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A Renormalization Group Method by Harmonic Extensions and the Classical Dipole Gas

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Abstract. In this paper, we develop a new renormalization group method, which is based on conditional expectations and harmonic extensions, to study functional integrals of small perturbations of Gaussian fields. In this new method, one integrates Gaussian fields inside domains at all scales conditioning on the fields outside these domains, and by the variation principle solves local elliptic problems. It does not rely on an a priori decomposition of the Gaussian covariance. We apply this method to the model of classical dipole gas on the lattice, and show that the scaling limit of the generating function with smooth test functions is the generating function of the renormalized Gaussian free field.

1. Introduction

In this paper, we develop a renormalization group (RG) method to estimate functional integrals, based on the ideas of conditional expectations and harmonic extensions. We demonstrate this method with the model of classical dipole gas, which has always been considered as a simple model to start with for this type of problems. For the classical dipole model, earlier important works are [29,32]. The renormalization group approach to this model originated from the works by Gawedzki and Kupiainen [34,35], based on Kadanoff spin blockings. A different method that uses the idea of decomposition of the covariance of the Gaussian field was initiated from [17], and was simplified and pedagogically presented in the lecture notes [18], see also [24]. The latter method has achieved several important applications in other problems such as the two-dimensional Coulomb gas model [26–28], ϕ^4-type field theories [7,9,10,21] and self-avoiding walks [8,11,20] (see also the recent works [5,6,12–15]). The ϕ^4 field theory problems are also studied in the p-adics setting by [1] which yields some strong consequences.

Our method is different from the above two methods, and may be as well regarded as a variation of the method by Brydges et al. Their decomposition of covariance scheme, which was also used by other people such as [33], could be implemented by Fourier analysis. In [19], a decomposition of Gaussian covariance with every piece of covariance having finite range was constructed using elliptic partial differential equation techniques, which also depends to some extent on Fourier analysis, and this decomposition is the foundation of the simplified version of their RG method (see also [2,4,16] for alternative constructions of such decompositions). We do not perform such a decomposition of covariance. Instead, we directly take harmonic extensions as our basic scheme and use the Poisson kernel to smooth the Gaussian field. We do not need Fourier analysis; instead, real space decay rates of Poisson kernels and (derivatives of) Green's functions are essential. Some complexities in [19] such as proof of elliptic regularity theorem on lattice are avoided. Many elements of this method such as the polymer expansions and so on are very close to the method by Brydges et al, especially to [18], while we also have many new features, such as simpler norms and regulators. We keep notations as close as possible to [18] for convenience of the readers who are familiar with [18].

Very roughly speaking, our method is aimed to study functional integrals of the form

$$Z = \mathbb{E}\big[\mathrm{e}^{V(\phi)}\big],$$

where ϕ is a Gaussian field and $\mathbb{E}$ is an expectation with respect to a Gaussian measure. Similarly with [18], we will rewrite the integrand into a local expansion over subsets X of an explicit part and an implicit remainder. For instance in the model considered in this paper, the above quantity Z will be rewritten into an expression of roughly the following form (more precisely, see Proposition 2):

$$\mathbb{E}\left[\sum_X \mathrm{e}^{\sigma \sum_{x\notin X} (\partial\phi(x))^2} K(X,\phi)\right],$$

where $K(X,\phi)$ depends only on $\phi(x)$ with x in (a neighborhood of) X. We will then take a family of conditional expectations at a sequence of scales parametrized by integer j—so our approach is a multi-scale analysis. To give a quick glance of the main idea, at a scale j we will have expressions, which up to several subtleties look as follows:

$$\mathbb{E}\left[\sum_Y \mathrm{e}^{\sigma_j \sum_{x\notin Y} \mathbb{E}\,[\,\partial\phi(x)\,|\,B_x^c\,]^2}\, \mathbb{E}\big[K_j'(Y,\phi)|Y^c\big]\right].$$

The actual expressions will be slightly different and more complicated and we refer to Sect. 2.4 for the exact expressions, but at this stage we point out that some conditional expectations have appeared inside the overall expectation. Indeed, for any function of the field $F(\phi)$, the notation $\mathbb{E}\,[F(\phi)|X^c]$ means integrating *all* the variables $\{\phi(x) : x\in X\}$ with $\{\phi(x) : x\in X^c\}$ *fixed* (X^c is the complement of X). Also, B_x is a block containing x, and σ_j is the most important dynamical parameter (which corresponds to renormalization of the dielectric constant in the dipole model). This idea of conditional expectation is close to Frohlich and Spencer's work on Kosterlitz–Thouless transition [30,31] where the authors take inside an expectation conditional integrations, each over *all* variables $\{\phi(x) : x \in \Omega\}$ where Ω is a bounded region around a charge density ρ with diameter $\sim 2^j$ at a scale j.

Such conditional expectations can be carried out by minimizing the quadratic form in the Gaussian measure with conditioning variables fixed. Since the Gaussian is associated with a Laplacian, these minimizers are harmonic extensions of ϕ from X^c into X. These harmonic extensions result in smoother dependence of the integrand of the expectation on the field. Some elliptic PDE methods along with random walk estimates will be used. We remark that this variational viewpoint also shows up in Balaban's RG method (see for instance [3] or Sects. 2.2–2.3 of [25]).

2. Outline of the Paper

2.1. Settings, Notations and Conventions

Let $\mathbb{Z}^d$ be the d dimensional lattice with $d \geq 2$. Denote the sets of lattice directions as $\mathcal{E}_+ = \{e_1, \ldots, e_d\}$ and $\mathcal{E}_- = \{-e_1, \ldots, -e_d\}$ where $e_k = (0, \ldots, 1, \ldots, 0)$ with only the kth element being 1. Let $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_-$. For $e \in \mathcal{E}$, $\partial_e f(x) = f(x+e) - f(x)$ is the lattice derivative. For $x, y \in \mathbb{Z}^d$, we say that (x, y) is a nearest neighbor pair and write $x \sim y$ if there exists an $e \in \mathcal{E}$ such that $x = y + e$. Denote $E(\mathbb{Z}^d)$ to be the set of all nearest neighbor pairs of $\mathbb{Z}^d$. For $X \subset \mathbb{Z}^d$, we define $E(X) := \{(x, y) \in E(\mathbb{Z}^d) : x, y \in X\}$.

Let L be a positive odd integer, and $N \in \mathbb{N}$. Let

$$\Lambda = \left[-L^N/2, L^N/2\right]^d \cap \mathbb{Z}^d,$$

and we will consider functions on Λ with periodic boundary condition. In other words, we view Λ as a torus by identifying the boundary points of Λ in the usual way.

For $x, y \in \Lambda$, define $d(x, y)$ to be the length of a shortest path of nearest neighbor sites in the torus Λ connecting x and y. Also define ∂X to be the "outer boundary": $\partial X = \{x \in \Lambda : d(x, X) = 1\}$. Write X^c to be the complement of X.

For a function ϕ on $\mathbb{Z}^d$, when it does not cause confusions, we write for short

$$\sum_X (\partial\phi)^2 = \sum_{x \in X} (\partial\phi(x))^2 := \frac{1}{2} \sum_{x \in X} \sum_{e \in \mathcal{E}} (\partial_e \phi(x))^2$$

and similarly for other such type of summations. If $\mathbb{E}$ is the expectation over ϕ, we will use a short-hand notation for conditional expectation

$$\mathbb{E}\left[-\middle|X\right] := \mathbb{E}\left[-\middle|\{\phi(x)|x \in X\}\right],$$

namely, the expectation with $\phi|_X$ fixed.

Unless we specify otherwise, Poisson kernels and Green's functions will be associated with the operator $-\Delta+m^2$ where m is a small mass regularization. For any set X, P_X or $P_X(x,y)$ $(x \in X,\ y \in \partial X)$ is the Poisson kernel for X. If $x \notin X$ then $P_X f(x) = f(x)$ is always understood. In other words, $P_X f$ is the harmonic extension of f from X^c into X with $f\big|_{X^c}$ unchanged.

2.2. The Dipole Gas Model and the Scaling Limit

Let μ be the Gaussian measure on the space of functions $\{\phi(x) : x \in \Lambda\}$ with mean zero and covariance $C_m = (-\Delta + m^2)^{-1}$ where $m > 0$. In other words, ϕ is the Gaussian free field on the Λ with covariance C_m. Let $\mathbb{E}$ be the expectation over ϕ. Then, the classical dipole gas model is defined by the following probability measure:

$$\nu(d\phi) = \frac{\mathrm{e}^{zW(\phi)}\mu(d\phi)}{\mathbb{E}\big(\mathrm{e}^{zW(\phi)}\big)},$$

where the denominator is the normalization constant and

$$W(\phi) := \sum_{x\in\Lambda}\sum_{e\in\mathcal{E}} \cos\left(\sqrt{\beta}\partial_e\phi(x)\right).$$

Such a measure is obtained by a definition of the model via the great canonical ensemble followed by a Sine–Gordon transformation, for instance, see [17].

We would like to study the problem of scaling limit. More precisely, let $\tilde{\Lambda} := [-\frac{1}{2},\frac{1}{2}]^d \subset \mathbb{R}^d$. Given a mean zero function $\tilde{f} \in C^\infty(\tilde{\Lambda})$, $\int_{\tilde{\Lambda}} \tilde{f} = 0$ with periodic boundary condition, we study the (real) generating function

$$Z_N(f) := \lim_{m\to 0} \frac{\mathbb{E}\left[\mathrm{e}^{\sum_{x\in\Lambda} f(x)\phi(x)}\mathrm{e}^{zW(\phi)}\right]}{\mathbb{E}\big[\mathrm{e}^{zW(\phi)}\big]}, \tag{2.1}$$

where

$$f(x) = f_N(x) := L^{-(d+2)N/2}\tilde{f}(L^{-N}x).$$

The main question is the scaling limit of $Z_N(f)$ as $N \to \infty$.

Remark 1. Our notion of scaling limit here is that we send the volume of Λ and the scaling factor in the definition of $f_N(x)$ to infinity together. This is easier than the stronger (and more commonly used) notion of scaling limit where one first takes the infinite volume limit and then sends the scaling factor in $f_N(x)$ to infinity. We also remark that since we take the limit $m^2 \to 0$ before the limit $N \to \infty$, in the sequel we will assume that m^2 is sufficiently small depending on N.

2.3. Some Preparative Steps Before RG

As the start of our strategy to study this problem, we perform an a priori tuning of the Gaussian measure. This tuning anticipates the fact that the best

Gaussian approximation to ν is not the Gaussian measure currently defined on ϕ. For any $X \subseteq \Lambda$, define

$$V(X,\phi) := \frac{1}{4} \sum_{x\in X, e\in\mathcal{E}} (\partial_e \phi(x))^2 . \tag{2.2}$$

The tuning is to split part of the quadratic form of the Gaussian measure into the integrand, so that the resulting Gaussian field has covariance $[\epsilon(-\Delta + m^2)]^{-1}$, with the associated expectation called $\mathbb{E}^\epsilon$:

$$Z_N(f) = \lim_{m\to 0} \frac{\mathbb{E}^\epsilon\left[e^{\sum_{x\in\Lambda} f(x)\phi(x)} e^{(\epsilon-1)V(\Lambda,\phi)+zW(\Lambda,\phi)}\right]}{\mathbb{E}^\epsilon\left[e^{(\epsilon-1)V(\Lambda,\phi)+zW(\Lambda,\phi)}\right]}. \tag{2.3}$$

Note that normalization factors caused by re-definition of Gaussian:

$$\mathbb{E}^\epsilon\left[\exp\left((\epsilon-1)V(\Lambda,\phi)\right)\right]$$

appear in both numerator and denominator and are thus cancelled.

We would like to make the expectation (and thus the RG maps which we will define later) independent of ϵ. So we rescale $\phi \to \phi/\sqrt{\epsilon}$ and let $\sigma = \epsilon^{-1} - 1$ and obtain

$$Z_N(f) = \lim_{m\to 0} \frac{\mathbb{E}\left[e^{\sum_{x\in\Lambda} f(x)\phi(x)/\sqrt{\epsilon}} \cdot e^{-\sigma V(\phi)+zW(\sqrt{1+\sigma}\phi)}\right]}{\mathbb{E}\left[e^{-\sigma V(\phi)+zW(\sqrt{1+\sigma}\phi)}\right]}. \tag{2.4}$$

We also shift the Gaussian field to get rid of the linear term $\sum f\phi/\sqrt{\epsilon}$. Write $-\Delta_m = -\Delta + m^2$ and make a translation $\phi \to \phi + \xi_m$ where $\xi_m = (-\sqrt{\epsilon}\Delta_m)^{-1} f$ in the numerator in (2.4). Since the function ξ_m appears frequently below, we will simply write $\xi = \xi_m$ without explicitly referring to its dependence on m. Then, one has

$$Z_N(f) = \lim_{m\to 0} e^{\frac{1}{2}\sum_{x\in\Lambda} f(x)(-\epsilon\Delta_m)^{-1} f(x)} Z'_N(\xi)/Z'_N(0), \tag{2.5}$$

where

$$Z'_N(\xi) = \mathbb{E}\left[e^{-\sigma V(\Lambda,\phi+\xi)+zW((\phi+\xi)/\sqrt{\epsilon})}\right]. \tag{2.6}$$

Let $-\tilde{\Delta}_m = -\tilde{\Delta} + m^2$, where $\tilde{\Delta}$ is the Laplacian acting on the functions on $\tilde{\Lambda}$, and $\tilde{C}_m := (-\tilde{\Delta}_m)^{-1}$ and $\tilde{\xi}_m := (-\sqrt{\epsilon}\tilde{\Delta}_m)^{-1}\tilde{f}$. We can verify that

$$L^{2N}\tilde{C}_{L^N m}(L^{-N}x) = C_m(x) + \mathfrak{e}_N \quad \text{and} \quad L^{-\frac{d-2}{2}N}\tilde{\xi}_{L^N m}(L^{-N}x) = \xi_m(x) + \bar{\mathfrak{e}}_N,$$

where $\mathfrak{e}_N$ and $\bar{\mathfrak{e}}_N$ are error terms due to lattice / continuum discrepancy which converge to zero as $N \to \infty$. Let $q < \frac{d}{d-1}$ and define

$$R := \sup_{m>0} \max\left(\|\tilde{C}_m\|_{L^q}, \|\partial\tilde{C}_m\|_{L^q}\right)$$

Note that $R < \infty$ since the worst local singularity is $O(|x|^{1-d})$ which is L^q integrable for any $q < \frac{d}{d-1}$. We will assume that $\|\tilde{f}\|_{L^p} \le h/R$ $(p > d)$, for a constant h to be specified later, so that for $\alpha = 0, 1$

$$\|\partial^\alpha \xi\|_{L^\infty} \le hL^{-\left(\frac{d-2}{2}+\alpha\right)N} \tag{2.7}$$

by Young's inequality.

Before the RG steps, we write both $Z'_N(\xi)$ and $Z'_N(0)$ into a form of "polymer expansion". For any set $X \subseteq \Lambda$, write

$$W(X,\phi) := \sum_{x\in X}\sum_{e\in\mathcal{E}} \cos\left(\sqrt{\beta}\partial_e\phi(x)\right).$$

Proposition 2. *With W defined above and $Z'_N(\xi)$ given by (2.6), we have*

$$Z'_N(\xi) = \mathbb{E}\left[\sum_{X\subseteq\Lambda} I_0(\Lambda\backslash X, \phi+\xi) K_0(X,\phi+\xi)\right], \tag{2.8}$$

where $I_0(X) = \prod_{x\in X} I_0(\{x\})$ and

$$I_0(\{x\},\phi+\xi) = \mathrm{e}^{-\frac{1}{4}\sigma\sum_{e\in\mathcal{E}}(\partial_e\phi(x)+\partial_e\xi(x))^2},$$

$$K_0(X,\phi+\xi) = \prod_{x\in X}\mathrm{e}^{-\frac{1}{4}\sigma\sum_{e\in\mathcal{E}}(\partial_e\phi(x)+\partial_e\xi(x))^2}\left(\mathrm{e}^{zW(\{x\},(\phi+\xi)/\sqrt{\epsilon})}-1\right).$$

The subscript 0 indicates that we are at the 0th RG step, and we will write $\sigma_0 = \sigma$. The quantity $Z'_N(0)$ has the same expansion with $\xi = 0$.

Proof. Consider Eq. (2.6): following Mayer expansion,

$$\begin{aligned} Z'_N(\xi) &= \mathbb{E}\left[\mathrm{e}^{zW(\Lambda)-\sigma V(\Lambda)}\right] \\ &= \mathbb{E}\left[\prod_{x\in\Lambda}\left(\mathrm{e}^{-\sigma V(\{x\})} + \left(\mathrm{e}^{zW(\{x\})}-1\right)\mathrm{e}^{-\sigma V(\{x\})}\right)\right]. \end{aligned}$$

Expanding the product amounts to associating a set $X \subseteq \Lambda$ to the second term and the complement $\Lambda\backslash X$ to the first term. This proves the statement (2.8). □

2.4. Outline of Main Ideas

Our renormalization group method is based on the idea of rewriting the expectation into an expectation of an expression involving many conditional expectations. We will carry out a multiscale analysis; an RG map will be iterated from one scale to the next one, during which we will re-arrange the conditional expectations. A basic algebraic structure and analytical bound will be propagated to every scale. To describe these structures and bounds, we first give some definitions.

2.4.1. Basics of Polymers.

1. We call blocks of size L^j j-blocks. These are translations of $\{x \in \mathbb{Z}^d : |x| < \frac{1}{2}(L^j - 1)\}$ by vectors in $\left(L^j\mathbb{Z}\right)^d$. In particular, a 0-block is a single site in $\mathbb{Z}^d$. A j-polymer X is a union of j-blocks. In particular, the empty set is also a j-polymer. The number of lattice sites in $X \subset \mathbb{Z}^d$ is denoted by $|X|$. The number of j-blocks in a j-polymer X is denoted by $|X|_j$.
2. $X \subset \mathbb{Z}^d$ is said to be connected if for any two points $x, y \in X$ there exists a path $(x_i : i = 0, \ldots, n)$ with $\left|x_{i+1} - x_i\right|_\infty = 1$ connecting x and y. Here, $|x|_\infty$ is the maximum of all coordinates of x; note that for instance $\{(0,0),(1,1)\}$ is connected if $d = 2$. Connected sets are not empty. Two sets X, Y are said to be strictly disjoint if there is no path from x to y when $x \in X$ and $y \in Y$; otherwise we say that they touch.
3. For any $X \subset \mathbb{Z}^d$, we let $\mathcal{C}(X)$ be the set of connected components of X.
4. For a j-polymer X, we have the following notations. $\mathcal{B}_j(X)$ is the set of all j-blocks in X. $\mathcal{P}_j(X)$ is the set of all j-polymers in X. $\mathcal{P}_{j,c}(X)$ is the set of all connected j-polymers in X. We sometimes just write $\mathcal{B}_j, \mathcal{P}_j, \mathcal{P}_{j,c}$ and so on when $X = \Lambda$.
5. Let $X \in \mathcal{P}_j$. Define for $j \geq 1$

$$\begin{aligned} \hat{X} &:= \cup\{B \in \mathcal{B}_j : B \text{ touches } X\}, \\ X^+ &:= \cup\left\{x \in \Lambda : d(x,X) \leq \frac{1}{3}L^j\right\}, \\ \ddot{X} &:= \cup\left\{x \in \Lambda : d(x,X) \leq \frac{1}{6}L^j\right\}, \\ \dot{X} &:= \cup\left\{x \in \Lambda : d(x,X) \leq \frac{1}{12}L^j\right\}. \end{aligned}$$

   Note that we have $X \subset \dot{X} \subset \ddot{X} \subset X^+ \subset \hat{X}$. Only $X, \hat{X}$ belong to $\mathcal{P}_j$.
6. When $j = 0$ and $X \in \mathcal{P}_0$, we define $\dot{X} = \ddot{X} = X^+ = \hat{X} = X$, and the Poisson kernel at scale 0 is understood as $P_{X^+} := id$.

We also have the following notations for functions of the fields.

1. Define $\mathcal{N}$ to be the set of functions of ϕ and ξ. Define $\mathcal{N}(X) \subseteq \mathcal{N}$ to be the set of functions of $\{\phi(x), \xi(x) \big| x \in X\}$. $\mathcal{N}^{\mathcal{P}_j}$ is the set of maps $K : \mathcal{P}_j \to \mathcal{N}$ such that $K(X) \in \mathcal{N}(\hat{X})$. We define $\mathcal{N}^{\mathcal{B}_j}$, $\mathcal{N}^{\mathcal{P}_{j,c}}$ similarly.
2. For $I \in \mathcal{N}^{\mathcal{B}_j}$, we write

$$I(X) = I^X := \prod_{B \in \mathcal{B}_j(X)} I(B) \qquad \text{for} \qquad X \in \mathcal{P}_j.$$

   For $K \in \mathcal{N}^{\mathcal{P}_j}$, we say that K factorizes over connected components if

$$K(X) = \prod_{Y \in \mathcal{C}(X)} K(Y). \tag{2.9}$$

In this case, K is determined by its value on connected polymers, so we can write $K \in \mathcal{N}^{\mathcal{P}_{j,c}}$.

The *basic structure* that we want to propagate to every scale of the RG iterations is, for $j \geq 0$

$$Z'_N(\xi) = \mathrm{e}^{\mathcal{E}_j}\, \mathbb{E}\left[\sum_{X \in \mathcal{P}_j(\Lambda)} I_j(\Lambda \backslash \hat{X}, \phi, \xi) K_j(X, \phi, \xi)\right]. \tag{2.10}$$

Here, $\mathrm{e}^{\mathcal{E}_j}$ is a ϕ, ξ independent constant factor. This constant will be shown to be the same for $Z'_N(\xi)$ and $Z'_N(0)$ and thus cancels. $K_j(X, \phi, \xi)$ only depends on the values of ϕ, ξ in a small neighborhood of X. Note that there is a "corridor" between each X and $\Lambda \backslash \hat{X}$ (namely, the union of X and $\Lambda \backslash \hat{X}$ is not the entire Λ, and we call this "missing part" $\hat{X} \backslash X$ heuristically as a "corridor"). These "corridors" will be important in our conditional expectation method.

