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Optimal Lower Bound of the Resonance Widths for the Helmholtz Resonator

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Abstract. Under a geometric assumption on the region near the end of its neck, we prove an optimal exponential lower bound on the widths of resonances for a general two-dimensional Helmholtz resonator. An extension of the result to the n-dimensional case, $n \leq 12$, is also obtained.

1. Introduction

A resonator consists of a bounded cavity (the chamber) connected to the exterior by a thin tube (the neck of the chamber). The frequencies of the sounds it produces are determined by the shape of the chamber, while their duration by the length and the width of the neck in a non-obvious way, and our goal is to understand these. Mathematically, this phenomenon is described by the resonances of the Dirichlet Laplacian $-\Delta_\Omega$ on the domain Ω consisting of the union of the chamber, the neck and the exterior (see Fig. 1).

This article extends our previous work [17], in that we are now able to handle regions where the shape of the exterior is quite general, although the shape of the neck stays the same. The main changes appear in Sects. 4, 5 and 6, where Carleman estimates are used, and Green's identity is replaced by an estimate to obtain a lower bound on the imaginary part of the resonances.

We recall that resonances are the eigenvalues of a complex deformation of $-\Delta_\Omega$; their real and imaginary parts are the frequencies and inverses of the half-lives, respectively, of the corresponding vibrational modes. It is of obvious physical interest to estimate these two quantities as precisely as possible. One practical way to do this involves studying this problem in the asymptotic limit when the width ε of the neck tends to zero. Those resonances with imaginary parts tending to zero converge to the eigenvalues of the Dirichlet Laplacian on

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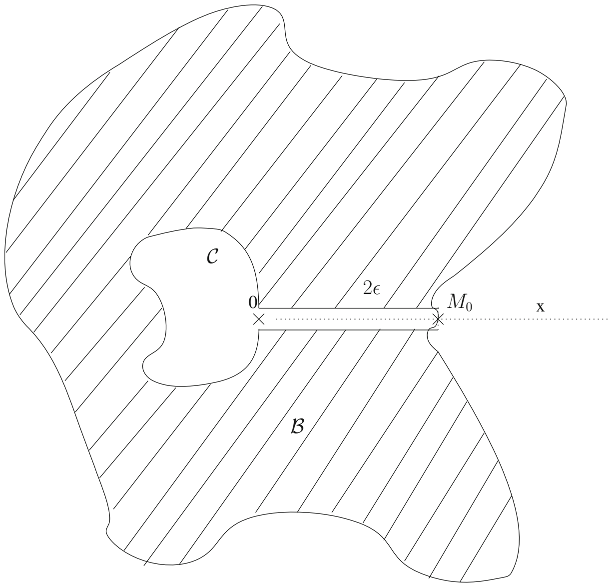

FIGURE 1. The Helmholtz resonator

the cavity, and there is an exponentially small upper bound for the absolute values of the imaginary parts (the widths) of the resonances [13]. However, without very restrictive hypotheses, no lower bound is known. We mention in particular that lower bounds are known in the one-dimensional case [9,10]. As for the higher dimensional case, we mention [5,8,11] which contain results concerning exponentially small widths of quantum resonances, but these do not apply to a Helmholtz resonator. We also mention that the semiclassical lower bound obtained in [11] is optimal (see also [7] for a generalization).

Here, we obtain an optimal lower bound (see Theorem 2.2) under a geometric condition concerning the external end part of the neck. Namely, we assume that the neck meets the boundary of the external region perpendicularly to it, and that the exterior region is concave and symmetric there [see (2.1) and Fig. 1]. This assumption is probably purely technical and should not be necessary. However, it permits us to adapt to this case some of the arguments of [17], to obtain the lower bound after reducing the problem to an estimate near the end part of the neck. This reduction itself is obtained using Carleman estimates up to the boundary, as in [14,15].

2. Geometrical Description and Results

Consider a Helmholtz resonator in $\mathbb{R}^2$ consisting of a regular bounded open set $\mathcal{C}$ (the cavity), connected to a regular unbounded open exterior domain $\mathbf{E}$ through a thin straight tube $\mathcal{T}(\varepsilon)$ (the neck) of radius $\varepsilon > 0$ (see Fig. 2). We shall suppose that ε is very small.

To state this more precisely, let $\mathcal{C}$ and $\mathcal{B}$ be two bounded domains in $\mathbb{R}^2$ with $\mathcal{C}^\infty$ boundary; their closures and boundaries are denoted as $\overline{\mathcal{C}}$, $\overline{\mathcal{B}}$ and $\partial\mathcal{C}$, $\partial\mathcal{B}$. We assume that Euclidean coordinates (x, y) can be chosen in such a way that, for some $L > 0$, one has,

$$\begin{aligned}&\overline{\mathcal{C}} \subset \mathcal{B}; \quad (0,0) \in \partial\mathcal{C}; \quad (L,0) \in \partial\mathcal{B}; \quad [0,L] \times \{0\} \subset \overline{\mathcal{B}}\backslash\mathcal{C};\\ &\text{Near } M_0 := (L,0),\ \mathcal{B} \text{ is convex and } \partial\mathcal{B} \text{ is symmetric with}\\ &\text{respect to } \{y = 0\}.\end{aligned} \tag{2.1}$$

Remark 2.1. This also contains the case where $\partial\mathcal{B}$ is flat near M_0, that is when $\{L\} \times [-\varepsilon_0, \varepsilon_0] \subset \partial\mathcal{B}$ for some $\varepsilon_0 > 0$.

Setting $\mathcal{T}(\varepsilon) := [-\varepsilon_0, L] \times (-\varepsilon, \varepsilon) \cap (\mathbb{R}^2\backslash\mathcal{C})$, $\mathcal{C}(\varepsilon) = \mathcal{C} \cup \mathcal{T}(\varepsilon)$ and $\mathbf{E} := \mathbb{R}^2\backslash\overline{\mathcal{B}}$, then the resonator is defined as,

$$\Omega(\varepsilon) := \mathcal{C}(\varepsilon) \cup \mathbf{E}.$$

As $\varepsilon \to 0^+$, the resonator $\Omega(\varepsilon)$ collapses to $\Omega_0 := \mathcal{C} \cup [0, M_0] \cup \mathbf{E}$, where M_0 is the point $(L, 0) \in \mathbb{R}^2$.

For any domain Q, let P_Q denote the Laplacian $-\Delta_Q$ with Dirichlet boundary conditions on ∂Q; for brevity, we write P_{Ω_ε} as P_ε.

The resonances of P_ε are defined as the eigenvalues of the operator obtained by performing a complex dilation with respect to the coordinates (x, y), for $|x| + |y|$ large. We are interested in those resonances of P_ε that are close to the eigenvalues of $P_\mathcal{C}$. Thus, let $\lambda_0 > 0$ be an eigenvalue of $P_\mathcal{C}$ with u_0 the corresponding (normalized) eigenfunction. We make the following Assumption (**H**):

$$\lambda_0 \text{ is the lowest eigenvalue of} - \Delta_\mathcal{C}.$$

By the arguments of [13], we know that there is a resonance $\rho(\varepsilon) \in \mathbb{C}$ of P_ε such that $\rho(\varepsilon) \to \lambda_0$ as $\varepsilon \to 0$. Furthermore, the lowest eigenvalue $\lambda(\varepsilon)$ of $P_{\mathcal{C}(\varepsilon)}$ is such that, for any $\delta > 0$,

$$|\rho(\varepsilon) - \lambda(\varepsilon)| \le C_\delta e^{-\pi(1-\delta)L/\varepsilon}, \tag{2.2}$$

for some $C_\delta > 0$ and all sufficiently small $\varepsilon > 0$. In particular, since $\lambda(\varepsilon) \in \mathbb{R}$, this gives

$$|\operatorname{Im} \rho(\varepsilon)| \le C_\delta e^{-\pi(1-\delta)L/\varepsilon}. \tag{2.3}$$

We now state our main result.

Theorem 2.2. *Under Assumption* (**H**)*, for any $\delta > 0$, there exists $C_\delta > 0$ such that, for all $\varepsilon > 0$ small enough, one has*

$$|\operatorname{Im} \rho(\varepsilon)| \ge \frac{1}{C_\delta} e^{-\pi(1+\delta)L/\varepsilon}.$$

Remark 2.3. We extend this result to the higher dimensional case in Sect. 11.

Remark 2.4. Gathering (2.3) and Theorem 2.2, we can reformulate the result as:

$$\lim_{\varepsilon \to 0_+} \varepsilon \ln |\operatorname{Im} \rho(\varepsilon)| = -\pi L. \tag{2.4}$$

3. Properties of the Resonant State

By definition, the resonance $\rho(\varepsilon)$ is an eigenvalue of the complex distorted operator,

$$P_\varepsilon(\mu) := U_\mu P_\varepsilon U_\mu^{-1},$$

where $\mu > 0$ is a small parameter, and U_μ is a complex distortion of the form,

$$U_\mu \varphi(x,y) := \varphi((x,y) + i\mu f(x,y)),$$

with $f \in C^\infty(\mathbb{R}^2;\mathbb{R}^2)$, $f = 0$ near $\overline{\mathcal{B}}$, $f(x,y) = (x,y)$ for $|(x,y)|$ large enough. (Observe that by Weyl Perturbation Theorem, the essential spectrum of $P_\varepsilon(\mu)$ is $e^{-2i\alpha}\mathbb{R}_+$, with $\alpha = \arctan \mu$.)

It is well known that such eigenvalues do not depend on μ (see, e.g., [12,19]), and that the corresponding eigenfunctions are of the form $U_\mu u_\varepsilon$ with u_ε independent of μ, smooth on $\mathbb{R}^2$ and analytic in a complex sector around $\mathbf{E}$. In other words, u_ε is a non-trivial analytic solution of the equation $-\Delta u_\varepsilon = \rho(\varepsilon) u_\varepsilon$ in $\Omega(\varepsilon)$, such that $u_\varepsilon \big|_{\partial\Omega(\varepsilon)} = 0$ and, for all $\mu > 0$ small enough, $U_\mu u_\varepsilon$ is well defined and is in $L^2(\Omega(\varepsilon))$ (in our context, this latter property will be taken as a definition of the fact that u_ε is *outgoing*). Moreover, u_ε can be normalized by setting, for some fixed $\mu > 0$,

$$\|U_\mu u_\varepsilon\|_{L^2(\Omega(\varepsilon))} = 1.$$

In that case, we learn from [13] (in particular Proposition 3.1 and formula (5.13)), that, for any $\delta > 0$, and for any $R > 0$ large enough, one has,

$$\|u_\varepsilon\|_{L^2(\Omega(\varepsilon)\cap\{|(x,y)|<R\})} \geq 1 - \mathcal{O}(e^{(\delta - \frac{\pi L}{2})/\varepsilon}), \tag{3.1}$$

and

$$\|u_\varepsilon\|_{H^1(\mathbf{E}\cap\{|(x,y)|<R\})} = \mathcal{O}(e^{(\delta - \frac{\pi L}{2}))/\varepsilon}). \tag{3.2}$$

Now, we take $R > 0$ such that $\overline{\mathcal{B}} \subset \{|(x,y)| < R\}$. Using the equation $-\Delta u_\varepsilon = \rho u_\varepsilon$ and Green's formula on the domain $\Omega(\varepsilon) \cap \{|(x,y)| < R\}$, and using polar coordinates (r,θ), we obtain,

$$\operatorname{Im} \rho \int_{\Omega(\varepsilon)\cap\{|(x,y)|<R\}} |u_\varepsilon|^2 dxdy = -\operatorname{Im} \int_0^{2\pi} \frac{\partial u_\varepsilon}{\partial r}(R,\theta)\overline{u}_\varepsilon(R,\theta) R d\theta,$$

and thus, by (3.1–3.2), and for some $\delta_0 > 0$,

$$\operatorname{Im} \rho = -(1 + \mathcal{O}(e^{(\delta - \pi L)/\varepsilon}))\ \operatorname{Im} \int_0^{2\pi} \frac{\partial u_\varepsilon}{\partial r}(R,\theta)\overline{u}_\varepsilon(R,\theta) R d\theta \tag{3.3}$$

where the $\mathcal{O}$ is locally uniform with respect to R.

