 \sim \partial_2 T_2,\ \partial_2 T_1 \sim \partial_0 T_4,\\
&\partial_3 T_1 \sim \partial_0 T_2,\ \partial_1 T_2 \sim \partial_1 T_4,\ \partial_3 T_2 \sim \partial_0 T_3,\ \partial_1 T_3 \sim \partial_3 T_4,\ \partial_2 T_3 \sim \partial_2 T_4.
\end{aligned} \qquad (95)
$$

As in the previous example, for a state $s \in \mathbb{R}^{\Delta_1(X)}$, we denote $s_{kl}^{(i)} \equiv s(kl_{T_i})$ and $s_{kl} \equiv s(kl_H)$. We have six edge variables

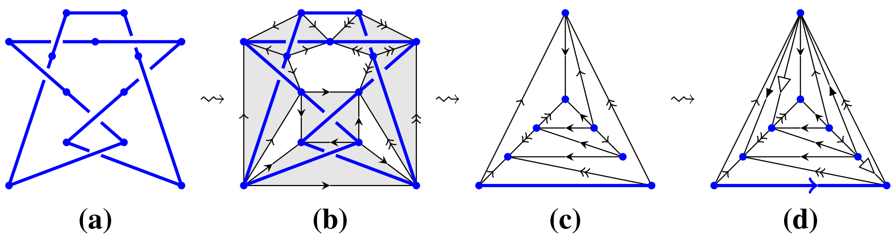

Figure 7. Construction of a one-vertex H-triangulation of the pair $(S^3, 6_1)$

$$
\begin{aligned}
x_0 &:= s({\color{blue}\bullet\!\!\longrightarrow\!\!\bullet}) = s_{01},\\
x_1 &:= s(\bullet\!\!-\!\!\rhd\!\!-\!\!\bullet) = s_{01}^{(1)} = s_{12}^{(2)} = s_{23}^{(3)} = s_{12}^{(4)},\\
x_2 &:= s(\bullet\!\!-\!\!\succ\!\!-\!\!\bullet) = s_{02} = s_{03} = s_{01}^{(3)} = s_{02}^{(3)} = s_{01}^{(4)},\\
x_3 &:= s(\bullet\!\!-\!\!\gg\!\!-\!\!\bullet) = s_{12} = s_{13} = s_{12}^{(1)} = s_{13}^{(1)} = s_{23}^{(2)} = s_{23}^{(4)},\\
x_4 &:= s(\bullet\!\!-\!\!\blacktriangleright\!\!-\!\!\bullet) = s_{23} = s_{02}^{(1)} = s_{23}^{(1)} = s_{01}^{(2)} = s_{13}^{(2)} = s_{12}^{(3)},\\
x_5 &:= s(\bullet\!\!\longrightarrow\!\!\bullet) = s_{03}^{(1)} = s_{02}^{(2)} = s_{03}^{(2)} = s_{03}^{(3)} = s_{13}^{(3)} = s_{02}^{(4)} = s_{03}^{(4)} = s_{13}^{(4)}.
\end{aligned}
\tag{96}
$$

Denoting by $\alpha \in \mathbb{R}^{\square(H)}$, $\beta_i \in \mathbb{R}^{\square(T_i)}$, $i \in \{1,2,3,4\}$, the dihedral angle arrangements, we have the following total angles around the edges of X:

$$
\begin{aligned}
\epsilon &:= \omega_X({\color{blue}\bullet\!\!\longrightarrow\!\!\bullet}) = \alpha_1,\\
\epsilon_1 &:= \omega_X(\bullet\!\!-\!\!\rhd\!\!-\!\!\bullet) = \beta_{11} + \beta_{23} + \beta_{31} + \beta_{43},\\
\epsilon_2 &:= \omega_X(\bullet\!\!-\!\!\succ\!\!-\!\!\bullet) = 2\pi - \alpha_1 - \beta_{33} + \beta_{41},\\
\epsilon_3 &:= \omega_X(\bullet\!\!-\!\!\gg\!\!-\!\!\bullet) = 2\pi - \alpha_1 - \beta_{11} + \beta_{21} + \beta_{41},\\
\epsilon_4 &:= \omega_X(\bullet\!\!-\!\!\blacktriangleright\!\!-\!\!\bullet) = 2\pi + \alpha_1 - \beta_{13} - \beta_{23} + \beta_{33},\\
\epsilon_5 &:= \omega_X(\bullet\!\!\longrightarrow\!\!\bullet) = 10\pi - \epsilon - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4,
\end{aligned}
\tag{97}
$$

where $\alpha_j = \alpha(q_{H,j})$, $\beta_{ij} := (\beta_i)_j := \beta_i(q_{T_i,j})$. The Boltzmann weight function reads as follows:

$$
\begin{aligned}
B(X,s) = {} & \mathcal{B}(\Delta\beta_{41} + i\nabla(x_5 - x_1), \Delta\beta_{43} + i\nabla(x_2 + x_3 - 2x_5))\\
& \times \mathcal{B}(\Delta\beta_{33} + i\nabla(x_5 - x_1), \Delta\beta_{32} + i\nabla(x_1 + x_2 - x_4 - x_5))\\
& \times \mathcal{B}(\Delta\beta_{23} + i\nabla(x_5 - x_3), \Delta\beta_{22} + i\nabla(x_3 + x_4 - x_1 - x_5))\\
& \times \mathcal{B}(\Delta\beta_{13} + i\nabla(x_3 - x_1), \Delta\beta_{11} + i\nabla(x_5 - x_4))\\
& \times \mathcal{B}(\Delta\alpha_1, \Delta\alpha_2 + i\nabla(x_2 + x_3 - x_4 - x_0))
\end{aligned}
\tag{98}
$$

