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Weakly Nonlinear Time-Adiabatic Theory

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Abstract. We revisit the time-adiabatic theorem of quantum mechanics and show that it can be extended to weakly nonlinear situations, that is to nonlinear Schrödinger equations in which either the nonlinear coupling constant or, equivalently, the solution is asymptotically small. To this end, a notion of criticality is introduced at which the linear bound states stay adiabatically stable, but nonlinear effects start to show up at leading order in the form of a slowly varying nonlinear phase modulation. In addition, we prove that in the same regime a class of nonlinear bound states also stays adiabatically stable, at least in terms of spectral projections.

1. Introduction

The time-adiabatic theorem of quantum mechanics is concerned with systems governed by a *slowly varying* time-dependent (self-adjoint) Hamiltonian operator $H = H(\varepsilon\tau)$, where $0 < \varepsilon \ll 1$ is a small adiabatic parameter, controlling the timescales on which H varies. The associated Cauchy problem, governing the time evolution of the quantum mechanical wave function $\Psi = \Psi(\tau, x)$, with $x \in \mathbb{R}^d$, reads

$$i\partial_\tau \Psi = H(\varepsilon\tau)\Psi, \quad \Psi_{|\tau=\tau_0} = \Psi_{\text{in}}(x). \tag{1.1}$$

In the following, it will be more convenient to rewrite the system using the (slow) *macroscopic time variable* $t = \varepsilon\tau$. In this case, (1.1) becomes a singularly perturbed problem of the form

$$i\varepsilon\partial_t \Psi^\varepsilon = H(t)\Psi^\varepsilon, \quad \Psi^\varepsilon_{|t=t_0} = \Psi^\varepsilon_{\text{in}}(x), \tag{1.2}$$

where $\Psi^\varepsilon(t, x) \equiv \Psi(t/\varepsilon, x)$. A typical example for the time-dependent Hamiltonian $H(t)$, and the one we will be concerned with, is given by

$$H(t) := -\frac{1}{2}\Delta + V(t, x), \tag{1.3}$$

where $V(t, x)$ describes some time-dependent (real valued) potential.

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It is well known that in the case where $V = V(x)$ is *time independent*, the spectral theorem of self-adjoint operators allows for a precise description of the time evolution associated to (1.2). In particular, it implies that if the initial data $\Psi^\varepsilon_{\mathrm{in}}$ is concentrated in a given spectral subspace of H, then it will remain so for all times. However, as soon as $H = H(t)$, the spectral subspaces (in general) start to mix during the time evolution, and thus we do not have any precise information on the solution $\Psi(t, \cdot)$.

However, one might hope that for small $0 < \varepsilon \ll 1$ there is a remedy to the situation. To this end, let us assume that the spectral subspaces of $H(t)$ vary smoothly in time for $t \in [0, T]$, and that the initial wave function $\Psi^\varepsilon_{\mathrm{in}}$ is concentrated in one of these subspaces. Then, the classical time-adiabatic theorem of quantum mechanics states that, for sufficiently small $\varepsilon \ll 1$, the solution $\Psi^\varepsilon(t, \cdot)$ approximately (i.e., up to a certain error which vanishes as $\varepsilon \to 0$) remains within the same subspace, provided the latter stays isolated from the rest of the spectrum of $H(t)$ for all $t \in [0, T]$, see below. In this situation, the spectral subspace is said to be *adiabatically stable* under the time evolution. Note that in the unscaled variable τ this result corresponds to an approximation on timescales of order $\tau \sim O(1/\varepsilon)$. The first adiabatic result for quantum systems appeared as early as 1928, cf. [5]. Since then, many mathematical extensions and developments have taken place, see, e.g., [1,2,14–17,25], and the references therein. For a general introduction to this subject, we refer to [32].

A possible way of introducing the slow parameter ε is to think about a quantum mechanical experiment in which the experimentalist is allowed to slowly tune the external potential $V = V(\varepsilon\tau, x)$. With this in mind, it is worth noting that modern quantum mechanical experiments are often performed on ultra-cold quantum gases in the state of their *Bose–Einstein condensation* [21]. Indeed, ultra-cold quantum gases offer a superb level of control, unprecedented in several respects, which has triggered a vast amount of scientific activity, both theoretical and experimental. It is well known that within a mean-field approximation the (macroscopic) wave function of the condensate is accurately described by a *nonlinear* Schrödinger (or, Gross–Pitaevskii) equation, cf. [21] for a general discussion [9,19], and the references therein for a rigorous mathematical justification. It therefore seems a natural question to ask, whether one can extend the results of time-adiabatic perturbation theory to the case of nonlinear Schrödinger equations (NLS). This work is a first, modest attempt in this direction, although one should mention that there exist some non-rigorous works in the physics literature, cf. [36]. Moreover, one should distinguish our time-adiabatic setting from the one in [24], which studies solitary wave solutions to nonlinear Schrödinger equations in a *space-adiabatic* situation, i.e., with a potential of the form $V = V(t, \varepsilon x)$. In addition, we mention [7,8] both of which include rigorous results for related nonlinear adiabatic situations.

To be more concrete, we shall study the Cauchy problem corresponding to the following class of NLS:

$$i\varepsilon\partial_t\Psi^\varepsilon = -\frac{1}{2}\Delta\Psi^\varepsilon + V(t, x)\Psi^\varepsilon + \lambda|\Psi^\varepsilon|^{2\sigma}\Psi^\varepsilon, \quad \Psi^\varepsilon_{|t=t_0} = \Psi^\varepsilon_{\mathrm{in}}(x), \qquad (1.4)$$

where $\sigma \in \mathbb{N}$, and where $\lambda \in \mathbb{R}$ denotes a nonlinear coupling constant, describing either focusing or defocusing behavior, cf. [27] for a broad discussion of these terms. The cubic case $\sigma = 1$ corresponds to the classical Gross–Pitaevskii equation. Clearly, an extension of the time-adiabatic theorem to such nonlinear models is not straightforward, in particular due to the lack of a spectral theory for general nonlinear operators. The basic idea in the present paper is to work in an asymptotic regime for which the nonlinearity can be considered as a small perturbation of the associated linear problem. A possible way to do so is to restrict ourselves to asymptotically small solutions of the form

$$\Psi^\varepsilon(t, x) = \varepsilon^{1/(2\sigma)}\psi^\varepsilon(t, x), \tag{1.5}$$

where, as $\varepsilon \to 0$, we formally regard $\psi^\varepsilon \sim O(1)$, say in $L^2(\mathbb{R}^d)$. Note that the size of the original wave function is then $\Psi^\varepsilon \sim O(\varepsilon^{1/(2\sigma)})$ and hence it becomes asymptotically larger as $\sigma \in \mathbb{N}$ becomes larger. Rewriting (1.4) in terms of the new unknown ψ^ε yields

$$i\varepsilon\partial_t\psi^\varepsilon = -\frac{1}{2}\Delta\psi^\varepsilon + V(t, x)\psi^\varepsilon + \lambda\varepsilon|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon, \quad \psi^\varepsilon_{|t=t_0} = \psi^\varepsilon_{\text{in}}(x), \tag{1.6}$$

with an effective nonlinear coupling constant of size $|\lambda^\varepsilon| = |\lambda|\varepsilon \ll 1$. The Cauchy problem (1.6) can thus be considered *weakly nonlinear*.

As we shall see below, a nonlinear coupling constant of order $O(\varepsilon)$ will be critical for our analysis, since it corresponds to the threshold for which nonlinear effects are present at the leading order description of ψ^ε. In particular, if λ^ε were even smaller, the problem would become essentially linearizable (as we will show below, cf., Remark 2.2). The main result of this work can now be stated as follows:

Theorem 1.1. *Let $\sigma \in \mathbb{N}$, $\lambda \in \mathbb{R}$, $I \subseteq \mathbb{R}$ be an open time interval containing $t_0 \in \mathbb{R}$, and $V \in C^1_{\mathrm{b}}(I; \mathcal{S}(\mathbb{R}^d))$. Assume that there exists a simple eigenvalue $E(t) \in \mathrm{spec}(H(t))$ which stays separated from the rest of the spectrum by some $\delta > 0$, i.e.,*

$$\inf_{t \in I} \mathrm{dist}\big(E(t), \mathrm{spec}(H(t))\backslash\{E(t)\}\big) = \delta,$$

and choose an associated normalized eigenfunction $\chi \in C^1_{\mathrm{b}}(I; H^s(\mathbb{R}^d))$ for $s \geqslant 0$. Finally, let $k > \frac{d}{2}$ and assume that at $t = t_0$, the initial data is concentrated in the eigenspace corresponding to $E(t_0)$, such that

$$\|\psi^\varepsilon_{\mathrm{in}} - \chi(t_0, \cdot) - \varepsilon\gamma^\varepsilon\|_{H^k(\mathbb{R}^d)} \leqslant C_0\varepsilon^{k+1},$$

where $\gamma^\varepsilon \in H^k(\mathbb{R}^d)$ is a corrector which is constructed according to (3.5) and satisfies $\langle\chi(t_0, \cdot), \gamma^\varepsilon\rangle_{L^2} = 0$.

Then, for any compact time interval $J \subset I$ containing t_0, there exist $\varepsilon_0(J) < 1$, and a constant $C > 0$, such that for any $0 < \varepsilon \leqslant \varepsilon_0(J)$ the unique solution $\psi^\varepsilon \in C(J; H^k(\mathbb{R}^d))$ to the nonlinear Schrödinger equation (1.6) exists, and, in addition,

$$\sup_{t \in J}\left\|\psi^\varepsilon(t, \cdot) - \chi(t, \cdot)e^{-i\varphi^\varepsilon(t)}\right\|_{L^2(\mathbb{R}^d)} \leqslant C\varepsilon,$$

where the phase $\varphi^\varepsilon(t) \in \mathbb{R}$ is given by

$$\varphi^\varepsilon(t) = \frac{1}{\varepsilon} \int_{t_0}^{t} E(s)\mathrm{d}s + \lambda \int_{t_0}^{t} \|\chi(s,\cdot)\|_{L^{2\sigma+2}}^{2\sigma+2}\,\mathrm{d}s - i\beta(t),$$

with $\beta(t) \in i\mathbb{R}$ the Berry phase, defined in (2.8).

We see that the presence of the nonlinearity shows up in the form of a slowly varying phase modulation within the leading order approximation of ψ^ε. An immediate consequence is the following corollary for the associated spectral projectors (for which we use Dirac's notation):

Corollary 1.2. *Under the same assumptions as in Theorem 1.1, we obtain*

$$\sup_{t \in J} \big\| |\psi^\varepsilon(t,\cdot)\rangle\langle\psi^\varepsilon(t,\cdot)| - |\chi(t,\cdot)\rangle\langle\chi(t,\cdot)| \big\|_{L^2 \to L^2} \leqslant C\varepsilon,$$

In other words, in terms of spectral projections, the *linear* time-adiabatic theorem is still valid under weakly nonlinear perturbations of the form (1.6).