Furthermore, for $j < N$, the function I_j will have a local form in the sense that it factorizes over j-blocks $I_j(X, \phi, \xi) = \prod_{B \in \mathcal{B}_j(X)} I_j(B, \phi, \xi)$ and

$$I_j(B, \phi, \xi) = \mathrm{e}^{-\frac{1}{4}\sigma_j \sum_{x \in B, e \in \mathcal{E}} (\partial_e P_B + \phi(x) + \partial_e \xi(x))^2}. \tag{2.11}$$

$I_j(B)$ is essentially determined by the dynamical parameter σ_j. On the other hand, K_j will only factorize over "connected components of polymer".

The *basic bounds* that hold on every scale about K_j, whose form will not be explicit, are as follows. For X connected,

$$\sum_{n=0}^{4} \frac{1}{n!} \left\| K_j^{(n)}(X, \phi, \xi) \right\| \leq \|K\|_j\, A^{-|X|_j} G\left(\ddot{X}, X^+\right). \tag{2.12}$$

Here, $K_j^{(n)}$ is an nth derivative of K_j; the precise definition of it and the norm will be given later. For any two sets $X \subset Y$, $G(X, Y)$ is a normalized conditional expectation called the "regulator"

$$G(X, Y) = \mathbb{E}\left[\mathrm{e}^{\frac{\kappa}{2} \sum_X (\partial\phi)^2} \middle| \phi_{Y^c}\right] / N(X, Y) \tag{2.13}$$

and the normalization factor is

$$N(X, Y) = \mathbb{E}\left[\mathrm{e}^{\frac{\kappa}{2} \sum_X (\partial\phi)^2} \middle| \phi_{Y^c} = 0\right]. \tag{2.14}$$

This form of regulator is different from the one defined in [18]; in particular, it is itself a conditional expectation. It will be shown to have some interesting properties.

Now, we outline the steps to go from scale j to scale $j + 1$ while the structure (2.10) is preserved.

(1) Extraction and Reblocking. Reblocking is a procedure which rewrites (2.10) into an expansion over "$j+1$ scale polymers", and we extract the components that grow too fast under this reblocking.

Proposition 3. *Suppose that L is sufficiently large. If at the scale j one has*

$$Z'_N(\xi) = \mathrm{e}^{\mathcal{E}_j}\, \mathbb{E}\left[\sum_{X\in\mathcal{P}_j} I_j^{\Lambda\backslash\hat{X}}(\phi,\xi) K_j(X,\phi,\xi)\right] \tag{2.15}$$

with $I_j \in \mathcal{N}^{\mathcal{B}_j}$ given by (2.11), then there exist $\mathcal{E}_{j+1}, I_{j+1} \in \mathcal{N}^{\mathcal{B}_{j+1}}$ and $K_j^\sharp \in \mathcal{N}^{\mathcal{P}_{j+1,c}}$(namely $K_j^\sharp$ factorizes over connected components in the sense of (2.9)), so that the following expansion at the scale $j+1$ holds

$$Z'_N(\xi) = \mathrm{e}^{\mathcal{E}_{j+1}}\, \mathbb{E}\left[\sum_{U\in\mathcal{P}_{j+1}} I_{j+1}^{\Lambda\backslash\hat{U}}(\phi,\xi)\, K_j^\sharp(U,\phi,\xi)\right] \tag{2.16}$$

where $\mathcal{E}_{j+1}$ is a constant independent of ϕ, ξ, and for every $D \in \mathcal{B}_{j+1}$,

$$I_{j+1}(D) = \mathrm{e}^{-\frac{1}{4}\sigma_{j+1}\sum_{x\in D, e\in\mathcal{E}}\left(\partial_e P_{D^+}\phi(x)+\partial_e\xi(x)\right)^2}$$

for some constant σ_{j+1}.

We will prove this Lemma in Sect. 3.

(2) Conditional Expectation. This step is the main difference between this new method and [18]. First of all, we observe that in (2.16), the sets $\Lambda\backslash\hat{U}$ and U do not touch. In other words, there exists a corridor $\hat{U}\backslash U$ around the set U where $K_j^\sharp$ evaluates on, and this corridor has width L^{j+1}. We then take conditional expectation and thus re-write the expectation in (2.16) as follows:

$$\mathbb{E}\left[\sum_{U\in\mathcal{P}_{j+1}} I_{j+1}^{\Lambda\backslash\hat{U}}(\phi,\xi)\, \mathbb{E}\Big[K_j^\sharp(U,\phi,\xi)\big|(U^+)^c\Big]\right] \tag{2.17}$$

where $U \subset U^+ \subset \hat{U}$. For notation conventions, see Sect. 2.1. To obtain (2.17), one switches the expectation and the sum in (2.16), then take the conditional expectation right inside the expectation. Since $I_{j+1}^{\Lambda\backslash\hat{U}}$ only depends on the values of ϕ being fixed, the conditional expectation can be taken only on the $K_j^\sharp$ factor. One then switches back the expectation and the sum.

This followed by factoring out ϕ, ξ independent constant gives K_{j+1} and we are back to the form (2.10) with all j replaced by $j+1$. In case $U = \Lambda$, we just integrate (unconditionally): $\mathbb{E}\big[K_j^\sharp(\Lambda,\phi)\big]$, but to streamline expressions we still write (2.17) keeping in mind the special treatment for the $U = \Lambda$ term.

Remark 4. Our discussion below will frequently involve Laplacian operators (with a tiny mass m) acting on functions on a set U with zero Dirichlet boundary condition on ∂U, so we simply refer to them as *Dirichlet Laplacian* for U. Similarly, for the Green's function of the (massive) Laplacian on ∂U with zero Dirichlet boundary condition on ∂U, we simply call it *Dirichlet Green's function* for U. Finally, if ζ is a Gaussian field on U with Dirichlet Green's function for U as its covariance, then we simply say that ζ is the *Dirichlet Gaussian field* on U.

We point out two important facts about the conditional expectation step. The first one is that we can write the Gaussian field ϕ into a sum of two decoupled parts. Let P_U be the Poisson kernel for U and recall our convention that $P_U\phi(x) = \phi(x)$ for $x \notin U$ as in Sect. 2.1.

Proposition 5. *Let $U \subset V \subset \Lambda$. Define ζ via $\phi(x) = P_U\phi(x) + \zeta(x)$. Then, the quadratic form*

$$-\sum_{x\in V} \phi(x)\Delta_m\phi(x) = -\sum_{x\in U} \zeta(x)\Delta^D_{U,m}\zeta(x) - \sum_{x\in V} P_U\phi(x)\Delta_m P_U\phi(x), \quad (2.18)$$

where $-\Delta^D_{U,m} = -\Delta^D_U + m^2$ and Δ^D_U is the Dirichlet Laplacian for U.

Notice that $x \in U$ does not contribute to the last summation since $\Delta_m P_U\phi(x) = 0$ in U. By this proposition, taking expectation of a function $K(\phi)$ conditioned on $\{\phi(x)\big|x \in U^c\}$ is simply integrating out a Gaussian field ζ:

$$\mathbb{E}\left[K(\phi,\xi)\big|U^c\right] = \mathbb{E}_\zeta\left[K(P_U\phi + \zeta, \xi)\right], \quad (2.19)$$

where the covariance of ζ is the C^D_U - the Dirichlet Green's function for U. In particular, we observe that I_j defined in (2.11) has an alternative representation

$$I_j(B,\phi,\xi) = \mathrm{e}^{-\frac{1}{4}\sigma_j \sum_{x\in B, e\in\mathcal{E}} \mathbb{E}\left[\partial_e\phi(x)+\partial_e\xi(x)\big|(B^+)^c\right]^2}.$$

(Note that the conditional expectation is taken before the square.) It is conceptually helpful to keep in mind that we are just re-arranging the following structure [comparing with (2.8)]

$$\mathbb{E}\left[\sum_{X\in\mathcal{P}_j} \mathrm{e}^{-\frac{1}{4}\sigma_j \sum_{x\notin \hat{X}, e\in\mathcal{E}} \mathbb{E}\left[\partial_e\phi(x)+\partial_e\xi(x)\big|(B^+)^c\right]^2} \mathbb{E}\left[\cdots\big|(X^+)^c\right]\right], \quad (2.20)$$

namely an outmost (unconditional) expectation of a simple combination of many conditional expectations.

Remark 6. In the paper, $P_U\phi$ will always be well-defined: by Prop 1.11 of [37], if the probability that the random walk starting from any point in U exits U in finite time is 1, then the harmonic extension exists and is unique. Domains $U \subsetneq \Lambda$ will always satisfy this condition because the random walk (with a tiny killing rate) hits any point in Λ in finite time with probability one.

The next fact is as follows:

Proposition 7. *Let $d \geq 2$, $x \in X \subset U \subset \Lambda$. If $d(x,\partial X) \geq cL^j$, then*

$$|(\partial_x P_X)C^D_U(\partial_x P_X)^\star(x,x)| \leq O(1)L^{-dj}, \quad (2.21)$$

where $O(1)$ depends on c, and C^D_U is the Dirichlet Green's function for U, and $(\partial_x P_X)^\star$ is the adjoint of $\partial_x P_X$.

For the proof, see Proposition 16. This result gives the scaling for the covariance of $\partial P_X \zeta$ where P_X is a Poisson kernel obtained from the previous RG step. We take a heuristic test to see the necessity of this proposition: setting $\xi = 0$, for $X \subset U$, if we perform an expectation conditioned on $\{\phi(x) \big| x \in X^c\}$, followed by another expectation conditioned on $\{\phi(x) \big| x \in U^c\}$, by (2.19),

$$\mathbb{E}_{\zeta_U} \mathbb{E}_{\zeta_X} \left[K\Big(P_X (P_U \phi + \zeta_U) + \zeta_X \Big) \right] = \mathbb{E}_{\zeta_U} \mathbb{E}_{\zeta_X} \left[K\big(P_U \phi + P_X \zeta_U + \zeta_X \big) \right], \tag{2.22}$$

then we need this proposition to deal with $P_X \zeta_U$ when integrating over ζ_U.

Proofs of the above two results are in the following sections.

Linearization and Stable Manifold Theorem. We have just outlined a single RG map

$$(\sigma_j, \sigma_{j+1}, E_{j+1}, K_j) \to K_{j+1}.$$

We will show smoothness of this map in Sect. 5. Note that two issues have not been discussed: (1) choice of σ_{j+1}, E_{j+1}, which should be a function of (σ_j, K_j), so that the RG map becomes $(\sigma_j, K_j) \to (\sigma_{j+1}, K_{j+1})$ (notice that we will not regard E_{j+1} as dynamical parameter and we will factorize it out); (2) choice of σ in the a priori tuning step. We will outline how to treat these two issues now.

Clearly, $(\sigma, K) = (0, 0)$ is a fixed point of the RG map. In Sect. 6, we show that the linearization of the map $(\sigma_j, \sigma_{j+1}, E_{j+1}, K_j) \to K_{j+1}$ around $(0, 0, 0, 0)$ has a form $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ where $\mathcal{L}_1$ captures the "large polymers" contributions to K_{j+1}, and $\mathcal{L}_2$ involves the remainder of second order Taylor expansion of conditionally expected K_j on "small polymers", both of which will be shown contractive with arbitrarily small norm by suitable choices of constants L and A introduced above. Furthermore, $\mathcal{L}_3(D)$ will roughly have a form

$$L^d E_{j+1} + \sigma_{j+1} \sum_{x \in D} (\partial P_{D^+} \phi(x))^2 - \sigma_j \left(\sum_{x \in D} (\partial P_{D^+} \phi(x))^2 + \delta E_j \right) + Tay \tag{2.23}$$

where Tay is the second-order Taylor expansion of conditionally expected K_j on small polymers, which consists of constant and quadratic terms, and D is a $j+1$ block. Now, it is easy to see that there is a way to choose E_{j+1} and σ_{j+1} so that $\mathcal{L}_3$ is almost 0, up to a localization procedure for "Tay". For proofs, see Sect. 6.

Once we have shown a way to choose the constants σ_{j+1}, E_{j+1} to ensure contractivity of the above linear map, a stable manifold theorem can be applied to prove that there exists a suitable tuning of σ so that

$$|\sigma_j| \lesssim 2^{-j} \qquad \|K_j\|_j \lesssim 2^{-j}. \tag{2.24}$$

Main Result: the Scaling Limit.

Theorem 8. *For any $p > d$, there exist constants $M > 0$ and $z_0 > 0$ so that: for all $\|\tilde{f}\|_{L^p} \leq M$ and all $|z| \leq z_0$ there exists a constant ϵ depending on z and*

$$\lim_{N\to\infty} Z_N(f) = \exp\left(\frac{1}{2}\int_{\tilde{\Lambda}} \tilde{f}(x)(-\epsilon\tilde{\Delta})^{-1}\tilde{f}(x)d^dx\right), \tag{2.25}$$

where $\tilde{\Delta}$ is the Laplacian in continuum, and $Z_N(f)$ is defined in (2.5).

The main ingredient of the proof is that with $j = N-1$, by Eqs. (2.10) and (2.24), one can bound $Z'_N(\xi)$ essentially by

$$e^{\mathcal{E}_{N-1}} \sum_{X\in\mathcal{P}_{N-1}} \left(1+2^{-N}\right)^{\Lambda\setminus\hat{X}} 2^{-N}. \tag{2.26}$$

Bounding the number of terms by 2^{L^d}, we see that it is almost $e^{\mathcal{E}_{N-1}}$ as N becomes large. The constant $e^{\mathcal{E}_{N-1}}$ will be the same for $Z'_N(\xi)$ and $Z'_N(0)$. So only the exponential factor in Eq. (2.5) survives in the $N\to\infty$ limit and it goes to the right-hand side of (2.25). The details are given in Sect. 7. We remark that the assumption on $\tilde{f}$, which makes f smooth at the scale N, is for simplicity of the demonstration of the method.

3. The Renormalization Group Steps

3.1. Some Additional Definitions

1. A j-polymer X is called a small set or small polymer if it is connected and $|X|_j \leq 2^d$. Otherwise, it is called large. We denote by $\mathcal{S}_j(X)$ the set of all small j-polymers in X.
2. Define $\hat{\mathcal{S}}_j$ to be the set of pairs (B,X) so that $X\in\mathcal{S}_j$ and $B\in\mathcal{B}_j(X)$.
3. We also introduce a notation $Y\in_X \mathcal{P}_j$ which means $Y\in\mathcal{P}_j$ and that if $X=\emptyset$ then $Y=\emptyset$.
4. Let $X\in\mathcal{P}_j$. Define its closure $\bar{X}\in\mathcal{P}_{j+1}$ to be the smallest (j+1)-polymer that contains X.
5. We define a notation $\chi^j_{\mathcal{A}}$ where $\mathcal{A}$ is a set of polymers: $\chi^j_{\mathcal{A}} = 1$ if any two polymers in $\mathcal{A}$ are strictly disjoint as j-polymers and $\chi^j_{\mathcal{A}} = 0$ otherwise. Also, if $\mathcal{A}$ is a set of polymers, we write $X_{\mathcal{A}}$ to be the union of all elements of $\mathcal{A}$.

3.2. Renormalization Group Steps

Now we focus on a single RG map from scale j to $j+1$. For simpler notations, we omit the subscript j and objects at scale $j+1$ will be labelled by a prime, e.g. K', $\mathcal{P}'$. The guidance principle will be that for all kinds of I's below, $I-1$ and their difference δI and K will be small, so their products will be higher order small quantities. These remarks will make more sense after we discuss the linearization of the smooth RG map in Sect. 6.

Extraction and Reblocking. We start to prove Proposition 3. Before the proof, we describe here the main ideas in the strategies we use below, and a reader may find helpful to read the proof along with these descriptions. The way to construct $I', K^\sharp$ is certainly not unique. However, our construction [see (3.9) below] must have the foresight that $K^\sharp$ will be smooth in its arguments (which will be shown in Sect. 5), with respect to certain norms defined in Sect. 4. Due to the nature of these norms, the proof of smoothness in Sect. 5 will reply on some separation properties of different factors appeared in the $K^\sharp$ finally constructed in (3.9). Ensuring these separation properties complicates the proof.

The proof of Proposition 3 then consists of two steps. The first step is called an extraction step, in which we extract $\delta I(B)$ from $I(B)$, see the third line of (3.2), resulting in a new quantity $\tilde{I}(B)$ defined as (3.1). The extracted quantities $\delta I(B)$ will show up as factors multiplying with K in (3.3).

The second step is called a reblocking step. In this step, summations over various sets in (3.3) will eventually become one single sum over next scale polymers $U \in \mathcal{P}'$ as in (3.8). During this reblocking, some $\tilde{I}$ factors will also become factors multiplying with K [see (3.9)].

There are two subtleties which one has to take care and thus complicates the proof. The first subtlety is that $\delta I(B)$ and $\tilde{I}(B)$ involve a Poisson kernel for $(\bar{B})^+$ which is a set of length size $O(L^{j+1})$. When these factors δI and $\tilde{I}$ show up as factors multiplying with $K(X)$ as discussed above, the factor $K(X)$ actually only has a corridor $\hat{X}\backslash X$ of width L^j (formed from the previous RG step), so the sets $(\bar{B})^+$ may intersect with X. This intersection would be disastrous when we estimate the norm of the product of these δI, $\tilde{I}$ and K factors. Therefore, in the proof we actually only extract $\delta I(B)$ for those B far enough from X, that is, outside the set $\langle X\rangle$ defined below. Inside $\langle X\rangle$, we do different extractions as in the second line of (3.2), so that the L^j width corridor of $K(X)$ is sufficient to ensure separation.

The other subtlety is that according to the conclusion of Proposition 3, one has to ensure existence of a corridor around U in (3.8). This is not ensured in the "naive reblocking" (3.4) below (though in (3.4) we do obtain one single sum over next scale polymers $V \in \mathcal{P}'$). Therefore, as an intermediate step between extraction and reblocking, we will perform another expansion by $\tilde{I} = (\tilde{I} - \mathrm{e}^{E'}) + \mathrm{e}^{E'}$ right after (3.5), and arrange such that some of the $\tilde{I} - \mathrm{e}^{E'}$ also show up as multiplying factors in (3.9), and the other $\tilde{I}$ will be separated away by a corridor (between $\Lambda\backslash\hat{U}$ and U in the last line of the proof).

Remark 9. We also have a remark on notations. The hats in the notation for a set of pairs such as the $\hat{\mathcal{S}}_j$ defined above in Sect. 3.1 and the $\hat{\mathcal{Y}}$ in the following proof are simply symbols, which have nothing to do with the hat operation ˆ on a single polymer defined in Sect. 2.4.1.

Proof of Proposition 3. Define $\tilde{I} \in \mathcal{N}^{\mathcal{B}_j}$ as

$$\tilde{I}(B) = \mathrm{e}^{E' - \frac{1}{4}\sigma' \sum_{x\in B, e\in\mathcal{E}} \left(\partial_e P_{(\bar{B})^+}\phi(x) + \partial_e \xi(x)\right)^2}, \tag{3.1}$$

where E' and σ' will be chosen later. Note that the above quantity $\tilde{I}(B)$ differs from the quantity $I(B)$ defined in (2.11) by the new constants E', σ' and the Poisson kernel P_{B+} is replaced by the Poisson kernel $P_{(\bar{B})^+}$. For a j-polymer X, denote

$$\langle X\rangle := \cup\{B \in \mathcal{B}_j : (\bar{B})^+ \cap \hat{X} \neq \emptyset\},$$

where the $+$ operation is on the scale $j+1$ and the hat is on the scale j. Then, we let

$$\begin{cases} 1(B) = (1 - \mathrm{e}^{E'}) + \mathrm{e}^{E'} & \text{if} \quad B \subseteq \hat{X}\backslash X \\ I(B) = (I(B) - \mathrm{e}^{E'}) + \mathrm{e}^{E'} & \text{if} \quad B \subseteq \langle X\rangle \backslash \hat{X} \\ I(B) = \delta I(B) + \tilde{I}(B) & \text{if} \quad B \subseteq \langle X\rangle^c \\ K(X) = \sum_{B\in\mathcal{B}(X)} \frac{1}{|X|_j} K(B,X) & \text{if} \quad X \in \mathcal{S}, \end{cases} \tag{3.2}$$

where δI is defined implicitly, and $K(B,X) := K(X)$. Insert these summations into the product factors in (2.15), and expand. We obtain

$$\begin{aligned} Z'_N(\xi) &= \mathrm{e}^{\mathcal{E}}\, \mathbb{E}\Big[\sum_X I^{\Lambda\backslash\hat{X}} 1^{\hat{X}\backslash X} \prod_{Y\in\mathcal{C}(X)\backslash\mathcal{S}} K(Y) \prod_{Y\in\mathcal{C}(X)\cap\mathcal{S}} K(Y)\Big] \\ &= \mathrm{e}^{\mathcal{E}}\, \mathbb{E}\Bigg[\sum_{\mathcal{X},\hat{\mathcal{Y}}} \chi_{\mathcal{X}\cup\mathcal{Y}} \sum_{P,Q,Z} (1-\mathrm{e}^{E'})^P (I - \mathrm{e}^{E'})^Q (\mathrm{e}^{E'})^{(\langle X\rangle\backslash X)\backslash(P\cup Q)} \\ &\qquad \cdot \delta I^Z \tilde{I}^{\langle X\rangle^c\backslash Z} \prod_{Y\in\mathcal{X}} K(Y) \prod_{(B,Y)\in\hat{\mathcal{Y}}} \frac{1}{|Y|_j} K(B,Y)\Bigg], \end{aligned} \tag{3.3}$$

where the first summation is over $\mathcal{X}$ which is a family of connected large polymers, and $\hat{\mathcal{Y}}$ which is a family of elements in $\hat{\mathcal{S}}$, i.e. $\hat{\mathcal{Y}} = \{(B_i, Y_i) \in \hat{\mathcal{S}}_j\}_{1\le i\le n}$ for some $n \ge 0$, and we have defined $\mathcal{Y} := \{Y_i\}_{1\le i\le n}$. In the above equation and in the sequel of this proof,

$$X := X_{\mathcal{X}\cup\mathcal{Y}}$$

and the second summation above is over $P \in \mathcal{P}(\hat{X}\backslash X)$, $Q \in \mathcal{P}(\langle X\rangle \backslash \hat{X})$, and $Z \in \mathcal{P}(\langle X\rangle^c)$.