so that we have a decomposition (65) with

$$
\begin{aligned}
K(X,s,u) = {} & \mathcal{K}(x_5 - x_1, x_2 + x_3 - 2x_5, u_{41}, u_{42})\\
& \times \mathcal{K}(x_5 - x_1, x_1 + x_2 - x_4 - x_5, u_{31}, u_{32})\\
& \times \mathcal{K}(x_5 - x_3, x_3 + x_4 - x_1 - x_5, u_{21}, u_{22})\\
& \times \mathcal{K}(x_3 - x_1, x_5 - x_4, u_{11}, u_{12})\\
& \times \mathcal{K}(0, x_2 + x_3 - x_4 - x_0, h_1, h_2)
\end{aligned}
\tag{99}
$$

and

$$
\begin{aligned}
M(X,u) = {} & \mathcal{M}_{\beta_{43},\beta_{42}}(u_{41}, u_{42}) \mathcal{M}_{\beta_{32},\beta_{31}}(u_{31}, u_{32})\\
& \times \mathcal{M}_{\beta_{22},\beta_{21}}(u_{21}, u_{22}) \mathcal{M}_{\beta_{11},\beta_{12}}(u_{11}, u_{12}) \mathcal{M}_{\alpha_2,\alpha_3}(h_1, h_2)
\end{aligned}
\tag{100}
$$

where $(h_1, h_2) := u(H)$, $(u_{i1}, u_{i2}) := u(T_i)$. Choosing the gauge fixing map λ so that $\langle \lambda_v, s\rangle = x_5/2$, we calculate

$$
L(X,u,\lambda) = \int_{\mathbb{R}^5} K(X,s,u)\delta(x_5/2)\mathrm{d}x_0 \ldots \mathrm{d}x_5 = 2\delta(h_1)\delta(h_2)\check{L}(X,u,\lambda) \tag{101}
$$

with

$$\begin{aligned}\check{L}(X,u,\lambda) &= \int_{\mathbb{R}^3} \mathcal{K}(-x_1, x_2+x_3, u_{41}, u_{42})\mathcal{K}(-x_1, x_1+x_2-x_4, u_{31}, u_{32}) \\ &\quad \times \mathcal{K}(-x_3, x_3+x_4-x_1, u_{21}, u_{22}) \\ &\quad \times \mathcal{K}(x_3-x_1, -x_4, u_{11}, u_{12})\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 \\ &= D(u_{11}, u_{22}, u_{32}, u_{41})\bar{D}(u_{12}, u_{21}, u_{31}, u_{42}) \end{aligned} \tag{102}$$

where

$$D(x_1,x_2,x_3,x_4) = \mathrm{e}^{\pi i x_1^2}\delta(x_1-x_2+x_3)\delta(x_3+x_4). \tag{103}$$

Now, corresponding to the factorization in the second equality of (101), the partition function also straightforwardly factorizes as in (82) with the RPF of the form

$$\begin{aligned}&\check{W}_b(X,\lambda) \\ &= \left| \int_{\mathbb{R}^2} \mathrm{e}^{\pi i x^2} \bar{\varphi}_{\beta_{11},\beta_{12}}(x)\varphi_{\beta_{22},\beta_{21}}(x-y)\varphi_{\beta_{32},\beta_{31}}(-y)\bar{\varphi}_{\beta_{43},\beta_{42}}(y)\mathrm{d}x\mathrm{d}y \right|^2 \\ &= \left| \int_{\mathbb{R}^2} \frac{\mathrm{e}^{\pi i x^2 - 2ic_b(x(\beta_{11}+\beta_{22})-y(\beta_{22}+\beta_{32}-\beta_{43}))}\Phi_b\left(w(\beta_{13})+x\right)\Phi_b\left(y+w(\beta_{41})\right)}{\Phi_b\left(x-y-w(\beta_{23})\right)\Phi_b\left(-y-w(\beta_{33})\right)}\mathrm{d}x\mathrm{d}y \right|^2 \\ &= \left| \int_{(\mathbb{R}+i0)^2} \frac{\mathrm{e}^{\pi i x^2 - 2\pi i x w(\beta_{13}-\beta_{11}-\beta_{22})+2ic_b y(\beta_{22}+\beta_{32}-\beta_{43})}\Phi_b\left(x\right)\Phi_b\left(y\right)}{\Phi_b\left(x-y-w(\beta_{23}+\beta_{13}-\beta_{41})\right)\Phi_b\left(-y-c_b\frac{\beta_{41}-\beta_{33}}{\pi}\right)}\mathrm{d}x\mathrm{d}y \right|^2 \\ &= \left| \int_{(\mathbb{R}+i0)^2} \frac{\mathrm{e}^{\pi i x^2 - 2\pi i x w(\pi+\epsilon+\epsilon_3-\epsilon_2-\epsilon_4)+2ic_b y(2\pi+\epsilon_2-\epsilon_1-\epsilon_3)}\Phi_b\left(x\right)\Phi_b\left(y\right)}{\Phi_b\left(x-y-w(4\pi-\epsilon_2-\epsilon_4)\right)\Phi_b\left(-y+c_b\frac{2\pi-\epsilon-\epsilon_2}{\pi}\right)}\mathrm{d}x\mathrm{d}y \right|^2 \end{aligned} \tag{104}$$