The proof of Theorem 1.1 relies on a multiple scales expansion which yields an approximate solution to (1.6). This approximate solution is proven to be well defined (under our assumptions) and, in a second step, used to infer the existence of a true solution ψ^ε which is asymptotically close to the approximate one for times of order $O(1)$. Unfortunately, to control the nonlinear effects within our method of proof, we require that the initial data is sufficiently well prepared (in the sense described above), even if one is only interested in the leading order approximation. In the language of, e.g., [32], we require the initial data to be concentrated in a *super-adiabatic* subspace. This is very similar to the situation encountered in [8], where the semiclassical asymptotics for weakly nonlinear Schrödinger equations with highly oscillatory periodic potentials is studied. In fact, the basic strategy for the proof of Theorem 1.1 is similar to the one used in [8].

Clearly, Theorem 1.1 can be reformulated in terms of Ψ^ε, yielding a time-adiabatic result for asymptotically small solutions of order $O(\sqrt{\varepsilon})$. In particular, we have

$$\sup_{t \in J} \big\| |\Psi^\varepsilon(t,\cdot)\rangle\langle\Psi^\varepsilon(t,\cdot)| - \varepsilon|\chi(t,\cdot)\rangle\langle\chi(t,\cdot)| \big\|_{L^2 \to L^2} \leqslant C\varepsilon^2, \qquad (1.7)$$

where $\Psi^\varepsilon = \sqrt{\varepsilon}\psi^\varepsilon$. In this case, a connection to the theory of *nonlinear bound states* for NLS equations becomes apparent. To this end, consider the "stationary" Schrödinger equation associated to (1.4), i.e.,

$$-\frac{1}{2}\Delta\Phi + V(t,x)\Phi + \lambda|\Phi|^{2\sigma}\Phi = E_*\Phi, \qquad (1.8)$$

where $E_* \equiv E_*(t) \in \mathbb{R}$ is a nonlinear energy-eigenvalue. Now, let $t \in \mathbb{R}$ be *fixed* assume that the potential $V(t,x)$ is such that $H(t)$ has a discrete (linear) eigenvalue/eigenfunction pair (E, χ). Then, classical bifurcation theory (cf., [18]) implies that for $E_* \approx E$ a nonlinear bound state solution Φ exists, which is approximately given by a small multiple of χ (see Sect. 5.2 for more details). In the context of NLS, this has been rigorously proved in a number of papers, cf. [20,23,33–35]. Combining this fact with the result in Theorem 1.1 will allow us

to prove that in the regime of $O(\sqrt{\varepsilon})$-solutions, these nonlinear bound states are also adiabatically stable, at least in terms of spectral projections. More precisely, we have the following result.

Corollary 1.3. *In addition to the assumptions of Theorem 1.1, assume that for a compact $J \subset I$ there is an $M_0(J) > 0$ such that for all $M \leqslant M_0$, there exists a family $\Phi \in C(J, H^2(\mathbb{R}^d))$ of solutions to (1.8) with $\|\Phi(t, \cdot)\|_{L^2} = M$, for all $t \in J$. Let $\varepsilon \in (0, \min(\varepsilon_0(J), M_0(J))]$ and denote by $\Phi^\varepsilon(t, x)$ a family of bound states with norm $\|\Phi^\varepsilon(t, \cdot)\|_{L^2} = \sqrt{\varepsilon}$. Furthermore, let Ψ^ε be the solution to (1.4) with initial data $\Psi^\varepsilon_{\text{in}} = \sqrt{\varepsilon}\psi^\varepsilon_{\text{in}}$, where ψ^ε is as in Theorem 1.1.*
Then, there exists a constant $K > 0$ such that

$$\sup_{t \in J} \left\| |\Psi^\varepsilon(t, \cdot)\rangle\langle\Psi^\varepsilon(t, \cdot)| - |\Phi^\varepsilon(t, \cdot)\rangle\langle\Phi^\varepsilon(t, \cdot)| \right\|_{L^2 \to L^2} \leqslant K\varepsilon^{\min(2, \sigma+1/2)}.$$

Note that for $\sigma = 1$ (cubic nonlinearity) this estimate is not as good as the one in (1.7). In addition, Corollary 1.3 does not provide any information on the phase, an issue which will be discussed in more detail in Sect. 5.2.

We finally want to mention that in the recent PhD thesis [11] (see also the upcoming paper [10]) a similar result has been proven (for $d = 3$, $\lambda > 0$ and $\sigma = 1$). Namely, that there exists a family of nonlinear bound states $t \mapsto \Phi(t, \cdot)$ with $\|\Phi(t, \cdot)\|_{L^2} = M \ll 1$ sufficiently small, such that for $\varepsilon \ll 1$ the solution Ψ^ε of (1.4) satisfies

$$\sup_{t \in [0,1]} \left\| |\Psi^\varepsilon(t, \cdot)\rangle\langle\Psi^\varepsilon(t, \cdot)| - |\Phi(t, \cdot)\rangle\langle\Phi(t, \cdot)| \right\|_{L^2 \to L^2} \lesssim \varepsilon.$$

Indeed, the author of [11] proves a slightly stronger statement in terms of the wave function, not only the spectral projections (their phase information, however, is not as precise as the one we have in Theorem 1.1). Note that Corollary 1.3 is consistent with the one from [11], if one specifies the latter to solutions of size $O(\sqrt{\varepsilon})$ (but we require more assumptions on the initial data due to the fact that we need to go through the proof of Theorem 1.1). The main difference between the two results seems to be that in our case there is only a single small parameter ε (induced by the dimensionless form of the NLS itself) in which the size of the solution (or, equivalently, the strength of the nonlinearity) is measured. In comparison to that, [11] considers both M and ε to be small and independent, so that for a given $M \ll 1$ one can choose an ε sufficiently small to derive a nonlinear time-adiabatic result.

The paper is now organized as follows: In Sect. 2, we shall show how to obtain the leading order approximation by means of formal asymptotic expansions. These expansions will then be made mathematically rigorous in Sect. 3. The nonlinear stability of our approximation is proved in Sect. 4, yielding the proof of Theorem 1.1. Possible extensions and variations of our results, in particular, the proof of Corollary 1.3 are then discussed in Sect. 5.

2. Formal Construction of the Approximate Solution

In this section, we shall first recall the standard adiabatic expansion for linear equations of Schrödinger type and show (in a second step) how to include the

case of (sub-)critical nonlinearities. To this end, we consider (1.6) in the case $\lambda = 0$ and seek a solution in the form

$$\psi^\varepsilon(t, x) = e^{-i\varphi(t)/\varepsilon}\mathcal{U}^\varepsilon(t, x),$$

where $\varphi(t) \in \mathbb{R}$ is some sufficiently smooth phase function, and the complex-valued amplitude $\mathcal{U}^\varepsilon$ is assumed to be of the form

$$\mathcal{U}^\varepsilon(t, x) \underset{\varepsilon \to 0}{\sim} \sum_{n \geqslant 0} \varepsilon^n U_n(t, x), \tag{2.1}$$

in the sense of formal asymptotic expansions. Plugging this into (1.6) yields

$$H(t)\mathcal{U}^\varepsilon = \dot{\varphi}(t)\mathcal{U}^\varepsilon + i\varepsilon\partial_t\mathcal{U}^\varepsilon,$$

where $H(t)$ is given by (1.3). Next, we plug in (2.1) and equate powers in ε. At leading order, i.e., by equating terms of order $O(1)$, we find:

$$H(t)U_0(t, x) = \dot{\varphi}(t)U_0(t, x). \tag{2.2}$$

This can be seen as an eigenvalue problem for the operator $H(t)$ with eigenvalue $E(t) = \dot{\varphi}(t)$ and we consequently conclude that

$$\varphi(t) = \int_{t_0}^{t} E(s)\,\mathrm{d}s, \tag{2.3}$$

the so-called dynamic phase. Let $I \subseteq \mathbb{R}$ be an open time interval containing t_0. Assuming for the moment that $E(t)$ is a simple eigenvalue for all $t \in I$, with associated normalized eigenfunction $\chi(t, \cdot) \in L^2(\mathbb{R}^d)$, we infer $U_0(t, x) = u_0(t)\chi(t, x)$, for some yet to be determined coefficient function $u_0(t) \in \mathbb{C}$.

   Next, by equating terms of order $O(\varepsilon)$, we find the following inhomogeneous equation

$$H(t)U_1(t, x) = \dot{\varphi}(t)U_1(t, x) + i\partial_t U_0(t, x). \tag{2.4}$$

Using the information from the step before, this can be rewritten as

$$L_E(t)U_1(t, x) = i\big(\dot{u}_0(t)\chi(t, x) + u_0(t)\partial_t\chi(t, x)\big). \tag{2.5}$$

where from now on, we shall denote

$$L_E(t) = H(t) - E(t).$$

The kernel of $L_E(t)$ is given by $\mathrm{span}(\chi(t, \cdot))$ and we consequently decompose

$$U_1(t, x) = u_1(t)\chi(t, x) + v_1(t, x), \tag{2.6}$$

where $\langle v_1(t, \cdot), \chi(t, \cdot)\rangle_{L^2} = 0$, for all $t \in I$.

   To guarantee that (2.5) has a solution, Fredholm's alternative asserts that the right hand side of (2.5) has to be orthogonal to $\chi(t, \cdot)$, for all $t \in I$. Taking the $L^2(\mathbb{R}^d)$ inner product of (2.5) with χ gives

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} + u_0\langle\partial_t\chi(t, \cdot), \chi(t, \cdot)\rangle_{L^2} = 0, \tag{2.7}$$

and thus (up to a multiplicative constant which we shall choose equal to 1 for simplicity), we find

$$u_0(t) = e^{-\beta(t)},$$

where

$$\beta(t) = \int_{t_0}^{t} \langle \partial_t \chi(s, \cdot), \chi(s, \cdot) \rangle_{L^2} \, \mathrm{d}s, \tag{2.8}$$

denotes the Berry phase term [2,3]. Note that $\beta(t) \in i\mathbb{R}$, for all $t \in I$, as one can easily see from differentiating the normalization condition $\langle \chi(t, \cdot), \chi(t, \cdot) \rangle_{L^2} = 1$. As a consequence, we also have that solutions to (2.7) satisfy

$$|u(t, x)| = |u(t_0, x)|, \quad \forall t \in I.$$

In summary, we find the well-known leading order approximation of linear time-adiabatic theory. Namely, that for $\varepsilon \to 0$ the solution ψ^{ε} behaves like

$$\psi^{\varepsilon}(t, x) \underset{\varepsilon \to 0}{\sim} \exp\left(-\frac{i}{\varepsilon} \int_{t_0}^{t} E(s) \, \mathrm{d}s - \beta(t) \right) \chi(t, x).$$