Therefore, to prove our result, it is sufficient to obtain a lower bound on $\operatorname{Im} \int_0^{2\pi} \frac{\partial u_\varepsilon}{\partial r}(R,\theta)\overline{u}_\varepsilon(R,\theta) R d\theta$. Note that, using (3.2), we immediately obtain (2.3).

4. Estimate Outside a Large Disc

The goal of this section is to prove,

Proposition 4.1. *Let $R_1 > R_0 > 0$ be fixed in such a way that $\overline{\mathcal{B}} \subset \{|(x,y)| < R_0\}$. Then, for any $C > 0$, there exists a constant $C' = C'(R_0, R_1, C) > 0$ such that, for all $\varepsilon > 0$ small enough, one has,*

$$|\operatorname{Im}\rho| \geq \frac{1}{C'}\|u_\varepsilon\|^2_{L^2(R_0<|(x,y)|<R_1)} - C'e^{-C/\varepsilon}.$$

Proof. Working in polar coordinates (r,θ), for $r \geq R_0$, we can represent $u = u_\varepsilon$ as,

$$u(r,\theta) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} u_k(r)e^{ik\theta},$$

where $u_k(r) := \int_0^{2\pi} u(r,\theta)e^{-ik\theta}d\theta = a_k H_k(r\sqrt{\rho})$, H_k being the outgoing Hankel function, defined for $k \geq 0$ as

$$H_k(t) := \frac{e^{i(t-\frac{k\pi}{2}-\frac{\pi}{4})}}{\Gamma(k+\frac{1}{2})}\sqrt{\frac{2}{\pi t}}\int_0^\infty e^{-s}s^{k-\frac{1}{2}}\left(1+\frac{is}{2t}\right)^{k-\frac{1}{2}}ds,$$

for $k < 0$ by $H_k = (-1)^k H_{-k}$, and solution to,

$$t^2H_k''(t) + tH_k'(t) + (t^2-k^2)H_k(t) = 0.$$

In particular, for all k, the function $h_k := H_k(r\sqrt{\rho})$ is an analytic function, solution to

$$-h_k'' - \frac{1}{r}h_k' + \frac{k^2}{r^2}h_k = \rho h_k, \tag{4.1}$$

and for any $\mu > 0$ fixed small enough, one has,

$$h_k(re^{i\mu}) \in H^2([R_0, +\infty)). \tag{4.2}$$

By (3.3), for any $R \in [R_0, R_1]$ we also have,

$$\operatorname{Im}\rho = -(1+\mathcal{O}(e^{(\delta-\pi L)/\varepsilon}))\sum_{k\in\mathbb{Z}}\alpha_k(R) = -(1+\mathcal{O}(e^{(\delta-\pi L)/\varepsilon}))\sum_{k\in\mathbb{Z}}\beta_k(R)|a_k|^2, \tag{4.3}$$

with

$$\alpha_k(R) := \operatorname{Im} Ru_k'(R)\overline{u}_k(R); \quad \beta_k(R) := \operatorname{Im} Rh_k'(R)\overline{h}_k(R). \tag{4.4}$$

We set,

$$\lambda(R) := \sum_{k\in\mathbb{Z}}\alpha_k(R) = \sum_{k\in\mathbb{Z}}\beta_k(R)|a_k|^2,$$

and, for $C > 0$ arbitrary large, we write,

$$\lambda(R) = \sum_{|k|\leq C/\varepsilon}\alpha_k(R) + \sum_{|k|>C/\varepsilon}\alpha_k(R) =: \lambda_-(R,C) + \lambda_+(R,C).$$

We first prove,

Lemma 4.2. *There exists $\delta > 0$ such that, for any $C > 0$, one has,*

$$\lambda_+(R, C) = \mathcal{O}(e^{-\delta C/\varepsilon}),$$

uniformly as $\varepsilon \to 0_+$.

Proof. In view of (4.4), it is enough to prove that $|u_k(R)|+|u_k'(R)| = \mathcal{O}(e^{-\delta|k|})$ for some $\delta = \delta(R) > 0$, uniformly as $|k| \to \infty$. From (4.1), we know that u_k is solution to,

$$-k^{-2}u_k'' - \frac{1}{k^2 r}u_k' + \frac{1}{r^2}u_k - \frac{\rho}{k^2}u_k = 0,$$

that can be considered as a semiclassical differential equation with small parameter $h := |k|^{-1}$ and principal symbol $a(r, r^*) := (r^*)^2 + r^{-2}$. In particular, this symbol is locally elliptic, and since u is locally bounded together with all its derivatives, we also know that u_k is locally uniformly bounded (together with all its derivatives) as $|k| \to \infty$. Then, we can apply standard techniques of semiclassical analysis (in particular Agmon estimates: see, e.g., [16]) to prove that $|u_k| + |u_k'|$ is locally $\mathcal{O}(e^{-\delta|k|})$ for some $\delta > 0$, and the result follows. □

Next, we show,

Lemma 4.3. *For any $C > 0$ and any $\sigma \in (0, \pi L/2)$, there exists $C' = C'(C, \delta_1) > 0$ such that*

$$\lambda_-(R, C) \geq \frac{1}{C'} \sum_{|k| \leq C/\varepsilon} |a_k|^2 - C'|\operatorname{Im}\rho|e^{-2\sigma/\varepsilon},$$

uniformly as $\varepsilon \to 0_+$.

Proof. For $|k| \leq C/\varepsilon$, let $\mu_k = \mu_{k,R} \in C^\infty(\mathbb{R}_+; \mathbb{R}_+)$ be a real non-decreasing function verifying,

$$\mu_k(r) = 0 \quad \text{for} \quad r \leq r_k := \max(C_0|k|, R)\ ;\ \mu_k(r) = \frac{\mu_0}{1+|k|} \quad \text{for} \quad r \geq r_k + 1,$$

where $\mu_0 > 0$ is fixed small enough, and $C_0 > 0$ will be chosen sufficiently large later on. We set,

$$\nu_k(r) := re^{i\mu_k(r)}; \quad g_k(r) = U_k h_k(r) := h_k(\nu_k(r)). \tag{4.5}$$

By (4.2) we have,

$$g_k \in H^2([R_0, +\infty)). \tag{4.6}$$

Moreover, by construction we also have,

$$\beta_k(R) = \operatorname{Im}\frac{\nu_k(R)}{\nu_k'(R)}g_k'(R)\overline{g}_k(R),$$

and using (4.1), we see that g_k is solution to,

$$-g_k'' - \left(\frac{\nu_k'}{\nu_k} - \frac{\nu_k''}{\nu_k'}\right)g_k' + \frac{k^2(\nu_k')^2}{\nu_k^2}g_k = \rho(\nu_k')^2 g_k. \tag{4.7}$$

Then, using (4.6–4.7),we can write,

$$\begin{aligned}\beta_k(R) =& -\operatorname{Im}\int_R^\infty \frac{d}{dr}\left(\frac{\nu_k(r)}{\nu_k'(r)}g_k'(r)\overline{g}_k(r)\right)dr\\ =& -\operatorname{Im}\int_R^\infty\left[\left(1-\frac{\nu_k\nu_k''}{(\nu_k')^2}\right)g_k'\overline{g}_k+\frac{\nu_k(r)}{\nu_k'(r)}g_k''(r)\overline{g}_k(r)+\frac{\nu_k(r)}{\nu_k'(r)}|g_k'(r)|^2\right]dr\\ =& -\operatorname{Im}\int_R^\infty\left[\left(\frac{k^2\nu_k'}{\nu_k}-\rho\nu_k\nu_k'\right)|g_k|^2+\frac{\nu_k(r)}{\nu_k'(r)}|g_k'(r)|^2\right]dr.\end{aligned}$$

Since $\nu_k'/\nu_k = r^{-1}+i\mu_k'$ and $\nu_k\nu_k' = r(1+ir\mu_k')e^{2i\mu_k}$, we obtain,

$$\beta_k(R) = \int_R^\infty \left(\gamma_k(r)|g_k'(r)|^2+\delta_k(r)|g_k(r)|^2\right)dr,$$

with,

$$\begin{aligned}\gamma_k(r) :=& \frac{\mu_k'}{r^{-2}+(\mu_k')^2};\\ \delta_k(r) :=& r\operatorname{Re}\rho\sin 2\mu_k + r\operatorname{Im}\rho\cos 2\mu_k\\ &+r^2\mu_k'[(\operatorname{Re}\rho)\cos 2\mu_k-(\operatorname{Im}\rho)\sin 2\mu_k]-k^2\mu_k'.\end{aligned}$$

In particular, $\gamma_k \geq 0$. Since $\mu_k \leq \mu_0(1+|k|)^{-1}$, $\operatorname{Im}\rho \leq 0$, and $\operatorname{Re}\rho \to \lambda_0 > 0$ as $\varepsilon \to 0$, we also have,

$$\delta_k \geq \delta_0 r\sin 2\mu_k + r\operatorname{Im}\rho\cos 2\mu_k + \mu_k'(\delta_0 r^2 - k^2),$$

where δ_0 is any positive constant such that $\delta_0 < \lambda_0\cos 2\mu_0$. But, by construction, we have $\mu_k'(r) = 0$ when $r \leq C_0|k|$. Therefore, $\mu_k'(r)(\delta_0 r^2 - k^2) \geq \mu_k'(r)(\delta_0 C_0^2 - 1)k^2 \geq 0$ if we choose $C_0 \geq \delta_0^{-1/2}$. Then, we obtain,

$$\begin{aligned}\beta_k(R) \geq& \int_R^\infty r\left(\delta_0(\sin 2\mu_k(r)+\operatorname{Im}\rho\cos 2\mu_k(r)\right)|g_k(r)|^2dr\\ \geq& \delta_0\sin\left(\frac{\mu_0}{1+|k|}\right)\int_{r_k+1}^\infty r|g_k(r)|^2dr - |\operatorname{Im}\rho|\int_R^\infty r|g_k(r)|^2dr. \quad (4.8)\end{aligned}$$

Since $|k| \leq C/\varepsilon$ and $|\operatorname{Im}\rho| = \mathcal{O}(e^{-c_1/\varepsilon})$ for some $c_1 > 0$, we also have $|\operatorname{Im}\rho| \leq \frac{1}{2}\delta_0\sin(\frac{\mu_0}{1+|k|})$ for $\varepsilon > 0$ small enough, and therefore,

$$\beta_k(R) \geq \frac{1}{2}\delta_0\sin\left(\frac{\mu_0}{1+|k|}\right)\int_{r_k+1}^\infty r|g_k(r)|^2dr - |\operatorname{Im}\rho|\int_R^{r_k+1} r|g_k(r)|^2dr.$$