Thus, in the fully balanced limit $\epsilon \to 0$ and $\epsilon_i \to 2\pi$, $i \in \{1,2,3\}$, we obtain the BRPF

$$\tilde{W}_b(X,\lambda) = \left| \int_{(\mathbb{R}+i0)^2} \frac{\mathrm{e}^{\pi i x^2 - 4\pi i c_b x}\Phi_b\left(x\right)\Phi_b\left(y\right)}{\Phi_b\left(x-y-c_b\right)\Phi_b\left(-y\right)}\mathrm{d}x\mathrm{d}y \right|^2. \tag{105}$$

5. Application to 3d Supersymmetric Field Theories

5.1. 3d Supersymmetric Theories Living on a Squashed Three-Sphere

Following the work of Pestun [36], the partition functions of 3d $\mathcal{N}=2$ supersymmetric theories, defined on a squashed three-sphere S_b^3, were calculated in the papers [22,23,26,27] using the localization method. These partition functions are given in the form of integrals with the integrands composed of hyperbolic gamma functions [16,55]. For any 3d $\mathcal{N}=2$ supersymmetric theory defined on S_b^3 with a gauge group G and a flavor group F, the corresponding partition function has the following structure

$$Z(\underline{f}) = \int_{-i\infty}^{i\infty} \prod_{j=1}^{\mathrm{rank}G} \mathrm{d}u_j \, J(\underline{u})Z^{vec}(\underline{u}) \prod_I Z_{\Phi_I}^{\mathrm{chir}}(\underline{f},\underline{u}). \tag{106}$$

Here, the integral is taken over u_j variables which are associated with the Weyl weights for the Cartan subalgebra of the gauge group G and the f_ks denote the chemical potentials for the flavor symmetry group F.[3] For CS theory, one has $J(\underline{u}) = \mathrm{e}^{-\pi i k \sum_{j=1}^{\mathrm{rank}G} u_j^2}$, where k is the level of the CS term, while for SYM theories one has $J(\underline{u}) = \mathrm{e}^{2\pi i \lambda \sum_{j=1}^{\mathrm{rank}G} u_j}$, where λ is the Fayet–Illiopoulos term. The terms $Z^{\mathrm{vec}}(\underline{u})$ and $Z^{\mathrm{chir}}_{\Phi_I}(\underline{f}, \underline{u})$ in (106) come from the vector superfield and the matter fields, respectively, and are given in terms of the hyperbolic gamma function.

The result of localization allows us to relate the physical theory with some matrix integral of the form (106). Also, we can invert the logic: having some matrix integral of the type (106) one can find a 3d $\mathcal{N} = 2$ supersymmetric field theory whose partition function is given by this matrix integral [16]. Thus, all the partition functions which we get by considering (9) can be interpreted as partition functions for some 3d $\mathcal{N} = 2$ supersymmetric field theories. Moreover, as the expression (9) corresponds to some triangulation of a three-dimensional manifold M, we obtain a link between three-manifolds and 3d $\mathcal{N} = 2$ supersymmetric field theories defined on S_b^3. This is known as a 3d/3d duality considered recently in [12,13,49,50] (see also [48] for the relation of the objects to four-dimensional supersymmetric field theories).

In [13], the state variables live in the faces, while in our case the state variables live on the edges. To get a 3d theory from 3d manifold M, one has to triangulate this manifold and calculate its partition function (9) and then interpret this expression as a partition function (106). One should notice that every common edge corresponds to abelian gauge group.

Let us start from our building block: the tetrahedral Boltzmann weight composed of three hyperbolic gamma functions, each corresponding to the contribution coming from the 3d $\mathcal{N} = 2$ chiral hypermultiplet. Namely,

$$\mathcal{B}(T, x) = \prod_{j=1}^{3} \gamma^{(2)} \left(\Delta\alpha_j + i\nabla(x_{j+1} + x'_{j+1} - x_{j-1} - x'_{j-1}); \omega_1, \omega_2 \right), \quad (107)$$

corresponds to three chiral superfields Q_j, $j = 1, 2, 3$, with $SU(3)$ global symmetry group (since $\sum_{j=1}^{3}(x_{j+1} + x'_{j+1} - x_{j-1} - x'_{j-1}) = 0$) and a superpotential

$$W \sim Q_1 Q_2 Q_3,$$

which has a correct R charge. This can be easily seen from the fact that the dihedral angles α_j, $j = 1, 2, 3$, correspond to R charges of three chiral superfields. And since $\sum_{j=1}^{3} \alpha_j = \pi$ then the R_W charge of the superpotential W is given as

$$R_W = \sum_{j=1}^{3} R_{Q_j} = \sum_{j=1}^{3} 2\alpha_j/\pi = 2.$$

---

[3] From physical point of view, f_k's are linear combinations of the R-charge, the masses of the hypermultiplets, and the Fayet–Illiopoulos terms associated to the additional Abelian global symmetries.