Remark 2.1. Concerning the significance of the Berry phase, we first note that the eigenvalue equation

$$H(t)\chi(t, \cdot) = E(t)\chi(t, \cdot),$$

does not uniquely determine the eigenfunction $\chi(t, \cdot)$, even if one imposes the normalization condition $\|\chi(t, \cdot)\|_{L^2} = 1$. One still has the freedom to change

$$\chi(t, \cdot) \to \tilde{\chi}(t, \cdot) = \chi(t, \cdot)e^{iS(t)}$$

with some (smooth) phase $S(t) \in \mathbb{R}$. Under such a gauge transformation, the Berry phase changes by

$$\beta(t) \to \tilde{\beta}(t) = \beta(t) + iS(t) - iS(t_0).$$

This allows to choose an appropriate S such that $\tilde{\beta} \equiv 0$, which is the reason why the Berry phase has historically been ignored for quite some time. However, in the case of a periodic Hamiltonian, i.e., $H(t_0) = H(t_0 + T)$ for some $T > 0$, one usually imposes (for physical reasons) the additional assumption that $\chi(t_0, \cdot) = \chi(t_0 + T, \cdot)$. In this case, the fact that $\tilde{\chi}$ should be single valued implies

$$S(t_0) - S(t_0 + T) = 2\pi n, \quad n \in \mathbb{Z}.$$

This shows that $\beta(t_0 + T)$ can only be changed by an integer multiple of $2\pi i$ and thus cannot be gauged away in general, as has been noted in [2,3]. Note that this problem remains, even in the case where $H(t)$ is real (as it is true in the present work) and thus $H(t)$ commutes with the operator of complex conjugation Θ. In this case, we infer that $[P(t), \Theta] = 0$, and hence there exists a real-valued eigenfunction $\chi(t, \cdot) \in \operatorname{ran} P(t)$ which consequently satisfies $\langle \partial_t \chi(t, \cdot), \chi(t, \cdot) \rangle_{L^2} \equiv 0$. This eigenfunction, however, does not necessarily satisfy the periodicity condition imposed before. In the present work, we do not want to exclude the possibility of a periodic $H(t)$ and thus refrain from gauging away the Berry phase. For a general discussion of the physical significance of geometric phases (of which the Berry phase is one particular example), we refer to [4,37].

With this in hand, it is possible to determine v_1 through (2.5). At least formally, this yields

$$v_1(t, x) = iL_E^{-1}(t)\Big(\dot{u}_0(t)\chi(t, x) + u_0(t)\partial_t\chi(t, x)\Big) = iL_E^{-1}(t)\Big(u_0(t)\partial_t\chi(t, x)\Big),$$

where we denote the partial inverse (or, partial resolvent) of $L_E(t)$ by

$$L_E^{-1}(t) := (1 - P(t))(H(t) - E(t))^{-1}(1 - P(t)), \tag{2.9}$$

with $P(t) = |\chi(t, \cdot)\rangle\langle\chi(t, \cdot)|$ being the projection onto the eigenspace corresponding to $E(t) \in \mathbb{R}$. Note that this also shows that initially $v_1(t_0, x) \neq 0$, in general.

The remaining unknown u_1 appearing in (2.6) can then be obtained by equating terms of order $O(\varepsilon^2)$. Indeed, by looking at the solvability condition for

$$L_E(t)U_2(t, x) = i\partial_t U_1 \equiv i\left(\dot{u}_1(t)\chi + u_1(t)\partial_t\chi(t, x)\right) + i\partial_t v_1,$$

one finds that $u_1(t)$ solves the following differential equation

$$\dot{u}_1 + \beta(t)u_1 + \langle\partial_t v_1(t, \cdot), \chi(t, \cdot)\rangle_{L^2} = 0.$$

Choosing, for simplicity, $u_1(t_0) = 0$, we get

$$u_1(t) = -e^{-\beta(t)}\int_{t_0}^t \langle\partial_t v_1(s, \cdot), \chi(s, \cdot)\rangle_{L^2}\, e^{\beta(s)}\,\mathrm{d}s.$$

By repeating these steps, one easily finds that all amplitudes $U_n(t, x)$, $n \geqslant 1$, appearing in (2.1), are of the form

$$U_n(t, x) = u_n(t)\chi(t, x) + v_n(t, x), \tag{2.10}$$

where every $u_n(t)$ is determined through an ordinary differential equation obtained from the solvability condition at order $O(\varepsilon^{n+1})$, together with the initial data $u_n(t_0) = 0$.

Next, we want to understand how to take into account a (sub-)critical nonlinearity in our asymptotic expansion. To this end, we first note that (2.1) yields

$$\varepsilon|\mathcal{U}^\varepsilon|^{2\sigma}\mathcal{U}^\varepsilon \sim \varepsilon|U_0|^{2\sigma}U_0 + \varepsilon^2\left((\sigma + 1)|U_0|^{2\sigma}U_1 + \varepsilon\sigma|U_0|^{2\sigma-2}U_0^2\overline{U}_1\right) + O(\varepsilon^3).$$

Thus, the leading order eigenvalue problem (2.2) does not change. The nonlinearity enters only in the expressions of order $O(\varepsilon)$ or higher. For the former, we find the following analog of (2.4):

$$L_E(t)U_1(t, x) = i\partial_t U_0(t, x) - \lambda|U_0|^{2\sigma}U_0. \tag{2.11}$$

Here, we can use our knowledge from before to make the following ansatz for U_0:

$$U_0(t, x) = \chi(t, x)e^{-\beta(t) - i\theta(t)}, \tag{2.12}$$

where $\beta(t)$ is defined in (2.8) and $\theta(t) \in \mathbb{R}$ is some other phase yet to be determined. By doing so, the solvability condition requiring that the right hand side of (2.11) has to be orthogonal to $\ker L_E(t)$ yields

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \lambda\langle|\chi(t, \cdot)|^{2\sigma}\chi(t, \cdot), \chi(t, \cdot)\rangle_{L^2} = \lambda\int_{\mathbb{R}^d} |\chi(t, x)|^{2\sigma+2}\,\mathrm{d}x,$$

where we have used the fact that $\beta(t) \in i\mathbb{R}$. Assuming, for the moment, that $\chi(t, \cdot) \in L^{2\sigma+2}(\mathbb{R}^d)$, we thus find

$$\theta(t) = \lambda \int_{t_0}^{t} \|\chi(s, \cdot)\|_{L^{2\sigma+2}}^{2\sigma+2} \, ds. \tag{2.13}$$

In view of (2.12), we see that the nonlinearity contributes at leading order by adding an additional nonlinear phase modulation, i.e.,

$$\psi^\varepsilon(t, x) \underset{\varepsilon \to 0}{\sim} \exp\left(-\frac{i}{\varepsilon} \int_{t_0}^{t} E(s) \, ds - i\lambda \int_{t_0}^{t} \|\chi(s, \cdot)\|_{L^{2\sigma+2}}^{2\sigma+2} \, ds - \beta(t) \right) \chi(t, x).$$

Note that even though the nonlinear phase modulation is slowly varying, one should still think of it as a small (i.e., of order $O(\varepsilon)$) nonlinear modification of the dynamical phase.

Remark 2.2. It is clear by now that the choice (1.5) is critical with respect to our asymptotic expansion. Indeed, if instead of (1.5) we set

$$\Psi^\varepsilon(t, x) = \varepsilon^{\alpha/(2\sigma)} \psi^\varepsilon(t, x),$$

then instead of (1.6) we would obtain

$$i\varepsilon \partial_t \psi^\varepsilon = -\frac{1}{2}\Delta\psi^\varepsilon + V(t, x)\psi^\varepsilon + \lambda\varepsilon^\alpha |\psi^\varepsilon|^{2\sigma}\psi^\varepsilon \tag{2.14}$$

Performing the same asymptotic expansion as before, we see that if $\alpha \geqslant 2$, then *no* nonlinear effects are present in the leading order asymptotics. The problem thus becomes essentially linearizable, and can be considered sub-critical with respect to our asymptotic analysis. A somewhat intermediate regime is obtained in the case where α is no longer a natural number and such that $1 < \alpha < 2$. This situation will be discussed in more detail in Sect. 5.1.3. Finally, if $0 \leqslant \alpha < 1$, the problem can be considered super-critical with respect to our asymptotic expansion. The case $\alpha = 0$ is probably the most relevant from the physics point of view, but clearly also mathematically much more challenging and thus beyond the scope of the present work. One can expect this problem to be intimately related to the modulation stability of nonlinear ground states studied in [35] (see also [28]).

3. A Mathematical Framework for Asymptotic Expansions

In this section, we will prove that the solution obtained through the formal multiple scales approximation above is indeed well defined and furnishes an approximate solution to (1.6). To this end, we shall impose the following basic assumptions on the time-dependent potential:

Assumption 1. *The potential $V(t, x)$ satisfies $V \in C_b^1(I; \mathcal{S}(\mathbb{R}^d))$, where $\mathcal{S}$ denotes the space of smooth and rapidly decaying functions.*

Remark 3.1. This assumption is mainly imposed for the sake of a simple and clean presentation but certainly far from optimal concerning the regularity of V. Indeed, all of our results can be reformulated for potentials $V(t, \cdot) \in C_b^k(\mathbb{R}^d)$

and vanishing at infinity. However, it turns out that at different stages of our proofs we require different (and relatively strong) bounds on k, which result in a somewhat tedious regularity count that we want to avoid.

Nest, we fix $t \in I$. Then, it is well known (see, e.g., [31, Chapter 10.1]) that for $V(t, \cdot)$ bounded, i.e., $k = 0$, and decaying at infinity, the Hamiltonian $H(t)$ is a self-adjoint operator with $\mathrm{dom}(H(t)) = H^2(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. Moreover, for any fixed $t \in I$, the spectrum of $H(t)$ is of the standard form, i.e.,

$$\mathrm{spec}(H(t)) = [0, \infty) \cup \{E_j(t) \mid -E_j(t) > 0, \quad j = 0, 1, \dots \}.$$

see, e.g., [13]. Of course as these eigenvalues vary in time, they might cross each other, or disappear into the continuous spectrum. Our main assumption is that the eigenvalue $E(t)$ we are interested in stays separated from the rest of the spectrum by a spectral gap.

Assumption 2. *There exists a simple eigenvalue $E(t) \in \mathrm{spec}(H(t))$ and a constant $\delta > 0$, satisfying*

$$\inf_{t \in I} \mathrm{dist}(E(t), \mathrm{spec}(H(t) \backslash E(t))) = \delta. \tag{3.1}$$

Note that this implies $E(t) \leqslant -\delta$, for all $t \in I$. Denoting by $\chi(t, \cdot) \in L^2(\mathbb{R}^d)$ a normalized eigenfunction corresponding to such a well-separated eigenvalue $E(t)$, we have the following regularity result.