Equivalently, setting $v_k(r) := u_k(\nu_k(r)) = a_k g_k(r)$, we have proved,

$$\alpha_k(R) \geq \frac{1}{2}\delta_0|a_k|^2\sin\left(\frac{\mu_0}{1+|k|}\right)\int_{r_k+1}^\infty r|g_k(r)|^2dr - |\operatorname{Im}\rho|\int_R^{r_k+1} r|v_k(r)|^2dr \tag{4.9}$$

Now, considering a cutoff function $\chi = \chi(r) \in C^\infty(\mathbb{R}_+;[0,1])$ such that $\chi = 1$ on $r \geq R_0$, $\chi = 0$ on $r \leq R_0 - \delta_0$ ($\delta_0 > 0$ small enough), we see that the function $w := \chi u$ satisfies $(-\Delta - \rho)w = [-\Delta, \chi]u$ on all of $\mathbb{R}^2$, and is outgoing. Then, standard estimates on the outgoing resolvent of the Laplacian (or, equivalently, on the Green function of the Helmholtz equation in $\mathbb{R}^n$, $n \geq 2$) show that, for all $\delta > 0$ arbitrarily small, one has $w = \mathcal{O}(e^{\delta r}||[-\Delta,\chi]u||_{L^2})$ uniformly as $r \to$

∞. Actually, such estimates remain valid for the complex distorted Laplacian $U_0\Delta U_0^{-1}$ [where U_0 is as in (4.5) with some arbitrary $\mu_0 \geq 0$ small enough], and since $||[-\Delta, \chi]u||_{L^2} = \mathcal{O}(e^{-\delta_1/\varepsilon})$ for any $\delta_1 \in (0, \pi L/2)$, we obtain: $u(r) = \mathcal{O}(e^{\delta r - \delta_1/\varepsilon})$ uniformly on $\{r \in \mathbb{C}\,; \operatorname{Re} r \geq R_0\,,\, |\operatorname{Im} r| \leq \mu_0(\operatorname{Re} R - R_0)\}$, where $\delta > 0$ is arbitrary. In particular, this gives us: $r|v_k(r)|^2 = \mathcal{O}(e^{\delta r - 2\delta_1/\varepsilon})$, and therefore,

$$\sum_{|k|\leq C/\varepsilon} \int_R^{r_k+1} r|v_k(r)|^2 dr = \mathcal{O}\left(\frac{C}{\varepsilon} e^{\delta C/\varepsilon - 2\delta_1/\varepsilon}\right) = \mathcal{O}(e^{-2\delta_1'/\varepsilon}),$$

where $\delta_1' = \delta_1 - \delta C$ can be taken arbitrarily close to δ_1 (and thus, to $\pi L/2$) by choosing $\delta << 1/C$. Inserting into (4.9) and taking the sum over k, we obtain,

$$\lambda_-(R, C) \geq \frac{1}{2}\delta_0 \sum_{|k|\leq C/\varepsilon} |a_k|^2 \sin\left(\frac{\mu_0}{1+|k|}\right) \int_{r_k+1}^{\infty} r|g_k(r)|^2 dr - C'|\operatorname{Im}\rho| e^{-\delta_1'/\varepsilon} \tag{4.10}$$

with $C' = C'(C) > 0$.

To complete the proof, we need to estimate the quantity $J_k := \int_{r_k+1}^{\infty} r|g_k(r)|^2 dr$ as $|k| \to \infty$. Setting $r = |k|s$, for $|k|$ large enough we find,

$$J_k \geq |k|^2 \int_{2C_0}^{\infty} |w_k(se^{i\mu_0/(1+|k|)})|^2 ds \tag{4.11}$$

where $w_k(z) := z^{1/2} h_k(|k|z)$ ($z \in \mathbb{C}$, $|z| \geq C_0$, $|\arg z| \leq \mu_0$). Using (4.1), we see that w_k is solution to,

$$-\frac{1}{k^2} w_k'' + \left(\frac{1}{z^2} - \frac{1}{4k^2 z^2} - \rho\right) w_k = 0.$$

This is a semiclassical Schrödinger equation, with small parameter $h := |k|^{-1}$, and we can apply to it the standard WKB complex method to find the asymptotic of w_k, both as $k \to \infty$ and $\operatorname{Re} z \to +\infty$. Using also that w_k must be outgoing, we immediately obtain,

$$w_k(z) \sim \frac{\tau_k}{(\rho - z^{-2})^{\frac{1}{4}}} \exp\left(i|k| \int_{2C_0}^{z} (\rho - t^{-2})^{\frac{1}{2}} dt\right) \tag{4.12}$$

as $|k| + \operatorname{Re} z \to \infty$, uniformly with respect to $\varepsilon > 0$. Here, $\tau_k \in \mathbb{C}$ is a complex constant of normalization that we have to compute. To do so, we use the well-known asymptotic of $H_k(t)$ as $\operatorname{Re} t \to +\infty$,

$$H_k(t) \sim \sqrt{\frac{2}{\pi t}} \exp\left(i\left(t - \frac{k\pi}{2} - \frac{\pi}{4}\right)\right),$$

that gives,

$$w_k(r) = r^{\frac{1}{2}} H_k\left(|k| r\sqrt{\rho}\right) \sim \sqrt{\frac{2}{\pi|k|}} \exp\left(i\left(|k| r\sqrt{\rho} - \frac{k\pi}{2} - \frac{\pi}{4}\right)\right) \quad (r \to +\infty).$$

Comparing with (4.12), we obtain,

$$\tau_k = \rho^{\frac{1}{4}} \sqrt{\frac{2}{\pi |k|}} e^{-i(\frac{k\pi}{2}+\frac{\pi}{4})} e^{i|k|L}$$

where

$$\begin{aligned} L &:= \lim_{r\to+\infty} \left(r\sqrt{\rho} - \int_{2C_0}^{r} (\rho - t^{-2})^{\frac{1}{2}} \, dt \right) \\ &= \lim_{r\to+\infty} \left(r\sqrt{\rho} - \left[\sqrt{\rho t^2 - 1} - \tan^{-1} \sqrt{\rho t^2 - 1} \right]_{2C_0}^{r} \right) \end{aligned}$$

that is,

$$L = \frac{\pi}{2} + \sqrt{4\rho C_0^2 - 1} - \tan^{-1} \sqrt{4\rho C_0^2 - 1}.$$

In particular,

$$\operatorname{Im} L = \operatorname{Im} \sqrt{4\rho C_0^2 - 1} + \frac{1}{2} \int_{\operatorname{Im}\sqrt{4\rho C_0^2 - 1}}^{-\operatorname{Im}\sqrt{4\rho C_0^2 - 1}} \frac{1}{1 + (\operatorname{Re}\sqrt{4\rho C_0^2 - 1} + it)^2} dt,$$

and thus

$$\operatorname{Im} L = (1 + \mathcal{O}(C_0^{-1})) \operatorname{Im} \sqrt{4\rho C_0^2 - 1} \le 0$$

if C_0 has been taken sufficiently large. As a consequence,

$$|\tau_k| \ge |\rho|^{\frac{1}{4}} \sqrt{\frac{2}{\pi|k|}},$$

and then, by (4.12), and for $s \ge 2C_0$, we deduce,

$$|k|^2 |w_k(se^{i\mu_0/(1+|k|)})|^2 \ge \delta_2 |k| e^{-\delta s},$$

where $\delta_2 > 0$ is a constant (independent both of k and ε). Going back to (4.11), for $|k|$ large enough we finally obtain,

$$J_k \ge \frac{|k|}{C_1},$$

where C_1 is a positive constant. Then, inserting into (4.10), we obtain

$$\lambda_-(R, C) \ge \frac{\delta_0}{3C_1} \sum_{|k| \le C/\varepsilon} |a_k|^2 - C' |\operatorname{Im} \rho| e^{-\delta_1'/\varepsilon},$$

and Lemma 4.3 follows. □

Now, for any $K \ge 0$, we have,

$$||u||_{r=R}^2 = R \sum_{k\in\mathbb{Z}} |a_k|^2 |h_k(R)|^2 \le C_K \sum_{|k|\le K} |a_k|^2 + R \sum_{|k|>K} |a_k|^2 |h_k(R)|^2,$$

with $C_K := \sup_{|k|\le K\,;\, R\in[R_0,R_1]} R|h_k(R)|^2$. Then, in the same spirit as in [4], we use an estimate on the outgoing Hankel functions that will permit us to compare its values at two different points.

Lemma 4.4. *One has,*

$$h_k(R) = -i\sqrt{\frac{2}{\pi}}\, k^{k-\frac{1}{2}} \left(\frac{2}{eR\sqrt{\rho}}\right)^k \left(1 + \mathcal{O}(k^{-1})\right),$$

uniformly with respect to $R \in [R_0, R_1]$, $\varepsilon > 0$ *small enough, and* $k \geq 1$ *large enough.*

Proof. See Appendix. □

It follows from this lemma that, for any $R \in [R_0, R_1]$, we have,

$$\frac{|h_k(R)|}{|h_k(R_0)|} = \mathcal{O}\left((R_0/R)^{|k|}\right)$$

uniformly as $|k| \to \infty$. Therefore, we obtain,

$$||u||^2_{r=R} \leq C_K \sum_{|k|\leq K} |a_k|^2 + CR \sum_{|k|>K} |a_k|^2 |h_k(R_0)|^2 R_0^{2|k|} R^{-2|k|} \tag{4.13}$$

where $C > 0$ does not depend on K, R. Integrating with respect to R on the interval $[R_0, R_1]$, we obtain,

$$||u||^2_{R_0\leq R\leq R_1} \leq C'_K \sum_{|k|\leq K} |a_k|^2 + C \sum_{|k|>K} |a_k|^2 |h_k(R_0)|^2 R_0^{2|k|} \frac{R_0^{2-2|k|}}{2|k|-2},$$

and thus,

$$||u||^2_{R_0\leq R\leq R_1} \leq C'_K \sum_{|k|\leq K} |a_k|^2 + \frac{CR_0}{2K-2} ||u||^2_{r=R_0}. \tag{4.14}$$

Moreover, for all $S \in [R_0, R_1]$, we have,

$$||u||^2_{r=R_0} = ||u||^2_{r=S} - \int_S^{R_0} (||u(r)||^2_{L^2(0,2\pi)} + 2r\,\mathrm{Re}\langle \partial_r u, u\rangle_{L^2(0,2\pi)}) dr,$$

that gives,

$$||u||^2_{r=R_0} = ||u||^2_{r=S} + \mathcal{O}(||\partial_r u||^2_{R_0\leq r\leq R_1} + ||u||^2_{R_0\leq r\leq R_1}),$$

and thus, using the equation $-\Delta u = \rho u$ and standard Sobolev estimates,

$$||u||^2_{r=R_0} = ||u||^2_{r=S} + \mathcal{O}(||u||^2_{R_0\leq r\leq R_1}).$$

Inserting this into (4.14), and taking K sufficiently large, we obtain,

$$||u||^2_{R_0\leq R\leq R_1} \leq C'_K \sum_{|k|\leq K} |a_k|^2 + \frac{C'}{K-1} ||u||^2_{r=S}, \tag{4.15}$$

where $C', C'_K > 0$ are constants, and C' is independent of K. Finally, integrating in S on $[R_0, R_1]$, and increasing again the value of K, we arrive at,

$$||u||^2_{R_0\leq r\leq R_1} \leq 2C'_K \sum_{|k|\leq K} |a_k|^2. \tag{4.16}$$

Then, Proposition 4.1 directly follows from (4.3), Lemmas 4.2 and 4.3 and (4.16). □

Remark 4.5. By integrating with respect to R on any bounded interval of $[R_0, +\infty)$, and using the equation $-\Delta u_\varepsilon = \rho u_\varepsilon$ and standard estimates on the Laplacian, we easily deduce from this proposition that, for any bounded open set $V \subset \{|(x,y)| \geq R_0\}$ and any $s \geq 0$, one has $\|u_\varepsilon\|^2_{H^s(V)} = \mathcal{O}(|\operatorname{Im}\rho| + e^{-C/\varepsilon})$ for any $C > 0$.