The first non-trivial case of a 3d theory with a non-trivial gauge group is the pentagon identity (47) (which realizes the 2-3 Pachner move) when we take two positive tetrahedra and glue them together over the common face. The partition function of two glued tetrahedra having vertices $(0,1,2,4)$ and $(0,2,3,4)$ is

$$\begin{aligned} W_{b,A} = \mathcal{B}(&\Delta\alpha_1+i\nabla(x_{02}+x_{34}-x_{03}-x_{24}), \Delta\alpha_2+i\nabla(x_{03}+x_{24}-x_{04}-x_{23})) \\ &\times\mathcal{B}(\Delta\beta_1+i\nabla(x_{01}+x_{24}-x_{02}-x_{14}), \Delta\beta_2+i\nabla(x_{02}+x_{14}-x_{04}-x_{12})) \end{aligned} \tag{108}$$

where

$$\alpha_1 = a_2 + b_1, \quad \alpha_2 = a_3 + b_2, \quad \beta_1 = a_1 + b_2, \quad \beta_2 = a_3 + b_1, \tag{109}$$

and $\sum_{j=1}^{3}(a_j + b_j) = \omega_1 + \omega_2$. This is the partition function for a theory A which consists of six 3d $\mathcal{N} = 2$ free chiral hypermultiplets with $F = SU(3) \times SU(3) \times U(1)$ global symmetry group. Each group $SU(3)$ corresponds to a separate tetrahedron and $U(1)$ group distinguishes the two tetrahedra.

On the other hand, using $2-3$ Pachner move, two glued tetrahedra can be considered as three tetrahedra with the vertices $(0,1,2,3), (0,1,3,4)$ and $(1,2,3,4)$ having a common edge x_{04} whose partition function is

$$\begin{aligned} &W_{b,B} \\ &\quad= \int_{\mathbb{R}} \mathcal{B}(\Delta a_1+i\nabla(x_{01}+x_{23}-x_{02}-x_{13}), \Delta b_1+i\nabla(x_{02}+x_{13}-x_{03}-x_{12})) \\ &\qquad\times\mathcal{B}(\Delta a_2+i\nabla(x_{12}+x_{34}-x_{13}-x_{24}), \Delta b_2+i\nabla(x_{13}+x_{24}-x_{14}-x_{23})) \\ &\qquad\times\mathcal{B}(\Delta a_3+i\nabla(x_{01}+x_{34}-x_{03}-x_{14}), \Delta b_3+i\nabla(x_{03}+x_{14}-x_{04}-x_{13}))\mathrm{d}x_{13}. \end{aligned} \tag{110}$$

It gives the partition function for a 3d $\mathcal{N} = 2$ SQED theory B (which has $U(1)$ gauge group) with 3 flavors and overall $F = SU(3) \times SU(3) \times U(1)$ global symmetry group and 2 singlet baryons. There are three tetrahedra in this picture so one can think of $SU(3)^3$ global symmetry group but a part of this, namely, $U(1)$ becomes a gauge group leaving $SU(3) \times SU(3) \times U(1)$ global symmetry group.

Since the partition functions in (108) and (110) are equal, one can say that the theories A and B are the same or dual to each other. Generally, different triangulations of three-manifolds produce different phases of the same theory, in other words, we get dual descriptions for 3d supersymmetric field theories related to a given three-dimensional manifold.

As a next example, we consider four tetrahedra T_1, T_2, T_3, T_4 built, respectively, from vertices $(0,1,2,5)$, $(0,2,3,5)$, $(0,3,4,5)$, $(0,1,4,5)$. They are glued together to form an octahedron and share a common edge x_{05}. From the face identification

$$\partial_1 T_1 \sim \partial_3 T_2, \quad \partial_2 T_2 \sim \partial_4 T_3, \quad \partial_3 T_3 \sim \partial_1 T_4, \quad \partial_2 T_1 \sim \partial_4 T_4, \tag{111}$$

we get the state variables

$$\begin{aligned}
&x_{05}^{(1)}=x_{05}^{(2)}=x_{05}^{(3)}=x_{05}^{(4)}, \quad x_{25}^{(1)}=x_{25}^{(2)}, \quad x_{02}^{(1)}=x_{02}^{(2)}, \quad x_{35}^{(2)}=x_{35}^{(3)},\\
&x_{03}^{(2)}=x_{03}^{(3)}, \quad x_{45}^{(3)}=x_{45}^{(4)}, \quad x_{04}^{(3)}=x_{04}^{(4)}, \quad x_{15}^{(1)}=x_{15}^{(4)}, \quad x_{01}^{(1)}=x_{01}^{(4)}, \qquad (112)
\end{aligned}$$

so that the partition function is equal to

$$\begin{aligned}
W_{b,\text{Octahedron}} = \int &\mathrm{d}x_{05}^{(1)}\\
&\times \mathcal{B}(\Delta\alpha_1 + i\nabla(x_{02}^{(1)}+x_{15}^{(1)}-x_{12}^{(1)}-x_{05}^{(1)}),\\
&\quad \Delta\alpha_2 + i\nabla(x_{12}^{(1)}-x_{01}^{(1)}-x_{25}^{(1)}+x_{05}^{(1)}))\\
&\times \mathcal{B}(\Delta\beta_1 + i\nabla(x_{03}^{(2)}+x_{25}^{(1)}-x_{23}^{(2)}-x_{05}^{(1)}),\\
&\quad \Delta\beta_2 + i\nabla(x_{23}^{(2)}-x_{02}^{(1)}-x_{35}^{(2)}+x_{05}^{(1)}))\\
&\times \mathcal{B}(\Delta\gamma_1 + i\nabla(x_{04}^{(3)}+x_{35}^{(2)}-x_{34}^{(3)}-x_{05}^{(1)}),\\
&\quad \Delta\gamma_2 + i\nabla(x_{34}^{(3)}-x_{03}^{(2)}-x_{45}^{(3)}+x_{05}^{(1)}))\\
&\times \mathcal{B}(\Delta\delta_1 - i\nabla(x_{14}^{(4)}-x_{01}^{(1)}-x_{45}^{(3)}+x_{05}^{(1)}),\\
&\quad \Delta\delta_2 - i\nabla(x_{04}^{(3)}+x_{15}^{(1)}-x_{14}^{(4)}-x_{05}^{(1)})), \qquad (113)
\end{aligned}$$