Lemma 3.2. *Let Assumptions 1 and 2 hold, then we can choose $\chi : I \to H^k(\mathbb{R}^d)$, such that $\chi \in C_{\mathrm{b}}^1(I, H^k(\mathbb{R}^d))$ for any $k \geqslant 0$, and such that $\|\chi(t, \cdot)\|_{L^2} = 1$, for all $t \in I$.*

Proof. The proof follows from standard arguments. Indeed, we first notice that, for any fixed $t \in \mathbb{R}$, $\chi(t, \cdot)$ satisfies the Schrödinger eigenvalue problem

$$\left(-\frac{1}{2} \Delta + V(t, x) \right) \chi(t, x) = E(t) \chi(t, x),$$

which, in view of Assumption 1 and [13, Proposition 1.2], implies the asserted regularity in $H^k(\mathbb{R}^d)$ for any $k \geqslant 0$. Thus, it only remains to prove the differentiability property in time. This follows from the fact that as long as $E(t)$ stays separated from the rest of the spectrum, the associated orthogonal projector $P(t)$ can be expressed via Riesz' formula as

$$P(t) = \frac{i}{2\pi} \oint_{\Gamma(t)} (H(t) - z)^{-1} \, \mathrm{d}z,$$

where $\Gamma(t) \subset \mathbb{C}$ is a continuous (positively oriented) curve encircling $E(t)$ once, such that

$$\inf_{t \in I} \mathrm{dist}(E(t), \mathrm{spec}(H(t))) = \delta/2,$$

i.e., no other points within $\mathrm{spec}(H(t))$ are enclosed by $\Gamma(t)$. Using this, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} P(t) = \frac{i}{2\pi} \oint_{\Gamma(t)} \frac{\mathrm{d}}{\mathrm{d}t} (H(t) - z)^{-1} \, \mathrm{d}z,$$

whenever $(H(t) - z)^{-1} \in C_{\mathrm{b}}^1(I; \mathcal{L}(L^2(\mathbb{R}^d)))$. The latter is proved, for example, in [32, Lemma 2.4]. Hence, $t \mapsto P(t)$ is in $C_{\mathrm{b}}^1(I)$.

To obtain a corresponding eigenfunction $\chi \in C_{\mathrm{b}}^1(I; H^k(\mathbb{R}^d))$, we can follow the classical idea of Kato [15] (see also [16,17]), which starts from the definition of the following operators

$$K(t) := i[P(t), \dot{P}(t)],$$

and $A(t)$ given by

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t) = iK(t)A(t), \quad A(t_0) = \mathbb{I}.$$

Then, it is easily seen that $K(t)$ is self-adjoint and thus $A^*(t) = A^{-1}(t)$. In addition, one checks (after some calculations invoking the properties of projections)

$$\frac{\mathrm{d}}{\mathrm{d}t} (A^*(t)P(t)A(t)) = 0,$$

so that the intertwining property holds:

$$P(t) = A(t)P(t_0)A^*(t), \quad \forall t \in I. \tag{3.2}$$

We now choose $\chi(t_0)$ normalized such that $P(t_0)\chi(t_0, \cdot) = \chi(t_0, \cdot)$ and define $\chi(t, \cdot) = A(t)\chi(t_0, \cdot)$. In view of (3.2), this implies that $P(t)\chi(t, \cdot) = \chi(t, \cdot)$ for all $t \in I$. In addition, $\|\chi(t, \cdot)\|_{L^2} = 1$ and since $A \in C_{\mathrm{b}}^1(I; \mathcal{L}(L^2(\mathbb{R}^d)))$ we also get $\chi(t, \cdot) \in C_{\mathrm{b}}^1(I; H^k(\mathbb{R}^d))$. $\qquad \square$

By Sobolev imbedding, we also have $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, provided $k > \frac{d}{2}$. In particular, we have $\chi \in L^q(\mathbb{R}^d)$ for any $q \in [2, \infty]$ and thus the expression for the nonlinear phase modulation $\theta(t)$ given by (2.13) is well defined. Moreover, for $k > \frac{d}{2}$, the Sobolev space $H^k(\mathbb{R}^d)$ is in fact an algebra, i.e., if $f, g \in H^k(\mathbb{R}^d)$ then $fg \in H^k(\mathbb{R}^d)$. This can be used to prove the following regularity result:

Lemma 3.3. *Let $\sigma \in \mathbb{N}$, $\lambda \in \mathbb{R}$, and Assumptions 1 and 2 hold. Then, the expressions appearing in the asymptotic expansion (2.1) satisfy $U_n \in C_{\mathrm{b}}^1(I; H^k(\mathbb{R}^d))$ for all $n \in \mathbb{N}$ and $k \geqslant 0$.*

Proof. Each U_n is of the form given in (2.10), i.e., $U_n(t, x) = u_n(t)\chi(t, x) + v_n(t, x)$, with $v_0 \equiv 0$. In view of the asymptotic expansion above, we know that each $u_n(t)$ solves an ordinary differential equation of the form

$$\dot{u}_n + \beta(t)u_n + \langle \partial_t v_n(t, \cdot), \chi(t, \cdot) \rangle_{L^2}$$
$$= i\lambda \left\langle \frac{\mathrm{d}^n}{\mathrm{d}s^n} F \left(U_0 + \sum_{\ell=1}^n s^\ell U_\ell \right) \Big|_{s=0}, \chi(t, \cdot) \right\rangle_{L^2},$$

where we denote the nonlinearity by $F(z) = |z|^{2\sigma}z$, which for $\sigma \in \mathbb{N}$ is smooth. Note that due to the orthogonality of χ with every v_ℓ, the right hand side in fact only involves u_ℓ's. Moreover, we see that for $n \geqslant 1$, these differential equations are indeed linear. Together with the fact that $\frac{\mathrm{d}}{\mathrm{d}t}|u_0(t, x)|^2 = 0$ we infer that there is no restriction on the existence time of $u_n(t)$. In view with Lemma 3.2, we thus have $u_n\chi \in C_{\mathrm{b}}^1(I; H^k(\mathbb{R}^d))$ for all $k \geqslant 0$.

On the other hand, we know that all v_n, for $n \geqslant 1$, are determined by inverting an elliptic equation for any fixed $t \in I$, i.e.,

$$v_n(t,x) = L_E^{-1}(t)\left(i\partial_t U_{n-1}(t,x) + \lambda \frac{\mathrm{d}^{n-1}}{\mathrm{d}s^{n-1}} F\left(U_0 + \sum_{\ell=1}^{n-1} s^\ell U_\ell\right)\Big|_{s=0}\right). \quad (3.3)$$

The fact that $H^k(\mathbb{R}^d)$ for $k > \frac{d}{2}$ forms an algebra implies that the right hand side, which is a sum of products of U_ℓ's, is in $H^k(\mathbb{R}^d)$ for all $k \geqslant 0$ (see also the proof of Proposition 3.4 below). Since $L_E^{-1}(t) : L^2(\mathbb{R}^d) \to \mathrm{dom}(H(t)) = H^2(\mathbb{R}^d)$, in view of (2.9), the assertion follows by induction over n. □

With this result in hand, we set

$$\psi_N^\varepsilon(t,x) := e^{-i\varphi(t)/\varepsilon} \sum_{n=0}^{N} \varepsilon^n U_n(t,x), \quad (3.4)$$

where $\varphi(t)$ is the dynamic phase given by (2.3) and $N \in \mathbb{N}$. Note that at $t = t_0$, the U_n can in general not be chosen arbitrarily, since parts of it need to be determined recursively as given in (3.3). In particular, we have

$$\psi_N^\varepsilon(t_0,x) = \chi(t_0,x) + \varepsilon\gamma^\varepsilon(x)$$

where, due to the regularity of v_n, the corrector $\gamma^\varepsilon \in H^k(\mathbb{R}^d)$ of Theorem 1.1 is of the form

$$\gamma^\varepsilon(x) = \sum_{n=1}^{N} \varepsilon^{n-1} v_n(t_0,x), \quad (3.5)$$

with $v_n(t_0,x)$ as above. This definition of ψ_N^ε then yields an *approximate solution* of the nonlinear Schrödinger equation (1.6) in the following sense:

Proposition 3.4. *Let $\sigma \in \mathbb{N}$, $\lambda \in \mathbb{R}$, and Assumptions 1 and 2 hold. Then, ψ_N^ε defined by (3.4) satisfies $\psi_N \in C_b^1(I; H^k(\mathbb{R}^d))$ for all $k \geqslant 0$ and*

$$i\varepsilon\partial_t\psi_N^\varepsilon + H(t)\psi_N^\varepsilon + \lambda\varepsilon|\psi_N^\varepsilon|^{2\sigma}\psi_N^\varepsilon = r_N^\varepsilon(t,x),$$

where the remainder is bounded by

$$\sup_{t \in I} \|r_N^\varepsilon(t,\cdot)\|_{H^k(\mathbb{R}^d)} \leqslant C\varepsilon^{N+1}.$$

Proof. By plugging ψ_N^ε into the nonlinear Schrödinger equations, the asymptotic expansion above shows that

$$r_N^\varepsilon(t,x) = \varepsilon^{N+1} e^{-i\varphi(t)}\left(i\partial_t U_N(t,x) + \lambda\widetilde{r}_N^\varepsilon(t,x)\right),$$

where

$$\widetilde{r}_N^\varepsilon = \sum_{j=N}^{(2\sigma+1)N} \varepsilon^{j-N} \sum_{\ell_1+\cdots+\ell_\sigma+m_1+\cdots+m_\sigma+r=j} U_{\ell_1}\ldots U_{\ell_\sigma}\bar{U}_{m_1}\ldots\bar{U}_{m_\sigma}U_r.$$

In view of the regularity result established in Lemma 3.3, and the algebra property of $H^k(\mathbb{R}^d)$, for $k > d/2$, we directly obtain the estimate on the remainder stated above. □

    This result, however, is not sufficient to conclude that the exact solution ψ^ε will stay close to the approximate solution ψ_N^ε for times of order $O(1)$. We shall show in the next section that this is indeed the case.

4. Nonlinear Stability of the Approximation

4.1. Preliminaries

Before we can prove stability of our asymptotic expansion, we need a basic (local in-time) existence result for solutions to nonlinear Schrödinger equations of the form (1.6).

Lemma 4.1. *Let $\sigma \in \mathbb{N}$, $\lambda \in \mathbb{R}$, and $\mathbb{N} \ni k > \frac{d}{2}$. Moreover, let $\psi_{\mathrm{in}}^\varepsilon \in H^k(\mathbb{R}^d)$ and the potential satisfies $V \in C_\mathrm{b}^1(I; \mathcal{S}(\mathbb{R}^d))$. Then, there exist $T_1^\varepsilon, T_2^\varepsilon > 0$, and a unique solution $\psi^\varepsilon \in C\big([t_0 - T_1^\varepsilon, t_0 + T_2^\varepsilon]; H^k(\mathbb{R}^d)\big)$ to (1.6). Furthermore,*

$$\|\psi^\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^d)} = \|\psi_{\mathrm{in}}^\varepsilon(t_0, \cdot)\|_{L^2(\mathbb{R}^d)}, \quad \forall t \in [t_0 - T_1^\varepsilon, t_0 + T_2^\varepsilon] \subset I.$$

Proof. The proof is a straightforward extension of the one given in, e.g., [30, Proposition 3.8] for the case without potential. We rewrite the NLS using Duhamel's principle

$$\psi^\varepsilon(t, \cdot) = e^{-it\frac{\Delta}{2\varepsilon}} \psi_{\mathrm{in}}^\varepsilon - i \int_{t_0}^t e^{i(s-t)\frac{\Delta}{2\varepsilon}} \left(\lambda |\psi^\varepsilon(s, \cdot)|^{2\sigma} + \frac{1}{\varepsilon} V(s, \cdot) \right) \psi^\varepsilon(s, \cdot) \, \mathrm{d}s$$

$$=: \Xi(\psi^\varepsilon)(t).$$

Clearly, the free Schrödinger group $e^{-it\frac{\Delta}{2\varepsilon}}$ is an isometry on $H^k(\mathbb{R}^d)$ for any $k \in \mathbb{R}$, and our assumptions on V guarantee that there is a constant $C = C(k, d) > 0$ such that

$$\|V\psi^\varepsilon\|_{H^k} \leqslant \sum_{|\alpha| \leqslant k} \|\partial^\alpha V\|_{L^\infty} \|\psi^\varepsilon\|_{H^{k-\alpha}} \leqslant C \|V\|_{C_\mathrm{b}^k} \|\psi^\varepsilon\|_{H^k} < \infty.$$

Moreover, for $\sigma \in \mathbb{N}$, the nonlinearity $F(z) = |z|^{2\sigma} z$ is smooth which, together with the fact that $H^k(\mathbb{R}^d)$ for $k > \frac{d}{2}$ forms an algebra, allows us to estimate

$$\|\psi^\varepsilon(t, \cdot)\|_{H^k} \leqslant \|\psi_{\mathrm{in}}^\varepsilon\|_{H^k} + C^\varepsilon \int_{t_0}^t \|\psi^\varepsilon(s, \cdot)\|_{H^k}^{2\sigma+1} + \|\psi^\varepsilon(s, \cdot)\|_{H^k} \, \mathrm{d}s, \qquad (4.1)$$

where $C^\varepsilon = C(k, d, \lambda, V, \varepsilon) > 0$.