Now observe that one can re-arrange the above summations in the following way:

$$\sum_{\mathcal{X},\hat{\mathcal{Y}}} \chi_{\mathcal{X}\cup\mathcal{Y}} \sum_{P,Q,Z} = \sum_{V\in\mathcal{P}'} \sum_{(P,Q,Z,\mathcal{X},\hat{\mathcal{Y}})\to V}, \tag{3.4}$$

where the second summation on the right-hand side means

$$\sum_{(P,Q,Z,\mathcal{X},\hat{\mathcal{Y}})\to V} := \sum_{\mathcal{X},\hat{\mathcal{Y}}} \chi_{\mathcal{X}\cup\mathcal{Y}} \sum_{\substack{P\in\mathcal{P}(\hat{X}\backslash X) \\ Q\in\mathcal{P}(\langle X\rangle\backslash\hat{X})}} \sum_{Z\in\mathcal{P}(\langle X\rangle^c)} 1_{\overline{P\cup Q\cup Z\cup(\cup_{i=1}^n B_i)\cup X_{\mathcal{X}}}=V}.$$

We would like to write the factors $\tilde{I}$ and $\mathrm{e}^{E'}$ into parts in V and outside V:

$$\begin{aligned} \tilde{I}^{\langle X\rangle^c \backslash Z} &= \tilde{I}^{V^c \cap \langle X\rangle^c}\, \tilde{I}^{V \cap (\langle X\rangle^c \backslash Z)}, \\ (\mathrm{e}^{E'})^{(\langle X\rangle \backslash X)\backslash(P\cup Q)} &= (\mathrm{e}^{E'})^{V^c \cap (\langle X\rangle \backslash X)}\, (\mathrm{e}^{E'})^{V \cap (\langle X\rangle \backslash X)\backslash(P\cup Q)}. \end{aligned} \tag{3.5}$$

Note that $V^c \cap \langle X\rangle^c$ (where some $\tilde{I}$ live on) could possibly touch V, so our next step is to make a corridor so that such touchings will be avoided. Write $\tilde{I} = (\tilde{I} - \mathrm{e}^{E'}) + \mathrm{e}^{E'}$ and expand,

$$\tilde{I}^{V^c \cap \langle X\rangle^c} = \sum_{W \in \mathcal{P}'(V^c)} (\tilde{I} - \mathrm{e}^{E'})^{W \cap \langle X\rangle^c} (\mathrm{e}^{E'})^{(V^c \backslash W)\cap \langle X\rangle^c}.$$

For each V and W, define U to be the smallest union of connected components of $V \cup W$ that contains V:

$$U = U_{W,V} := \cap\{T \big| T \in \mathcal{UC}(V \cup W), T \supseteq V\} \in \mathcal{P}',$$

where $\mathcal{UC}(V \cup W)$ is the set of unions of ($j+1$ scale) connected components of $V \cup W$. Observe that if L is sufficiently large, one has $\langle X\rangle \subseteq \hat{V} \subseteq \hat{U}$. So

$$\begin{aligned} \tilde{I}^{V^c \cap \langle X\rangle^c} = \sum_{W \in \mathcal{P}'(V^c)} & (\tilde{I} - \mathrm{e}^{E'})^{W \backslash \hat{U}} (\tilde{I} - \mathrm{e}^{E'})^{W \cap U \cap \langle X\rangle^c} \\ & \times (\mathrm{e}^{E'})^{(V^c \backslash W)\backslash \hat{U}} (\mathrm{e}^{E'})^{(V^c \backslash W)\cap \hat{U} \cap \langle X\rangle^c}. \end{aligned}$$

Let $R := W\backslash U = W \backslash \hat{U}$. Note that one has the following identities for the sets appearing in the above equation: $W \cap U = U \backslash V$ and

$$\begin{aligned} (V^c \backslash W)\backslash \hat{U} &= (\hat{U})^c \backslash R, \\ (V^c \backslash W) \cap \hat{U} &= \hat{U} \backslash U. \end{aligned}$$

The summation over W amounts to a summation over U and R:

$$\begin{aligned} \tilde{I}^{V^c \cap \langle X\rangle^c} &= \sum_{U \in_V \mathcal{P}', U \supseteq V} \ \sum_{R \in \mathcal{P}'(\Lambda \backslash \hat{U})} (\tilde{I} - \mathrm{e}^{E'})^{R} (\tilde{I} - \mathrm{e}^{E'})^{(U \backslash V)\cap \langle X\rangle^c} \\ & \quad \times (\mathrm{e}^{E'})^{(\hat{U})^c \backslash R} (\mathrm{e}^{E'})^{(\hat{U} \backslash U)\cap \langle X\rangle^c} \\ &= \sum_{U \in_V \mathcal{P}', U \supseteq V} \tilde{I}^{\Lambda \backslash \hat{U}} (\tilde{I} - \mathrm{e}^{E'})^{(U \backslash V)\cap \langle X\rangle^c} (\mathrm{e}^{E'})^{(\hat{U} \backslash U)\cap \langle X\rangle^c}. \end{aligned} \tag{3.6}$$

The factor $(\mathrm{e}^{E'})^{V^c \cap (\langle X\rangle \backslash X)}$ appearing in (3.5) is treated as follows. Since $\langle X\rangle \subseteq \hat{U}$

$$\begin{aligned} (\mathrm{e}^{E'})^{V^c \cap (\langle X\rangle \backslash X)} &= (\mathrm{e}^{E'})^{V^c \cap \langle X\rangle} (\mathrm{e}^{-E'})^{V^c \cap X} \\ &= (\mathrm{e}^{E'})^{(\hat{U} \backslash U)\cap \langle X\rangle} (\mathrm{e}^{E'})^{V^c \cap \langle X\rangle \cap U} (\mathrm{e}^{-E'})^{V^c \cap X}. \end{aligned} \tag{3.7}$$

Combine (3.3–3.7),

$$Z'_N(\xi) = \mathrm{e}^{\mathcal{E}}\, \mathbb{E}\Big[\sum_{U \in \mathcal{P}'} \tilde{I}^{\Lambda \backslash \hat{U}} (\mathrm{e}^{E'})^{\hat{U}} K^{\sharp}(U) \Big], \tag{3.8}$$

where for $U \neq \emptyset$

$$
\begin{aligned}
K^\sharp(U) := \sum_{V\subseteq U, V\neq\emptyset} \sum_{(P,Q,Z,\mathcal{X},\hat{\mathcal{Y}})\to V} &(1-\mathrm{e}^{E'})^P (I-\mathrm{e}^{E'})^Q \delta I^Z (\tilde{I}-\mathrm{e}^{E'})^{(U\backslash V)\cap\langle X\rangle^c} \\
&\times (\mathrm{e}^{E'})^{(\langle X\rangle\backslash X)\cap U\backslash(P\cup Q)} (\mathrm{e}^{-E'})^{U\cup X}\, \tilde{I}^{V\cap(\langle X\rangle^c\backslash Z)} \\
&\times \prod_{Y\in\mathcal{X}} K(Y) \prod_{(B,Y)\in\hat{\mathcal{Y}}} \frac{1}{|Y|_j} K(B,Y).
\end{aligned}
\tag{3.9}
$$

Factorizing the constant $\mathrm{e}^{E'}$ by letting

$$
\mathcal{E}' = \mathcal{E} + E'|\Lambda|_j
$$

$$
I'(D) = \mathrm{e}^{-L^d E'} \prod_{B\in\mathcal{B}(D)} \tilde{I}(B) = \mathrm{e}^{-\frac{1}{4}\sigma_{j+1}\sum_{x\in D, e\in\mathcal{E}}\left(\partial_e P_{D^+}\phi(x)+\partial_e\xi(x)\right)^2}
$$

for $D \in \mathcal{B}'$, we obtain

$$
Z_N'(\xi) = \mathrm{e}^{\mathcal{E}'}\mathbb{E}\Big[\sum_{U\in\mathcal{P}'} (I')^{\Lambda\backslash\hat{U}}\, K^\sharp(U)\Big].
$$

This is precisely the statement (2.16). □

Conditional Expectation.

Lemma 10. *$K^\sharp$ factorizes over $j+1$ scale connected components, namely*

$$
K^\sharp(U) = \prod_{V\in\mathcal{C}_{j+1}(U)} K^\sharp(V), \tag{3.10}
$$

where $\mathcal{C}_{j+1}(U)$ is the set of connected components of U as a $j+1$ polymer.

Proof. Let $V_1,\ldots,V_{|\mathcal{C}(U)|}$ be all the connected components of U. For any E which may stand for U, Z, P, Q, elements of $\mathcal{X}\cup\mathcal{Y}$, one of the B_i, or $X = X_{\mathcal{X}\cup\mathcal{Y}}$, let $E^{(p)} = E\backslash\cup_{q\neq p} V_q$. It is easy to check that for $i \neq j$, $E^{(i)}$ and $E^{(j)}$ are strictly disjoint on scale j. Then, the lemma is proved by the factorization property of I, K on scale j. □

We are now ready to take the expectation of $K^\sharp(V)$ conditioned on ϕ outside V^+ for each $V \in \mathcal{C}(U)\backslash\{\Lambda\}$, because $\Lambda\backslash\hat{V}$ and V^+ do not touch. In the case $V = \Lambda$, we just take expectation of $K^\sharp(V)$ without conditioning, but write $\mathbb{E}\big[K^\sharp(\Lambda)\big|(\Lambda^+)^c\big] := \mathbb{E}\big[K^\sharp(\Lambda)\big]$ to shorten the notations. So we obtain the following structure as announced in (2.17):

$$
\begin{aligned}
Z_N'(\xi) &= \mathrm{e}^{\mathcal{E}_{j+1}}\mathbb{E}\Big[\sum_{U\in\mathcal{P}_{j+1}} I_{j+1}^{\Lambda\backslash\hat{U}} K_{j+1}(U)\Big], \\
K_{j+1}(U) &:= \prod_{V\in\mathcal{C}(U)} \mathbb{E}\left[K_j^\sharp(V)\big|(V^+)^c\right].
\end{aligned}
\tag{3.11}
$$

Now, we have come back to the basic structure (2.10) with j replaced by $j+1$. Obviously, $K_{j+1}(U) \in \mathcal{P}_{j+1,c}$. In Sect. 4, we give precise definitions for norms and spaces of the K_j above, and in Sect. 5 we prove smoothness of the above map $(\sigma_j, E_{j+1}, \sigma_{j+1}, K_j) \mapsto K_{j+1}$.

3.3. Properties of the Conditional Expectation

The variation principle. One of our main ideas is to write the Gaussian field ϕ into a sum of two decoupled parts. This is important for the conditional expectation.

Fact. *Given any positive definite quadratic form $Q(v)$ for vector v, if $v = (x, y)$, one can write $Q(v) = Q_1(x) + L(x, y) + Q_2(y)$ where $Q_{1,2}$ are positive definite quadratic forms and $L(x, y)$ is the cross term. Let $\tilde{x}(y)$ be the minimizer of $Q(v) = Q(x, y)$ with y fixed. Then, one can cancel $L(x, y)$ by shifting x by $\tilde{x}$:*

$$Q(v) = Q_1(x - \tilde{x}) + Q\left((\tilde{x}, y)\right). \tag{3.12}$$

Before introducing the next proposition, let us recall our convention that $P_U\phi(x) = \phi(x)$ for $x \notin U$ as in Sect. 2.1.

Proposition 11. *Let $U \subset V \subset \mathbb{Z}^d$ be two finite sets. Let ϕ_U and ϕ_{U^c} be the restriction of ϕ to U and U^c. Let P_U be the Poisson kernel for U and write $\phi(x) = P_U\phi(x) + \zeta(x)$. Then,*

$$-\sum_{x\in V} \phi(x)\Delta_m\phi(x) = -\sum_{x\in U} \zeta(x)\Delta^D_{U,m}\zeta(x) - \sum_{x\in V} P_U\phi(x)\Delta_m P_U\phi(x), \tag{3.13}$$

where $\Delta^D_{U,m}$ is the Dirichlet Laplacian for U.

Proof. We can apply the Fact (3.12) for $\phi = (\phi_U, \phi_{U^c})$, and

$$\begin{aligned} Q(\phi) &= -\sum_{x\in V} \phi(x)\Delta_m\phi(x) \\ &= -\sum_{x\in U} \phi_U(x)\Delta^D_{U,m}\phi_U(x) + L(\phi_U, \phi_{U^c}) - \sum_{x\in U^c} \phi_{U^c}(x)\Delta^D_{U^c,m}\phi_{U^c}(x), \end{aligned}$$

where L is the cross term, and $\Delta^D_{U^c,m}$ is the Dirichlet Laplacian for U^c. Since the minimizer of $Q(\phi)$ with ϕ_{U^c} fixed is the harmonic extension of ϕ from U^c into U, and the harmonic field is equal to $P_U\phi$, one has

$$\begin{aligned} Q(\phi) &= -\sum_{x\in U} \left(\phi_U - P_U\phi\right)(x)\Delta^D_{U,m}\left(\phi_U - P_U\phi\right)(x) - Q\left((P_U\phi, \phi_{U^c})\right) \\ &= -\sum_{x\in U} \zeta(x)\Delta^D_{U,m}\zeta(x) - \sum_{x\in V} P_U\phi(x)\Delta_m P_U\phi(x). \end{aligned}$$

This completes the proof. We remark that in the last term, the points $x \in U$ do not actually contribute to the sum since $\Delta_m P_U\phi = 0$ in U. □

By this proposition, taking expectation of a function $K(\phi)$ conditioned on $\{\phi(x) \big| x \in U^c\}$ is equivalent to simply integrating out ζ:

$$\mathbb{E}\left[K(\phi)\big|U^c\right] = \mathbb{E}_\zeta\left[K(P_U\phi + \zeta)\right], \tag{3.14}$$

where the covariance of ζ is the Dirichlet Green's function for U.

As another important fact, we note that $K(X, \phi, \xi)$ constructed above [see (3.11)] has a "special structure": it only depends on ϕ, ξ via $P_{X^+}\phi + \xi$; in other words, there exists a function $\widetilde{K}(X, \psi)$ so that

$$K(X, \phi, \xi) = \widetilde{K}(X, P_{X^+}\phi + \xi). \tag{3.15}$$

In fact, we have the following lemma.

Lemma 12. *Let $U \subset \Lambda$ be a given set. For every $k = 1, \ldots, m$, let $Y_k \subset U$, and $H_k(\phi, \xi)$ be a given function of ϕ and ξ. Suppose that there exist functions $\widetilde{H}_k$ such that*

$$H_k(\phi, \xi) = \widetilde{H}_k(P_{Y_k}\phi + \xi),$$

namely H_k only depends on ϕ, ξ via $P_{Y_k}\phi + \xi$. Then, the function $\mathbb{E}\left[\prod_k H_k(\phi, \xi)\big|U^c\right]$ only depends on ϕ, ξ via $P_U\phi + \xi$.

Proof. We write the expectation conditioned on $\phi\big|_{U^c}$ as expectation over the Dirichlet Gaussian field ζ on U, and then exploit the assumption on H_k:

$$\begin{aligned}\mathbb{E}\Big[\prod_k H_k(\phi, \xi)\big|U^c\Big] &= \mathbb{E}_\zeta\Big[\prod_k H_k(P_U\phi + \zeta, \xi)\Big] \\ &= \mathbb{E}_\zeta\Big[\prod_k \widetilde{H}_k\Big(P_{Y_k}(P_U\phi + \zeta) + \xi\Big)\Big].\end{aligned} \tag{3.16}$$

The last quantity depends on ϕ, ξ via $P_U\phi+\xi$ by noting that $P_{Y_k}P_U = P_U$. $\square$

Note that $K_0(X, \phi, \xi)$ is actually a function of $\phi + \xi$. By our convention, when $j = 0$, P_{X^+} is understood as the identity operator, so we do start from functions with this special structure [(3.15)]. Together with the above lemma and (3.9), (3.11), we see that for every $j \geq 0$, the fact (3.15) holds:

Corollary 13. *Let $K_j(X, \phi, \xi)$ be the functions constructed in (3.11). Then for every $j \geq 0$, there exists a function $\widetilde{K}_j(X)$ such that $K_j(X, \phi, \xi) = \widetilde{K}_j(X, P_{X^+}\phi + \xi)$.*

In the following, it will be helpful to have this point of view in mind.

The Important Scaling. Our main result in this subsection is Proposition 16. We first collect some general results about harmonic functions on the lattice. These will include derivative estimates and "mean value" type bounds.

Lemma 14. *Let $\mathcal{B}_R$ be the discrete ball of radius R centered on the origin, namely $\mathcal{B}_R = \{x \in \mathbb{Z}^d : |x| < R\}$. There exists a constant $c > 0$ such that the following holds for every R sufficiently large.*

- *If g is harmonic in $\mathcal{B}_R$, then for every $e \in \mathcal{S}$,*

$$|\partial_e g(0)| \leq cR^{-1} \sup_{x \in \mathcal{B}} |g(x)|. \tag{3.17}$$

- *If f is harmonic and non-negative in $\mathcal{B}_R$, then for every $e \in \mathcal{S}$,*

$$|\partial_e f(0)| \leq cR^{-1} f(0). \tag{3.18}$$

Proof. This is [38, Theorem 6.3.8 of Sect. 6.3]. The statement of that theorem is about harmonic functions related with general "$\mathcal{P}_d$ class" (i.e. symmetric, finite range) random walks. In particular, it is true for harmonic functions associated with standard Laplacian related with simple random walks. The large R requirement was used to deal with the lattice effect on the boundary of the ball in their proof. $\square$

Note that the constant c in the above lemma does not depend on the function g or f. In the second statement, non-negativity condition is necessary: the linear function $f(x) = x$ on $[-1, 1]$ would violate the bound (3.18).

The next result is a mean value type bound. For $R > 0$ and $a \in \mathbb{Z}^d$, we define a cube of size R centered at a by

$$\mathcal{K}_R := \left\{ y \in \mathbb{Z}^d \middle| \, |y - a|_\infty \leq R \right\}. \tag{3.19}$$

Lemma 15. *Given real numbers s, t such that $0 < 3s < r < 1$. Let $\mathcal{K}_R$ and $\mathcal{K}_{rR}$ be cubes of sizes R and rR, respectively, centered at the same point. Assume that u is harmonic in the cube $\mathcal{K}_R$. Let $X = \mathcal{K}_R \backslash \mathcal{K}_{rR}$, $x \in \mathcal{K}_{rR}$ and $d(x, \partial\mathcal{K}_{rR}) > sR$. Then*

$$|u(x)| \leq O(R^{-d}) \sum_{y \in X} |u(y)|, \tag{3.20}$$

$$u(x)^2 \leq O(R^{-d}) \sum_{y \in X} u(y)^2. \tag{3.21}$$

Here, the constants in the big-O notation depend on s, t.

Proof. For any integer b such that $rR \leq b < R$, let $\mathcal{K}_b$ be cubes of sizes b co-centered with $\mathcal{K}_R$. Then, since u is harmonic, and the Poisson kernel $0 \leq P_{\mathcal{K}_b}(x, y) \leq c\, b^{-(d-1)}$ for some constant $c > 0$ by the assumption on x, one has

$$|u(x)| = \Big| \sum_{y \in \partial\mathcal{K}_b} P_{\mathcal{K}_b}(x, y) u(y) \Big| \leq c\, b^{-(d-1)} \sum_{y \in \partial\mathcal{K}_b} |u(y)|.$$

Multiply both sides by b^{d-1} and sum over $rR \leq b \leq R$, we have

$$R^d |u(x)| \leq c' \sum_{y \in X} |u(y)| \tag{3.22}$$

for some constant $c' > 0$ which proves (3.20). By Cauchy-Schwartz inequality,

$$|u(x)| \leq O(R^{-d}) \left(\sum_{y \in X} u(y)^2 \right)^{1/2} |X|^{1/2}.$$

This together with $|X| = O(R^d)$ proves (3.21). $\square$

The next Proposition plays an important role in controlling the fundamental scaling. See the paragraph below Proposition 7 for a motivation.

Proposition 16. *Let $x \in X \subset U \subset \Lambda$. If $d(x, \partial X) \geq cL^j$, then*

$$\sum_{y_1, y_2 \in \partial X} (\partial_{x,e} P_X)(x, y_1)\, C_U(y_1, y_2)\, (\partial_{x,e} P_X)(x, y_2) \leq O(1) L^{-dj} \tag{3.23}$$

for all $e \in \mathcal{E}$ where the constant $O(1)$ only depends on the constant c. Here, $\partial_{x,e}$ is the discrete derivative w.r.t. the argument x to the direction e.