which corresponds to the partition function of a 3d $\mathcal{N}=2$ SQED theory with 4 flavors and four singlet baryons with the overall global symmetry $SU(3)^3 \times U(1)$. Octahedron can be also represented by gluing five tetrahedra, and in the next subsection we interpret that move using the Bailey tree technique. As we will show, the triangulation with five tetrahedra gives a dual description of the starting theory in terms of a quiver gauge theory with $U(1)\times U(1)$ gauge group.

In a similar manner, gluing F tetrahedra sharing one common edge, one gets the partition function for a 3d $\mathcal{N}=2$ SQED theory with F flavors and F additional singlet baryons which has $SU(3)^{F-1}\times U(1)$ global symmetry group (since $U(1)$ becomes a gauge group).

5.2. Bailey Tree Technique

There is an alternative way to see the constructions of the previous subsection based on the application of Bailey tree technique for hyperbolic integrals (very much in the spirit of [45]). This approach gives an algebraic way of getting the partition functions and relates the triangulated three-dimensional manifolds from one side, and 3d supersymmetric field theories defined on a squashed three-sphere, from the other. The Bailey tree technique is useful for tracking different triangulations related to each other by 2-3 Pachner move from the algebraic viewpoint.

Definition 1. We say that two functions $\alpha(z)$ and $\beta(z)$, $z\in\mathbb{C}$, form an integral hyperbolic Bailey pair (the hyperbolic level) with respect to a parameter $t\in\mathbb{C}$ if

$$\beta(w) = \int \mathcal{B}(t+w-z, t-w+z)\alpha(z)\mathrm{d}z. \qquad (114)$$

To indicate explicitly the parameter t, we will write $\alpha(z,t)$ and $\beta(z,t)$ though the functions can also depend on other parameters.

Theorem 2. (Follows from Theorem 1 [45]) *Whenever two functions $\alpha(z,t)$ and $\beta(z,t)$ form an integral hyperbolic Bailey pair with respect to t, the new functions*

$$\alpha'(w,s+t)=\mathcal{B}(t+u+w,2s)\alpha(w,t) \tag{115}$$

and

$$\beta'(w,s+t)=\int\mathcal{B}(s+w-x,u+x)\mathcal{B}(s+2t+u+w,s-w+x)\beta(x,t)\mathrm{d}x, \tag{116}$$

form an integral hyperbolic Bailey pair with respect to parameter $s+t$.

Proof. The proof is similar to the proof in the elliptic case [45]. We start from the definition for $\beta'(w,s+t)$:

$$\beta'(w,s+t)=\int\mathcal{B}(s+w-x,u+x)\mathcal{B}(s+2t+u+w,s-w+x)\beta(x,t)\mathrm{d}x,$$

where we substitute $\beta(x,t)$ from equation (114):

$$\begin{aligned}\beta'(w,s+t)=\int&\mathcal{B}(s+w-x,u+x)\mathcal{B}(s+2t+u+w,s-w+x)\\&\times\mathcal{B}(t+x-y,t-x+y)\alpha(y,t)\mathrm{d}y\mathrm{d}x.\end{aligned} \tag{117}$$

In the latter expression, we can apply formula

$$\int\prod_{j=1}^{3}\gamma^{(2)}(a_j-u;\omega_1,\omega_2)\gamma^{(2)}(b_j+u;\omega_1,\omega_2)\mathrm{d}u=\prod_{k,l=1}^{3}\gamma^{(2)}(a_k+b_l;\omega_1,\omega_2), \tag{118}$$

where $\sum_{j=1}^{3}(a_j+b_j)=\omega_1+\omega_2$, so that we get

$$\beta'(w,s+t)=\int\mathcal{B}(s+t+w-x,s+t-w+x)\alpha'(x,s+t)\mathrm{d}x. \tag{119}$$

□

From identity (118), one gets the following Bailey pair

$$\alpha(z,t)=\prod_{j=1}^{2}\mathcal{B}(\alpha_j-z,\beta_j+z), \tag{120}$$

where $2t+\sum_{j=1}^{2}(\alpha_j+\beta_j)=\omega_1+\omega_2$, and

$$\beta(w,t)=\prod_{j=1}^{2}\mathcal{B}(t+w+\alpha_j,t-w+\beta_{3-j}). \tag{121}$$

The pentagon identity permits us to define a particular Bailey pair thus giving to Bailey pairs a topological interpretation in terms of the 2-3 Pachner moves.

In other words, the construction of new Bailey pairs through Theorem 2 corresponds to changing a triangulation by the 2-3 Pachner move.