    Now, denote by $X := C(([t_0 - T_1^\varepsilon, t_0 + T_2^\varepsilon]; H^k(\mathbb{R}^d))$ for some $T_1^\varepsilon, T_2^\varepsilon > 0$ to be chosen later on and $s > \frac{d}{2}$. Further, let $R > 1$ be such that $\|\psi_{\mathrm{in}}^\varepsilon\|_{H^k} \leqslant R$. Then, we can show that the $u \mapsto \Xi(u)$ maps the ball $B_{2R}(0) \subset X$ into itself. Indeed, the estimate (4.1) implies

$$\begin{aligned}
\|\Xi(u)\|_X &\leqslant \|\psi_{\mathrm{in}}^\varepsilon\|_{H^k} + C^\varepsilon \max(T_1^\varepsilon, T_2^\varepsilon)\big(\|u\|_X^{2\sigma+1} + \|u\|_X\big) \\
&\leqslant R + C^\varepsilon \max(T_1^\varepsilon, T_2^\varepsilon)\big((2R)^{2\sigma+1} + 2R\big) \\
&\leqslant R + 2^{2\sigma+2} C^\varepsilon \max(T_1^\varepsilon, T_2^\varepsilon) R^{2\sigma+1}.
\end{aligned}$$

Hence, we can choose $T_1^\varepsilon, T_2^\varepsilon \leqslant \frac{R^{-2\sigma}}{2^{2\sigma+2}C^\varepsilon}$ and such that $[t_0 - T_1^\varepsilon, t_0 + T_2^\varepsilon] \subset I$. The same type of estimate shows that $u \mapsto \Xi(u)$ is a contraction on $B_{2R}(0) \subset X$

and hence there exists a unique fixed point $u = \psi^\varepsilon \in X$. The conservation of the L^2-norm of the solution then follows from the fact that $H(t)$ is self-adjoint and the nonlinearity is gauge invariant. $\qquad\square$

Remark 4.2. By carefully tracking the ε-dependence of $T^\varepsilon_{1,2}$, one finds that, in general, $T^\varepsilon_{1,2}$ will go to zero, as $\varepsilon \to 0$. However, the stability proof below actually shows that for our choice of initial data, one can find $T^\varepsilon_{1,2} > 0$ independent of ε.

We will also need the following Moser-type lemma, proved in, e.g., [22].

Lemma 4.3. *Let* $R > 0$, $s \in \mathbb{N}$ *and* $F(z) = |z|^{2\sigma} z$, *with* $\sigma \in \mathbb{N}$. *Then, there exists* $K = K(R, s, \sigma)$ *such that if* w *satisfies*

$$\left\| \partial^\beta w \right\|_{L^\infty(\mathbb{R}^d)} \leqslant R, \quad |\beta| \leqslant s,$$

and η *satisfies* $\|\eta\|_{L^\infty(\mathbb{R}^d)} \leqslant R$, *then*

$$\sum_{|\beta| \leqslant s} \left\| \partial^\beta \big(F(w + \eta) - F(w) \big) \right\|_{L^2(\mathbb{R}^d)} \leqslant K \sum_{|\beta| \leqslant s} \left\| \partial^\beta \eta \right\|_{L^2(\mathbb{R}^d)}.$$

In [22], this lemma was proved for ε-scaled derivatives. For our purposes, we can set $\varepsilon = 1$ but note that the estimate above is linear in the H^s norm of η, which subsequently allows the use of Grownwall's lemma (see the proof of Proposition 4.4 below).

4.2. Nonlinear Stability

We are now in the position to prove the desired stability result for the asymptotic expansion obtained above.

Proposition 4.4. *Let* $\sigma \in \mathbb{N}$, $\lambda \in \mathbb{R}$, $\mathbb{N} \ni k > \frac{d}{2}$, *and Assumptions* 1 *and* 2 *hold. Given an approximate solution* ψ^ε_N *of the form* (3.4) *with* $N > k$, *we assume that, at* $t = t_0$, *the initial data* $\psi^\varepsilon_{\text{in}} \in H^k(\mathbb{R}^d)$ *is such that*

$$\|\psi^\varepsilon_{\text{in}} - \psi^\varepsilon_{N-1}(t_0, \cdot)\|_{H^k(\mathbb{R}^d)} \leqslant C_0 \varepsilon^N.$$

Then, for any compact time interval $J \subset I$ *containing* t_0, *there exist an* $\varepsilon_0(J) > 0$, *and a constant* $C > 0$, *such that for any* $0 < \varepsilon \leqslant \varepsilon_0(J)$ *the unique solution* $\psi^\varepsilon \in C(J; H^k(\mathbb{R}^d))$ *to* (1.6) *exists and, in addition,*

$$\sup_{t \in J} \|\psi^\varepsilon(t, \cdot) - \psi^\varepsilon_{N-1}(t, \cdot)\|_{H^k(\mathbb{R}^d)} \leqslant C \varepsilon^{N-k}.$$

Note that this result in particular implies that the solution ψ^ε to (1.6) cannot exhibit blow-up on any finite time interval $J \subset I \subseteq \mathbb{R}$.

Proof. Let $J = [t_0 - T_1, t_0 + T_2] \subset I$, for some $T_1, T_2 > 0$ independent of ε. From Lemma 4.1 we obtain the existence of a unique solution $\psi^\varepsilon \in C([t_0 - T^\varepsilon_1, t_0 + T^\varepsilon_2], H^k(\mathbb{R}^d))$ with $k > \frac{d}{2}$, to (1.6). We denote the difference between the exact and the approximate solution by

$$\eta^\varepsilon := \psi^\varepsilon - \psi^\varepsilon_N.$$

Proposition 3.4 then implies that $\eta^\varepsilon \in C([t_0 - \tau^\varepsilon_1, t_0 + \tau^\varepsilon_2], H^k(\mathbb{R}^d))$, where $\tau^\varepsilon_j = \min(T^\varepsilon_j, T_j)$, with $j = 1, 2$. We prove that for ε sufficiently small, η^ε

may be extended up to the time interval $J \subset I$, with $\eta^\varepsilon \in C(J, H^k(\mathbb{R}^d))$. For simplicity, we shall only show the argument for the times bigger than t_0. A similar argument applies on the time interval $[t_0 - T_1, t_0]$.

Take $\varepsilon_0 > 0$ so that $C_0 \varepsilon_0 \leqslant \frac{1}{2}$, and for $\varepsilon \in]0, \varepsilon_0]$, let

$$t^\varepsilon := \sup \left\{ t \geqslant t_0 \mid \sup_{t' \in [t_0, t]} \|\eta^\varepsilon(t')\|_{H^k(\mathbb{R}^d)} \leqslant 1 \right\}.$$

We already know that $t^\varepsilon > 0$ by the local existence result for ψ^ε. By possibly reducing $\varepsilon_0 > 0$ even further, we shall show that $t^\varepsilon \geqslant t_0 + T_2$. The error satisfies

$$i\varepsilon \partial_t \eta^\varepsilon = H(t)\eta^\varepsilon + \lambda \varepsilon \left(|\psi_N^\varepsilon + \eta^\varepsilon|^{2\sigma}(\psi_N^\varepsilon + \eta^\varepsilon) - |\psi_N^\varepsilon|^{2\sigma}\psi_N^\varepsilon \right) + r_N^\varepsilon, \qquad (4.2)$$

subject to $\eta^\varepsilon_{|t=t_0} = \eta^\varepsilon_{\text{in}}(x)$, where $\|\eta^\varepsilon_{\text{in}}\|_{H^k} = O(\varepsilon^{N+1})$ by assumption.

Next, we multiply (4.2) by $\overline{\eta}^\varepsilon$, integrate over $\mathbb{R}^d$, and take the real part of the resulting expression. Since $H(t)$ is self-adjoint, this yields

$$\partial_t \|\eta^\varepsilon\|_{L^2} \lesssim \left\| |\psi_N^\varepsilon + \eta^\varepsilon|^{2\sigma}(\psi_N^\varepsilon + \eta^\varepsilon) - |\psi_N^\varepsilon|^{2\sigma}\psi_N^\varepsilon \right\|_{L^2} + \frac{1}{\varepsilon}\|r_N^\varepsilon\|_{L^2}.$$

In view of Proposition 3.4, we have $\|r_N^\varepsilon\|_{H^k} = O(\varepsilon^{N+1})$. On the other hand, for $k > \frac{d}{2}$ the Gagliardo–Nirenberg inequality implies

$$\|\eta^\varepsilon\|_{L^\infty} \lesssim \|\eta^\varepsilon\|_{H^k} \lesssim 1 \quad \forall t \in [t_0, t^\varepsilon],$$

and we will show, that in fact $\|\eta^\varepsilon\|_{L^\infty}$ is (asymptotically) small. To this end, we first recall that since $\psi_N(t, \cdot) \in H^m(\mathbb{R}^d)$ for all $m \geqslant 0$ we also have that $\psi_N(t, \cdot) \in W^{s,\infty}(\mathbb{R}^d)$, for all $s \leqslant k$. Applying Lemma 4.3 with $s = 0$, we consequently obtain

$$\partial_t \|\eta^\varepsilon\|_{L^2} \leqslant K\|\eta^\varepsilon\|_{L^2} + C\varepsilon^N,$$

for $t \in [t_0, t^\varepsilon]$ and, using Grownwall's lemma, we thus find

$$\|\eta^\varepsilon\|_{L^2} \leq C_1 \varepsilon^N, \quad \forall t \in [t_0, t^\varepsilon]. \qquad (4.3)$$