Remark 4.6. The result of Proposition 4.1 can easily be generalized to any dimension $n \geq 2$ by working with the complex measure $(\nu_k(r)/\nu_k'(r))^{n-1}dr$ instead of $(\nu_k(r)/\nu_k'(r))dr$ in the proof of Lemma 4.3.

Remark 4.7. As pointed out to us by J. Sjöstrand, an alternative (and probably more conceptual) proof of Proposition 4.1 may consist in making the change of scale $r \mapsto r/h$, where $h > 0$ is an extra small parameter, and to apply the techniques of semiclassical analysis as $h \to 0_+$. The fact that u is outgoing means that it lives around the outgoing trajectories starting from the obstacle, and thus in a microlocal weighted space where $-h^2\Delta - \rho$ can be written as the product of an elliptic pseudodifferential operator with $\partial_r - iA$, where the selfadjoint operator A acts on the tangent variable θ only, and is positive. Such arguments are developed in [18], Section 4.

5. Estimate Near the Obstacle

Now, reasoning by contradiction, assume the existence of $\delta_0 > 0$ such that, along a sequence $\varepsilon \to 0^+$, one has

$$|\operatorname{Im}\rho| = \mathcal{O}(e^{-(\pi L+\delta_0)/\varepsilon}). \tag{5.1}$$

In the rest of the proof, it will always be assumed that ε tends to zero along this sequence. Then, Proposition 4.1 (added to standard Sobolev estimates) tells us that for any $R_1 > R_0 > 0$ such that $\overline{\mathcal{B}} \subset \{|(x,y)| < R_0\}$, we have,

$$\|u_\varepsilon\|^2_{H^1(R_0<|(x,y)|<R_1)} = \mathcal{O}(e^{-(\pi L+\delta_0)/\varepsilon}). \tag{5.2}$$

To propagate this estimate up to an arbitrarily small neighborhood of $\overline{\mathcal{B}}$, we use the Carleman estimate in [14, Theorem 3.5].

First, fix a point (x_0, y_0) in $\mathbf{E} = \mathbb{R}^2\backslash\overline{\mathcal{B}}$, and assume there exists a real function f defined on a small open neighborhood V_0 of (x_0, y_0) in $\mathbf{E}$, with $f(x_0, y_0) = 0$, $\nabla f(x_0, y_0) \neq 0$, and such that for any $\delta > 0$ small enough, there exists $\delta' = \delta'(\delta) > 0$, such that,

$$\|u_\varepsilon\|^2_{H^1(V\cap\{f\geq\delta\})} = \mathcal{O}(e^{-(\pi L+\delta')/\varepsilon}), \tag{5.3}$$

uniformly as $\varepsilon \to 0_+$. (For instance, in view of (5.2), (x_0, y_0) could be any point of $\mathbf{E}$ such that $|(x_0, y_0)| = R_-$, with $R_- := \inf\{R > 0\,; \overline{\mathcal{B}} \subset \{|(x,y)| \leq R\}$, and $f(x,y) = x^2 + y^2 - R_-^2$.)

For $\lambda > 0$ fixed large enough and (x,y) in V_0, following [14,15], we consider the function,

$$\varphi(x,y) := e^{\lambda(f(x,y)-(x-x_0)^2-(y-y_0)^2)}.$$

Then, setting,

$$p_\varphi(x,y,\xi,\eta) := \xi^2+\eta^2-|\nabla\varphi(x,y)|^2+2i\langle\nabla\varphi(x,y),(\xi,\eta)\rangle = q_1+iq_2,$$

it is easy to check that, if λ has been taken large enough, then there exists a constant $C_0>0$ such that one has the implication,

$$p_\varphi(x,y,\xi,\eta)=0 \Rightarrow \{q_1,q_2\}(x,y,\xi,\eta)\geq \frac{1}{C_0},$$

where $\{q_1,q_2\}$ is the Poisson bracket of the real-valued functions q_1 and q_2. Moreover, possibly by shrinking V_0 around (x_0,y_0), we see that $\nabla\varphi\neq 0$ on V. In particular, Assumption 3.1 of [14] is satisfied, and if $\chi\in C_0^\infty(V_0\,;[0,1])$ is such that $\chi=1$ near (x_0,y_0), we can apply Theorem 3.5 of [14] to the function $w:=\chi u_\varepsilon$, and with small parameter $h:=\varepsilon/\mu$, where $\mu>0$ is an extra-parameter that will be fixed small enough later on. Then, for ε/μ small enough, we obtain,

$$\|e^{\mu\varphi/\varepsilon}w\|_{L^2}^2+\mu^{-2}\varepsilon^2\|e^{\mu\varphi/\varepsilon}\nabla w\|_{L^2}^2\leq C\mu^{-3}\varepsilon^3\|e^{\mu\varphi/\varepsilon}\Delta w\|_{L^2}^2 \tag{5.4}$$

where $C>0$ is a constant. Then, writing $-\Delta w=\rho w-[\Delta,\chi]u_\varepsilon$, and observing that, for ε/μ small enough, the term involving ρw in the right-hand side of (5.4) can be absorbed by the first term of the left-hand side, we led to,

$$\|e^{\mu\varphi/\varepsilon}w\|_{L^2}^2+\mu^{-2}\varepsilon^2\|e^{\mu\varphi/\varepsilon}\nabla w\|_{L^2}^2\leq C\mu^{-3}\varepsilon^3\|e^{\mu\varphi/\varepsilon}[\Delta,\chi]u_\varepsilon\|_{L^2}^2,$$

with a new constant $C>0$. Now, setting $m_0:=\sup_{V_0}\varphi$, $V_0':=\{\chi=1\}$, $S_\delta:=\mathrm{Supp}\nabla\chi\cap\{f<\delta\}$ ($\delta>0$ small enough), and using (5.3), we deduce,

$$\begin{aligned}&\|e^{\mu\varphi/\varepsilon}u_\varepsilon\|_{L^2(V_0')}^2+\mu^{-2}\varepsilon^2\|e^{\mu\varphi/\varepsilon}\nabla u_\varepsilon\|_{L^2(V_0')}^2\\&\quad=\mathcal{O}(\mu^{-3}\varepsilon^3\|e^{\mu\varphi/\varepsilon}[\Delta,\chi]u_\varepsilon\|_{L^2(S_\delta)}^2+e^{(\mu m_0-\pi L-\delta')/\varepsilon}).\end{aligned} \tag{5.5}$$

On the other hand, we have $S_\delta\subset\{f<\delta\}\cap\{|(x,y)-(x_0,y_0)|\geq\delta_1\}$ for some $\delta_1>0$ independent of δ, and thus, by construction, for $\delta>0$ sufficiently small, there exists a constant $\delta_2>0$ such that,

$$S_\delta\subset\{\varphi(x,y)\leq 1-\delta_2\}. \tag{5.6}$$

As a consequence, we obtain,

$$\begin{aligned}&\|e^{\mu\varphi/\varepsilon}u_\varepsilon\|_{L^2(V_0')}^2+\mu^{-2}\varepsilon^2\|e^{\mu\varphi/\varepsilon}\nabla u_\varepsilon\|_{L^2(V_0')}^2\\&\quad=\mathcal{O}(\mu^{-3}\varepsilon^3e^{\mu(1-\delta_2)/\varepsilon}\|u_\varepsilon\|_{H^1(S_\delta)}^2+e^{(\mu m_0-\pi L-\delta')/\varepsilon}).\end{aligned} \tag{5.7}$$

Since $S_\delta\subset\mathbf{E}$, we also know [see (3.2)] that $\|u_\varepsilon\|_{H^1(S)}$ is not exponentially larger than $e^{-\pi L/2\varepsilon}$. Moreover, since $\varphi(x_0,y_0)=1$, if B_r stands for the ball of radius r centered at (x_0,y_0), we have $\varphi\leq 1-\theta(r)$ on B_r, with $\theta(r)\to 0$ as $r\to 0$. Therefore, for $r>0$ small enough, we deduce from (5.7),

$$\begin{aligned}&\|u_\varepsilon\|_{L^2(B_r)}^2+\mu^{-2}\varepsilon^2\|\nabla u_\varepsilon\|_{L^2(B_r)}^2\\&\quad=\mathcal{O}(\mu^{-3}\varepsilon^3e^{(\mu(\theta(r)-\frac{1}{2}\delta_2)-\pi L)/\varepsilon}+e^{(\mu(m_0-1+\theta(r))-\pi L-\delta')/\varepsilon}).\end{aligned} \tag{5.8}$$

Now, we first fix $\delta > 0$ such that (5.6) is satisfied, and then $r > 0$ and $\mu > 0$ sufficiently small, in such a way that $\theta(r) \leq \frac{1}{4}\delta_2$ and $(\mu(m_0 - 1 + \theta(r)) \leq \frac{1}{2}\delta'$. We obtain,

$$\|u_\varepsilon\|^2_{L^2(B_r)} + \varepsilon^2\|\nabla u_\varepsilon\|^2_{L^2(B_r)} = \mathcal{O}(e^{-\pi L/\varepsilon}(e^{-\frac{\mu}{4}\delta_2/\varepsilon} + e^{-\frac{1}{2}\delta'/\varepsilon}))$$

In other words, we have extended the estimate (5.3) across the boundary $\{f = 0\}$ near (x_0, y_0). Our argument can be performed near any point $(x_0, y_0) \in \mathbf{E}$ where an estimate like (5.3) is valid, and thus, starting from the points of the circle $\{|(x, y)| = R_-\}$ [where the estimate is valid thanks to the Proposition 4.1 and the assumption (5.1)], and deforming continuously this circle up to make it become the boundary of $\mathcal{B}$, a standard covering argument leads to,

Proposition 5.1. *Under assumption* (5.1)*, for any compact set* $K \subset \mathbf{E}$*, there exists* $\delta = \delta(K) > 0$ *such that,*

$$\|u_\varepsilon\|^2_{H^1(K)} = \mathcal{O}(e^{-(\pi L+\delta)/\varepsilon}),$$

uniformly as $\varepsilon \to 0_+$.

Remark 5.2. Using the equation, we deduce that, actually, in the previous estimate H^1 can be replaced by any H^m, $m \geq 0$.