In the case of an octahedron triangulated into four tetrahedra which we considered in the previous subsection, the partition function (119) can be written as

$$Z_{4\Delta's} = \int \mathcal{B}(s+t+w-x, s+t-w+x)\mathcal{B}(t+u+x, 2s) \times \prod_{j=1}^{2} \mathcal{B}(\alpha_j - x, \beta_j + x)\mathrm{d}x, \tag{122}$$

where $2t + \sum_{j=1}^{2}(\alpha_j + \beta_j) = \omega_1 + \omega_2$. On the other hand, this expression is equal to (117)

$$Z_{5\Delta's} = \int \mathcal{B}(s+w-x, u+x)\mathcal{B}(s-w+x, 2t+s+u+w) \times \mathcal{B}(t+x-y, t-x+y) \prod_{j=1}^{2} \mathcal{B}(\alpha_j - y, \beta_j + y)\mathrm{d}y\mathrm{d}x, \tag{123}$$

which corresponds to triangulation of the octahedron in terms of five tetrahedra. Repeating this procedure, one can further increase the number of tetrahedra thus obtaining new equalities for dual 3d supersymmetric field theories related to these triangulations.

6. Relationship to Representations to PSL$(2, \mathbb{C})$

In this section, we briefly discuss the relationship between the invariant $W_b(X, t, \lambda)$ and representations of the corresponding fundamental group into PSL$(2, \mathbb{C})$ and simplicial Chern–Simons theory.

6.1. Angle Structures and Representations of Fundamental Groups to PSL$(2, \mathbb{C})$

For simplicity, let us assume that $X = (M, \mathcal{T})$ is an oriented triangulated closed pseudo-three-manifold, i.e., $\partial X = \emptyset$, where M is the underlying pseudo-three-manifold and $\mathcal{T}$ is the triangulation. Let $\square(\mathcal{T})$ be the set of all quads in $\mathcal{T}$. Recall that $\mathbb{Z}/3\mathbb{Z} = \{1, \tau, \tau^2\}$ acts on $\square(\mathcal{T})$ corresponding to the cyclic order of three edges around each vertex. For $q \in \square(\mathcal{T})$, we will use q' and q'' to denote $\tau(q)$ and $\tau^2(q)$ below. A *shaped structure* on X (or $\mathcal{T}$) is a function $\alpha : \square(\mathcal{T}) \to (0, \pi)$ so that $\alpha(q) + \alpha(q') + \alpha(q'') = \pi$ for all $q \in \square(\mathcal{T})$. The *weight* of a shape structure α is the function $f : \Delta_1(\mathcal{T}) \to \mathbb{R}$ sending each edge e to $f(e) = \sum_{q \sim e} \alpha(q)$ where $q \sim e$ means the quad q faces the edge e. In particular, an *angle structure* is a shaped structure whose weight at each edge is 2π.

The invariant $W_b(X)$ in Theorem 1 is defined for each shaped triangulation, i.e., $W_b(X) = W_b(M; \mathcal{T}, \alpha)$. Theorem 1 implies that $W_b(M; \mathcal{T}_3, \alpha) = W_b(M; \mathcal{T}_2, \beta)$ if $\mathcal{T}_3$ is obtained from $\mathcal{T}_2$ by a 2-3 Pachner move so that β is the angle structure on $\mathcal{T}_2$ induced by the angle structure α on $\mathcal{T}_3$. (The equation for defining β from α is indicated in Fig. 1). In general, there are many different (or may be none) angle structures on $\mathcal{T}_3$ inducing the same angle structure

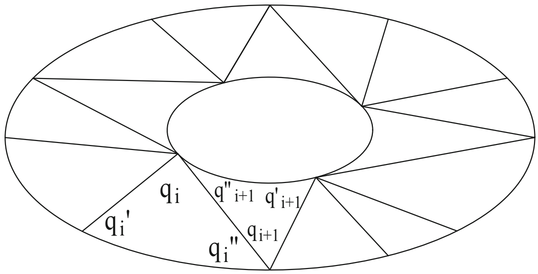

Figure 8. Edge loop in a vertex link

on $\mathcal{T}_2$. These different angles structures are related by a gauge transformation induced by the degree 3 edge in $\mathcal{T}_3$. Theorem 1 says that $W_b(M;\mathcal{T},\alpha)$ depends only on the (edge type) gauge equivalence class of the shaped structure α.

We will describe briefly the edge type gauge equivalence class now. Recall that a *tangential angle structure* on $\mathcal{T}$ (see [35]) is a map $x : \square(\mathcal{T}) \to \mathbb{R}$ so that for each $q \in \square(\mathcal{T})$, $x(q)+x(q')+x(q'')=0$ and for each edge $e \in \Delta_1(\mathcal{T})$, $\sum_{q\sim e} x(q) = 0$. Thus the space of all tangential angle structures is a vector space, denoted by $\mathrm{TAS}(\mathcal{T})$. For any shape structure α, $v \in \mathrm{TAS}(\mathcal{T})$ and small t, $\beta = \alpha + tv$ is still a shape structure so that β and α have the same weight. A generating set of vectors for $\mathrm{TAS}(\mathcal{T})$ was well known and can be described as follows. Consider the vertex link $lk(v)$ of a vertex $v \in \Delta_0(\mathcal{T})$. Let s be an edge loop in the dual CW decomposition of the triangulated surface $lk(v)$. The loop s can be described as a sequence of triangles $\{t_1,...,t_n\}$ and edges $\{\epsilon_1,...,\epsilon_n\}$ in $lk(v)$ so that ϵ_i is adjacent to t_i and t_{i+1} ($t_{n+1}=t_1$). Since each t_i corresponds to a tetrahedron T_i and each ϵ_i corresponds to a co-dimension-1 face F_i in $\mathcal{T}$, each T_i contains a unique quad q_i facing the edge $F_{i-1}\cap F_i$ in T_i (Fig. 8).