The idea is now to obtain a similar estimate for (weak) derivatives of η^ε, to close the argument in $H^k(\mathbb{R}^d)$. To this end, we first note that

$$i\varepsilon \partial_t (\nabla \eta^\varepsilon) = H(t)(\nabla \eta^\varepsilon) + [\nabla, H(t)]\eta^\varepsilon + \lambda \varepsilon \nabla \left(F(\psi_N^\varepsilon + \eta^\varepsilon) - F(\psi_N^\varepsilon) \right) + \nabla r_N^\varepsilon,$$

and the same type of argument as before, together with the Cauchy Schwarz inequality, yields

$$\partial_t \|\nabla \eta^\varepsilon\|_{L^2} \lesssim \|\nabla \left(F(\psi_N^\varepsilon + \eta^\varepsilon) - F(\psi_N^\varepsilon) \right)\|_{L^2} + \frac{1}{\varepsilon}\|[\nabla, H(t)]\eta^\varepsilon\|_{L^2} + \frac{1}{\varepsilon}\|\nabla r_N^\varepsilon\|_{L^2}.$$

Now, $[\nabla, H(t)] = \nabla V(t, x)$, which is bounded by assumption, and so

$$\partial_t \|\nabla \eta_N^\varepsilon\|_{L^2} \lesssim \|\nabla \left(F(\psi_N^\varepsilon + \eta^\varepsilon) - F(\psi_N^\varepsilon) \right)\|_{L^2} + \frac{1}{\varepsilon}\|\eta^\varepsilon\|_{L^2} + C\varepsilon^N.$$

Invoking again Lemma 4.3 with $s = 1$, and the bound (4.3), we infer that $\forall t \in [t_0, t^\varepsilon]$ it holds

$$\partial_t \|\nabla \eta^\varepsilon\|_{L^2} \lesssim \|\nabla \eta^\varepsilon\|_{L^2} + \varepsilon^{N-1},$$

and Grownwall's lemma, together with (4.3), then yields

$$\|\eta^\varepsilon\|_{H^1} \lesssim \varepsilon^{N-1}, \quad \forall t \in [t_0, t^\varepsilon],$$

Iterating in $s \leqslant k$, we obtain, more generally

$$\|\eta^\varepsilon\|_{H^k} \lesssim \varepsilon^{N-k}, \quad \forall t \in [t_0, t^\varepsilon]. \tag{4.4}$$

and the Gagliardo–Nirenberg inequality consequently implies

$$\|\eta^\varepsilon\|_{L^\infty} \lesssim \|\eta^\varepsilon\|_{H^k} \lesssim \varepsilon^{N-k} \quad \forall t \in [t_0, t^\varepsilon],$$

provided $k > \frac{d}{2}$. For $N - k > 0$, continuity of $\|\eta^\varepsilon(\cdot, t)\|_{H^k}$ implies that $t^\varepsilon \geqslant t_0 + T_2$, for $\varepsilon \leqslant \varepsilon_0(T_2)$ sufficiently small, since if $t^\varepsilon < t_0 + T_2$ the estimate

$$\sup_{t \in [0, t^\varepsilon]} \|\eta^\varepsilon\|_{L^\infty} \lesssim \|\eta^\varepsilon\|_{H^k} \lesssim \varepsilon^{N-k} < \frac{1}{2},$$

(after possibly reducing ε_0) shows that η^ε can be continued beyond t^ε, a contradiction. In particular, we obtain that η^ε, and hence ψ^ε, is well defined for all $t \in [t_0, t_0 + T_2]$, thus showing $T_2^\varepsilon \geqslant T_2$. Since the same argument can be applied for times smaller than t_0, we finally conclude that ψ^ε is well defined for all $t \in J = [t_0 - T_1, t_0 + T_2] \subset I$ and $0 < \varepsilon \leqslant \varepsilon_0(J)$.

To complete the proof of the theorem, we note that (4.4) implies

$$\sup_{t \in J} \|\psi^\varepsilon - \psi_N^\varepsilon\|_{H^k} \lesssim \varepsilon^{N-k},$$

and since $N > k$, we also have

$$\sup_{t \in J} \|\psi_N^\varepsilon - \psi_{N-1}^\varepsilon\|_{H^k} \lesssim \varepsilon^N = o(\varepsilon^{N-k}).$$

Thus, we can use the triangle inequality to replace ψ_N^ε with ψ_{N-1}^ε in our estimate, which yields the desired result. □

Proposition 4.4 directly implies the result stated in Theorem 1.1. Due to our method of proof, Proposition 4.4 yields a loss in accuracy for the obtained error estimates, which we are unable to overcome at this point.

5. Possible Extensions and Variations

5.1. Remarks on Closely Related Cases

In this section, we collect several remarks on how to extend Theorem 1.1 to other, closely related, situations.

5.1.1. Degenerate Eigenvalues. The results above readily generalize to the case of an M-fold degenerate eigenvalue $E(t)$, satisfying the gap condition (3.1). In this case, there exists a smooth basis $\chi_\ell(t, \cdot) \in L^2(\mathbb{R}^d)$, where $\ell = 1, \ldots, L$, of the associated eigenspace. The associated projection onto the eigenspace corresponding to $E(t)$ then becomes

$$P(t) = \sum_{\ell=1}^{L} |\chi_\ell(t, \cdot)\rangle \langle \chi_\ell(t, \cdot)|.$$

Using this, one can proceed along the same lines as above to obtain that

$$\sup_{t \in \tilde{I}} \left\| \psi^\varepsilon(t, \cdot) - e^{i\varphi(t)/\varepsilon} \sum_{\ell=1}^{L} u_{0,\ell}(t) \chi_\ell(t, \cdot) \right\|_{H^k} \leqslant C\varepsilon.$$

However, the formulas in general become more complicated, since the coefficient functions, $u_{\ell,0}(t)$, are now determined by an $L \times L$ system of ordinary differential equations (leading to matrix-valued Berry phases and analogous nonlinear phase modulations). This consequently leads to rather tedious computations in the subsequent steps of our asymptotic expansion, which is why we shall not go into further details here. In addition, the global (on I) existence of the solutions to the $L \times L$ system is not clear, a priori. However, Proposition 4.4 applies on any interval $\tilde{I} \subseteq I$ on which the u_ℓ's are defined.

5.1.2. Quadratic Potentials. In view of a possible application to Bose–Einstein condensates, the assumption that $V(t,x)$ vanishes as $|x| \to \infty$ seems unrealistic, since one typically considers trapping potentials of the form

$$V(t,x) = \sum_{j=1}^{d} \Omega_j(t) x_j^2, \quad \Omega(t) \in \mathbb{R},$$

i.e., a time-dependent harmonic oscillator. There is, however, no fundamental difficulty in extending our result to such a situation. Indeed, as long as $\Omega_j(t) > 0$, the existence of eigenvalues $E(t)$ together with their associated smooth (and rapidly decaying) eigenfunctions is guaranteed (see, e.g., [31]), and the asymptotic expansion stays (at least formally) exactly the same as before. Only from the point of view of rigorous estimates, one needs to shift from the usual Sobolev space setting $H^k(\mathbb{R}^d)$, to weighted spaces of the form

$$\Sigma^k = H^k(\mathbb{R}^d) \cap \{|x|^k f \in L^2(\mathbb{R}^d)\}.$$

The basic existence and well-posedness theory for NLS in such weighted spaces has been established in [6], yielding a unique solution $\psi^\varepsilon \in C(\tilde{I}; \Sigma^k)$ on some $\tilde{I} \subseteq I$, provided $\psi_{\mathrm{in}} \in \Sigma^k$. Moreover, an extension of the Schauder Lemma 4.3 to Σ^k is straightforward. The only extra work needed is in the proof of the nonlinear stability, where now $[\nabla, H(t)] = \nabla V(t,x)$ is no longer bounded. However, since $\|\nabla V \eta^\varepsilon\|_{L^2} \simeq \|x \eta^\varepsilon\|_{L^2}$, and since $[x, H(t)] = \nabla$, a closed set of estimates for the *combined L^2-norms* of $x\eta^\varepsilon$ and $\nabla \eta^\varepsilon$ (and thus for the Σ^1-norm of η^ε) can be obtained, cf. [8] for more details. Iterating this then yields a stability result in Σ^k.

5.1.3. The Intermediate Regime $1 < \alpha < 2$. We go back to the discussion started in Remark 2.2 and consider the slightly more general situation of

$$i\varepsilon \partial_t \psi^\varepsilon = -\frac{1}{2}\Delta \psi^\varepsilon + V(t,x)\psi^\varepsilon + \lambda \varepsilon^\alpha |\psi^\varepsilon|^{2\sigma} \psi^\varepsilon, \quad \alpha \geqslant 1.$$

We already know that if $\alpha = 1$ the problem is critical, and that if $\alpha \geqslant 2$, the problem is sub-critical (i.e., linearizable). The intermediate regime $1 < \alpha < 2$, however, is slightly more complicated, since the asymptotic expansion used before fails to match the size of the nonlinearity. One way to overcome this problem is to include the nonlinearity in the equation of order $O(\varepsilon)$, which yields

$$L_E(t) U_1(t,x) = i\partial_t U_0(t,x) + \varepsilon^{\alpha-1} \lambda |U_0|^{2\sigma} U_0,$$

instead of (2.11). For $\varepsilon^{\alpha-1} \ll 1$, this can be seen as a regular perturbation problem of the associated linear situation. The corresponding solvability condition now yields an ε-dependent leading order amplitude of the form

$$U_0^\varepsilon(t,x) = \chi(t,x)e^{-\beta(t)+i\varepsilon^{\alpha-1}\theta(t)},$$

with $\theta(t)$ given by (2.12). The nonlinear phase modulation appearing in this expression is obviously rather weak, due to the small $\varepsilon^{\alpha-1} \ll 1$ factor in front. The price to pay is that now all the U_n^ε become ε-dependent. An alternative approach would be to consider a modified asymptotic expansion, which includes powers of ε^α. This can be done in principle, but is rather cumbersome and we will leave the details to the reader.