6. Estimate at the Boundary

Now, we plan to propagate the estimates of the previous section up to the boundary of $\mathcal{B}$ (but away from any arbitrarily small neighborhood of M_0), by making use of the Carleman estimate at the boundary as stated in [15], Proposition 2 [see also [14], Theorem 7.6, applied to $e^{-\rho t}u_\varepsilon(x, y)$].

We consider an arbitrary point (x_0, y_0) on the boundary $\partial\mathcal{B}$ of $\mathcal{B}$, with $(x_0, y_0) \neq (L, 0)$, and a small enough open neighborhood V of (x_0, y_0) in $\mathbb{R}^2$. We also consider a compact neighborhood $K \subset V$ of (x_0, y_0), and we denote by f a function defining $\partial\mathcal{B}$ near (x_0, y_0), in the sense that one has,

$$\mathcal{B} \cap V = \{(x, y) \in V\,;\, f(x, y) < 0\},$$

and $\nabla f \neq 0$ on V. Finally, as in following [14, 15], one sets,

$$\varphi(x, y) := e^{\lambda(f(x,y)-(x-x_0)^2-(y-y_0)^2)},$$

where $\lambda > 0$ is fixed sufficiently large and $C_0 > \sup_V(f(x, y) - (x - x_0)^2 - (y - y_0)^2))$. In particular, if V has been taken sufficiently small, we see (e.g., as in [14], Lemma A.1) that φ satisfies Assumption (8) of [15]. Moreover, since the outward pointing unit normal to $\mathbf{E}$ in V is $n := -\nabla f/|\nabla f|$, we also have $\partial_n\varphi\,|_{\partial\mathbf{E}\cap V} < 0$. Therefore, we can apply Proposition 2 of [15] (or, alternatively, Theorem 7.6 of [14]), and we obtain the existence of a constant $C > 0$ such that, for any $\mu, \varepsilon > 0$ with ε/μ small enough,

$$\begin{aligned}&\|e^{\mu\varphi/\varepsilon}\chi u_\varepsilon\|^2_{L^2(\mathbf{E}\cap V)} + \mu^{-2}\varepsilon^2\|e^{\mu\varphi/\varepsilon}\nabla(\chi u_\varepsilon)\|^2_{L^2(\mathbf{E}\cap V)}\\ &\quad\leq C\mu^{-3}\varepsilon^3\|e^{\mu\varphi/\varepsilon}\Delta(\chi u_\varepsilon)\|^2_{L^2(\mathbf{E}\cap V)},\end{aligned}$$

where $\chi \in C_0^\infty(V\,;\,[0,1])$ is some fixed cutoff function such that $\chi = 1$ on K. Using that $-\Delta u_\varepsilon = \rho u_\varepsilon$, for ε small enough, we deduce,

$$\begin{aligned}&\|e^{\mu\varphi/\varepsilon}u_\varepsilon\|^2_{L^2(\mathbf{E}\cap K)} + \mu^{-2}\varepsilon^2\|e^{\mu\varphi/\varepsilon}\nabla u_\varepsilon\|^2_{L^2(\mathbf{E}\cap K)}\\ &\quad\le 2C\mu^{-3}\varepsilon^3\|e^{\mu\varphi/\varepsilon}[\Delta,\chi]u_\varepsilon)\|^2_{L^2(\mathbf{E}\cap V)}.\end{aligned}$$

Now, for all $\delta > 0$ small enough, on $\mathrm{Supp}\nabla\chi \cap \{f \le \delta\} \cap V$, we have,

$$\varphi \le \varphi(x_0, y_0) - \delta',$$

with $\delta' = \delta'(\delta) > 0$. On the other hand, on $\{f \ge \delta\} \cap V$, by Proposition 5.1 we have,

$$\|u_\varepsilon\|^2_{L^2(\{f\ge\delta\}\cap V)} = \mathcal{O}(e^{-(\pi L+\delta')/\varepsilon}).$$

Therefore, using also (3.2), and fixing $\mu > 0$ in a convenient way as before, we obtain the existence of $\delta_1 > 0$, such that,

$$\|e^{\mu\varphi/\varepsilon}u_\varepsilon\|^2_{L^2(\mathbf{E}\cap K)} + \varepsilon^2\|e^{\mu\varphi/\varepsilon}\nabla u_\varepsilon\|^2_{L^2(\mathbf{E}\cap K)} = \mathcal{O}(e^{(\mu\varphi(x_0,y_0)-\pi L-\delta_1)/\varepsilon}),$$

and if $V' \subset K$ is a sufficiently small neighborhood of (x_0, y_0), we finally obtain,

$$\|u_\varepsilon\|^2_{H^1(\mathbf{E}\cap V')} = \mathcal{O}(e^{-(\pi L+\frac{1}{2}\delta_1)/\varepsilon}).$$

Since (x_0, y_0) was arbitrary on $\partial\mathcal{B}\backslash\{M_0\}$ (where $M_0 = (L, 0)$), we have proved,

Proposition 6.1. *Under the assumption (5.1), for any neighborhood $\mathcal{U}$ of M_0 and any compact set $K \subset \mathbb{R}^2$, there exists $\delta > 0$ such that,*

$$\|u_\varepsilon\|^2_{H^1(\mathbf{E}\cap K\backslash\mathcal{U})} = \mathcal{O}(e^{-(\pi L+\delta)/\varepsilon}),$$

uniformly as $\varepsilon \to 0_+$.

Remark 6.2. Using the equation and a standard result of regularity on the Dirichlet Laplacian (see, e.g., [2]), we can deduce that, in the previous estimate, H^1 can be replaced by any H^m, $m \ge 0$.

7. Estimate Near the Aperture

Now, we concentrate our attention to a small neighborhood of M_0 in $\overline{\mathbf{E}}$. More precisely, we fix $\varepsilon_1 \in (0, \varepsilon_0]$, such that,

$$\frac{\pi^2}{4\varepsilon_1^2} > \lambda_0,$$

and we consider the rectangle,

$$Q := [L_\varepsilon, L + \varepsilon_1] \times [-\varepsilon_1, \varepsilon_1],$$

where $L_\varepsilon = L - \mathcal{O}(\varepsilon^2)$ is defined as the unique value such that $(L_\varepsilon, \pm\varepsilon) \in \partial\mathcal{B}$.

In particular, the point $M_\varepsilon := (L_\varepsilon, 0)$ belongs to ∂Q, and, if ε_1 is taken sufficiently small, then,

$$Q\backslash(\{L_\varepsilon\} \times [-\varepsilon_1, \varepsilon_1]) \subset \Omega(\varepsilon).$$

Moreover, by Proposition 6.1, we know the existence of some $\delta > 0$ such that u_ε is $\mathcal{O}(e^{-(\pi L+\delta)/\varepsilon})$ near $\partial Q\backslash(\{L_\varepsilon\} \times [-\varepsilon_1, \varepsilon_1])$.

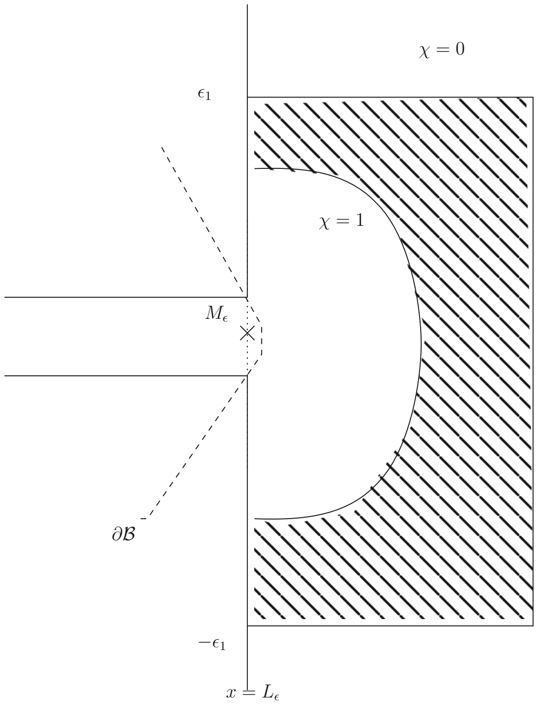

FIGURE 2. The aperture

Let $\chi \in C_0^\infty(\mathbb{R}^2; [0,1])$ such that (see Fig. 2),

- $\chi = 1$ on $[L_\varepsilon, L + \frac{1}{2}\varepsilon_1] \times [-\frac{1}{2}\varepsilon_1, \frac{1}{2}\varepsilon_1]$;
- $\chi = 0$ on $([L + \varepsilon_1, +\infty) \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty, -\varepsilon_1]) \cup (\mathbb{R} \times [\varepsilon_1, +\infty))$.

We set,

$$v := \chi u_\varepsilon.$$

In particular, $v \in H^2(Q)$ and $v\big|_{|y|=\varepsilon_1} = 0$. Therefore, on Q, we can expand v as,

$$v(x,y) = \sum_{j\geq 1} v_j(x)\varphi_j(y), \tag{7.1}$$

where φ_js are the eigenfunctions of the Dirichlet realization of $-d^2/dy^2$ on $[-\varepsilon_1, \varepsilon_1]$, namely,

$$\varphi_{2j}(y) = \frac{1}{\sqrt{\varepsilon_1}} \sin(\alpha_{2j} y/\varepsilon_1)\,;\ \varphi_{2j-1}(y) = \frac{1}{\sqrt{\varepsilon_1}} \cos(\alpha_{2j-1} y/\varepsilon_1)\,;\ \alpha_j := \frac{j\pi}{2},$$

and $v_j \in H^2([L_\varepsilon, L + \varepsilon_1])$. Moreover, using Proposition 6.1 and Remark 6.2, on Q we have,

$$-\Delta v = \rho v + r$$

where $\|r\|^2_{H^m(Q)} = \|[\Delta,\chi]u_\varepsilon\|^2_{H^m(Q)} = \mathcal{O}(e^{-(\pi L+\delta)/\varepsilon})$, and $r\left|_{|y|=\varepsilon_1}\right. = 0$ ($m \geq 0$ arbitrary, and $\delta = \delta(m) > 0$). We deduce that the v_js verify,

$$-v_j'' + \beta_j v_j = r_j, \tag{7.2}$$

where we have set $\beta_j := \frac{\alpha_j^2}{\varepsilon_1^2} - \rho$, and $r_j := \int_{-\varepsilon_1}^{\varepsilon_1} r(x,y)\varphi_j(y)dy$, so that we have,

$$\sum_{j\geq 1} j^m \|r_j\|^2_{H^m([L,L+\varepsilon_1])} = \mathcal{O}(e^{-(\pi L+\delta)/\varepsilon}). \tag{7.3}$$

By construction, we also have $v_j = 0$ on $[L+\varepsilon_1, +\infty)$.

Proposition 7.1. *Assume (5.1). Then, for all $j \geq 1$, there exist $b_j \in \mathbb{C}$ and $s_j \in \cap_{m\geq 0} H^m([L, L+\varepsilon_1]$, such that,*

$$v_j(x) = b_j e^{-(x-L_\varepsilon)\sqrt{\beta_j}} + s_j(x);$$
$$\sum_{j\geq 1} j^m \|s_j\|^2_{H^m([L_\varepsilon,L+\varepsilon_1])} = \mathcal{O}(e^{-(\pi L+\delta_m)/\varepsilon}),$$

with $\delta_m > 0$ and uniformly with respect to ε small enough.