Proof. Notice that $C_U \leq C_\Lambda$ as quadratic forms, so it is enough to prove the statement with C_U replaced by C_Λ. Since $y_2 \in \partial X$ and $C_\Lambda(x - y_2)$ is $-\Delta_m$-harmonic in $x \in X$, one has

$$\sum_{y_1 \in \partial X} P_X(x, y_1) C_\Lambda(y_1, y_2) = C_\Lambda(x, y_2).$$

Taking derivative w.r.t. x on the above equation, we obtain that the left-hand side of Eq. (3.23) is equal to

$$\sum_{y_2 \in \partial X} \partial_{x,e} C_\Lambda(x, y_2)\, \partial_{x,e} P_X(x, y_2). \tag{3.24}$$

By Corollary 43 (for decay rate of ∇C_Λ) and the assumption $d(x, \partial X) \geq cL^j$, one has

$$|\partial_{x,e} C_\Lambda(x, y_2)| \leq O(L^{-(d-1)j}).$$

Using again the same assumption, there exists a discrete ball $\mathcal{B}_R(x) \subset X$ centered on x with radius $R = \frac{c}{2} L^j$ (and R is independent of x). For every $y_2 \in \partial X$, $P_X(x, y_2)$ is harmonic and non-negative in $\mathcal{B}_R(x)$. Applying (3.18),

$$\left|\partial_{x,e} P_X(x, y_2)\right| \leq c_1 R^{-1} P_X(x, y_2)$$

with c_1 depending on c but independent of x and y_2 (since it is independent of the harmonic function). (The above bound holds for $P_X^{(0)}$, the Poisson kernel with mass regularization $m = 0$, and as long as $m > 0$ is sufficiently small so that $|(P_X - P_X^{(0)})(x, y_2)| \leq R^{-1} P_X(x, y_2)$ it is still valid.) So (3.24) is bounded by

$$O(L^{-(d-1)j}) O(L^{-j}) \sum_{y_2 \in \partial X} P_X(x, y_2).$$

Since $\sum_{y_2 \in \partial X} P_X(x, y_2) \leq 1$ for all $m > 0$, the above quantity is bounded by $O(L^{-dj})$. $\square$

Remark 17. One may find that our method also resembles Gawedzki and Kupiainen's approach [34,35] because the Poisson kernel here plays a similar role as their spin blocking operator. However, there are many differences. For example, our fluctuation fields ζ have finite range covariances; the integrands at different scales do not have to be in Gibbsian forms; and our polymer arrangements are closer to Brydges [18].

4. Norms

Before we define the norms, we have a remark about the choices of four important constants: L, A, κ and h where L has already appeared above and A, κ and h will appear in the definitions of norms below.

We will first fix $L > L_0(d)$ large enough which satisfies all the largeness requirements in Lemma 25 (a geometric result), Lemma 32 and Proposition 37. These results establish contractivity of the three linear maps defined in Proposition 28, and L has to be large to overwhelm some $O(1)$ constants appearing in the estimates of the norms of these linear maps.

We then choose $A > A_0(d, L)$ large enough which satisfies all the largeness requirements in Proposition 26 (smoothness of RG) and Proposition 29 (contractivity of the linear map $\mathcal{L}_1$ defined in Proposition 28).

After this, we choose $0 < \kappa < \kappa_0(d, L, A)$ small enough which satisfies all the smallness requirements in Lemma 24 [integrating "regulators" defined in (4.4)] and Lemma 31. Finally, we choose $h > h_0(d, L, A, \kappa)$ large enough for the arguments in the proof of Lemma 31.

4.1. Definitions of Norms

We now define the norm of the fields, the norm of a function of the fields (i.e. elements in $\mathcal{N}$) at a fixed field, and the norm of a function in $\mathcal{N}^{\mathcal{P}_j}$. For $j > 0$, the definitions are as follows:

1. Define $h_j = hL^{-(d-2)j/2}$ for constant $h > 0$. We first define the seminorm for the fields. Let us recall that ξ is the field introduced in Sect. 2. For $X \subset Y$ and $\lambda \in \mathbb{R}$, we define
$$\|(f, \lambda\xi)\|_{\Phi_j(X,Y)} := h_j^{-1} \sup_{x\in X, e} \left| L^j \partial_e (P_Y f(x) + \lambda\xi(x)) \right|. \tag{4.1}$$
The notation $\|f\|_{\Phi_j(X,Y)}$ where ξ part is dropped will be understood as $\|(f,0)\|_{\Phi_j(X,Y)}$. As a special case, if $X \in \mathcal{P}_j$ then we simply write
$$\|(f, \lambda\xi)\|_{\Phi_j(X)} := \|(f, \lambda\xi)\|_{\Phi_j(\dot{X}, X^+)}. \tag{4.2}$$
The space $\Phi_j(X, Y)$ is then defined as:
$$\Phi_j(X, Y) := (\mathbb{R}^\Lambda \times \mathbb{R})/\{\|(f, \lambda\xi)\|_{\Phi_j(X,Y)} = 0\},$$
and this normed vector space is complete (i.e. Banach space), and $\Phi_j(X) := \Phi_j(\dot{X}, X^+)$.
2. We then define differentials for functions of the fields, and their norm. Let $K(\phi, \xi)$ be a function of ϕ, ξ. For test functions
$$(f, \lambda)^{\times n} := (f_1, \lambda_1\xi, \ldots, f_n, \lambda_n\xi),$$
the nth differential of $K(\phi, \xi)$ is
$$K^{(n)}(\phi, \xi; (f, \lambda)^{\times n}) := \frac{\partial^n}{\partial t_1 \ldots \partial t_n} K(\phi + \sum_{i=1}^n t_i f_i, \xi + \sum_{i=1}^n t_i \lambda_i \xi)\Big|_{t_i=0}.$$

It is normed with a space of test functions Φ by

$$\|K^{(n)}(\phi,\xi)\|_{T^n_\phi(\Phi)} := \sup_{\|(f_i,\lambda_i\xi)\|_\Phi \le 1} \left|K^{(n)}(\phi,\xi;(f,\lambda)^{\times n})\right|.$$

We then measure the amplitude of $K(\phi,\xi)$ at a fixed field ϕ by incorporating all its derivatives at ϕ that we want to control:

$$\|K(\phi,\xi)\|_{T_\phi(\Phi)} := \sum_{n=0}^{4} \frac{1}{n!} \|K^{(n)}(\phi,\xi)\|_{T^n_\phi(\Phi)} \tag{4.3}$$

In most of the discussions, we actually have a function $K(X,\phi,\xi)$ which is element in $\mathcal{N}^{\mathcal{P}_j}$. Then, the above $T_\phi(\Phi)$ norm is taken for every $X \in \mathcal{P}_j$, and Φ will be chosen to be $\Phi_j(X)$ defined in (4.2).

3. For $\kappa > 0$, we define "regulators":

$$G(X,Y) := \mathbb{E}\left[\mathrm{e}^{\frac{\kappa}{2}\sum_{x\in X, e\in\mathcal{E}}(\partial_e\phi(x))^2} \middle| Y^c\right] / N(X,Y) \tag{4.4}$$

for $X \subset Y$ where the normalization factor is defined by

$$N(X,Y) := \mathbb{E}\left[\mathrm{e}^{\frac{\kappa}{2}\sum_{x\in X, e\in\mathcal{E}}(\partial_e\phi(x))^2} \middle| \phi_{Y^c} = 0\right].$$

For $K \in \mathcal{N}^{\mathcal{P}_j}$, define

$$\|K(X)\|_j := \sup_\phi \|K(X,\phi,\xi)\|_{T_\phi(\Phi_j(X))} G(\ddot{X}, X^+)^{-1}. \tag{4.5}$$

Finally, for $A > 0$,

$$\|K\|_j := \sup_{X\in\mathcal{P}_j} \|K(X)\|_j A^{|X|_j}. \tag{4.6}$$

The space $(\mathcal{N}^{\mathcal{P}_j}, \|\cdot\|_j)$ is then a Banach space.

For the case $j = 0$: (4.1)–(4.3) are still defined for $j = 0$ with $P_Y = id$ and $\dot{X} = X$ (recall these conventions made in Sect. 2). (4.5) is defined with G replaced by

$$G_0(X) := \mathrm{e}^{\frac{\kappa}{2}\sum_{x\in X, e\in\mathcal{E}}(\partial_e\phi(x))^2}.$$

4.2. Properties

Lemma 18. *Let F be function of ϕ, ξ, and $X \subset Y \subset U$. We have the following property for the $T_\phi(\Phi)$ norms:*

$$\|F^{(n)}(\phi,\xi)\|_{T^n_\phi(\Phi_j(Y,U))} \le \|F^{(n)}(\phi,\xi)\|_{T^n_\phi(\Phi_j(X,U))} \tag{4.7}$$

which also holds without n.

Proof. The proof is immediate because $\|f\|_{\Phi_j(Y,U)} \ge \|f\|_{\Phi_j(X,U)}$. □

Before the discussion on further properties, we recall that our functions of the fields have the special structure (3.15). It turns out that in view of this structure, it is sometimes more convenient to consider a type of function spaces $\widetilde{\Phi}_j(X,Y)$ for $X \subset Y$ defined as follows:

$$\widetilde{\Phi}_j(X,Y) := \{g : \Delta_m g = 0 \text{ on } Y\} \oplus \mathbb{R}\xi$$

equipped with semi-norm

$$\|g \oplus \lambda\xi\|_{\widetilde{\Phi}_j(X,Y)} := h_j^{-1} \sup_{x\in X,e} \Big| L^j \partial_e (g(x) + \lambda\xi(x)) \Big|.$$

Note that the above sum is really a direct sum since the test function f in (2.1) is not identically zero and, therefore, ξ is not Δ_m-harmonic. Now if a function $F(\phi,\xi) = \tilde{F}(\psi)$ with $\psi = P_Y\phi + \xi$, one actually has

$$\|F^{(n)}(\phi,\xi)\|_{T_\phi^n(\Phi_j(X,Y))} = \sup_{\|g_i\oplus\lambda_i\xi\|_{\widetilde{\Phi}_j(X,Y)}\le 1} \Big|\partial_{t_i}^n\big|_{t_i=0} \tilde{F}(\psi + \sum_{i=1}^n t_i(g_i + \lambda_i\xi))\Big|$$

for any subset $X \subset Y$ since in this situation, varying ϕ by $t_i f_i$ for generic functions f_i is equivalent with varying $P_Y\phi$ by harmonic functions on Y.

Lemma 19. *Assume the setting of Lemma 12. For every $k = 1,\ldots,m$, let $X_k \subset Y_k \subset U$. Define $X := \cup_{k=1}^m X_k$. Then, one has*

$$\Big\|\mathbb{E}\Big[\prod_{k=1}^m H_k(\phi,\xi)\big|U^c\Big]\Big\|_{T_\phi(\Phi_j(X,U))} \le \mathbb{E}\Big[\prod_{k=1}^m \|H_k(\phi,\xi)\|_{T_\phi(\Phi_j(X_k,Y_k))}\,\big|U^c\Big]. \tag{4.8}$$

Remark 20. Lemma 19 is stated in terms of generic functions H_k. The typical situation in which we apply this lemma is that $Y_k = X_k^+$, and $H_k(\phi,\xi) = K_k(X_k,\phi,\xi)$ with each $K_k(X_k,\phi,\xi)$ satisfying (3.15).

Remark 21. Lemma 19 is analogous with [18, Lemma 6.7] (the norm of a product bounded by product of norms) and [18, Lemma 6.9] (the norm of an expectation bounded by expectation of the norm). The difference is that in our approach we combine the two results; in fact, here both sides of (4.8) have the conditional expectation with the same conditioning, so that the two sides are comparable.

Proof of Lemma 19. Let $\zeta = \phi - P_U\phi$ and define

$$F(\phi,\xi) := \mathbb{E}_\zeta\Big[\prod_k H_k(P_U\phi + \zeta,\xi)\Big].$$

Lemma 12 states that there exists $\widetilde{F}$ such that $F(\phi,\xi) = \tilde{F}(P_U\phi + \xi)$. Write $\langle t,f\rangle_n := \sum_{i=1}^n t_i f_i$. By the discussion before this lemma, the $T_\phi^n(\Phi_j(X,U))$ norm of $F^{(n)}(\phi,\xi)$ is equal to

$$\sup_{\|g_i\oplus\lambda_i\xi\|_{\widetilde{\Phi}_j(X,U)}\le 1} \Big|\partial_{t_i}^n\big|_{t_i=0} \mathbb{E}_\zeta\Big[\prod_k H_k\Big(P_U\phi + \langle t,g\rangle_n + \zeta,\ \xi + \langle t,\lambda\xi\rangle_n\Big)\Big]\Big|.$$

This is bounded by taking the $\mathbb{E}_\zeta$ outside the supremum, and we apply the product rule of derivatives. We then obtain factors of the form

$$\sup_{g_i\oplus\lambda_i\xi}\left|\partial_{t_i}^{r}\big|_{t_i=0}H_k\Big(\phi+\langle t,g\rangle_r,\,\xi+\langle t,\lambda\xi\rangle_r\Big)\right|$$

with the sup over the same set as above. Since g_i are harmonic on Y_k and by Lemma 18, the supremum can be replaced by one taken over all $g_i\oplus\lambda_i\xi$ such that g_i are harmonic on Y_k and $\|g_i\oplus\lambda_i\xi\|_{\widetilde{\Phi}_j(X_k,Y_k)}\le 1$. By the assumption on H_k, and $P_{Y_k}g=g$, the above function H_k is equal to $\widetilde{H}_k\big(P_{Y_k}\phi+\langle t,g\rangle_r+\xi+\langle t,\lambda\xi\rangle_r\big)$. Again by the discussion before this lemma, the above quantity is actually bounded by $\|H_k(\phi,\xi)\|_{T_\phi^r(\Phi_j(X_k,Y_k))}$. Summing over multi-indices $(r_1,\ldots,r_m)$ with $|r|=n$, followed by summing over n, one obtains the desired bound. $\square$

Before the next lemma, we introduce a short notation

$$(\partial_m f)^2 := (\partial f)^2 + m^2 f^2 \tag{4.9}$$

Lemma 22. *We have the following properties for the regulator.*

1. $G(X,Y,\phi=0)=1$.
2. *If $X_1\subset Y_1$, $X_2\subset Y_2$, and $Y_1\cup\partial Y_1, Y_2\cup\partial Y_2$ are disjoint, then*
$$G(X_1,Y_1)G(X_2,Y_2)=G(X_1\cup X_2,Y_1\cup Y_2). \tag{4.10}$$
3. *We have an alternative representation of $G(X,Y)$*
$$G(X,Y)=\exp\left(\frac{\kappa}{2}\sum_X(\partial\psi_1)^2-\frac{1}{2}\sum_Y(\partial_m\psi_1)^2+\frac{1}{2}\sum_Y(\partial_m\psi_2)^2\right), \tag{4.11}$$
where ψ_1 is the minimizer of $\sum_Y(\partial_m\phi)^2-\kappa\sum_X(\partial\phi)^2$ with ϕ_{Y^c} fixed, and ψ_2 is the minimizer of $\sum_Y(\partial_m\phi)^2$ with ϕ_{Y^c} fixed.
4. *Fixing Y, $G(X,Y)$ is monotonically increasing in X for all $X\subset Y$.*
5. *With $\psi_{1,2}$ defined in (3),*
$$\exp\left(\frac{\kappa}{2}\sum_X(\partial\psi_2)^2\right)\le G(X,Y)\le\exp\left(\frac{\kappa}{2}\sum_X(\partial\psi_1)^2\right). \tag{4.12}$$

Proof. (1)(2) hold by definition and the fact that $G(X,Y)$ is a function of ϕ on ∂Y. For (3),

$$G(X,Y)=\frac{\int e^{\frac{\kappa}{2}\sum_X(\partial\phi)^2-\frac{1}{2}\sum_\Lambda(\partial_m\phi)^2}d^Y\phi\Big/\int e^{-\frac{1}{2}\sum_\Lambda(\partial_m\phi)^2}d^Y\phi}{\int e^{\frac{\kappa}{2}\sum_X(\partial\phi)^2-\frac{1}{2}\sum_Y(\partial_m^D\phi)^2}d^Y\phi\Big/\int e^{-\frac{1}{2}\sum_Y(\partial_m^D\phi)^2}d^Y\phi}, \tag{4.13}$$

where $d^Y\phi$ is the Lebesgue measure on $\{\phi(x):x\in Y\}\cong\mathbb{R}^Y$, ∂^D takes Dirichlet boundary condition on ∂Y. Using Fact (3.12) for both quadratic forms $-\frac{\kappa}{2}\sum_X(\partial\phi)^2 + \frac{1}{2}\sum_\Lambda(\partial_m\phi)^2$ and $\frac{1}{2}\sum_\Lambda(\partial_m\phi)^2$, we obtain (3), where the quantity

$$\int e^{\frac{\kappa}{2}\sum_X(\partial\phi)^2 - \frac{1}{2}\sum_Y(\partial_m^D\phi)^2} d^Y\phi$$

appears in both numerator and denominator and thus cancels, and so does the quantity

$$\int e^{-\frac{1}{2}\sum_Y(\partial_m^D\phi)^2} d^Y\phi.$$

(4) holds because of (3) and that

$$\inf_\phi \left\{ \sum_Y (\partial_m\phi)^2 - \kappa\sum_X(\partial\phi)^2 \middle| Y^c \right\} \tag{4.14}$$

is monotonically decreasing in X. The two inequalities in (5) hold by replacing ψ_1 by ψ_2 or replacing ψ_2 by ψ_1, and using definitions of ψ_1,ψ_2. $\square$

Remark 23. The regulator in [18] has the form $e^{\kappa\sum(\partial\phi')^2+\text{the other terms}}$; since the smoothed field ϕ' there is analogous to our ψ, the last property above implies that our regulator has about the same amplitude as the one in [18], except that we no longer need the other terms.

Before proving a further property, we recall a formula. If U is a finite set and $\psi = \{\psi(x) : x \in U\}$ is a family of centered Gaussian random variables with covariance identity, and $T : l^2(U) \to l^2(U)$ satisfies $\|T\| < 1$ then

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}(\psi, T\psi)_{l^2(U)}\Big)\Big] = \det(1-T)^{-1/2} = \exp\left(\frac{1}{2}\sum_{n=1}^\infty \frac{1}{n}Tr(T^n)\right) \tag{4.15}$$

The next lemma shows that the conditional expectations almost automatically do the work when one wants to see how the regulators undergo integrations, except that we need to manually control a ratio of normalizations.

Lemma 24. *Suppose that $\kappa > 0$ is sufficiently small. For $X \subset Y \subset U$, and $d(X, Y^c) = c_0L^j$, one has the bound*

$$\mathbb{E}\left[G(X,Y)\middle|U^c\right] \le c^{L^{-dj}|X|}G(X,U) \tag{4.16}$$

if $U \neq \Lambda$, for some constant c only depending on c_0. One also has, as the special case, the short-hand notation and bound

$$\mathbb{E}\left[G(X,Y)\middle|(\Lambda^+)^c\right] := \mathbb{E}\left[G(X,Y)\right] \le c^{L^{-dj}|X|}.$$

In particular, if $X = \ddot{X}_0$ for some $X_0 \in \mathcal{P}_j$, then the factor $c^{L^{-dj}|X|}$ can be written as $\bar{c}^{|X_0|_j}$ for some constant $\bar{c}$. Furthermore, G_0 also satisfies the same bound.

Proof. By definition, one has

$$\mathbb{E}\left[G(X,Y)\middle|U^c\right] = \mathbb{E}\left[\mathrm{e}^{\frac{\kappa}{2}\sum_{x\in X, e\in\mathcal{E}}(\partial_e\phi(x))^2}\middle|U^c\right]/N(X,Y)$$
$$= G(X,U)\frac{N(X,U)}{N(X,Y)}.$$

So it remains to estimate the last ratio. Recall that the factor $N(X,Y)$ is an expectation of the exponential weight over the Dirichlet Gaussian field ϕ. For this Dirichlet Gaussian field ϕ, define $\phi = C_Y^{1/2}\psi$ so that ψ has covariance identity, where C_Y is the Dirichlet Green's function for Y. Then, define $T_Y = \frac{1}{2}\sum_{e\in\mathcal{E}}(\partial_e C_Y^{1/2})^\star 1_X(\partial_e C_Y^{1/2})$ as an operator on $l^2 = l^2(\Lambda)$. We define similar operators C_U, T_U in the same way for U. Let ∂_e^D, $-\Delta_Y$ take Dirichlet boundary condition on ∂Y. Because C_Y is the inverse of $-\Delta_Y + m^2$,

$$\begin{aligned}(f, T_Y f)_{l^2} &= \frac{1}{2}\sum_{x\in X, e\in\mathcal{E}}(\partial_e C_Y^{1/2} f(x))^2\\
&\le \frac{1}{2}\sum_{x\in Y,e}(\partial_e^D C_Y^{1/2} f(x))^2 + \frac{m^2}{2}\sum_{x\in Y,e}(C_Y^{1/2}f(x))^2\\
&\le \sum_{x\in Y} C_Y^{1/2} f(x)(-\Delta_Y + m^2)C_Y^{1/2} f(x)\\
&\le (f,f)_{l^2}.\end{aligned} \tag{4.17}$$

So $\|T_Y\| \le 1$. Similarly, $\|T_U\| \le 1$. By (4.15)

$$\frac{N(X,U)}{N(X,Y)} = \frac{\mathbb{E}\left[\mathrm{e}^{\frac{\kappa}{2}(\psi, T_U\psi)}\right]}{\mathbb{E}\left[\mathrm{e}^{\frac{\kappa}{2}(\psi, T_Y\psi)}\right]} = \left(\frac{\det(1-\kappa T_U)}{\det(1-\kappa T_Y)}\right)^{-1/2} \tag{4.18}$$

Taking logarithm, we need to compute

$$Tr\left(\log(1-\kappa T_U) - \log(1-\kappa T_Y)\right) \le O(1) Tr\left(\kappa T_U - \kappa T_Y\right),$$

where we have used $\|T_Y\| \le 1$, $\|T_U\| \le 1$, κ is small, and $\log(1-x)$ is Lipschitz on $x \in [-\frac{1}{2}, \frac{1}{2}]$. Since $C_U - C_Y = P_Y C_U$,

$$\begin{aligned}Tr\left(T_U - T_Y\right) &= \frac{1}{2}\sum_{e\in\mathcal{E}, x\in X}\partial_e(C_U - C_Y)\partial_e^\star(x,x)\\
&= \frac{1}{2}\sum_{e\in\mathcal{E}, x\in X}\sum_{y\in\partial Y}\partial_{x,e}P_Y(x,y)\partial_{x,e}C_U(y,x)\end{aligned} \tag{4.19}$$

By Lemma 15 and proceed similarly as Eq. (3.24) in proof of Proposition 16, making use of the $O(L^j)$ distance between x and y, the above expression is bounded by $O(L^{-jd})\,|X|$ which concludes the proof.