5.2. Connection to Nonlinear Bound States

We finally turn to the connection with nonlinear bound states and thus, to the proof of Corollary 1.3. To this end, we first recall that, for fixed $t \in \mathbb{R}$, nonlinear bound states are solutions to the stationary Schrödinger equation

$$-\frac{1}{2}\Delta\Phi + V(t,x)\Phi + \lambda|\Phi|^{2\sigma}\Phi = E_*\Phi. \tag{5.1}$$

Denoting by $E(t) < 0$ a simple eigenvalue of the linear Hamiltonian $H(t)$, standard bifurcation theory (cf. [18]) then ensures the existence of such nonlinear bound states Φ bifurcating from the zero solution at the eigenvalues $E(t)$. Depending on the type of eigenvalue $E(t)$, these bound states are called nonlinear ground states, or nonlinear excited states, respectively (see, e.g., [29,33]). To be more precise, we recall the following result:

Lemma 5.1. *Let $t \in I$ be fixed, $V(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)$ and denote by $E(t) < 0$ a simple eigenvalue separated from the rest of $\mathrm{spec}\,(H(t))$. For $\lambda > 0$ let $E_*(t) \in (E(t), 0)$ and for $\lambda < 0$, let $E_*(t) < E(t)$. Then, there exists a solution $(E_*(t), \Phi(t,\cdot))$ to (5.1), such that $E_*(t) \mapsto \|\Phi(t,\cdot)\|_{H^2}$ is smooth for $E_* \neq E$ and*

$$\lim_{E_* \to E} \|\Phi(t,\cdot)\|_{H^2} = 0.$$

In addition, for $\varepsilon_1 < 1$ sufficiently small and $\frac{1}{\lambda}(E_* - E) < \varepsilon_1$, we have

$$\left\|\Phi(t,\cdot) - \left(\frac{E_*(t)-E(t)}{\mu(t)}\right)^{\frac{1}{2\sigma}}\chi(t,\cdot)\right\|_{H^2(\mathbb{R}^d)} = O(E_*(t) - E(t)),$$

where $\chi(t,\cdot) \in L^2(\mathbb{R}^3)$ is a normalized eigenfunction associated to the linear eigenvalue $E(t)$ and

$$\mu(t) := \lambda\|\chi(t,\cdot)\|_{L^{2\sigma+2}}^{2\sigma+2}.$$

This result (for time-independent potentials $V = V(x)$) is stated in [28, Theorem 2.1] in the case where the linear Hamiltonian has exactly one eigenvalue $E < 0$. The formula for Φ is a result of a perturbation calculation and already stated in [23]. A more detailed proof in the case where H admits exactly two simple eigenvalues, $d = 3$, $\sigma = 1$, and $\lambda > 0$, can be found in [34,

Lemma 2.1]. The situation for several linear eigenvalues is discussed in, e.g., [33], while [12] studies the case of degenerate eigenvalues.

Remark 5.2. One should note that in several of the aforementioned works (cf. [29,33,34]), the additional assumption that 0 is neither an eigenvalue nor a resonance for the linear Hamiltonian $H = -\Delta + V$ is imposed. This condition, however, is not used in the existence proof of nonlinear bound states (it ensures the applicability of certain dispersive estimates for the associated Schrödinger group [20]).

Hence, for any fixed $t \in I$, there is an $\varepsilon_1(t) > 0$, such that as long as $\frac{1}{\lambda}(E_*(t) - E(t)) < \varepsilon_1(t)$, we have a nonlinear bound state of size

$$M(t) := \|\Phi(t,\cdot)\|_{L^2} = O(\varepsilon_1^{1/(2\sigma)}).$$

Unfortunately, Lemma 5.1 does not provide any assertion on the time dependence of $t \mapsto M(t)$. In fact, it has been shown in [11], that for a given compact time interval $J \subset I$, and $M > 0$ sufficiently small there exists a unique family of bound states $t \mapsto \Phi(t,\cdot) \in H^2(\mathbb{R}^d))$ continuously depending on time, and with constant mass $\|\Phi(t,\cdot)\|_{L^2}^2 = M$. However, since [11] is currently only published in a PhD thesis (the corresponding paper [10] is being finalized), we shall not take this result for granted but only assume that such a property holds. More precisely, we impose:

Assumption 3. *Let $J \subset I$ be a compact time interval and assume that there is an $M_0(J) > 0$ such that for all $M \leqslant M_0$, there exists a unique family of solutions $t \mapsto \Phi(t,x)$ to (5.1) with $\Phi \in C(J; H^2(\mathbb{R}^d))$ and $\|\Phi(t,\cdot)\|_{L^2}^2 = M$.*

This allows us to prove the following result, in which $\varepsilon_0(J) > 0$ denotes the same constant as in Theorem 1.1.

Proposition 5.3. *Let $\sigma \in \mathbb{N}$, $\lambda \in \mathbb{R}$, and Assumptions 1, 2 and 3 hold. Moreover, let $\varepsilon \in (0, \min(\varepsilon_0(J), M_0(J))]$ and denote by $\Phi^\varepsilon(t,x)$ a family of bound states such that $\|\Phi^\varepsilon(t,\cdot)\|_{L^2} = \sqrt{\varepsilon}$. Finally, let Ψ^ε be the solution to (1.4) with initial data $\Psi_{\mathrm{in}}^\varepsilon = \sqrt{\varepsilon}\psi_{\mathrm{in}}^\varepsilon$ where $\psi_{\mathrm{in}}^\varepsilon$ satisfies the conditions of Theorem 1.1. Then, there exists a constant $K > 0$ such that*

$$\sup_{t \in J} \left\| \Psi^\varepsilon(t,\cdot) - \Phi^\varepsilon(t,\cdot)e^{-i\varphi^\varepsilon(t)} \right\|_{L^2(\mathbb{R}^3)} \leqslant K\varepsilon^{\min(3/2,\sigma)},$$

where φ^ε is given by

$$\varphi^\varepsilon(t) = \frac{1}{\varepsilon}\int_{t_0}^t E(s)\mathrm{d}s + \lambda\int_{t_0}^t \|\chi(s,\cdot)\|_{L^{2\sigma+2}}^{2\sigma+2}\,\mathrm{d}s - i\beta(t).$$

Proof. Using the triangle inequality, we obtain

$$\left\| \Psi^\varepsilon(t,\cdot) - \Phi^\varepsilon(t,\cdot)e^{-i\varphi^\varepsilon(t)} \right\|_{L^2} \leqslant \left\| \Psi^\varepsilon(t,\cdot) - \sqrt{\varepsilon}\chi(t,\cdot)e^{-i\varphi^\varepsilon(t)} \right\|_{L^2}$$
$$+ \left\| \sqrt{\varepsilon}\chi(t,\cdot) - \Phi^\varepsilon(t,\cdot) \right\|_{L^2}$$

where in the second term on the right hand side we have used the fact that φ^ε is purely time dependent. In view of Theorem 1.1, the first term on the right hand side is $O(\varepsilon^{3/2})$, uniformly for $t \in J$, where J is any compact time interval

$J \subset I$. To estimate the second term, we use the triangle inequality once more to obtain

$$\left\| \sqrt{\varepsilon}\chi(t,\cdot) - \Phi^\varepsilon(t,\cdot) \right\|_{L^2} \leqslant \left\| \left(\frac{E_*(t) - E(t)}{\mu(t)} \right)^{\frac{1}{2\sigma}} \chi(t,\cdot) - \Phi^\varepsilon(t,\cdot) \right\|_{L^2}$$
$$+ \left| \left(\frac{E_*(t) - E(t)}{\mu(t)} \right)^{\frac{1}{2\sigma}} - \sqrt{\varepsilon} \right|,$$

since $\|\chi(t,\cdot)\|_{L^2} = 1$, by assumption. From Lemma 5.1, we find that for all $t \in J$:

$$\left| \|\Phi^\varepsilon\|_{L^2} - \left(\frac{E_*(t) - E(t)}{\mu(t)} \right)^{\frac{1}{2\sigma}} \right| \leqslant O(E_*(t) - E(t)),$$

which together with the fact that $\|\Phi^\varepsilon(t,\cdot)\|_{L^2} = \sqrt{\varepsilon}$, implies

$$\varepsilon = \left(\frac{E_*(t) - E(t)}{\mu(t)} \right)^{\frac{1}{\sigma}} + O\big((E_*(t) - E(t))^{1 + \frac{1}{2\sigma}} \big). \qquad (5.2)$$

In particular (5.2) together with Lemma 5.1 implies

$$\left\| \left(\frac{E_*(t) - E(t)}{\mu(t)} \right)^{\frac{1}{2\sigma}} \chi(t,\cdot) - \Phi^\varepsilon(t,\cdot) \right\|_{L^2} \lesssim \varepsilon^\sigma.$$

On the other hand, using $\sqrt{a} - \sqrt{b} = (a - b)/(\sqrt{a} + \sqrt{b})$, we also have

$$\left| \left(\frac{E_*(t) - E(t)}{\mu(t)} \right)^{\frac{1}{2\sigma}} - \sqrt{\varepsilon} \right| \lesssim \varepsilon^\sigma.$$

In summary, we find that for $t \in J$ it holds

$$\left\| \Psi^\varepsilon(t,\cdot) - \Phi(t,\cdot)e^{-i\varphi^\varepsilon(t)} \right\|_{L^2} \lesssim \varepsilon^{3/2} + \varepsilon^\sigma \lesssim \varepsilon^{\min(3/2,\sigma)},$$

which yields the desired result. $\square$

Remark 5.4. Interestingly, the proof shows that in the asymptotic regime considered and in the case $\sigma = 1$ (cubic nonlinearity), the linear eigenfunction (endowed with the appropriate phase factor) satisfies a better approximation estimate than the nonlinear one. However, for $\sigma \geqslant 2$, the approximation rate is the same for both the linear and the nonlinear subspace.

Proof of Corollary 1.3. To estimate the operator norm of

$$|\Psi^\varepsilon\rangle\langle\Psi^\varepsilon| - |\Phi^\varepsilon\rangle\langle\Phi^\varepsilon| = |\Psi^\varepsilon\rangle\langle\Psi^\varepsilon| - |\Phi^\varepsilon e^{-i\varphi^\varepsilon}\rangle\langle\Phi^\varepsilon e^{-i\varphi^\varepsilon}|,$$

it is enough to consider $f \in L^2(\mathbb{R}^d)$, with $\|f\|_{L^2} = 1$ and estimate

$$\||\Psi^\varepsilon\rangle\langle\Psi^\varepsilon, f\rangle - |\Phi^\varepsilon e^{-i\varphi^\varepsilon}\rangle\langle\Phi^\varepsilon e^{-i\varphi^\varepsilon}, f\rangle\|_{L^2}$$
$$\leqslant \||\Psi^\varepsilon\rangle\langle\Psi^\varepsilon, f\rangle - |\Psi^\varepsilon\rangle\langle\Phi^\varepsilon e^{-i\varphi^\varepsilon}, f\rangle\|_{L^2}$$
$$+ \||\Psi^\varepsilon\rangle\langle\Phi^\varepsilon e^{-i\varphi^\varepsilon}, f\rangle - |\Phi^\varepsilon e^{-i\varphi^\varepsilon}\rangle\langle\Phi^\varepsilon e^{-i\varphi^\varepsilon} f\rangle\|_{L^2}.$$

Cauchy–Schwarz implies that the first term on the right hand side can be estimated by

$$\big\| |\Psi^\varepsilon\rangle\langle\Psi^\varepsilon, f\rangle - |\Psi^\varepsilon\rangle\langle\Phi^\varepsilon e^{-i\varphi^\varepsilon}, f\rangle \big\|_{L^2} \leqslant \|\Psi^\varepsilon\|_{L^2}\|\Psi^\varepsilon - \Phi^\varepsilon e^{-i\varphi^\varepsilon}\|_{L^2},$$

since $\|f\|_{L^2} = 1$. Analogously, we get

$$\big\| |\Psi^\varepsilon\rangle\langle\Phi^\varepsilon e^{-i\varphi^\varepsilon}, f\rangle - |\Phi^\varepsilon e^{-i\varphi^\varepsilon}\rangle\langle\Phi^\varepsilon e^{-i\varphi^\varepsilon} f\rangle \big\|_{L^2} \leqslant \|\Phi^\varepsilon\|_{L^2}\|\Psi^\varepsilon - \Phi^\varepsilon e^{-i\varphi^\varepsilon}\|_{L^2}.$$

Having in mind that $\|\Phi^\varepsilon\|_{L^2} \simeq \|\Psi^\varepsilon\|_{L^2} = O(\sqrt{\varepsilon})$, this, together with the result of Proposition 5.3, then yields

$$\big\| |\Psi^\varepsilon(t,\cdot)\rangle\langle\Psi^\varepsilon(t,\cdot)| - |\Phi^\varepsilon(t,\cdot)\rangle\langle\Phi^\varepsilon(t,\cdot)| \big\|_{L^2 \to L^2} \lesssim \varepsilon^{\min(2,\sigma+1/2)},$$

which implies the assertion of Corollary 1.3.   □

The main drawback of Proposition 5.3 is the fact that the phase φ^ε involves information of the linear eigenvalue problem. In particular, the dynamical phase involves the linear eigenvalue $E(t)$ instead of $E_*(t)$. This, however, can be remedied in the case of sufficiently strong nonlinearities.