Proof. Set,

$$W_j := \begin{pmatrix} v_j \\ v_j' \end{pmatrix}.$$

Then, by (7.2), W_j is the solution of,

$$\begin{cases} W_j' = A_j W_j - R_j; \\ W_j(L+\varepsilon_1) = 0, \end{cases}$$

with $A_j := \begin{pmatrix} 0 & 1 \\ \beta_j & 0 \end{pmatrix}$ and $R_j := \begin{pmatrix} 0 \\ r_j \end{pmatrix}$. Therefore,

$$W_j(x) = \int_x^{L+\varepsilon_1} e^{(x-t)A_j} R_j(t)dt,$$

and, diagonalizing A_j and re-writing the solution in a basis of eigenvectors of A_j, we obtain in particular,

$$v_j'(x) + \sqrt{\beta_j} v_j(x) = \int_x^{L+\varepsilon_1} e^{(x-t)\sqrt{\beta_j}} r_j(t)dt.$$

Using again that $v(L+\varepsilon_1) = 0$, we deduce,

$$v_j(x) = -\int_x^{L+\varepsilon_1}\int_{x_1}^{L+\varepsilon_1} e^{(2x_1-t-x)\sqrt{\beta_j}} r_j(t)dtdx_1.$$

Then, the results follow with $b_j := -\int_{L_\varepsilon}^{L+\varepsilon_1}\int_{x_1}^{L+\varepsilon_1} e^{(2x_1-t-L_\varepsilon)\sqrt{\beta_j}} r_j(t)dtdx_1$ and $s_j(x) := \int_{L_\varepsilon}^{x}\int_{x_1}^{L+\varepsilon_1} e^{(2x_1-t-x)\sqrt{\beta_j}} r_j(t)dtdx_1$, by observing that $\mathrm{Re}((2x_1 - t - x)\sqrt{\beta_j}) < 0$ on the domain of integration of $s_j(x)$ and using (7.3). □

Remark 7.2. Let $\varepsilon_2 \in (0, \frac{1}{2}\varepsilon_1)$ arbitrary. By Proposition 5.1, we know that there exists a constant $\delta = \delta(\varepsilon_2) > 0$ such that,

$$\|v\|_{L^2((L+\varepsilon_2, L+\varepsilon_1)\times(-\varepsilon_1,\varepsilon_1))} = \mathcal{O}(e^{-(\pi L+\delta)/2\varepsilon}).$$

On the other hand, using (7.1) and Proposition 7.1, on $(L_\varepsilon, L+\varepsilon_1)\times(-\varepsilon_1,\varepsilon_1))$, we have,

$$v(x,y) = \sum_{j\geq 1} b_j e^{-(x-L_\varepsilon)\sqrt{\beta_j}}\varphi_j(y) + s(x,y),$$

with $\|s\|_{L^2((L_\varepsilon, L+\varepsilon_1)\times(-\varepsilon_1,\varepsilon_1))} = \mathcal{O}(e^{-(\pi L+\delta_0)/2\varepsilon})$ for some constant $\delta_0 > 0$. Since $\sqrt{\beta}_j \sim \frac{j\pi}{2\varepsilon_1}$ as $j \to \infty$, and ε_2 is arbitrarily small, we immediately deduce that, for any $\nu > 0$, there exists $\delta = \delta(\nu) > 0$, such that,

$$\sum_{j\geq 1} |b_j|^2 e^{-\nu j} = \mathcal{O}(e^{-(\pi L+\delta)/\varepsilon}), \tag{7.4}$$

uniformly as $\varepsilon \to 0_+$.

8. Representations at the Aperture

In this section, we consider the trace of v on $\{x = L_\varepsilon\}$. By construction, it also coincides with the trace u_ε as long as $|y| < \frac{1}{2}\varepsilon_1$. Now, as in [17], there are two ways of taking this trace, depending if one takes the limit $x \to (L_\varepsilon)_+$ or $x \to (L_\varepsilon)_-$.

Considering first the limit $x \to (L_\varepsilon)_-$, we can just apply the results of [17], Sections 4 & 6 (in particular (4.2), (4.3) and Lemma 6.1), and for $x < L_\varepsilon$ close to L_ε and $|y| < \varepsilon$, we obtain,

$$v(x,y) = \sum_{k=1}^{\infty} \left(a_{k,+}e^{\theta_k x/\varepsilon} + a_{k,-}e^{-\theta_k x/\varepsilon}\right)\psi_k(y), \tag{8.1}$$

where we have used the notations,

$$\psi_{2k}(y) = \frac{1}{\sqrt{\varepsilon}}\sin(\alpha_{2k}y/\varepsilon)\,;\ \psi_{2k-1}(y) = \frac{1}{\sqrt{\varepsilon}}\cos(\alpha_{2k-1}y/\varepsilon)\,;\ \alpha_k := \frac{k\pi}{2};$$
$$\theta_k := \sqrt{\alpha_k^2 - \varepsilon^2\rho(\varepsilon)},$$

(here, $\sqrt{\cdot}$ stands for the principal square root), and where $a_{k,\pm}$ are (ε-dependent) constant complex numbers. Moreover, the sum converges in $H^2((L-\varepsilon_1, L_\varepsilon)\times(-\varepsilon,\varepsilon))$, and the limit $x \to (L_\varepsilon)_-$ gives (see [17], Lemma 6.1),

$$v(L_\varepsilon,y) = \sum_{k=1}^{\infty} \left(a_{k,+}e^{\theta_k L_\varepsilon/\varepsilon} + a_{k,-}e^{-\theta_k L_\varepsilon/\varepsilon}\right)\psi_k(y), \tag{8.2}$$

together with [see [17], formula (6.7)],

$$\partial_x v(L,y) = \frac{1}{\varepsilon}\sum_{k\geq 1}\theta_k\left(a_{k,+}e^{\theta_k L_\varepsilon/\varepsilon} - a_{k,-}e^{-\theta_k L_\varepsilon/\varepsilon}\right)\psi_k(y) \text{ in } H^{1/2}(|y|\leq\varepsilon). \tag{8.3}$$

Then, starting from(7.1), and using similar arguments, the limit $x \to (L_\varepsilon)_+$ can be taken in the same way, and using Proposition 7.1, we obtain,

$$v(L_\varepsilon,y) = \sum_{j=1}^{\infty}(b_j + s_j(L_\varepsilon))\,\varphi_j(y), \tag{8.4}$$

together with,

$$\partial_x v(L_\varepsilon,y) = \sum_{j=1}^{\infty}(-\sqrt{\beta_j}b_j + s_j{}'(L_\varepsilon))\varphi_j(y) \text{ in } H^{1/2}(|y|\leq\varepsilon_1). \tag{8.5}$$

Moreover, still by Proposition 7.1, we have,

$$\sum_{j\geq 1}\left(|s_j(L_\varepsilon)|^2 + |s_j'(L_\varepsilon)|^2\right) = \mathcal{O}(e^{-(\pi L+\delta)/\varepsilon}), \tag{8.6}$$

for some constant $\delta > 0$.

9. Estimates on the Coefficients

At this point, we can proceed as [17], Section 7 (but working with v instead of u_ε), with the difference that, in our present case, the index j_0 appearing in [17], formula (6.8), is just 0 (that is, all the sums over $\{j\leq j_0\}$ become null). For the sake of completeness, we briefly reproduce these arguments here.

The main idea consists in computing in two different ways the three following quantities:

$$\langle v,\partial_x v\rangle_{\{L_\varepsilon\}\times[-\varepsilon,\varepsilon]}\,,\ \langle v,\varphi_1\rangle_{\{L_\varepsilon\}\times[-\varepsilon,\varepsilon]}\,,\ \langle\partial_x v,\psi_1\rangle_{\{L\}\times[-\varepsilon,\varepsilon]}.$$

We set

$$A_{k,\pm} := a_{k,\pm}e^{\pm\theta_k L/\varepsilon}.$$

In view of (8.2–8.6), the two computations of $\langle v,\partial_x v\rangle_{\{L_\varepsilon\}\times[-\varepsilon,\varepsilon]}$ give the identity

$$\frac{1}{\varepsilon}\sum_{k\geq 1}\theta_k(|A_{k,+}|^2 - |A_{k,-}|^2 + 2i\,\mathrm{Im}(A_{k,+}\overline{A}_{k,-})) = -\sum_{j\geq 1}(\sqrt{\beta_j})|b_j|^2 + r(\varepsilon),$$

with

$$\begin{aligned} r(\varepsilon) &= \mathcal{O}\left(e^{-(\pi L+\delta)/\varepsilon} + e^{-(\pi L+\delta)/2\varepsilon}\left(\sum_{j\geq 1}|b_j|^2\right)^{\frac{1}{2}}\right)\\ &= \mathcal{O}\left(e^{-(\pi L+\frac{\delta}{2})/\varepsilon} + e^{-\delta/\varepsilon}\sum_{j\geq 1}|b_j|^2\right). \end{aligned} \tag{9.1}$$

Taking the real part, and using the fact that $\operatorname{Re}\theta_k \sim k\pi/2$ as $k \to \infty$, while $|\operatorname{Im}\theta_k| = \mathcal{O}(k^{-1}e^{-\delta/\varepsilon})$ for some constant $\delta > 0$, we obtain,

$$\begin{aligned}\frac{1}{\varepsilon}\sum_{k\geq 1}(\operatorname{Re}\theta_k)(|A_{k,+}|^2 - |A_{k,-}|^2) + \frac{1}{\varepsilon}\sum_{k\geq 1}\mathcal{O}(k^{-1}e^{-\delta/\varepsilon})|A_{k,+}A_{k,-}| \\ = -\sum_{j\geq 1}(\operatorname{Re}\sqrt{\beta_j})|b_j|^2 + r(\varepsilon).\end{aligned}$$

In particular, since $\operatorname{Re}\sqrt{\beta_j} = \frac{\pi j}{2\varepsilon_1}(1 + \mathcal{O}(\varepsilon^2 j^{-2}))$, we see that there exists a constant $C > 0$ such that

$$\begin{aligned}\sum_{k\geq 1}\operatorname{Re}\theta_k(|A_{k,+}|^2 - |A_{k,-}|^2) \leq C\sum_{k\geq 1}k^{-1}e^{-\delta/\varepsilon}|A_{k,+}A_{k,-}| \\ -\frac{\pi}{2}\frac{\varepsilon}{\varepsilon_1}\sum_{j\geq 1}j(1 - C\varepsilon^2 j^{-2})|b_j|^2 + r(\varepsilon). \qquad (9.2)\end{aligned}$$

Moreover, by Appendix A in [17], there exists a constant $c > 0$, such that,

$$\sum_{k\geq 1}k|a_{k,-}e^{-c\theta_k}|^2 = \mathcal{O}(\varepsilon^{-1/2}), \qquad (9.3)$$

and thus, for ε small enough,

$$\sum_{k\geq 2}k|A_{k,-}|^2 = \sum_{k\geq 2}k|a_{k,-}e^{-c\theta_k}|^2 e^{-2\theta_k(\frac{L}{\varepsilon}-c)} = \mathcal{O}(\varepsilon^{-1/2}e^{-2\pi L/\varepsilon}). \qquad (9.4)$$