The bound on $\mathbb{E}\left[G(X,Y)\middle|(\Lambda^+)^c\right]$ is similar. The only modification is to replace C_U by C_Λ which satisfies periodic instead of Dirichlet boundary condition. For G_0, we can directly bound $\mathbb{E}\left[\mathrm{e}^{\frac{\kappa}{2}\sum_{x\in X,e\in\mathcal{E}}(\partial_e\phi(x))^2}\middle|U^c\right]$ by $c^{|X|}$. $\square$

5. Smoothness of RG

In this section, we prove that the RG map constructed in Sect. 3 is smooth. First of all, we need some geometric results from [18].

Lemma 25 (*Brydges* [18]). *There exists an $\eta = \eta(d) > 1$ such that for all $L \geq 2^d + 1$ and for all large connected sets $X \in \mathcal{P}_j$, $|X|_j \geq \eta|\bar{X}|_{j+1}$. In addition, for all $X \in \mathcal{P}_j$, $|X|_j \geq |\bar{X}|_{j+1}$, and*

$$|X|_j \geq \frac{1}{2}(1+\eta)|\bar{X}|_{j+1} - \frac{1}{2}(1+\eta)2^{d+1}|\mathcal{C}(X)| \tag{5.1}$$

Proof. The lemma is the same with [18] (Lemma 6.15 and 6.16), so we omit the proof. $\square$

In the following result, assuming $j \geq 0$, we omit subscript j for objects at scale j and write a prime for objects at scale $j+1$, as in Sect. 3. Recall that the spaces $\mathcal{N}^{\mathcal{P}_c}$, $\mathcal{N}^{\mathcal{P}'_c}$ are defined in Sect. 2.4.1, and they are equipped with norms defined in Sect. 4.6.

Proposition 26. *Let $B'(\mathcal{N}^{\mathcal{P}'_c})$ be a ball centered on the origin in $\mathcal{N}^{\mathcal{P}'_c}$. There exists $A(d,L,B')$ and $A^\star(d,A)$ such that for $A > A(d,L,B')$ and $A^\star > A^\star(d,A)$, the map $(\sigma_j, E_{j+1}, \sigma_{j+1}, K_j) \mapsto K_{j+1}$ defined in Sect. 3.2 is smooth from $(-A^{\star-1}, A^{\star-1})^3 \times B_{A^{\star-1}}(\mathcal{N}^{\mathcal{P}_c})$ to $B'(\mathcal{N}^{\mathcal{P}'_c})$ where $B_{A^{\star-1}}(\mathcal{N}^{\mathcal{P}_c})$ is a ball centered on the origin in $\mathcal{N}^{\mathcal{P}_c}$ with radius $A^{\star-1}$.*

Proof. Let $A^{\star-1} \ll \kappa$. For $U \in \mathcal{P}'_c$, by definition of $K^\sharp$,

$$\begin{aligned} K'(U) = \sum_{V\subseteq U, V\neq\emptyset} \sum_{(P,Q,Z,\mathcal{X},\hat{\mathcal{Y}})\to V} &\mathbf{E}^{U^+} \\ &\times \underbrace{(1-\mathrm{e}^{E'})^P\,(\mathrm{e}^{E'})^{(\langle X\rangle\setminus X)\cap U\setminus(P\cup Q)}\,(\mathrm{e}^{-E'})^{U\cup X}}_{\leq (A^\star/2)^{-|P|_j}\,2^{|(\langle X\rangle\setminus X)\cap U\setminus(P\cup Q)|_j}\,2^{|U\cup X|_j}}, \end{aligned} \tag{5.2}$$

where, with $\prod K := \prod_{Y\in\mathcal{X}} K(Y) \prod_{(B,Y)\in\hat{\mathcal{Y}}} \frac{1}{|Y|_j} K(B,Y)$ as a short-hand notation,

$$\begin{aligned} \mathbf{E}^{U^+} &:= \mathbb{E}\left[(\tilde{I}-\mathrm{e}^{E'})^{(U\setminus V)\cap\langle X\rangle^c}\,\tilde{I}^{V\cap(\langle X\rangle^c\setminus Z)}\,\delta I^Z (I-\mathrm{e}^{E'})^Q \prod K\middle|(U^+)^c\right] \\ &= \mathbb{E}\Bigg[\underbrace{\mathbb{E}\left[(\tilde{I}-\mathrm{e}^{E'})^{(U\setminus V)\cap\langle X\rangle^c}\,\tilde{I}^{V\cap(\langle X\rangle^c\setminus Z)}\,\delta I^Z (I-\mathrm{e}^{E'})^Q\middle|(W^+)^c\right]}_{=:\mathbf{E}^{W^+}} \prod K\middle|(U^+)^c\Bigg], \end{aligned} \tag{5.3}$$

where $W = U\backslash\hat{X}$ (recall that $X := X_{\mathcal{X}\cup\mathcal{Y}}$) and the last step used the corridors around $K(Y)$ to make sense of the $(W^+)^c$ conditional expectation. In the above W^+ is a + operation at scale j and U^+ is a + operation at scale $j+1$.

We first control $\mathbf{E}^{W^+}$. With $\phi = P_{W^+}\phi + \zeta$ and the inequality $(a+b)^2 \le 2a^2 + 2b^2$, and using assumption $A^{\star-1} \ll \kappa$, Lemma 44, we list the estimates for each factor.

$$\|(I - \mathrm{e}^{E'})(B)\|_{T_\phi(\Phi_j(B))} \le 5(\kappa A^\star)^{-1}\mathrm{e}^{\frac{\kappa}{2}\sum_B(\partial P_{W^+}\phi)^2 + \frac{\kappa}{2}\sum_B(\partial P_{B^+}\zeta)^2}$$

for all $B \in Q$, where $B^+ \subseteq W^+$ since $Q \subseteq \langle X\rangle \backslash \hat{X}$; and,

$$\|(\tilde{I} - \mathrm{e}^{E'})(B)\|_{T_\phi(\Phi_j(B))} \le 5(\kappa A^\star)^{-1}\mathrm{e}^{\frac{\kappa}{2}\sum_B(\partial P_{W^+}\phi)^2 + \frac{\kappa}{2}\sum_B(\partial P_{(\bar{B})^+}\zeta)^2}$$

for all $B \in \mathcal{B}_j((U\backslash V) \cap \langle X\rangle^c)$, where $(\bar{B})^+ \subseteq W^+$ since $\langle X\rangle$ is designed to ensure that; and

$$\|\tilde{I}(B)\|_{T_\phi(\Phi_j(B))} \le 2\mathrm{e}^{\frac{\kappa}{2}\sum_B(\partial P_{W^+}\phi)^2 + \frac{\kappa}{2}\sum_B(\partial P_{(\bar{B})^+}\zeta)^2}$$

for all $B \in \mathcal{B}_j(V \cap (\langle X\rangle^c \backslash Z))$, where $(\bar{B})^+ \subseteq W^+$ since $B \subseteq \langle X\rangle^c$; and

$$\begin{aligned}
&\|\delta I(B)\|_{T_\phi(\Phi_j(B))}\\
&\quad\le \|I(B) - 1\|_{T_\phi(\Phi_j(B))} + \|\tilde{I}(B) - 1\|_{T_\phi(\Phi_j(B))}\\
&\quad\le 8(\kappa A^\star)^{-1}\mathrm{e}^{\frac{\kappa}{2}\sum_B(\partial P_{W^+}\phi)^2}\mathrm{e}^{\frac{\kappa}{2}\sum_B(\partial P_{B^+}\zeta)^2 + \frac{\kappa}{2}\sum_B(\partial P_{(\bar{B})^+}\zeta)^2}
\end{aligned}$$

by $\mathrm{e}^a + \mathrm{e}^b \le 2\mathrm{e}^{a+b}$ $(a, b > 0)$ for all $B \in \mathcal{B}_j(Z)$, where $(\bar{B})^+ \subseteq W^+$ since $Z \subseteq \langle X\rangle^c$. Combining all the above estimates, together with Lemma 19, we have

$$\|\mathbf{E}^{W^+}\|_{T_\phi(\Phi_j(W))} \le (\kappa A^\star/8)^{-|Q\cup Z\cup((U\backslash V)\backslash\langle X\rangle)|_j}\mathrm{e}^{\frac{\kappa}{2}\sum_W(\partial P_{W^+}\phi)^2}\mathcal{M}, \tag{5.4}$$

where

$$\mathcal{M} \le \mathbb{E}_\zeta\left[\mathrm{e}^{\frac{\kappa}{2}\sum_{B\in\mathcal{B}_j(W)}\sum_B(\partial P_{B^+}\zeta)^2}\mathrm{e}^{\frac{\kappa}{2}\sum_{B\in\mathcal{B}_j(W)}\sum_B(\partial P_{(\bar{B})^+}\zeta)^2}\right]. \tag{5.5}$$

In the next Lemma, we show that $\mathcal{M} \le c^{|U|_j}$.

Now we proceed to control $\mathbf{E}^{U^+}$. Instead of $(a+b)^2 \le 2a^2 + 2b^2$ we use properties of the regulator established in Sect. 4. Since for all $X \in \mathcal{P}_{j,c}$

$$\|K_j(X)\|_{T_\phi(\Phi_j(X))} \le A^{\star-1}G(\ddot{X}, X^+)A^{-|X|_j}.$$

By Lemma 19, Lemma 22 (2)(4)(5) and Lemma 24,

$$\begin{aligned}
\|\mathbf{E}^{U^+}\|_{T_\phi(\Phi_j(U))} &\le c^{|U|_j}\,(\kappa A^\star/8)^{-|Z\cup Q\cup\left((U\backslash V)\backslash\langle X\rangle\right)|_j - |\mathcal{X}| - |\mathcal{Y}|}A^{-|X_{\mathcal{X}\cup\mathcal{Y}}|_j}\\
&\quad\times\mathbb{E}\Big[\mathrm{e}^{\frac{\kappa}{2}\sum_W(\partial P_{W^+}\phi)^2}\prod_{Y\in\mathcal{X}}G(\ddot{Y}_k, Y_k^+)\prod_{Y\in\mathcal{Y}}G(\ddot{Y}_i, Y_i^+)\big|(U^+)^c\Big]\\
&\le c^{|U|_j}\cdot(\kappa A^\star/8)^{-|Z\cup Q\cup((U\backslash V)\backslash\langle X\rangle)|_j - |\mathcal{X}| - |\mathcal{Y}|}G(\ddot{U}, U^+)c'^{|W|_j}(A/c')^{-|X_{\mathcal{X}\cup\mathcal{Y}}|_j}.
\end{aligned} \tag{5.6}$$

We can bound the number of terms in the summation in (5.2) by $k^{|U|_j}$ with $k = 2^7$, because every j-block in U either belongs to V or V^c, and the same statement is true if V is replaced by $P, Q, Z, X_{\mathcal{X}}, Y_{\mathcal{Y}}$, and if it is in $Y \in \mathcal{Y}$ it is either the B of $(B, Y) \in \hat{\mathcal{Y}}$ or not. By Lemma 25, for $a = \frac{1}{2}(1+\eta)$, with $\mathcal{X} = \{X_k\}$, $\hat{\mathcal{Y}} = \{(B_i, Y_i)\}$, the quantity $a|U|_{j+1}$ can be bounded by

$$\begin{aligned}
&a|\bar{Z}|_{j+1} + a|\cup_i \bar{B}_i|_{j+1} + a|\cup_k \bar{X}_k|_{j+1} + a|\bar{Q}|_{j+1} + a|(U\backslash V) \cap \langle X\rangle^c|_{j+1} \\
&\quad \le \left(|Z|_j + a2^{d+1}|\mathcal{C}(Z)|\right) + a|\hat{\mathcal{Y}}| + \left(\sum_k |X_k|_j + a2^{d+1}|\mathcal{X}|\right) \\
&\qquad + \left(|Q|_j + a2^{d+1}|\mathcal{C}(Q)|\right) + aL^d|(U\backslash V) \cap \langle X\rangle^c|_j \\
&\quad \le (1 + a2^{d+1})\left(|Z|_j + |Q|_j\right) + a|\hat{\mathcal{Y}}| \\
&\qquad + (|X_{\mathcal{X}}|_j + a2^{d+1}|\mathcal{X}|) + aL^d|(U\backslash V) \cap \langle X\rangle^c|_j.
\end{aligned}$$

Then, we can easily check that with $A, A^\star$ sufficiently large as assumed in the proposition

$$\|K'\|_{j+1} = \sup_{U \in \mathcal{P}'} \|K'(U)\|_{j+1} A^{a|U|_{j+1}} A^{(1-a)|U|_{j+1}} < r,$$

where r is the radius of $B'(\mathcal{N}_{j+1}^{\mathcal{P}_{j+1}})$, because $A^{|X_{\mathcal{X}}|_j}$ is cancelled by its inverse in (5.6), and

$$\begin{aligned}
&\lim_{A\to\infty} A^{(1-a)|U|_{j+1}} \cdot A^{-|X_{\mathcal{Y}}|_j} \cdot k^{|U|_j} \cdot c^{|U|_j} \cdot c'^{\,|W|_j + |X_{\mathcal{X}\cup\mathcal{Y}}|_j} \\
&\quad \times 2^{|(\langle X\rangle\backslash X) \cap U\backslash(P\cup Q)|_j} \cdot 2^{|U\cup X|_j} = 0 && (5.7) \\
&\lim_{A^\star\to\infty} (\kappa A^\star/8)^{-|Q\cup Z\cup((U\backslash V)\backslash\langle X\rangle)|_j - |\mathcal{X}| - |\mathcal{Y}|} \\
&\quad \cdot A^{(1+a2^{d+1})|Q\cup Z|_j + a|\hat{\mathcal{Y}}| + a2^{d+1}|\mathcal{X}| + aL^d|(U\backslash V)\cap\langle X\rangle^c|_j} = 0. && (5.8)
\end{aligned}$$

The derivatives of the map $(\sigma_j, E_{j+1}, \sigma_{j+1}, K_j) \mapsto K_{j+1}$ with respect to σ_j, E_{j+1}, σ_{j+1} and K_j are bounded similarly. □

Lemma 27. *Let $\mathcal{M}$ be the quantity introduced in the proof of Proposition 26. There exists a constant c independent of $L, A, A^\star$ such that*

$$\mathcal{M} \le c^{|U|_j}. \tag{5.9}$$

Proof. Defining $\zeta = C_{W^+}^{1/2}\psi$ where C_{W^+} is the Dirichlet Green's function for W^+ and $\psi \in L^2(W^+)$, $\mathcal{M}$ is bounded by

$$\mathbb{E}_\psi \exp\left\{4\kappa \sum_{x\in W} \psi(x) T\psi(x)\right\}, \tag{5.10}$$

where ψ has identity covariance and

$$\begin{aligned} T &= \frac{1}{4} \sum_{B\in\mathcal{B}_j(W), e\in\mathcal{E}} \big(C_{U^+}^{1/2} P_{B^+}^{\star} \partial_e^{\star} 1_B \partial_e P_{B^+} C_{U^+}^{1/2} \\ &\quad + C_{U^+}^{1/2} P_{(\bar{B})^+}^{\star} \partial_e^{\star} 1_B \partial_e P_{(\bar{B})^+} C_{U^+}^{1/2}\big) \\ &=: T_1 + T_2 \end{aligned} \tag{5.11}$$

is a linear map from $L^2(W^+)$ to itself. T_1, T_2 are defined to be the two terms, respectively. We have by Proposition 16,

$$\begin{aligned} Tr(T) &= \frac{1}{4} \sum_{B\in\mathcal{B}_j(W), e\in\mathcal{E}} \Big(\sum_{x\in B} \partial_e P_{B^+} C_{U^+} \left(\partial_e P_{B^+}\right)^{\star}(x,x) \\ &\quad + \sum_{x\in B} \partial_e P_{(\bar{B})^+} C_{U^+} \big(\partial_e P_{(\bar{B})^+}\big)^{\star}(x,x)\Big) \\ &\le O(1)(L^{-dj} + L^{-d(j+1)})|W| \\ &\le O(1)|W|_j. \end{aligned} \tag{5.12}$$

For the next step, we bound $\|T\|$. In fact,

$$\begin{aligned} (f, T_1 f)_{l^2} &= \frac{1}{4} \sum_{B\in\mathcal{B}_j(W)} \sum_{x\in B, e} \left(\partial_e P_{B^+} C_{U^+}^{\frac{1}{2}} f(x)\right)^2 \\ &\le \frac{1}{4} \sum_{B\in\mathcal{B}_j(W)} \sum_{x\in B^+, e} \left(\partial_e C_{U^+}^{\frac{1}{2}} f(x)\right)^2 \\ &\le c_d \sum_{x\in W, e} \left(\partial_e C_{U^+}^{\frac{1}{2}} f(x)\right)^2, \end{aligned} \tag{5.13}$$

where we used the fact that the harmonic extension minimizes the Dirichlet form to get rid of the Poisson kernels. The constant c_d comes from overlapping of B^+'s. Then, we can proceed as (4.17) to bound the above expression by $c_d(f,f)_{l^2}$. T_2 is bounded in the same way. Now by $|Tr(T^n)| \le |Tr(T)|\, \|T\|^{n-1}$, and formula (4.15) the proof of the lemma is completed. □

6. Linearized RG

Having established smoothness, in this section we study the linearization of the RG map in σ_j, K_j, E_{j+1} and σ_{j+1}.

In view of Lemma 19, we can show, by induction along all the RG steps, that $K_j(X)$ depends on ϕ, ξ via $P_{X^+}\phi + \xi$ (at scale 0, I_0, K_0 depend on ϕ, ξ via $\phi + \xi$). We write

$$Tay\, \mathbb{E}\left[K_j(X)|(U^+)^c\right]$$

to be the second-order Taylor expansion of $\mathbb{E}\left[K_j(X)|(U^+)^c\right]$ in $P_{U^+}\phi + \xi$.

Proposition 28. *The linearization of the map* $(\sigma_j, E_{j+1}, \sigma_{j+1}, K_j) \to K_{j+1}$ *around* $(0,0,0,0)$ *is* $\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ *where*

$$\mathcal{L}_1 K_j(U) = \sum_{X \in \mathcal{P}_{j,c} \setminus \mathcal{S}_j, \bar{X} = U} \mathbb{E}\left[K_j(X) \big| (U^+)^c\right], \tag{6.1}$$

$$\mathcal{L}_2 K_j(U) = \sum_{B \in \mathcal{B}_j, \bar{B} = U} \sum_{X \in \mathcal{S}_j, X \supseteq B} \frac{1}{|X|_j}(1 - Tay)\mathbb{E}\left[K_j(X) \big| (U^+)^c\right], \tag{6.2}$$

$$\begin{aligned} &\mathcal{L}_3\big(\sigma_j, E_{j+1}, \sigma_{j+1}, K_j\big)(U) \\ &\quad = \sum_{B \in \mathcal{B}_j, \bar{B} = U} \Bigg(\frac{\sigma_{j+1}}{4} \sum_{x \in B, e} \Big(\partial_e P_{(\bar{B})^+}\phi(x) + \xi(x)\Big)^2 \\ &\qquad + E_{j+1}(B) - \frac{\sigma_j}{4} \sum_{x \in B} \mathbb{E}\left[\big(\partial P_{B^+}\phi(x) + \xi(x)\big)^2 \,|(U^+)^c\right] \\ &\qquad + \sum_{X \in \mathcal{S}_j, X \supseteq B} \frac{1}{|X|_j} Tay\, \mathbb{E}\left[K_j(X) | (U^+)^c\right]\Bigg). \end{aligned} \tag{6.3}$$

Proof. In Proposition 26, we proved that the map $(\sigma_j, E_{j+1}, \sigma_{j+1}, K_j) \to K_{j+1}$ is smooth around $(0,0,0,0)$ so that we can linearize the map. In (3.9) since $V \neq \emptyset$, the $\tilde{I}_j - \mathrm{e}^{E_{j+1}}$ factor does not contribute to the linear order. Also if $X = \emptyset$ then $\hat{X} = \langle X \rangle = \emptyset$, so $1 - \mathrm{e}^{E_{j+1}}$ and $I_j - \mathrm{e}^{E_{j+1}}$ do not contribute to the linear order either. The terms that contribute to the linear order correspond to $(Z, |\mathcal{X}|, |\hat{\mathcal{Y}}|)$ equal to $(\emptyset, 0, 1)$ or $(\emptyset, 1, 0)$ or $(B, 0, 0)$ where $B \in \mathcal{B}_j$. Grouping these terms into large sets part and small sets part with Taylor leading terms and remainder, we obtain the above linear operators. $\square$

6.1. Large Sets

Proposition 29. *Let L be sufficiently large. Let A be sufficiently large depending on L. Then, $\mathcal{L}_1$ in Proposition 28 is a contraction. Moreover,* $\lim_{A \to \infty} \|\mathcal{L}_1\| = 0$.