Define a map $g_s : \square(\mathcal{T}) \to \mathbb{R}$ by $g_s(q_i')=1$, $g_s(q_i'')=-1$ and $g_s(q)=0$ for all other qs. One checks easily that $g_s \in \mathrm{TAS}(\mathcal{T})$. In particular, if s is the loop around a vertex u in $lk(v)$, then g_s is the gauge transformation associated to the edge e corresponding to u. Two shaped structures α and β on $\mathcal{T}$ are *edge type gauge equivalent* if their difference $\alpha-\beta$ is a linear combinations of g_ss for edge loops s which are around vertices in vertex links. Theorem 1 says that $W_b(M;\mathcal{T},\alpha)$ depends only on the edge type gauge equivalence classes. A theorem in [51] shows $\mathrm{TAS}(\mathcal{T})$ is generated by vectors g_s. Define the *angle holonomy* $\alpha(s)$ of a shaped structure α along an edge loop s in $lk(v)$ to be $\sum_{i=1}^n \alpha(q_i)$. The works of [1,51] show that two shaped structures are edge type gauge equivalent if and only if they have the same angle holonomy along any edge loop s in vertex links. This suggests a way to represent the edge type gauge equivalence class of shaped structures using volume optimization. Namely, given a shaped structure α, let A_α be the set of all shaped structures on $\mathcal{T}$ edge type gauge equivalent to α. The *volume* of a shape structure is the sum of the volume of the hyperbolic tetrahedra determined by the shape. It is well known that volume is a strictly concave function of shape structure α. In particular, there is at most one shape structure $\beta \in A_\alpha$ which has the maximum volume. Note that it may not exist in A_α, i.e., the maximum volume point may appear in the boundary of the closure of A_α. Suppose now that α is an angle structure and the maximum volume β exists in A_α. Then by the standard volume optimization method (see [19,38], or [35]), one sees that the complex shape parameter z_β given by (3) associated to β satisfies Thurston's gluing equation. Therefore, it produces a representation ρ of $\pi_1(M - \Delta_0(\mathcal{T}))$ to $\mathrm{PSL}(2,\mathbb{C})$ so that for any edge loop s in $lk(v)$, the eigenvalues of $\rho(s)$ are of the form $re^{\pm\sqrt{-1}\beta(s)/2}$ for $r \in \mathbb{R}_{>0}$. This shows if there exists an angle structure of the maximum volume edge type gauge equivalent to α, one can assign the invariant $W_b(M;\mathcal{T},\alpha)$ to the representation ρ, i.e., the invariant $W_b(M;\mathcal{T},\alpha)$ may be an invariant of a pair (M,ρ). The precise conjectural picture of $W_b(M;\mathcal{T},\alpha)$ is: if two angle structures $(\mathcal{T}_i,\alpha_i)$ $(i=1,2)$ are associated to the same representation ρ, then $W_b(M;\mathcal{T}_1,\alpha_1) = W_b(M;\mathcal{T}_2,\alpha_2)$.

6.2. Relationship with Simplicial PSL$(2,\mathbb{R})$ Chern–Simons Theory

In [34], we proposed a variational principle for finding real-valued solutions of Thurston's equation on a triangulated oriented closed pseudo-three-manifold $(M;\mathcal{T})$. Given $(M;\mathcal{T})$, we introduce the homogeneous Thurston's equation (HTE) as follows. A map $x : \square(\mathcal{T}) \to \mathbb{R}$ is said to solve HTE if for each $q \in \square(\mathcal{T})$, $x(q) + x(q') + x(q'') = 0$ and for each edge e in $\mathcal{T}$,

$$\prod_{q\sim e} x(q') = \prod_{q\sim e}(-x(q'')).$$

It can be proved that solutions to Thurston's equation over the real numbers on $(M,\mathcal{T})$ correspond to nowhere zero solutions to HTE. The main observation in [34] is that critical points of an entropy function of the form $\sum_{i=1}^n x_i \ln(|x_i|)$ are nowhere zero solutions to HTE. The converse also holds if M is a closed three-manifold.

Our pentagon relation (52) implies the following pentagon relation for the entropy. Namely, given five positive numbers a_1, a_2, b_1, b_2, b_3 so that $\sum_{i=1}^2 a_i + \sum_{j=1}^3 b_i = 1$ and $a_1a_2 = b_1b_2b_3$, then

$$\sum_{i,j}(a_i+b_j)\ln(a_i+b_j) = \sum_{i=1}^{2}(a_i\ln(a_i) + (1-a_i)\ln(1-a_i)) + \sum_{j=1}^{3} b_j\ln(b_j). \tag{124}$$

Identity (124) suggests there should exist a non-quantum topological invariant for three-manifold from simplicial $SL(2,\mathbb{R})$ Chern–Simons theory. Furthermore, this invariant should be the semi-classical limit of $W_b(M;\mathcal{T},\alpha)$ when b degenerates.