Therefore, we deduce from (9.2)(with some new positive constants C, δ),

$$\begin{aligned}&\sum_{k\geq 1}(k - Ck^{-1}e^{-\delta/\varepsilon})|A_{k,+}|^2 \\ &\quad\leq (1 + Ce^{-\delta/\varepsilon})|A_{1,-}|^2 - \frac{2\varepsilon}{\pi}(1 + r_1(\varepsilon))\sum_{j\geq 1}\operatorname{Re}\sqrt{\beta_j}|b_j|^2 + r_2(\varepsilon), \qquad (9.5)\end{aligned}$$

with

$$r_1(\varepsilon) = \mathcal{O}\left(e^{-\delta/\varepsilon}\right); \quad r_2(\varepsilon) = \mathcal{O}\left(e^{-(\pi L+\delta)/\varepsilon}\right). \qquad (9.6)$$

Now, computing $\langle v(L_\varepsilon,\cdot), \varphi_1\rangle_{L^2(|y|<\varepsilon)}$ and $\langle \partial_x v(L_\varepsilon,\cdot), \psi_1\rangle_{L^2(|y|<\varepsilon)}$ in two different ways [using (8.2–8.6)], we find

$$\sum_{k\geq 1}\mu_k(A_{k,+} + A_{k,-}) = b_1;$$

$$\frac{1}{\varepsilon}\theta_1(A_{1,+} - A_{1,-}) = -\sum_{j\geq 1}\nu_j(\sqrt{\beta_j}b_j - s_j'(L_\varepsilon)),$$

with

$$\mu_k := \int_{-\varepsilon}^{\varepsilon}\psi_k(y)\varphi_1(y)dy = \begin{cases}0 & \text{if } k \text{ is even;}\\ (-1)^{\frac{k-1}{2}}\frac{4k\sqrt{\varepsilon/\varepsilon_1}}{\pi(k^2-(\varepsilon/\varepsilon_1)^2)}\cos\frac{\pi}{2}\frac{\varepsilon}{\varepsilon_1} & \text{if } k \text{ is odd,}\end{cases}$$

and

$$\nu_j := \int_{-\varepsilon}^{\varepsilon} \varphi_j(y)\psi_1(y)dy = \begin{cases} 0 \quad \text{if } j \text{ is even;} \\ \frac{4\sqrt{\varepsilon/\varepsilon_1}\sin(((\varepsilon/\varepsilon_1)j-1)\pi/2)}{\pi((\varepsilon/\varepsilon_1)^2j^2-1)} \quad \text{if } j \neq \frac{\varepsilon_1}{\varepsilon} \text{ is odd;} \\ \sqrt{(\varepsilon/\varepsilon_1)} \quad \text{if } j = \frac{\varepsilon_1}{\varepsilon} \text{ is odd.} \end{cases}$$

Using (9.4) again and (7.4), we obtain

$$|A_{1,+} + A_{1,-}| \leq Ce^{-(\pi L+\delta)/2\varepsilon} + \sum_{k\geq 2} |\frac{\mu_k}{\mu_1} A_{k,+}| + \frac{C}{\sqrt{\varepsilon}} e^{-\pi L/\varepsilon}; \tag{9.7}$$

$$|A_{1,+} - A_{1,-}| \leq \frac{\varepsilon}{|\theta_1|} \sum_{j\geq 1} |\nu_j \sqrt{\beta_j} b_j| + Ce^{-(\pi L+\delta)/2\varepsilon}, \tag{9.8}$$

with some new constant $C > 0$.

Then, we observe that $|\mu_k/\mu_1| \leq (k - \frac{\varepsilon^2}{\varepsilon_1^2})^{-1}$ (k odd), thus by (9.5),

$$\begin{aligned} \sum_{k\geq 2} |\frac{\mu_k}{\mu_1} A_{k,+}| &\leq \left(\sum_{k\geq 3} \frac{1}{k(k-\frac{\varepsilon^2}{\varepsilon_1^2})^2} \right)^{\frac{1}{2}} \left(\sum_{k\geq 2} k|A_{k,+}|^2 \right)^{\frac{1}{2}} \\ &\leq \tau_1 \left(\alpha |A_{1,-}|^2 - \beta \frac{2\varepsilon}{\pi} \sum_{j\geq 1} \operatorname{Re}\sqrt{\beta_j}|b_j|^2 + r_2(\varepsilon) \right)^{\frac{1}{2}} + Ce^{-(\pi L+\delta)/2\varepsilon}, \end{aligned} \tag{9.9}$$

where τ_1 can be taken arbitrarily close to $(\sum_{k\geq 3} k^{-3})^{\frac{1}{2}} < \frac{1}{2}$, and α, β are positive numbers that tend to 1 as $\varepsilon \to 0$, and are such that $\alpha|A_{1,-}|^2 - \beta\frac{2\varepsilon}{\pi}\sum_{j\geq 1}\operatorname{Re}\sqrt{\beta_j}|b_j|^2 + r_2(\varepsilon)$ remains non-negative for all $\varepsilon > 0$ small enough. Inserting (9.9) into (9.7), we obtain

$$\begin{aligned} &|A_{1,+} + A_{1,-}| \\ &\quad \leq \tau_1 \left(\alpha |A_{1,-}|^2 - \beta \frac{2\varepsilon}{\pi} \sum_{j\geq 1} \operatorname{Re}\sqrt{\beta_j}|b_j|^2 + r_2(\varepsilon) \right)^{\frac{1}{2}} + 2Ce^{-(\pi L+\delta)/2\varepsilon}. \end{aligned} \tag{9.10}$$

On the other hand, going back to (9.8), the Cauchy–Schwarz inequality gives,

$$\frac{\varepsilon}{|\theta_1|} \sum_{j\geq 1} |\nu_j \sqrt{\beta_j} b_j| \leq \tau_2 \left(\frac{2\varepsilon}{\pi} \sum_{j\geq 1} |b_j|^2 |\sqrt{\beta_j}| \right)^{\frac{1}{2}} \tag{9.11}$$

with

$$\begin{aligned} \tau_2^2 &= \frac{\varepsilon\pi}{2|\theta_1|^2} \sum_{j\geq 1} j|\nu_j|^2 |\sqrt{\beta_j}| \\ &= \frac{16}{\pi^2}(1+\mathcal{O}(\varepsilon^2)) \sum_{j\geq 1,\, j \text{ odd}} \frac{\varepsilon}{\varepsilon_1} \frac{\frac{j\varepsilon}{\varepsilon_1}\sin^2\left((\frac{j\varepsilon}{\varepsilon_1}-1)\frac{\pi}{2}\right)}{\left((\frac{j\varepsilon}{\varepsilon_1})^2-1\right)^2}(1+\mathcal{O}(j^{-2})) \end{aligned} \tag{9.12}$$

In particular, when $\varepsilon \to 0$, then τ_2 tends to $\Gamma_2 := \frac{2\sqrt{2}}{\pi} \left(\int_0^\infty \frac{x\sin^2((x-1)\pi/2)}{(x^2-1)^2}dx\right)^{\frac{1}{2}}$, and we deduce from (9.8) and (9.11), plus the fact that $\operatorname{Im}\sqrt{\beta_j} = \mathcal{O}(e^{-\delta/\varepsilon})$ uniformly,

$$|A_{1,+} - A_{1,-}| \leq \tilde{\tau}_2 \left(\frac{2\varepsilon}{\pi}\sum_{j\geq 1} \operatorname{Re}\sqrt{\beta_j}|b_j|^2\right)^{\frac{1}{2}} + Ce^{-(\pi L+\delta)/2\varepsilon}, \tag{9.13}$$

where $\tilde{\tau}_2$ can be taken arbitrarily close to Γ_2. Actually, Γ_2 can be computed exactly, and one finds,

$$\Gamma_2 = \frac{2\sqrt{2}}{\pi}\left(-\frac{1}{2} + \frac{\pi}{4}\operatorname{Si}(\pi)\right)^{\frac{1}{2}} \approx 0,879.$$

(Here, $\operatorname{Si}(x) := \int_0^x \frac{\sin t}{t}dt$.)

Summing (9.10) with (9.13), and using the triangle inequality, we finally obtain

$$2|A_{1,-}| \leq \tau_1\sqrt{\alpha|A_{1,-}|^2 - \beta X + r_2(\varepsilon)} + \tau_2\sqrt{X} + 3Ce^{-(\pi L+\delta)/2\varepsilon}, \tag{9.14}$$

where we have set

$$X := \frac{2\varepsilon}{\pi}\sum_j \operatorname{Re}\sqrt{\beta_j}|b_j|^2.$$

Now, an elementary computation shows that the map

$$[0, A^2] \ni Y \mapsto \tau_1\sqrt{A^2 - \beta Y^2} + \tau_2 Y$$

reaches its maximum at $Y = \frac{\tau_2^2}{\beta\tau_1^2+\tau_2^2}A/\sqrt{\beta}$, and the maximum value is

$$\left(\sqrt{\tau_1^2 + \beta^{-1}\tilde{\tau}_2^2}\right)A.$$

Therefore, we deduce from (9.14),

$$\begin{aligned} 2|A_{1,-}| &\leq \left(\sqrt{\tau_1^2 + \beta^{-1}\tilde{\tau}_2^2}\right)\sqrt{\alpha|A_{1,-}|^2 + r_2(\varepsilon)} + 3Ce^{-(\pi L+\delta)/2\varepsilon} \\ &\leq \left(\sqrt{\alpha(\tau_1^2 + \beta^{-1}\tilde{\tau}_2^2)}\right)|A_{1,-}| + \mathcal{O}(e^{-(\pi L+\delta)/2\varepsilon}). \end{aligned} \tag{9.15}$$

Since $\sqrt{\alpha(\tau_1^2 + \beta^{-1}\tau_2^2)}$ tends to $\sqrt{\sum_{k\geq 2} k^{-3} + \Gamma_2^2}$ as $\varepsilon \to 0$, and

$$\sum_{k\geq 3} k^{-3} + \Gamma_2^2 \leq \frac{1}{4} + \frac{8}{10} < 4,$$

we have proved,

Proposition 9.1. *Under the assumption* (5.1), *there exist two constants* $C, \delta > 0$ *such that, for any* $\varepsilon > 0$ *small enough, one has,*

$$|A_{1,-}| \leq Ce^{-(\pi L+\delta)/2\varepsilon}. \tag{9.16}$$

10. End of the Proof

By Assumption **(H)**, we see that the Dirichlet eigenfunction u_0 satisfies the hypothesis of [3] Lemma 3.1. Then, following the arguments of [3] leading to (13) in that paper, and using again [13], Proposition 3.1 and Formula (5.13), we conclude that for any $\delta > 0$ and any $x \in (0, L)$, there exists C_1 such that the resonant state u_ε verifies (see [3], Formula (13)),

$$\|u_\varepsilon\|_{L^2([x,L_\varepsilon]\times[-\varepsilon,\varepsilon])} \geq \frac{1}{C_0}\varepsilon^{4.5+\delta}e^{-\pi x/2\varepsilon}. \tag{10.1}$$

Using this estimate, we can now prove as in [17], Proposition 8.2, the following proposition that contradicts the inequality (9.16), and thus completes the proof the theorem 2.2.