Proof. By Lemma 24

$$\|\mathcal{L}_1 K_j(U)\|_{j+1} \le \sum_{X \in \mathcal{P}_{j,c} \setminus \mathcal{S}_j, \bar{X} = U} \|K_j\|_j c^{|X|_j} A^{-|X|_j}, \tag{6.4}$$

therefore, by Lemma 25,

$$\begin{aligned} \|\mathcal{L}_1 K_j\|_{j+1} &= \sup_{U \in \mathcal{P}_{j+1}} \|\mathcal{L}_1 K_j(U)\|_{j+1} A^{|U|_{j+1}} \\ &\le \left[\sup_{U \in \mathcal{P}_{j+1}} A^{|U|_{j+1}} \sum_{X \in \mathcal{P}_{j,c} \setminus \mathcal{S}_j, \bar{X} = U} c^{|X|_j} A^{-|X|_j}\right] \|K_j\|_j \\ &\le \left[\sup_{U \in \mathcal{P}_{j+1}} A^{|U|_{j+1}} 2^{L^d |U|_{j+1}} (A/c)^{-\eta |U|_{j+1}}\right] \|K_j\|_j, \end{aligned} \tag{6.5}$$

where $\eta > 1$ is introduced in Lemma 25. The bracketed expression goes to zero as $A \to \infty$. □

6.2. Taylor Remainder

We prepare to show contractivity of $\mathcal{L}_2$. We first show that the Taylor remainder after the second derivative is bounded by the third derivative. It is a general result about the $T_\phi(\Phi)$ norm with no need to specify the test function space Φ.

Lemma 30. *For F a function of ϕ let Tay_n be its n-th order Taylor expansion about $\phi = 0$, and Φ be a space of test functions, then*

$$\|(1 - Tay_2)F(\phi)\|_{T_\phi(\Phi)} \le (1 + \|\phi\|_\Phi)^3 \sup_{\substack{t\in[0,1]\\ k=3,4}} \left\|F^{(k)}(t\phi)\right\|_{T^k_{t\phi}(\Phi)}. \tag{6.6}$$

Proof. By Taylor remainder theorem, with $f^{\times n} := (f_1, \ldots, f_n)$,

$$\begin{aligned}\|(1 - Tay_2)F(\phi)\|_{T_\phi(\Phi)} &= \sum_{n=0}^{4} \frac{1}{n!} \sup_{\|f_i\|_\Phi \le 1} \left|(F - Tay_2 F)^{(n)}(\phi; f^{\times n})\right| \\ &= \sum_{n=0}^{4} \frac{1}{n!} \sup_{\|f_i\|_\Phi \le 1} \left|\left(F^{(n)} - Tay_{2-n}(F^{(n)})\right)(\phi; f^{\times n})\right| \end{aligned}\tag{6.7}$$

where $Tay_{2-n} = 0$ for $n > 2$. The absolute valued quantity is equal to

$$\begin{aligned}&\left|1_{\{n<3\}} \int_0^1 \frac{(1-t)^{2-n}}{(2-n)!} \partial_t^{3-n} F^{(n)}(t\phi; f^{\times n}) + 1_{\{n\ge3\}} F^{(n)}(\phi; f^{\times n})\right| \\ &\quad = \left|1_{\{n<3\}} \int_0^1 \frac{(1-t)^{2-n}}{(2-n)!} F^{(3)}(t\phi; \phi^{\times(3-n)}, f^{\times n}) + 1_{\{n\ge3\}} F^{(n)}(\phi; f^{\times n})\right|,\end{aligned}\tag{6.8}$$

where $\phi^{\times(3-n)}$ means $3 - n$ test functions ϕ. Calculating the t integrals,

$$\begin{aligned}&\|(1 - Tay_2)F(\phi)\|_{T_\phi(\Phi)} \\ &\le \sum_{n=0}^{3} \frac{1}{n!} \sup_{\|f_i\|_\Phi \le 1} \left|\frac{1}{(3-n)!} \sup_{t\in[0,1]} F^{(3)}(t\phi; \phi^{\times(3-n)}, f^{\times n})\right| + \|F^{(4)}(\phi)\|_{T^4_\phi(\Phi)} \\ &\le (1 + \|\phi\|_\Phi)^3 \sup_{t\in(0,1),k=3,4} \|F^{(k)}(t\phi)\|_{T^k_{t\phi}(\Phi)},\end{aligned}\tag{6.9}$$

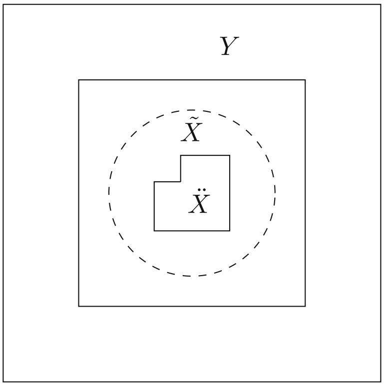
where in the last step binomial theorem is applied. □

Lemma 31. *Let $(B, X) \in \hat{\mathcal{S}}_j$, $\bar{B} = U$, if κ is small enough depending on L, and h is large enough depending on κ and L, then*

$$\left(2 + \|\phi\|_{\Phi_{j+1}(\dot{X},U^+)}\right)^3 G(\ddot{X}, U^+) \le qG(\ddot{U}, U^+) \tag{6.10}$$

for a constant q, where the dot(s) operations on X are at scale j, and the dots and $+$ operations on U are at scale $j + 1$.

Proof. Let $\psi = P_{U^+}\phi$. For each $e \in \mathcal{E}$, $\partial_e\psi$ is harmonic in $U^+ \cap (U^+ - e)$. Since X, U are j and $j+1$ scale small sets, respectively, and $d(X, \partial\ddot{U}) = O(L^{j+1})$, we can find a set $Y \subset \ddot{U}$, such that: 1) Y is of the form $\mathcal{K}_R \backslash \mathcal{K}_{rR}$ for some $r \in (0, \frac{1}{2})$ as in Lemma 15; 2) $Y \cap \ddot{X} = \emptyset$ and $d(\ddot{X}, Y) = O(L^j)$; 3) $d(Y, \partial\ddot{U}) = O(L^{j+1})$; 4) $R = \mathrm{diam}(Y) = O(L^j)$.



Then by (3.21) of Lemma 15,

$$\sup_{e\in\mathcal{E}, x\in\dot{X}} |\partial_e\psi(x)|^2 \le O(L^{-dj}) \sum_{e\in E(Y)} (\partial_e\psi)^2 . \tag{6.11}$$

By definition of the norm $\|\phi\|^2_{\Phi_{j+1}(\dot{X},U^+)} = h^{-2}L^{d(j+1)} \sup_{e,x\in\dot{X}} |\partial_e\psi(x)|^2$, if we choose h large enough such that $h^{-1}O(L^d) \le 1$, then

$$\|\phi\|^2_{\Phi_{j+1}(\dot{X},U^+)} \le h^{-1} \sum_{e\in E(Y)} (\partial_e\psi)^2 . \tag{6.12}$$

Since there exists a $q \ge 1$ such that for all $s \ge 0$, $(2+s)^3 \le q\mathrm{e}^{s^2/2}$, one has

$$\left(2 + \|\phi\|_{\Phi_{j+1}(\dot{X},U^+)}\right)^3 \le q \exp\left(\frac{h^{-1}}{2} \sum_{e\in E(Y)} (\partial_e\psi)^2\right) . \tag{6.13}$$

Apply (4.11) of Lemma 22 to $G(\ddot{X}, U^+)$, and use the fact that $\psi = P_{U^+}\phi$ together with (6.13), then the left-hand side of (6.10) is bounded by

$$q \exp\left\{\frac{\kappa}{2}\sum_{e\in E(U^+)} (a_e\partial_e\psi_1)^2 + \frac{h^{-1}}{2}\sum_{e\in E(Y)} (\partial_e\psi)^2 - \frac{1}{2}\sum_{U^+}(\partial_m\psi_1)^2 + \frac{1}{2}\sum_{U^+}(\partial_m\psi)^2\right\},$$

where the function $a_e = 1$ if $e \in E(\ddot{X})$ and decays to zero in a neighborhood of $\ddot{X}$, and the support of a_e, that is, $\tilde{X} := \mathrm{supp}(a) = \{x : \exists \bar{e} \in \mathcal{E} \text{ s.t. } a_{x,x+\bar{e}} \ne 0\}$, still satisfies $d(\tilde{X}, Y) = O(L^j)$, and $|\nabla^k a_e| \le O(L^{-kj})$ for $k = 0, \dots, 3$, and finally

$$\psi_1 \text{ maximizes } \quad \kappa \sum_{e\in E(U^+)} (a_e\partial_e\phi)^2 - \sum_{U^+}(\partial_m\phi)^2 \text{ fixing } \phi\big|_{(U^+)^c}. \tag{6.14}$$

Notice that applying (4.11) of Lemma 22 to $G(\ddot{X}, U^+)$ results in a term $\frac{\kappa}{2}$ times a Dirichlet form over $\ddot{X}$, and we "enlarged" the set $\ddot{X}$ to $\tilde{X}$ by smoothing out the coefficient a_e, followed by a replacement of that Dirichlet form with that of the maximizer ψ_1 solving the new elliptic problem (6.14)—this only makes the above exponential larger. In the following, we show that by choosing h large enough one has

$$\frac{h^{-1}}{2} \sum_{e \in E(Y)} (\partial_e \psi)^2 \le \frac{\kappa}{2} \sum_{e \in E(Y)} (\partial_e \psi_1)^2 . \tag{6.15}$$

Then, the left-hand side of (6.10) is bounded by

$$q \exp \left\{ \frac{\kappa}{2} \sum_{e \in E(\ddot{U})} (\partial_e \bar{\psi})^2 - \frac{1}{2} \sum_{U^+} (\partial_m \bar{\psi})^2 + \frac{1}{2} \sum_{U^+} (\partial_m \psi)^2 \right\} = qG(\ddot{U}, U^+)$$

which holds by replacing ψ_1 by $\bar{\psi}$ which is the maximizer of $\frac{\kappa}{2} \sum_{e \in E(\ddot{U})} (\partial_e \phi)^2 - \frac{1}{2} \sum_{U^+} (\partial_m \phi)^2$ with $\phi\big|_{(U^+)^c}$ fixed.

To show (6.15), let $\bar{a} = 1 - \kappa a$. We have

$$\begin{cases} (-\Delta + m^2)\psi = 0 & \text{in } U^+ \\ \psi = \phi & \text{on } \partial U^+ \end{cases} \qquad \begin{cases} (-\Delta_{\bar{a}} + m^2)\psi_1 = 0 & \text{in } U^+ \\ \psi_1 = \phi & \text{on } \partial U^+ \end{cases},$$

where $\Delta_{\bar{a}} f(x) = \sum_e \bar{a}_e (f(x+e) - f(x))$. Subtract them and we obtain a non-constant coefficient elliptic problem for $\psi_1 - \psi$

$$\begin{cases} (-\Delta_{\bar{a}} + m^2)(\psi_1 - \psi) = -\kappa \Delta_a \psi & \text{in } U^+ \\ \psi_1 - \psi_0 = 0 & \text{on } \partial U^+ \end{cases}$$

One has the following representation of derivative of the solution to the above equation (note that the support of a is $\tilde{X}$ so $\Delta_a \psi = 0$ outside $\tilde{X}$)

$$\partial_e (\psi_1 - \psi)(y) = \kappa \sum_{x \in \tilde{X}} \partial_{y,e} G_{\bar{a}}(y, x) \Delta_a \psi(x) \tag{6.16}$$

for $y \in Y, e \in \mathcal{E}$, where $G_{\bar{a}}$ is the Dirichlet Green's function associated with $-\Delta_{\bar{a}} + m^2$.

Our situation is that for a Laplacian with non-constant coefficient $\Delta_{\bar{a}}$, although one has desired bound for the Green's function $G_{\bar{a}}$ (i.e. bound with the decay rate as if the Laplacian was a constant coefficient one), the desired bound for $\partial_y G_{\bar{a}}(y, x)$ does not hold in general. However, we do have bound with desired scaling in an averaging sense, i.e. after a summation over y—the variable w.r.t. which $G_{\bar{a}}$ is differentiated. Consider

$$\begin{aligned} \sum_{e \in E(Y)} \Big(\partial_e (\psi_1 - \psi) \Big)^2 &= \kappa^2 \sum_{e \in E(Y)} \Big(\sum_{x \in \tilde{X}} \partial_{y,e} G_{\bar{a}}(y, x) \Delta_a \psi(x) \Big)^2 \\ &= \kappa^2 \sum_{x_1, x_2 \in \tilde{X}} \Delta_a \psi(x_1) \Delta_a \psi(x_2) \sum_{e \in E(Y)} \partial_{y,e} G_{\bar{a}}(y, x_1) \partial_{y,e} G_{\bar{a}}(y, x_2) \end{aligned}$$

$$\leq \frac{\kappa^2}{2} \sum_{x_1,x_2 \in \tilde{X}} \Big| \Delta_a \psi(x_1) \Delta_a \psi(x_2) \Big|$$
$$\times \sum_{e \in E(Y)} \Big(\big(\partial_{y,e} G_{\bar{a}}(y,x_1)\big)^2 + \big(\partial_{y,e} G_{\bar{a}}(y,x_2)\big)^2 \Big).$$

With this bound at hand, our proof of (6.15) now follows from two claims. The first claim is that for every $x \in \tilde{X}$,

$$\sum_{e \in E(Y)} \big(\partial_{y,e} G_{\bar{a}}(y,x)\big)^2 \leq O(L^{-2j}) \sum_{y \in \tilde{Y}} G_{\bar{a}}(y,x)^2 \leq O(L^{-(d-2)j}), \quad (6.17)$$

where $\tilde{Y}$ is such that $Y \subset \tilde{Y}$, $d(Y, \tilde{Y}^c) = O(L^j)$ and $d(\tilde{Y}, \tilde{X}) = O(L^j)$. Note that the last inequality follows from $G_{\bar{a}}(y,x) \leq O(L^{-(d-2)j})$ (this is a standard bound for Green's function of non-constant coefficient Laplacian, see for instance [23]) and $|Y| = O(L^{dj})$. Note that the right side of (6.17) does not depend on x_1, x_2, so it remains to bound $\Big(\sum_{x \in \tilde{X}} |\Delta_a \psi(x)| \Big)^2$.

The second claim is that for every $x \in \tilde{X}$,

$$|\Delta_a \psi(x)| \leq O(L^{-\frac{d+2}{2}j}) \Big(\sum_Y |\nabla \psi|^2 \Big)^{1/2} \quad (6.18)$$

so that one has

$$\left(\sum_{x \in \tilde{X}} |\Delta_a \psi(x)| \right)^2 \leq O\Big((L^{dj} \cdot L^{-\frac{d+2}{2}j})^2 \Big) \sum_Y (\nabla \psi)^2 = O(L^{(d-2)j}) \sum_Y (\nabla \psi)^2 \quad (6.19)$$

As a consequence of (6.16), (6.17) and (6.19), one has

$$\frac{1}{2} \sum_Y (\nabla \psi)^2 \leq \sum_Y (\nabla \psi - \nabla \psi_1)^2 + \sum_Y (\nabla \psi_1)^2 \leq O(1) \kappa^2 \sum_Y (\nabla \psi)^2 + \sum_Y (\nabla \psi_1)^2$$

Choosing h large enough such that $h^{-1} \leq \kappa(1/2 - O(1)\kappa^2)$, we obtain (6.15).

The proof to the first inequality of (6.17) is motivated by Cacciopoli's inequality in the continuum setting, which roughly states that for a solution u to an elliptic problem one can bound the L^2 norm of u by the L^2 norm (over a larger domain) of ∇u (as a reverse of Poincare inequality), under certain conditions [see for instance [36, Chapter 3]]. We do not provide the proof of its discrete counterpart in full generality, but only prove a weak version that is sufficient for our purpose.

Fixing $x \in \tilde{X}$, let $u(y) = G_{\bar{a}}(y,x)$, which is $(-\Delta_{\bar{a}} + m^2)$-harmonic in U^+ away from the singular point $y = x$: namely $\sum_{e \in \mathcal{E}} \bar{a}_e (u(y+e) - u(y)) - m^2 u(y) = 0$ for $y \in U^+ \backslash \{x\}$. Since κ and m^2 are sufficiently small, the function $\bar{a}_e$ is such that there exist $0 < \lambda < \bar{\lambda}$ and $\lambda < \bar{a}_e \pm m^2 < \bar{\lambda}$. Then, for every function v on $\tilde{Y}$, one has

$$\sum_{e\in E(\tilde{Y})} \bar{a}_e\, \partial_e u\, \partial_e v + \sum_{\tilde{Y}} m^2 uv = 0. \tag{6.20}$$

Let $v = u\varphi^2$ for some non-negative function φ such that $\mathrm{supp}(\varphi) \subset \tilde{Y}$, then one has (note that the lattice derivative does not exactly satisfy the product rule)

$$\partial_e v(x) = \partial_e u(x)\, \varphi(x+e)^2 + (\varphi(x) + \varphi(x+e))\, \partial_e \varphi(x)\, u(x).$$

Substituting this into the identity (6.20), one has

$$\begin{aligned}
&\lambda \sum_{y,y+e\in\tilde{Y}} \varphi(y+e)^2(\partial_e u(y))^2 + m^2 \sum_{y\in\tilde{Y}} \varphi(y)^2 u(y)^2 \\
&\quad \le -\sum_{y,y+e\in\tilde{Y}} (\varphi(y)+\varphi(y+e))u(y)\bar{a}_{(y,y+e)}\partial_e u(x)\partial_e\varphi(y) \\
&\quad \le \frac{\lambda}{2} \sum_{y,y+e\in\tilde{Y}} (\frac{\varphi(y)+\varphi(y+e)}{2})^2(\partial_e u(y))^2 + \frac{2}{\lambda} \sum_{y,y+e\in\tilde{Y}} \bar{a}^2_{(y,y+e)}(\partial_e\varphi(y))^2 u(y)^2,
\end{aligned}$$

where the first inequality used $\bar{a} > \lambda$ and the second inequality is by Cauchy-Schwartz. Subtracting both sides by the first term in the last line, as long as φ is chosen to vanish on $\{y : d(x, \partial\tilde{Y}) < 3\}$, we have

$$\begin{aligned}
&\frac{\lambda}{2} \sum_{y,y+e\in\tilde{Y}} \varphi(y)^2(\partial_e u(y))^2 \\
&\quad \le \frac{2\bar{\lambda}^2}{\lambda} \sum_{y,y+e\in\tilde{Y}} (\partial_e\varphi(y))^2 u(y)^2 - m^2 \sum_{y\in\tilde{Y}} \varphi(y)^2 u(y)^2.
\end{aligned}$$

Choosing $\varphi = 1$ on Y, and $|\nabla\varphi| \le O(L^{-j})$, and m^2 small enough such that $m^2 L^{2N} < \bar{\lambda}^2/\lambda$, we obtain the first inequality of (6.17).

The proof of (6.18) is based on the idea of writing $\Delta_a\psi$ in terms of (derivatives of) a and constant coefficient derivatives of ψ, in a way analogous to the relation $\nabla\cdot(a\nabla f) = \nabla a\cdot\nabla f + a\Delta f$ in continuum. Note that a_e above is defined on edges e. For a lattice site x, define $a(x) = (2d)^{-1}\sum_e a_{(x,x+e)}$. Then

$$\begin{aligned}
|\Delta_a\psi(x)| &= |\sum_{e\in\mathcal{E}} a_e(\psi(x+e) - \psi(x))| \\
&\le |\sum_{e\in\mathcal{E}} (a_{(x,x+e)} - a(x) + a(x))(\psi(x+e) - \psi(x))| \\
&\le \sup_{e\in\mathcal{E}} |a_{(x,x+e)} - a(x)||\nabla\psi(x)| + |a(x)||\Delta\psi(x)|
\end{aligned}$$

Note that the last term is zero since $\Delta\psi = 0$. The term $|a_{(x,x+e)} - a(x)|$ is bounded by $(2d)^{-1}\sum_{e'\in\mathcal{E}} |a_{(x,x+e)} - a_{(x,x+e')}|$ which by the choice of a is bounded by $O(L^{-j})$. Lemma 15 allows us to bound

$$|\nabla\psi(x)| \le O(L^{-dj/2}) \left(\sum_Y (\nabla\psi)^2\right)^{1/2}.$$

Therefore, we obtain (6.18). So (6.15) is shown and the proof of the lemma is completed. □

Before the next Lemma, we define

$$F_X(U,\phi,\xi) := \mathbb{E}\left[K_j(X,\phi,\xi)\middle| \left(U^+\right)^c\right]. \tag{6.21}$$

It depends on ϕ,ξ via $\psi := P_{U^+}\phi+\xi$, i.e. there exists a function $\widetilde{F}_X$ such that $F_X(U,\phi,\xi) = \widetilde{F}_X(U,\psi)$.