Corollary 5.5. *Under the same assumptions as in Proposition 5.3 but for $\sigma \geqslant 2$, there exists a solution Ψ^ε of* (1.4) *and a $K > 0$ such that*

$$\sup_{t \in J} \Big\| \Psi^\varepsilon(t,\cdot) - \Phi^\varepsilon(t,\cdot)e^{-i\varphi_*^\varepsilon(t)} \Big\|_{L^2(\mathbb{R}^3)} \leqslant K\varepsilon^{\sigma-1},$$

where φ_*^ε is given by

$$\varphi_*^\varepsilon(t) = \frac{1}{\varepsilon}\int_{t_0}^t E_*(s)\mathrm{d}s + \lambda\int_{t_0}^t \|\chi(s,\cdot)\|_{L^{2\sigma+2}}^{2\sigma+2}\,\mathrm{d}s.$$

In other words, for nonlinearities with powers larger than cubic, we can obtain the physically "correct" dynamical phase. Here, we also implicitly assume that χ is chosen to be real valued, and thus $\beta(t) \equiv 0$, which is always possible as long as the evolution is not cyclic in time (see Remark 2.1).

Proof. We choose a real-valued family of linear eigenfunctions $\chi(t,x)$, which implies $\beta(t) \equiv 0$. Using the triangle inequality, we have

$$\Big\| \Psi^\varepsilon(t,\cdot) - \Phi^\varepsilon(t,\cdot)e^{-i\varphi_*^\varepsilon(t)} \Big\|_{L^2} \leqslant \Big\| \Psi^\varepsilon(t,\cdot) - \Phi^\varepsilon(t,\cdot)e^{-i\varphi^\varepsilon(t)} \Big\|_{L^2}$$
$$+ \Big\| \Phi^\varepsilon(t,\cdot)\left(e^{-i\varphi^\varepsilon(t)} - e^{-i\varphi_*^\varepsilon(t)}\right) \Big\|_{L^2}.$$

Here, the first term on the right hand side is of order $O(\varepsilon^{\min(3/2,\sigma)})$, in view of Proposition 5.3. To estimate the second term, we write

$$\Big\| \Phi^\varepsilon(t,\cdot)\left(e^{-i\varphi^\varepsilon(t)} - e^{-i\varphi_*^\varepsilon(t)}\right) \Big\|_{L^2} = \Big\| \Phi^\varepsilon(t,\cdot)\left(e^{-i(\varphi^\varepsilon(t)-\varphi_*^\varepsilon(t))} - 1\right) \Big\|_{L^2}$$
$$\leqslant \|\Phi^\varepsilon(t,\cdot)\|_{L^2}\left| e^{-i(\varphi^\varepsilon(t)-\varphi_*^\varepsilon(t))} - 1 \right|.$$

Recalling the definition of the phases and the fact that $|e^{i\theta} - 1| \leqslant |\theta|$, we obtain

$$\left| e^{-i(\varphi^\varepsilon(t)-\varphi_*^\varepsilon(t))} - 1 \right| \leqslant \frac{1}{\varepsilon}\int_{t_0}^t |E(s) - E_*(s)|\mathrm{d}s \lesssim \varepsilon^{\sigma-1},$$

in view of (5.2).   □

In certain dimensions (and for certain $\sigma \geqslant 2$), a more careful analysis, using the result of Lemma 5.1 together with a Gagliardo–Nirenberg type inequality, would allow a similar statement in which the $\|\chi(s, \cdot)\|_{L^{2\sigma+2}}$ appearing in φ_*^ε is replaced by $\|\Phi^\varepsilon(s, \cdot)\|_{L^{2\sigma+2}}$. In this case, the phase function consequently only depends on information given by the nonlinear eigenvalue problem (5.1). In view of the many assumptions needed, however, this result seems to be very far from optimal and we shall therefore not pursue it any further.

Acknowledgements

References

[1] Avron, J.E., Elgart, A.: Adiabatic theorem without a gap condition. Commun. Math. Phys. **203**(2), 445–463 (1999)

[2] Berry, M.V.: Quantal phase factors accompanying adiabatic changes. Proc. R. Soc. Lond. A **392**, 45–57 (1984)

[3] Berry, M.V.: Histories of adiabatic quantum transitions. Proc. R. Soc. Lond. Ser. A **429**, 61–72 (1990)

[4] Bohm, A., Mostafazadeh, A., Koizumi, H., Niu, Q., Zwanziger, J.: The Geometric phase in quantum systems: foundations, mathematical concepts, and applications in molecular and condensed matter physics. Springer, Berlin (2003)

[5] Born, M., Fock, V.: Beweis des Adiabatensatzes. Z. Phys. A Hadrons Nuclei **51**(3-4), 165–180 (1928)

[6] Carles, R.: Nonlinear Schrödinger equation with time dependent potential. Commun. Math. Sci. **9**(4), 937–964 (2011)

[7] Carles, R., Fermanian Kammerer, C.: A nonlinear adiabatic theorem for coherent states. Nonlinearity **24**(8), 2143–2164 (2011)

[8] Carles, R., Markowich, P.A., Sparber, C.: Semiclassical asymptotics for weakly nonlinear Bloch waves. J. Stat. Phys. **117**, 343–375 (2004)

[9] Erdös, L., Schlein, B., Yau, H.-T.: Rigorous derivation of the Gross–Pitaevskii equation. Phys. Rev. Lett. **98**(4), 040404 (2007)

[10] Gang, Z., Grech, P.D.: An adiabatic theorem for the Gross–Pitaevskii equation. Preprint (2015)

[11] Grech, P.D.: Adiabatic dynamics in closed and open quantum systems. PhD thesis (ETH Zürich) (2011). doi:10.3929/ethz-a-6665029

[12] Gustafson, S., Phan, T.V.: Stable directions for degenerate excited states of nonlinear Schrödinger equations. SIAM J. Math. Anal. **43**(4), 1716–1758 (2011)

[13] Hislop, P.D.: Exponential decay of two-body eigenfunctions: A review. Electron. J. Differ. Eqs. Conf. **04**, 265–288 (2000)

[14] Joye, A.: General adiabatic evolution with a gap condition. Commun. Math. Phys. **275**(1), 139–162 (2007)

[15] Kato, T.: On the adiabatic theorem of quantum mechanics. J. Phys. Soc. Jpn. **5**, 435–439 (1950)

[16] Nenciu, G.: On the adiabatic theorem of quantum mechanics. J. Phys. A Math. Gen. **13**, L15–L18 (1980)

[17] Nenciu, G.: Linear adiabatic theory. Exponential estimates. Commun. Math. Phys. **152**(3), 479–496 (1993)

[18] Nirenberg, L.: Topics in Nonlinear Functional Analysis. Courant Institute, New York (1974)

[19] Pickl, P.: A simple derivation of mean field limits for quantum systems. Lett. Math. Phys. **97**(2), 151–164 (2011)

[20] Pillet, C.-A., Wayne, C.E.: Invariant manifolds for a class of dispersive, Hamiltonian equations. J. Diff. Equ. **141**(2), 310–326 (1997)

[21] Pitaevskii, L., Stringari, S.: Bose–Einstein Condensation. Internat. Series of Monographs on Physics vol 116, Clarendon Press, Oxford (2003)

[22] Rauch, J.: Hyperbolic Partial Differential Equations and Geometric Optics. Graduate Studies in Mathematics vol. 133, American Math. Soc. (2012)

[23] Rose, H.A., Weinstein, M.I.: On the bound states of the nonlinear Schrödinger equation with a linear potential. Physica D **30**, 207–218 (1998)

[24] Salem, W.K.A.: Solitary wave dynamics in time-dependent potentials. J. Math. Phys. **49**(3), 032101 (2008)

[25] Schmid, J.: Adiabatic theorems with and without spectral gap condition for non-semisimple spectral values. In: Mathematical Results in Quantum Mechanics (Proceedings of the QMath12 Conference), pp. 355–362, World Scientific Berlin (2013)

[26] Simon, B.: Holonomy, the quantum adiabatic theorem, and Berry's phase. Phys. Rev. Lett. **51**, 2167–2170 (1983)

[27] Sulem, C., Sulem, P.-L.: The Nonlinear Schrödinger Equation, Self-focusing and Wave Collapse. Springer, New York (1999)

[28] Soffer, A., Weinstein, M.I.: Multichannel nonlinear scattering for nonintegrable equations. Commun. Math. Phys. **133**(1), 119–146 (1990)

[29] Soffer, A., Weinstein, M.I.: Selection of ground states for nonlinear Schrödinger equations. Rev. Math. Phys. **16**(8), 977–995 (2004)

[30] Tao, T.: Nonlinear Dispersive Equations. CBMS Regional Conference Series in Mathematics 106, Amer. Math. Soc., Providence (2006)

[31] Teschl, G.: Mathematical Methods in Quantum Mechanics, With Applications to Schrödinger Operators. Graduate Studies in Mathematics 99, Amer. Math. Soc., Providence (2009)

[32] Teufel, S.: *Adiabatic Perturbation Theory in Quantum Dynamics*. Lecture Notes in Mathematics vol. 1821, Springer, Berlin, Heidelberg (2003)

[33] Tsai, T.-P.: Asymptotic dynamics of nonlinear Schrödinger equations with many bound states. J. Diff. Equ. **192**, 225–282 (2003)

[34] Tsai, T.-P., Yau, H.-T.: Asymptotic dynamics of nonlinear Schrödinger equations: resonance dominated and dispersion dominated solutions. Commun. Pure Appl. Math. **55**, 0153–0216 (2002)

[35] Weinstein, M.I.: Modulational stability of ground states of nonlinear Schrödinger equations. SIAM J. Math. Anal. **16**(3), 472–491 (1985)

[36] Yukalov, V.I.: Adiabatic theorems for linear and nonlinear Hamiltonians. Phys. Rev. A **79**, 052117 (2009)

[37] Xia, D., Chang, M.-C., Niu, Q.: Berry phase effects on electronic properties. Rev. Mod. Phys. **82**, 1959–2007 (2010)

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