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Appendix A. Special Functions

A.1. Faddeev's Quantum Dilogarithm

Faddeev's quantum dilogarithm $\Phi_b(z)$ is defined by the integral

$$\Phi_b(z) \equiv \exp\left(\int_{\mathbb{R}+i0} \frac{\mathrm{e}^{-2izw}\mathrm{d}w}{4\sinh(wb)\sinh(w/b)w}\right), \tag{125}$$

in the strip $|\mathrm{Im}z| < |\mathrm{Im}c_b|$, where

$$c_b := i(b+b^{-1})/2. \tag{126}$$

It is usefull to define

$$\zeta_{\mathrm{inv}} := \mathrm{e}^{\pi i(1+2c_b^2)/6} = \mathrm{e}^{\pi i c_b^2}\zeta_o^2, \quad \zeta_o := \mathrm{e}^{\pi i(1-4c_b^2)/12}. \tag{127}$$

$$\begin{aligned}
&\text{symmetry} && \Phi_b(z) = \Phi_{-b}(z) = \Phi_{1/b}(z), && (128)\\
&\text{functional equations} && \Phi_b(z - ib^{\pm 1}/2) = (1+\mathrm{e}^{2\pi b^{\pm 1}z})\Phi_b(z+ib^{\pm 1}/2), && (129)\\
&\text{inversion property} && \Phi_b(z)\Phi_b(-z) = \zeta_{\mathrm{inv}}^{-1}\mathrm{e}^{\pi i z^2}, && (130)\\
&\text{zeros} && z \in \left\{c_b + mib + nib^{-1}; m,n \in \mathbb{Z}^{\geq 0}\right\}, && (131)\\
&\text{poles} && z \in \left\{-c_b - mib - nib^{-1}; m,n \in \mathbb{Z}^{\geq 0}\right\}, && (132)\\
&\text{unitarity} && \overline{\Phi_b(z)} = 1/\Phi_b(\overline{z}). && (133)
\end{aligned}$$

A.2. The Elliptic Gamma Function

The elliptic gamma function is defined by the formula

$$\Gamma(z;p,q) \;=\; \prod_{i,j=0}^{\infty} \frac{1-z^{-1}p^{i+1}q^{j+1}}{1-zp^iq^j} \tag{134}$$

and which has the following properties:

$$\begin{aligned}
&\text{symmetry} && \Gamma(z;p,q) = \Gamma(z;q,p), && (135)\\
&\text{functional equations} && \Gamma(qz;p,q) = \theta(z;p)\Gamma(z;p,q), && (136)\\
& && \Gamma(pz;p,q) = \theta(z;q)\Gamma(z;p,q), && (137)\\
&\text{reflection property} && \Gamma(z;p,q)\,\Gamma\left(\frac{pq}{z};p,q\right) = 1, && (138)\\
&\text{zeros} && z \in \left\{p^{i+1}q^{j+1}; i,j \in \mathbb{Z}^{\geq 0}\right\}, && (139)
\end{aligned}$$

$$\text{poles} \quad z \in \left\{p^{-i}q^{-j}; i,j \in \mathbb{Z}^{\geq 0}\right\}, \tag{140}$$

$$\text{residue} \quad \operatorname*{Res}_{z=1} \Gamma(z;p,q) = -\frac{1}{(p;p)_\infty (q;q)_\infty}. \tag{141}$$

Here $\theta(z;p)$ is a theta-function $\theta(z;p) = (z;p)_\infty (pz^{-1};p)_\infty$.

A.3. Some Useful Formulas

Faddeev's quantum dilogarithm and the hyperbolic gamma functions are related via the formula

$$\gamma^{(2)}(-i\sqrt{\omega_1\omega_2}(x+c_b);\omega_1,\omega_2) = \frac{\mathrm{e}^{i\pi x^2/2}}{\sqrt{\zeta_{\text{inv}}}\Phi_b(x)}, \tag{142}$$

where $b := \sqrt{\frac{\omega_1}{\omega_2}}$.

Recall that the inversion relation (33) for $\gamma^{(2)}(x)$ is of the form

$$\gamma^{(2)}(x;\omega_1,\omega_2)\gamma^{(2)}(\omega_1+\omega_2-x;\omega_1,\omega_2) = 1 \tag{143}$$

and the complex conjugation property

$$\overline{\gamma^{(2)}(z)} = \gamma^{(2)}(\bar{z}). \tag{144}$$

If we define

$$\mathcal{B}(u,v) := \frac{\gamma^{(2)}(u;\omega_1,\omega_2)\gamma^{(2)}(v;\omega_1,\omega_2)}{\gamma^{(2)}(u+v;\omega_1,\omega_2)} \tag{145}$$

then it is easy to see that

$$\mathcal{B}\left(\sqrt{-\omega_1\omega_2}x, \sqrt{-\omega_1\omega_2}y\right) = \Psi\left(\frac{x}{2}+c_b, -\frac{x}{2}-c_b, y\right) \tag{146}$$

where

$$\Psi(u,v,w) := \int_{\mathbb{R}} \frac{\Phi_b(u+x)}{\Phi_b(v+x)} \mathrm{e}^{2\pi i w x} \mathrm{d}x, \tag{147}$$

which is calculated as follows [18]

$$\Psi(u,v,w) = \zeta_o \frac{\Phi_b(u-v-c_b)\Phi_b(w+c_b)}{\Phi_b(u-v+w-c_b)} \mathrm{e}^{-2\pi i w(v+c_b)}. \tag{148}$$

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