Lemma 32. *Let L be sufficiently large. Then, $\mathcal{L}_2$ in Proposition 28 is a contraction with the norm going to zero as $L \to \infty$.*

Proof. Let Tay be the second-order Taylor expansion in $\psi = P_{U^+}\phi+\xi$. With the F_X defined in (6.21), we aim to bound

$$\left\|(1-Tay)F_X(U,\phi,\xi)\right\|_{T_\phi(\Phi_{j+1}(U))}. \tag{6.22}$$

Recall that $\Phi_{j+1}(U)$ is short for $\Phi_{j+1}(\dot{U},U^+)$ and by Lemma 18 this can be replaced by $\Phi_{j+1}(\dot{X},U^+)$. Applying Lemma 30 with the test function space $\Phi := \widetilde{\Phi}_{j+1}(\dot{X},U^+)$, we can bound (6.22) by

$$\left\|(1-Tay)\widetilde{F}_X(U,\psi)\right\|_{T_\psi(\Phi)} \le \left(1+\|\psi\|_\Phi\right)^3 \sup_{k=3,4} \left\|\widetilde{F}_X^{(k)}(U,\psi)\right\|_{T_\psi^k(\Phi)} \tag{6.23}$$

Now by linearity of $\widetilde{F}_X^{(k)}$ in test functions,

$$\begin{aligned}\left\|\widetilde{F}_X^{(3)}(U,\psi)\right\|_{T_\psi^3(\widetilde{\Phi}_{j+1}(\dot{X},U^+))} &\le L^{-\frac{3}{2}d}\left\|\widetilde{F}_X^{(3)}(U,\psi)\right\|_{T_\psi^3(\widetilde{\Phi}_j(\dot{X},U^+))}\\ &\le L^{-\frac{3}{2}d}\cdot 3!\cdot \mathbb{E}\left[\|K_j(X,\phi,\xi)\|_{T_{\phi,\xi}(\Phi_j(X))}\middle|(U^+)^c\right]\\ &\le O(L^{-\frac{3}{2}d})\,\|K_j(X)\|_j\, c^{|X|_j}\, G(\ddot{X},U^+),\end{aligned} \tag{6.24}$$

where in the last step Lemma 24 is applied. We can prove analogously that $\tilde{F}_X^{(4)}(U,\psi)$ satisfies a similar bound with a factor $O(L^{-2d})$. Next, we estimate

$$\begin{aligned}\|\psi\|_\Phi &\le h_j^{-1}\sup_{x\in\dot{X},e}\left|L^j\partial_e P_{U^+}\phi(x)\right| + h_j^{-1}\sup_{x\in\dot{X},e}\left|L^j\partial_e\xi(x)\right|\\ &\le \|\phi\|_{\Phi_{j+1}(\dot{X},U^+)}+1\end{aligned} \tag{6.25}$$

by (2.7). Combining (6.23)–(6.25), and applying Lemma 31, followed by (4) of Lemma 22, one obtains

$$\begin{aligned}\|(1-Tay)F_X(U)\|_{j+1} &\le O(L^{-\frac{3d}{2}})\,c^{|X|_j}\,\|K_j(X)\|_j\\ &\le O(L^{-\frac{3d}{2}})(\frac{A}{c})^{-|X|_j}\,\|K_j\|_j.\end{aligned} \tag{6.26}$$

Note that the sum over B and X in the definition (6.2) of $\mathcal{L}_2$ gives a factor $O(L^d)$. Apply the geometric Lemma 25 to $|X|_j$, one then has

$$\begin{aligned}\|\mathcal{L}_2 K_j\|_{j+1} &\le O(L^{-3d/2})\left[\sup_{U\in\mathcal{P}_{j+1}} A^{|U|_{j+1}}O(L^d)A^{-|U|_{j+1}}c^{2^d}\right]\|K\|_j\\ &= O(L^{-d/2})\|K_j\|_j.\end{aligned}$$

As $L\to\infty$, the factor $L^{-d/2}$ overwhelms the constants hidden in the big-O notation and, therefore, $\mathcal{L}_2$ has arbitrarily small norm. □

6.3. $\mathcal{L}_3$ and Determination of Coupling Constants

We now localize the last term in $\mathcal{L}_3$, which is the second order Taylor expansion of $\widetilde{F}_X(U,\psi)$ in ψ [which are introduced in (6.21)]. To do this, we fix a point $z\in B$, and replace $\psi(x)$ by $x\cdot\partial\psi(z)$ (which according to our convention means $\frac{1}{2}\sum_{e\in\mathcal{E}} x_e\partial_e\psi(z)$), and then average over $z\in B$. We will show that the error of this replacement is irrelevant. Then

$$\frac{1}{2}\tilde{F}_X^{(2)}(U,0;\psi,\psi)=\mathrm{Loc}K_j(B,X,U)+(1-\mathrm{Loc})K_j(B,X,U),$$

where we have defined

$$\begin{aligned}&\mathrm{Loc}\,K_j(B,X,U)\\ &\quad:=\frac{1}{8|B|}\sum_{z\in B,\mu,\nu\in\mathcal{E}}\partial^2_{t_1t_2}\Big|_{t_i=0}\mathbb{E}_\zeta\left[K_j(X,t_1x_\mu+t_2x_\nu+\zeta)\right]\partial_\mu\psi(z)\partial_\nu\psi(z)\end{aligned}$$

and

$$\begin{aligned}(1-\mathrm{Loc})K_j(B,X,U)&:=\frac{1}{2|B|}\sum_{z\in B}\left(\partial^2_{t_1t_2}\Big|_{t_i=0}\mathbb{E}_\zeta\left[K_j(X,t_1\psi+t_2\psi+\zeta)\right]\right.\\ &\quad\left.-\partial^2_{t_1t_2}\Big|_{t_i=0}\mathbb{E}_\zeta\left[K(X,t_1x\cdot\partial\psi(z)+t_2x\cdot\partial\psi(z)+\zeta)\right]\right)\\ &=\frac{1}{2|B|}\sum_{z\in B}\left(\widetilde{F}_X^{(2)}(U,0;\psi-x\cdot\partial\psi(z),\psi)+\widetilde{F}_X^{(2)}(U,0;\psi-x\cdot\partial\psi(z),x\cdot\partial\psi(z))\right).\end{aligned}\tag{6.27}$$

We show that $\psi-x\cdot\partial\psi(z)$ gives additional contractive factors as going to the next scale:

Lemma 33. *If $\psi=P_{U^+}\phi+\xi\in\widetilde{\Phi}_j(\dot{X},U^+)$, then*

$$\|\psi-x\cdot\partial\psi(z)\|_{\widetilde{\Phi}_j(\dot{X},U^+)}\le O(L^{-\frac{d}{2}-1})\left(\|\phi\|_{\Phi_{j+1}(U)}+1\right).\tag{6.28}$$

Proof. Since $P_{U^+}x=x$, the left side of (6.28) is equal to

$$h_j^{-1}\sup_{x\in\dot{X},e}L^j\Big|\partial_eP_{U^+}\phi(x)+\partial_e\xi(x)-\partial_eP_{U^+}\phi(z)-\partial_e\xi(z)\Big|.\tag{6.29}$$

We apply Newton–Leibniz formula along a curve connecting x, z, and then apply (3.17) with $R = O(L^{j+1})$ using the distance $O(L^{j+1})$ between $\dot{X}$ and $\partial\dot{U}$,

$$\begin{aligned} &h_j^{-1} \sup{}_{x\in\dot{X},e} L^j \left|\partial_e P_{U^+}\phi(x) - \partial_e P_{U^+}\phi(z)\right| \\ &\quad\le h_j^{-1} \sup{}_{x\in\dot{U}} L^j \operatorname{diam}(\dot{X})\, O(L^{-j-1}) \left|\partial P_{U^+}\phi(x)\right| \\ &\quad\le O(L^{-\frac{d+2}{2}})\, \|\phi\|_{\Phi_{j+1}(U)}\,, \end{aligned}$$

where $\operatorname{diam}(\dot{X}) = O(L^j)$ since X is small. The second term in (6.29) can be bounded by

$$h_j^{-1} \sup_{x\in\dot{X},e} L^j \left|\partial_e\xi(x) - \partial_e\xi(z)\right| \le O(L^{-\frac{d}{2}(N-j)}) \le O(L^{-\frac{d+2}{2}})$$

as long as $j + 1 < N$, and by $d \ge 2$ and (2.7). Combining the above bounds completes the proof. □

Lemma 34. *If L be sufficiently large and define*

$$\mathcal{L}_3' K_j(U) = \sum_{\bar{B}=U} \sum_{X\in\mathcal{S}_j, X\supseteq B} (1 - Loc) K_j(B, X, U) \tag{6.30}$$

then $\mathcal{L}_3'$ is contractive with arbitrarily small norm; namely, $\|\mathcal{L}_3'\| \to 0$ as $L \to \infty$.

Proof. In view of the definition (6.27) of $(1 - \mathrm{Loc})K_j$, we let

$$H_{z,X}(U, \phi, \xi) = \tilde{F}_X^{(2)}(U, 0; \psi - x\cdot\partial\psi(z), \psi) \tag{6.31}$$

then with $\tilde{f} := P_{U^+} f + \lambda\xi$,

$$\begin{aligned} &H_{z,X}^{(1)}(U, \phi, \xi; (f, \lambda\xi)) \\ &\quad= \tilde{F}_X^{(2)}(U, 0; \psi - x\cdot\partial\psi(z), \tilde{f}) + \tilde{F}_X^{(2)}(U, 0; \tilde{f} - x\cdot\partial\tilde{f}(z), \psi)\,; \\ &H_{z,X}^{(2)}(U, \phi, \xi; (f_1, \lambda_1\xi), (f_2, \lambda_2\xi)) \\ &\quad= \tilde{F}_X^{(2)}(U, 0; \tilde{f}_1 - x\cdot\partial\tilde{f}_1(z), \tilde{f}_2) + \tilde{F}_X^{(2)}(U, 0; \tilde{f}_2 - x\cdot\partial\tilde{f}_2(z), \tilde{f}_1) \end{aligned} \tag{6.32}$$

and $H_{z,X}^{(3)} = H_{z,X}^{(4)} = 0$. In the calculations here, though z is fixed, $P_{U^+}\phi(z)$ should also participate in the differentiations: $P_{U^+}\phi(z) \to P_{U^+}(\phi + tf)(z)$.

We now bound the all the test functions appeared in (6.32). The bound for $\psi - x\cdot\partial\psi(z)$ is given in Lemma 33. Similarly, one can bound $\tilde{f} - x\cdot\partial\tilde{f}(z)$ by $O(L^{-\frac{d}{2}-1})\|(f, \lambda\xi)\|_{\Phi_{j+1}(U)}$. Since $\|-\|_{\Phi_j(U)} \le L^{-d/2}\,\|-\|_{\Phi_{j+1}(U)}$ we also have estimates

$$\begin{aligned} &\|\psi\|_{\Phi_j(\dot{X},U^+)} \le O(L^{-d/2})\big(\|\phi\|_{\Phi_{j+1}(U)} + 1\big)\,; \\ &\|\tilde{f}\|_{\Phi_j(\dot{X},U^+)} \le O(L^{-d/2})\|(f, \lambda\xi)\|_{\Phi_{j+1}(U)}. \end{aligned} \tag{6.33}$$

Therefore, we obtain the bound

$$\left|H_{z,X}^{(n)}(U,\phi,\xi;(f,\lambda\xi)^{\times n})\right| \le O(L^{-d-1})\big\|\tilde F_X^{(2)}(U,0)\big\|_{T_0^2(\tilde\Phi_j(\dot X,U^+))} \cdot \left(\|\phi\|_{\Phi_{j+1}(U)}+1\right)^{2-n}\prod_{i=1}^n \big\|(f_i,\lambda_i\xi)\big\|_{\Phi_{j+1}(U)}. \tag{6.34}$$

By the same arguments as (6.13) and Lemma 22(5), one can bound $(1+\|\phi\|_{\Phi_{j+1}(U)})^2$ by $G(\ddot U,U^+)$. Therefore,

$$\big\|H_{z,X}(U,\phi,\xi)\big\|_{T_\phi(\Phi_{j+1}(U))} \le O(L^{-d-1})\big\|\widetilde F_X^{(2)}(U,0)\big\|_{T_\phi^2(\widetilde\Phi_j(\dot X,U^+))}G(\ddot U,U^+). \tag{6.35}$$

By Lemma 24 followed by Lemma 22(1), together with $X\in\mathcal S_j$

$$\begin{aligned}
&\big\|\tilde F_X^{(2)}(U,0)\big\|_{T_\phi^2(\tilde\Phi_j(\dot X,U^+))}\\
&\quad\le \mathbb E\left[\big\|K_j(X,\phi,\xi=0)\big\|_{T_\phi(\Phi_j(\dot X,U^+))}\,|\phi_{(U^+)^c}=0\right]\\
&\quad\le \mathbb E\left[\big\|K_j(X)\big\|_j\,G(\ddot X,X^+)|\phi_{(U^+)^c}=0\right]\le \big\|K_j(X)\big\|_j\,c^{|X|_j}\\
&\quad\le O(1)A^{-1}\,\big\|K_j\big\|_j\,.
\end{aligned} \tag{6.36}$$

Combining the above inequalities, we obtain

$$\big\|H_{z,X}(U)\big\|_{j+1}\le O(L^{-d-1})A^{-1}\,\|K\|_j\,.$$

It can be shown analogously that the other term on the right side of (6.27) satisfies the same bound. Finally, the sum over B and X in (6.30) gives a factor $O(L^d)$, so one has

$$\big\|\mathcal L_3'K(U)\big\|_{j+1}\le O(L^{-1})A^{-1}\,\|K\|_j\,. \tag{6.37}$$

Since $\mathcal L_3'K_j(U)=0$ unless U is a block, $\big\|\mathcal L_3'K_j\big\|_{j+1}\le O(L^{-1})\,\|K\|_j$. □

Now we turn to $\operatorname{Loc}K_j$. We observe that the coefficient of $\partial_\mu\psi(z)\partial_\nu\psi(z)$ is

$$\alpha_{\mu\nu}(B):=\frac{1}{8|B|}\sum_{X\in\mathcal S_j,X\supseteq B}\partial_{t_1t_2}^2\Big|_{t_i=0}\mathbb E_\zeta\left[K_j(X,t_1x_\mu+t_2x_\nu+\zeta)\right]. \tag{6.38}$$

Note that each summand above is just derivative of $\mathbb E_\zeta K_j(X)$ at zero field with test functions x_μ and x_ν. Since $\|x_\mu\|_{\Phi_j(X)}\le h^{-1}L^{dj/2}$ (for this one needs the fact that the Poisson kernel in the definition of Φ_j norm acting on x_μ still gives x_μ), we have

$$|\alpha_{\mu\nu}(B)|\le O(1)h^{-2}\|K_j\|_jA^{-1}. \tag{6.39}$$

Note that for a fixed $D\in\mathcal B_{j+1}$, and for all $\bar B=D$, $\alpha_{\mu\nu}(B)$ depends on the position of B in D because ζ is not translation invariant. This problem was not present in the method [18]. We cure this problem by the following lemma.

Lemma 35. *Let $D \in \mathcal{B}_{j+1}$, and let $B_{ct} \in \mathcal{B}_j$ be the j-block at the center of D. Then, with definition (6.38),*

$$\left|\alpha_{\mu\nu}(B) - \alpha_{\mu\nu}(B_{ct})\right| \le O(L^{-d})h^{-4}\|K_j\|_j A^{-1} \tag{6.40}$$

for all $B \in \mathcal{B}_j$ such that $\bar{B} = D$.

Proof. Let T be a translation so that $TB = B_{ct}$, and $\zeta_{D^+}, \zeta_{TD^+}$ be Gaussian fields on D^+, TD^+ with Dirichlet Green's functions C_{D^+}, C_{TD^+} as covariances, respectively. Then, $\alpha_{\mu\nu}(B)$ can be rewritten as the right side of (6.38) with B replaced by B_{ct} and $\zeta = \zeta_{D^+}$ replaced by $\zeta = \zeta_{TD^+}$, so that

$$\begin{aligned}
&\left|\alpha_{\mu\nu}(B) - \alpha_{\mu\nu}(B_{ct})\right| \\
&\quad\le \frac{1}{8|B_{ct}|} \sum_{X\in\mathcal{S}_j, X\supseteq B_{ct}} \left| \partial^2_{t_1 t_2}\big|_{t_i=0} \Big(\mathbb{E}_{\zeta_{TD^+}} \left[K_j(X, t_1 x_\mu + t_2 x_\nu + \zeta_{TD^+})\right] \right. \\
&\qquad \left. - \mathbb{E}_{\zeta_{D^+}} \left[K_j(X, t_1 x_\mu + t_2 x_\nu + \zeta_{D^+})\right] \Big) \right|.
\end{aligned} \tag{6.41}$$

To estimate the difference of the two expectations, define

$$C(t) := tC_{D^+} + (1-t)C_{TD^+}$$

and recall that K_j depends on ζ via $\nabla\zeta$, let

$$\mathcal{K}(\nabla\zeta) := K_j(X, t_1 x_\mu + t_2 x_\nu + \zeta).$$

Then, one has the formula

$$\mathbb{E}_{\nabla^2 C(1)}\mathcal{K} - \mathbb{E}_{\nabla^2 C(0)}\mathcal{K} = \int_0^1 \frac{d}{dt}\mathbb{E}_{\nabla^2 C(t)}\mathcal{K}\mathrm{d}t = \frac{1}{2}\int_0^1 \mathbb{E}_{\nabla^2 C(t)}\left[\Delta_{\nabla^2 \dot{C}(t)}\mathcal{K}\right]\mathrm{d}t,$$

where for any covariance C (in our case $C = \nabla^2 \dot{C}(t)$) the Laplacian is defined as:

$$\Delta_C := \sum_{x,y} C(x,y)\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)}.$$

Now we aim to show a pointwise bound for $\nabla^2\dot{C}(t) = \nabla^2 C_{D^+} - \nabla^2 C_{TD^+}$. One has

$$\nabla^2 C_{\mathbb{Z}^d}(x,y) - \nabla^2 C_{D^+}(x,y) = \nabla^2 P_{D^+} C_{\mathbb{Z}^d}(x,y)$$

Observe that x, y have distance of $O(L^{j+1})$ from ∂D^+, because $\mathcal{K}$ only depends on the field on ∂X^+. We can proceed as the arguments following (3.24) in proof of Lemma 15, or the arguments following (4.19) in proof of Lemma 24, to show that $\nabla^2 P_{D^+} C_{\mathbb{Z}^d}(x,y)$ is bounded by $O(L^{-d(j+1)})$. Analogously, $\nabla^2 C_{\mathbb{Z}^d} - \nabla^2 C_{TD^+}$ satisfies the same bound. Therefore,

$$|\nabla^2 \dot{C}(t)| \le O(L^{-d(j+1)}). \tag{6.42}$$

Our situation is that we would like to bound the fourth derivative of K_j by $\|K_j\|_j$. This is the reason we incorporated the fourth derivative in the definition of $\|K_j\|_j$, see (4.3). Note that $\partial/\partial\phi(x_0)$ acting on $\mathcal{K}$ is equivalent with

$$\partial_s|_{s=0} K_j(X, t_1 x_\mu + t_2 x_\nu + \zeta + s\delta_{x_0}),$$

where δ_{x_0} is the Kronecker function at x_0. In fact, we have $\|\delta_{x_0}\|_{\Phi_j(X)} \leq h^{-1}L^{-dj/2}$ because the $\partial_e P_{X+}$ in the definition of $\Phi_j(X)$ norm acting on δ_{x_0} gives a factor $O(L^{-dj})$. Proceeding as in (6.39), we have $\|x_\mu\|_{\Phi_j(X)} \leq h^{-1}L^{dj/2}$, and $|B_{ct}|^{-1} = O(L^{-dj})$, and the sum $\sum_{x,y}$ gives a factor $O(L^{2dj})$. Combining these with (6.42), we then obtain the desired bound. □

Let $D \in \mathcal{B}_{j+1}$. Define $\alpha_{\mu\nu} := \alpha_{\mu\nu}(B_{ct})$ where $B_{ct} \in \mathcal{B}_j$ is at the center of D. Clearly, it is well defined (independent of D). By reflection and rotation symmetries, there exists an α so that $\alpha_{\mu\nu} = \frac{1}{2}\alpha(\delta_{\mu\nu} + \delta_{\mu,-\nu})$.

Lemma 36. *Let $\psi = P_{U+}\phi + \xi$ and L be sufficiently large. Then,*

$$\mathcal{L}_3'' := \frac{1}{4}\sum_{\bar{B}=D}\left(\sum_{x\in B,e\in\mathcal{E}} \alpha\,(\partial_e\psi(x))^2 - \sum_{x\in B,e\in\mathcal{E}} \alpha_{\mu\nu}\,(\partial_e\psi(x))^2\right) \tag{6.43}$$

is contractive with norm going to zero as $L \to \infty$.

Proof. This is essentially Lemma 10 of [24], so the proof is omitted. □

Proposition 37. *We can choose E_{j+1} and σ_{j+1} so that if L be sufficiently large then $\mathcal{L}_3$ in Proposition 28 is contractive, with arbitrarily small norm as $L \to \infty$.*

Proof. As the first step with $D = \bar{B} \in \mathcal{P}_{j+1}(\Lambda)$, $\phi = P_{D+}\phi + \zeta$ we compute

$$\mathbb{E}\Big[\sum_{x\in B,e\in\mathcal{E}} (\partial_e P_{B+}\phi + \partial_e\xi(x))^2 \big| (D^+)^c\Big] = \sum_{x\in B,e\in\mathcal{E}} (\partial_e P_{D+}\phi(x) + \partial_e\xi(x))^2 + \delta E_j, \tag{6.44}$$

where $\delta E_j = \sum_{x\in B,e\in\mathcal{E}} \mathbb{E}_\zeta\left[(\partial_e P_{B+}\zeta)^2\right] = O(1)$ by Proposition 16.

Let $\psi = P_{D+}\phi + \xi$. By Lemmas 34, 35 and 36, it remains to show the contractivity of

$$\tilde{\mathcal{L}}_3 = \sum_{\bar{B}=U}\Bigg[E_{j+1}(B) + \frac{\sigma_{j+1}}{4}\sum_{x\in B,e\in\mathcal{E}} (\partial_e\psi(x))^2 - \frac{\sigma_j}{4}\Bigg(\sum_{x\in B,e\in\mathcal{E}} (\partial_e\psi(x))^2 + \delta E_j\Bigg)$$
$$+\mathbb{E}_\zeta\left[K_j(X,\zeta)\right] + \frac{\alpha}{4}\sum_{x\in B,e\in\mathcal{E}} (\partial_e\psi(x))^2\Bigg], \tag{6.45}$$

where α is given before Lemma 36. Choose

$$\begin{aligned} \sigma_{j+1} &= \sigma_j - \alpha \\ E_{j+1} &= \sigma_j\delta E_j - \mathbb{E}_\zeta\left[K_j(X,\zeta)\right] \end{aligned} \tag{6.46}$$

then we actually have $\tilde{\mathcal{L}}_3 = 0$. □

By the above choice of E_{j+1}, we can easily see that it is the same number for $Z'_N(\xi)$ and $Z'_N(0)$. Therefore, $\mathrm{e}^{\mathcal{E}_j}$ is the same for $Z'_N(\xi)$ and $Z'_N(0)$, for all j.

7. Proof of Scaling Limit of the Generating Function

Proposition 38. *Let L be sufficiently large; A sufficiently large depending on L; κ sufficiently small depending on L, A; h sufficiently large depending on L, A, κ; and r sufficiently small depending on L, A, κ, h. Then, for $|z| < r$, there exists a constant σ depending on z so that the dynamic system*

$$\begin{aligned} \sigma_{j+1} &= \sigma_j + \alpha(K_j) \\ K_{j+1} &= \mathcal{L}K_j + f(\sigma_j, K_j) \end{aligned} \tag{7.1}$$

satisfies

$$\left|\sigma_j\right| \le r2^{-j} \qquad \left\|K_j\right\|_j \le r2^{-j} \tag{7.2}$$

Proof. By contractivity of $\mathcal{L}$, we apply Theorem 2.16 in [18] (i.e. the stable manifold theorem) to obtain a smooth function $\sigma = h(K_0)$ so that (7.2) hold. Since K_0 depends on z and σ, we solve σ from equation $\sigma - h(K_0(z, \sigma)) = 0$, using Lemma 47. Noting that this equation holds with $(\sigma, z) = 0$, and that $K_0(z = 0, \sigma) = 0$, the derivative of left-hand side w.r.t. σ is 1. So by implicit function theorem, there exists a σ depending on z so that $\sigma = h(K_0(z, \sigma))$. Therefore, the proposition is proved. $\square$

With the generating function $Z_N(f)$ defined in (2.1), we have

Theorem 39. *For any $p > d$, there exist constants $M > 0$ and $z_0 > 0$ so that for all $\|\tilde{f}\|_{L^p} \le M$, and all $|z| \le z_0$ there exists a constant ϵ depending on z so that*

$$\lim_{N\to\infty} Z_N(f) = \exp\left(-\frac{1}{2}\int_{\tilde{\Lambda}} \tilde{f}(x)(-\epsilon\bar{\Delta})^{-1}\tilde{f}(x)d^dx\right),$$

where $\bar{\Delta}$ is the Laplacian in continuum.

Proof. By (2.5),

$$Z_N(f) = \lim_{m\to 0} \mathrm{e}^{\frac{1}{2}\sum_{x\in\Lambda} f(x)(-\epsilon\Delta_m)^{-1}f(x)} Z'_N(\xi)/Z'_N(0). \tag{7.3}$$

In fact, since $\int_{\tilde{\Lambda}} \tilde{f} = 0$

$$\mathrm{e}^{\frac{1}{2}\sum_{x\in\Lambda} f(x)(\epsilon\Delta_m)^{-1}f(x)} \to \mathrm{e}^{-\frac{1}{2}\int_{\tilde{\Lambda}} \tilde{f}(x)(-\epsilon\bar{\Delta})^{-1}\tilde{f}(x)d^dx} \tag{7.4}$$

as $m \to 0$ followed by $N \to \infty$.

At scale $N-1$ (we do not want to continue all the way to the last step since it would be not clear how to define $\tilde{I}_{N-1}$ and I_N), by Proposition 38 and Lemma 24

$$
\begin{aligned}
\left|Z'_N(\xi) - \mathrm{e}^{\mathcal{E}_{N-1}}\right| &= \mathrm{e}^{\mathcal{E}_{N-1}}\left|\mathbb{E}\big[I_{N-1}(\Lambda\backslash\hat{X})K_{N-1}(X)\big] - 1\right| \\
&\le \mathrm{e}^{\mathcal{E}_{N-1}} \\
&\quad \times\Big[\sum\nolimits_{X\neq\emptyset}(1+2^{-N+1})^{|\Lambda\backslash\hat{X}|_{N-1}}\cdot 2^{-N+1}\mathbb{E}G(\ddot{X},X^+)\big| + \big|\mathbb{E}I^{\Lambda}_{N-1}-1\big|\Big] \\
&\le \mathrm{e}^{\mathcal{E}_{N-1}}\Big[2^{L^d}(1+2^{-N+1})^{L^d}\cdot 2^{-N+1}c^{L^d} + 2^{-N+1}\Big]. \qquad (7.5)
\end{aligned}
$$

where $X \in \mathcal{P}_{N-1}$. Since the constant $\mathrm{e}^{\mathcal{E}_{N-1}}$ is identical for $Z'_N(\xi)$ and $Z'_N(0)$, and $Z'_N(0)$ satisfies the same bound above, one has $Z'_N(\xi)/Z'_N(0) \to 1$. Therefore, the theorem is proved. □

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Appendix A. Decay of the Derivative of Green's Function

The decay rates of (derivatives of) Green's functions are essential in our method. In this section, we aim to show Corollary 43 on the torus. For $m > 0$, let G_m be the Green's function of $-\Delta_m = -\Delta + m^2$ on $\mathbb{Z}^d$, and $\bar{G}_m$ be the Green's function of $-\bar{\Delta}_m = -\bar{\Delta} + m^2$ on $\mathbb{R}^d$ where $\bar{\Delta}$ is the Laplacian in the continuum. We start with the following approximation result.

Lemma 40. *Let $d \ge 2$. For all $e \in \mathcal{E}$, as $|x| \to \infty$, one has*

$$\partial_e G_m(x) = \partial_e \bar{G}_{\bar{m}}(x) + O(|x|^{-(d+1)}) \qquad (A.1)$$

where $\partial_e \bar{G}_{\bar{m}}(x) = \bar{G}_{\bar{m}}(x+e) - \bar{G}_{\bar{m}}(x)$ is the discrete derivative, and $\bar{m}^2 = \log(m^2+1)$, and the big-O term does not depend on m.

Proof. The proof is essentially analogous to the proofs of Corollary 4.3.3 and Corollary 4.4.5 in Lawler's book [38], so we only sketch the proof. Writing

$$\partial_e G_m(x) = \sum_{n=0}^{\infty} \lambda^n \partial_e \bar{p}_n(x) + \sum_{n=0}^{\infty} \lambda^n (\partial_e p_n(x) - \partial_e \bar{p}_n(x)) \qquad (A.2)$$

where p (resp. $\bar{p}$) are discrete (resp. continuum) heat kernels, and $\lambda^{-1} - 1 = m^2$. Since $\bar{m}^2 = \log(\lambda^{-1})$, we have

$$\partial_e \bar{G}_{\bar{m}}(x) = \int_0^\infty \lambda^t \partial_e \bar{p}_t(x)\mathrm{d}t.$$

The difference between this integral and the discrete sum is bounded by $O(|x|^{-(d+3)})$ using standard error estimates of Riemann sums as in the proof of [38, Lemma 4.3.2]. The second term on the right-hand side of (A.2) can be bounded by $O(|x|^{-(d+1)})$ in the same way as in the proof of [38, Theorem 4.3.1], except that we apply the gradient estimates of heat kernels as mentioned in the Remark in the end of that subsection of the book. □

Lemma 41. *Let $d \geq 2$. For every $e \in \mathcal{E}$, every $x \in \Lambda$ where Λ is the torus of length size L^N defined in Sect. 2.2, and every sufficiently small $m > 0$,*

$$\left| \sum_{y \in \mathbb{Z}^d \setminus \{0\}} \partial_e G_m(x + L^N y) \right| \leq c_d L^{-(d-1)N} \tag{A.3}$$

where c_d only depends on d.

Remark 42. Note that the left-hand side is not absolutely summable uniformly in $m > 0$. The bound (A.3) appeared in [22, Eq. (3.5)], but we give our proof here. In fact by following our proof, one can also show that if $m = 0$ and we define the above sum as a symmetric sum, then the bound still holds; however, we do not need this since we always have a positive mass regularization.

Proof. Denote by D_μ the smooth derivative. Without loss of generality assume that $e = e_1$. The term $O(|x|^{-(d+1)})$ in (A.1) is summable, and noting that the sum avoids the origin $y = 0$ one has

$$\left| \sum_{y \in \mathbb{Z}^d \setminus \{0\}} O(|x + L^N y|^{-(d+1)}) \right| = O(L^{-(d+1)N}).$$

Up to this term, $\partial_{e_1} G_m(x + L^N y)$ is equal to (with $\bar{m}^2 = \log(m^2 + 1)$),

$$\begin{aligned}
&\bar{G}_{\bar{m}}(x + e_1 + yL^N) - \bar{G}_{\bar{m}}(x + yL^N) \\
&\quad = \Big(\bar{G}_{\bar{m}}(yL^N) + (x + e_1) \cdot D\bar{G}_{\bar{m}}(yL^N) + \frac{1}{2}(x + e_1)^2 \cdot D^2\bar{G}_{\bar{m}}(yL^N) + Err\Big) \\
&\qquad - \Big(\bar{G}_{\bar{m}}(yL^N) + x \cdot D\bar{G}_{\bar{m}}(yL^N) + \frac{1}{2}x^2 \cdot D^2\bar{G}_{\bar{m}}(yL^N) + Err\Big) \\
&\quad = D_1\bar{G}_{\bar{m}}(yL^N) + (x \cdot DD_1\bar{G}_{\bar{m}}(yL^N) + \frac{1}{2}D_1^2\bar{G}_{\bar{m}}(yL^N)) + Err
\end{aligned} \tag{A.4}$$

where we have performed Taylor expansions around yL^N and the error term

$$Err = O\left(L^{2N} \sup_{|z - yL^N| < \frac{L^N}{2}} |D^3\bar{G}_{\bar{m}}(z)| \right)$$

which comes from Taylor remainder theorem. The reason we do Taylor expansion up to this order is that $D^3\bar{G}_{\bar{m}}$ is absolutely summable uniformly in $m > 0$, and the sum over $y \neq 0$ of Err gives $O(L^{-(d-1)N})$.

Regarding the other terms in (A.4), for $\bar{m} > 0$ we note that $D_1\bar{G}_{\bar{m}}$ and $x_i D_i D_1 \bar{G}_{\bar{m}}$ with $i \neq 1$ are odd functions in the first coordinate, so the sum over $y \neq 0$ yields zero, and we are left with $(x_1 + \frac{1}{2})D_1^2\bar{G}_{\bar{m}}$. Letting $\mathbf{p}(z) = (z_2, \ldots, z_d, z_1)$ for $z = (z_1, \ldots, z_d)$, the sum of $D_1^2\bar{G}_{\bar{m}}$ at $z, \mathbf{p}(z), \ldots, \mathbf{p}^{d-1}(z)$ is precisely the Laplacian of $\bar{G}_{\bar{m}}$, which is equal to $\bar{m}^2\bar{G}_{\bar{m}}(z)$. The sum over

$y \neq 0$ of $\bar{m}^2 \bar{G}_{\bar{m}}(yL^N)$ can be bounded by $\bar{m}^2 O(L^{-(d-2)N}(\bar{m}L^N)^{-1})$ if $d \geq 3$ and by $\bar{m}^2 O(\log(L^N)(\bar{m}L^N)^{-1})$ if $d = 2$. Choosing $m > 0$ (and, therefore, $\bar{m}$) sufficiently small, we have obtain the desired bound. □

Corollary 43. *Let $d \geq 2$ and C_m be the Green's function of $-\Delta + m^2$ on the torus Λ. For all $e \in \mathcal{E}$, $x \in \Lambda$ and $m > 0$,*

$$|\partial_e C_m(x)| \leq c_d \|x\|_\Lambda^{-(d-1)} \tag{A.5}$$

where the constant c_d only depends on d, and $\|x\|_\Lambda = d(0,x)$ with $d(-,-)$ being the distance function on Λ defined in Sect. 2.1.

Note that the distant function $d(-,-)$ is not to be confused with the dimension d.

Proof. The statement is an immediate consequence of

$$\partial_e C_m(x) = \sum_{y \in \mathbb{Z}^d} \partial_e G_m(x + L^N y)$$

and Lemmas 40 and 41. □

Appendix B. Estimates

In Sect. 4, we defined norms for functions of the fields. In the Appendix, we give estimates in terms of these norms of some functions of interest.

Lemma 44. *There exists a constant $c > 0$ so that if $\sigma/\kappa < c$ and $h^2\sigma < c$, then for every $B \in \mathcal{B}_j$, $j < N-1$, one has*

$$\|\mathrm{e}^{-\frac{\sigma}{2}\sum_{x\in B,e}(\partial_e P_{B^+}\phi(x)+\partial_e\xi(x))^2}\|_{T_\phi(\Phi_j(B))} \leq 2\mathrm{e}^{\frac{\kappa}{4}\sum_B(\partial P_{B^+}\phi)^2}, \tag{B.1}$$

$$\|\mathrm{e}^{-\frac{\sigma}{2}\sum_{x\in B,e}(\partial_e P_{(\bar{B})^+}\phi(x)+\partial_e\xi(x))^2}\|_{T_\phi(\Phi_j(\dot{\bar{B}},\bar{B}^+))} \leq 2\mathrm{e}^{\frac{\kappa}{4}\sum_B(\partial P_{(\bar{B})^+}\phi)^2}, \tag{B.2}$$

$$\|\mathrm{e}^{-\frac{\sigma}{2}\sum_{x\in B,e}(\partial_e P_{B^+}\phi(x)+\partial_e\xi(x))^2}-1\|_{T_\phi(\Phi_j(B))} \leq 4c^{-1}h^2|\sigma|\mathrm{e}^{\frac{\kappa}{4}\sum_B(\partial P_{B^+}\phi)^2}, \tag{B.3}$$

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\sigma}{2}\sum_{x\in B,e}(\partial_e P_{(\bar{B})^+}\phi(x)+\partial_e\xi(x))^2} - 1\|_{T_\phi(\Phi_j(\dot{\bar{B}},\bar{B}^+))}\\ &\quad\leq 4c^{-1}h^2\mathrm{e}^{\frac{\kappa}{4}\sum_B(\partial P_{(\bar{B})^+}\phi)^2}.\end{aligned} \tag{B.4}$$

Remark 45. Note that the prefactors on the exponentials on the right-hand sides are always $\frac{\kappa}{4}$, whereas the prefactor in our regulator defined in Sect. 4 is $\frac{\kappa}{2}$.

Proof. To prove (B.1), let

$$V = -\frac{1}{2}\sum_{x\in B,e}(\partial_e P_{B^+}\phi(x) + \partial_e\xi(x))^2$$

and let $\|(f,\lambda\xi)^{\times n}\|_{\Phi_j(B)} \le 1$. By $|\partial\xi|^2 \le h^2 L^{-dN}$, it is straightforward to check that if σ/κ is sufficiently small, for $n = 0, 1, 2$,

$$\left|(\sigma V)^{(n)}(\phi, \xi; (f,\lambda\xi)^{\times n})\right| \le \frac{\kappa}{2^{n+4}} \sum_{x\in B,e} (\partial_e P_{B^+}\phi(x))^2 + 2\sigma h^2 \tag{B.5}$$

and for $n \ge 3$, $V^{(n)} = 0$. Therefore, for $n = 0, \dots, 4$,

$$\begin{aligned}\frac{1}{n!}\left|\left(\mathrm{e}^{\sigma V}\right)^{(n)}(\phi, \xi; (f,\lambda\xi)^{\times n})\right| &\le \mathrm{e}^{|\sigma V|}\mathrm{e}^{|\sigma V^{(1)}|+|\sigma V^{(2)}|} \\ &\le \mathrm{e}^{\frac{\kappa}{4}\sum_{x\in B,e}(\partial_e P_{B^+}\phi(x))^2 + 8\sigma h^2} \\ &\le 2\,\mathrm{e}^{\frac{\kappa}{4}\sum_{x\in B,e}(\partial_e P_{B^+}\phi(x))^2}\end{aligned}$$

if $h^2\sigma$ is sufficiently small, where we bounded the polynomials in $(\sigma V)^{(n)}$ by $\mathrm{e}^{|\sigma V^{(1)}|+|\sigma V^{(2)}|}$. So (B.1) is proved. (B.2) is proved in the same way.

To prove (B.3), note that similarly as above one can show that $\mathrm{e}^{\sigma V}$ is analytic in σ, so

$$\begin{aligned}\|\mathrm{e}^{\sigma V} - 1\|_{T_\phi(\Phi_j(B))} &= \left\|\frac{1}{2\pi i}\int_{|z|=ch^{-2}} \frac{\sigma \mathrm{e}^{zV}}{z(z-\sigma)} dz\right\|_{T_\phi(\Phi_j(B))} \\ &\le 4c^{-1}h^2|\sigma|\mathrm{e}^{\frac{\kappa}{4}\sum_B(\partial P_{B^+}\phi)^2}\end{aligned}$$

and (B.4) is proved in the same way. □

Another example is the estimate of the initial interaction. At step $j = 0$, a block B is a single lattice point x. Define

$$\tilde W(\{x\}, \phi, u) = \frac{1}{2}\sum_{e\in\mathcal{E}} \cos(u\,\partial_e\phi(x)).$$

We also write $W(\{x\}, \phi) = \tilde W(\{x\}, \phi, \sqrt{\beta(1+\sigma)})$. Recall that $\|-\|_0$ is the $\|-\|_j$ norm defined in 4.5 with $j = 0$.

Lemma 46. *If $\kappa \ge h^{-1}$, then* (1) *$\tilde W(\{x\}, \phi, u)$ satisfies*

$$\begin{aligned}&\sum_{n=0}^{3} \frac{1}{n!} \sup_{|\partial f(x)|\le h} \partial_{t_1\dots t_n}\big|_{t_i=0}\left|\partial_u^m W(\{x\}, \phi + \sum_{i=1}^n t_i f_i)\right| \\ &\quad\le C_{h,u}\mathrm{e}^{\frac{\kappa}{2}\sum_{e\in\mathcal{E}}(\partial_e\phi(x))^2}\end{aligned} \tag{B.6}$$

for $m = 0, 1, 2, \dots$, where $C_{h,u} = d(2h)^m \mathrm{e}^{hu}$.

(2) *Let $\|-\|_{00}$ be the $\|-\|_0$ norm with $G = 1$. For $|z|$ sufficiently small,*

$$\|\mathrm{e}^{zW(\{x\})}\|_{00} \le 2. \tag{B.7}$$

Proof. (1) The case $m = 0$ holds even without $\mathrm{e}^{\frac{\kappa}{2}\sum_{e\in\mathcal{E}}(\partial_e\phi(x))^2}$ by straightforward computations and thus is omitted. For $m > 0$,

$$\partial_u^m W = \pm\frac{1}{2}\sum_{e\in\mathcal{E}} {}^{\sin}_{\cos}(u\partial_e\phi(x))\,(\partial_e\phi(x))^m\,.$$

We then have the bound

$$\sum_{n=0}^{4} \frac{1}{n!} \sup_{|\partial f(x)|\leq h} \partial_{t_1\ldots t_n}\big|_{t_i=0}\Big(\partial_e(\phi(x)+\sum_{i=1}^{n} t_i f_i(x))\Big)^m \\ \leq (2h)^m \mathrm{e}^{\frac{\kappa}{2}\sum_{e\in\mathcal{E}}(\partial_e\phi(x))^2}.$$

The bound for $\partial_u^m W$ follows by the product rule of differentiations and the case $m=0$.

(2) For $|z|$ sufficiently small,

$$\|\mathrm{e}^{zW(B)}\|_{00} \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!}\|W(B)\|_{00}^n \leq \exp\left(4d|z|\mathrm{e}^h\right) \leq 2.$$

This is precisely the claimed bound. □

Lemma 47. *Let K_0 be the function defined in Proposition 2. Given $r>0$, if $|z|$ and $|\sigma|$ are sufficiently small, then $\|K_0\|_0 < r$. Furthermore, K_0 is smooth in z and σ.*

Proof. As in the proof of (B.7), one has

$$\|\mathrm{e}^{zW(\{x\})}-1\|_{00} \leq \exp\left(4d|z|\mathrm{e}^h\right) - 1 \leq c|z|$$

for some constant c. Write $V_0(\{x\}) = -\frac{1}{2}\sum_e(\partial_e\phi(x)+\partial_e\xi(x))^2$. By Lemma 44,

$$\|(\mathrm{e}^{zW(\{x\})}-1)\mathrm{e}^{\sigma V_0(\{x\})}\|_0 \leq 2c|z|,$$

therefore,

$$\|K_0\|_0 = \sup_{X\in\mathcal{P}_{0,c}} \|K_0(X)\|_0\, A^{|X|_0} \leq \sup_{X\in\mathcal{P}_{0,c}} (2c|z|A)^{|X|_0} < r.$$

The derivative of $\prod_{x\in X}(\mathrm{e}^{zW(\{x\})}-1)$ w.r.t. σ is equal to

$$\sum_{x\in X} zW'(\{x\})\frac{1}{2\sqrt{1+\sigma}} \prod_{y\in X\backslash\{x\}} (\mathrm{e}^{zW(\{y\})}-1),$$

therefore, its $\|-\|_0$ norm is bounded by $c'A|z|$ for some constant c'. The derivative of $\mathrm{e}^{\sigma V_0(\{x\})}$ and higher derivatives can be bounded similarly. The derivative of $\prod_{x\in X}(\mathrm{e}^{zW(\{x\})}-1)$ w.r.t. z is equal to

$$\sum_{x\in X} W(\{x\}) \prod_{y\in X\backslash\{x\}} (\mathrm{e}^{zW(\{y\})}-1)$$

which can be bounded in the same way. □

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