Proposition 10.1. *For any $\delta > 0$, there exists $C > 0$, such that*

$$|A_{1,-}| \geq \frac{1}{C}\varepsilon^{4.5+\delta}e^{-\pi L/2\varepsilon}, \tag{10.2}$$

for $\varepsilon > 0$ small enough.

Proof. Starting from (9.5), we see,

$$\sum_{k\geq 1}|A_{k,+}|^2 \leq (1 + Ce^{-\delta/\varepsilon})|A_{1,-}|^2 + Ce^{-(\pi L+\delta)/\varepsilon}. \tag{10.3}$$

Then, computing the quantity $\|u_\varepsilon\|_{L^2([x,L]\times[-\varepsilon,\varepsilon])}$ using the expression (8.1), we obtain (see [17], proof of Proposition 8.2),

$$\begin{aligned}\|u_\varepsilon\|^2_{L^2([x,L_\varepsilon]\times[-\varepsilon,\varepsilon])} \leq 4\sum_{k\geq 1}|A_{k,+}|^2 + 4\sum_{k>1}|a_{k,-}|^2e^{-2x\operatorname{Re}\theta_k/\varepsilon}\\ +\varepsilon|a_{1,-}|^2e^{-2x\operatorname{Re}\theta_1/\varepsilon}.\end{aligned} \tag{10.4}$$

Using (10.3) and (9.3), we deduce

$$\begin{aligned}\|u_\varepsilon\|^2_{L^2([x,L_\varepsilon]\times[-\varepsilon,\varepsilon])} \leq C\varepsilon|a_{1,-}|^2e^{-2x\operatorname{Re}\theta_1/\varepsilon} + C|a_{1,-}|^2e^{-2L\operatorname{Re}\theta_1/\varepsilon}\\ +C\varepsilon^{-C}e^{-2x\operatorname{Re}\theta_1/\varepsilon}e^{-2C_0x/\varepsilon} + Ce^{-(\pi L+\delta)/\varepsilon},\end{aligned} \tag{10.5}$$

and thus, using (10.1), we finally obtain,

$$\varepsilon^{9+2\delta} \leq C|a_{1,-}|^2, \tag{10.6}$$

and the result is proved. $\square$

11. An Extension to Larger Dimensions

Here, we consider the similar problem in dimension $n \geq 3$, obtained by taking tubes with square sections. That is, $\mathcal{C}$ is a regular bounded open subset of $\mathbb{R}^n$, and we have (in Euclidean coordinates $x = (x_1, \ldots, x_n) = (x_1, x') \in \mathbb{R}\times\mathbb{R}^{n-1}$),

$$\begin{aligned}&\overline{\mathcal{C}} \subset \mathcal{B};\\ &(0,0) \in \partial\mathcal{C}; (L,0) \in \partial\mathcal{B};\\ &[0,L] \times \{0\} \subset \overline{\mathcal{B}}\backslash\mathcal{C};\\ &\text{Near } M_0 := (L,0),\ \mathcal{B} \text{ is convex and } \partial\mathcal{B} \text{ is symmetric with}\\ &\text{respect to } \{x_j = 0\} \text{ for all } j \geq 2.\end{aligned} \tag{11.1}$$

Remark 11.1. In particular, this also contains the case where $\partial\mathcal{B}$ is flat near M_0, that is when $\{(L, x_2, \ldots, x_n)\,;\, |x_j| < \varepsilon_0,\, j = 2, \ldots, n\} \subset \partial\mathcal{B}$ for some $\varepsilon_0 > 0$.

Then, setting $Q_\varepsilon := \{(x_2, \ldots, x_n)\,;\, |x_j| < \varepsilon,\, j = 2, \ldots, n\}$, $\mathcal{T}(\varepsilon) := [-\varepsilon_0, L] \times Q_\varepsilon \cap (\mathbb{R}^n\backslash\mathcal{C})$, and $\mathbf{E} := \mathbb{R}^n\backslash\overline{\mathcal{B}}$, we consider the resonances of the resonator $\Omega(\varepsilon) := \mathcal{C} \cup \mathcal{T}(\varepsilon) \cup \mathbf{E}$.

As before, let λ_0 be an eigenvalue of $-\Delta_{\mathcal{C}}$, and let u_0 be the corresponding normalized eigenfunction.

In this situation, the lower estimate of [13] (see also [3]) becomes

$$\operatorname{Im}\rho(\varepsilon) = \mathcal{O}(e^{-(1-\delta)\pi L\sqrt{n-1}/\varepsilon}),$$

where $\rho(\varepsilon)$ stands for any resonance that tends to λ_0 as $\varepsilon \to 0_+$, and $\delta > 0$ is arbitrary.

We assume again,

Assumption (**H**):

$$\lambda_0 \text{ is the lowest eigenvalue of} - \Delta_{\mathcal{C}}\ .$$

Then, we have

Theorem 11.2. *Assume* (**H**) *and* $2 \leq n \leq 12$. *Then, for any* $\delta > 0$, *there exists* $C_\delta > 0$ *such that, the only resonance* $\rho(\varepsilon)$ *close to* λ_0 *satisfies,*

$$|\operatorname{Im}\rho(\varepsilon)| \geq \frac{1}{C_\delta} e^{-\pi(1+\delta)L\sqrt{n-1}/\varepsilon},$$

uniformly as $\varepsilon \to 0_+$.

Proof. The computations are very similar to those in dimension 2, and we highlight here only what is specific to dimension n. The notations are similar, but their meaning is modified as follows. For $k = (k_2, \ldots, k_n) \in \mathbb{N}^{n-1}$ (where $\mathbb{N} := \{1, 2, 3, \ldots\}$), we set

$$\begin{aligned}\alpha_k &:= \left(\frac{k_2\pi}{2}, \ldots, \frac{k_n\pi}{2}\right) \in \mathbb{R}^{n-1};\\ \theta_k &:= \sqrt{|\alpha_k|^2 - \varepsilon^2\rho(\varepsilon)};\\ \beta_k &:= |\alpha_k|^2\varepsilon_1^{-2} - \rho(\varepsilon);\\ \psi_k(x') &:= \psi_{k_2}(x_2)\ldots\psi_{k_n}(x_n);\\ \varphi_k(x') &:= \varphi_{k_2}\ldots(x_2)\varphi_{k_n}(x_n).\end{aligned}$$

(Here, $|k|$ stands for the Euclidean norm of k in $\mathbb{R}^{n-1}$.) With these notations, the formulas (8.1–8.6) remain valid with the following changes:

- $\sum_{k=1}^\infty$ must be replaced by $\sum_{k\in\mathbb{N}^{n-1}}$, and analog for $\sum_{j=1}^\infty$;
- y must be replaced by x';

- $(-\varepsilon,\varepsilon)$ and $(-\varepsilon_1,\varepsilon_1)$ must be, respectively, replaced by Q_ε and Q_{ε_1} (where ε_1 is taken such that $\frac{(n-1)\pi^2}{4\varepsilon_1^2}>\lambda_0$).

Computing in two ways the quantities $\langle v,\partial_x v\rangle_{\{L\}\times Q_\varepsilon}$, $\langle v,\varphi_{1,\dots,1}\rangle_{\{L\}\times Q_\varepsilon}$, and $\langle \partial_x v,\psi_{1,\dots,1}\rangle_{\{L\}\times Q_\varepsilon}$, we find the following analogs of (9.5–9.8):

$$\begin{aligned}
&\sum_{k\in\mathbb{N}^{n-1}}(|k|-C|k|^{-1}e^{-\delta/\varepsilon})|A_{k,+}|^2\\
&\quad\le(1+Ce^{-\delta/\varepsilon})|A_{1,\dots,1,-}|^2-\frac{2\varepsilon}{\pi}(1+r_1)\sum_{j\in\mathbb{N}^{n-1}}\operatorname{Re}\sqrt{\beta_j}|b_j|^2+r_2;\\
&\quad|A_{1,\dots,1,+}+A_{1,\dots,1,-}|\le Ce^{-(\pi L\sqrt{n-1}+\delta)/2\varepsilon}+\sum_{|k|>\sqrt{n-1}}\Big|\frac{\mu_k}{\mu_{1,1}}A_{k,+}\Big|\\
&\qquad\qquad+\frac{C}{\sqrt{\varepsilon}}e^{-\pi L\sqrt{4+(n-2)^2}/2\varepsilon};\\
&\quad|A_{1,1,+}-A_{1,1,-}|\le\frac{\varepsilon}{|\theta_{1,\dots,1}|}\sum_{j\in\mathbb{N}^{n-1}}|\nu_j\sqrt{\beta_j}b_j|+Ce^{-(\pi L\sqrt{n-1}+\delta)/2\varepsilon},
\end{aligned}$$

where we have set

$$\nu_j:=\nu_{j_2}\dots\nu_{j_n};\quad \mu_k:=\mu_{k_2}\dots\mu_{k_n},$$

and with,

$$r_1=\mathcal{O}(e^{-\delta/\varepsilon});\quad r_2=\mathcal{O}(e^{-(\pi L\sqrt{n-1}+\delta)/\varepsilon}).$$

Using the fact that $\mu_{k_2,\dots,k_n}/\mu_{1,\dots,1}\le(k_2-\frac{\varepsilon^2}{\varepsilon_1^2})^{-1}\dots(k_n-\frac{\varepsilon^2}{\varepsilon_1^2})^{-1}$ ($k_2,\dots,k_n$ odd), this also gives

$$\begin{aligned}
&|A_{1,\dots,1,+}+A_{1,\dots,1,-}|\\
&\quad\le\tilde{\tau}_1\left(|A_{1,\dots,1,-}|^2-\frac{\varepsilon}{\varepsilon_1}\sum_{j\in\mathbb{N}^{n-1}}|j||b_j|^2\right)^{\frac12}+Ce^{-(\pi L\sqrt{2}+\delta)/2\varepsilon},
\end{aligned}\tag{11.2}$$

where $\tilde{\tau}_1$ can be taken arbitrarily close to

$$\begin{aligned}
J_1&:=\Big(\sum_{|k|^2>n-1\,;\,k_j\text{ odd}}|k|^{-1}k_2^{-2}\dots k_n^{-2}\Big)^{\frac12}\\
&=\Big(\sum_{k_j\text{ odd}}|k|^{-1}k_2^{-2}\dots k_n^{-2}-\frac{1}{\sqrt{n-1}}\Big)^{\frac12}.
\end{aligned}\tag{11.3}$$

A rough estimate on J_1 can be obtained by writing,

$$J_1^2\le\frac{1}{\sqrt{n-1}}\left(\Big(\sum_{\ell\in\mathbb{N}\text{ odd}}\frac{1}{\ell^2}\Big)^{n-1}-1\right)\le\frac{1}{\sqrt{n-1}}\left(\Big(\frac{\pi^2}{8}\Big)^{n-1}-1\right).$$

In a similar way we obtain,

$$|A_{1,\dots,1,+} - A_{1,\dots,1,-}| \leq \tilde{\tau}_2 \left(\frac{\varepsilon}{\varepsilon_1} \sum_{j \in \mathbb{N}^{n-1}} |j||b_j|^2 \right)^{\frac{1}{2}} + Ce^{-\pi L(\sqrt{n-1}+\delta)/2\varepsilon}, \quad (11.4)$$

where $\tilde{\tau}_2$ can be taken arbitrarily close to the quantity

$$J_2 = \frac{4^{n-1}}{(\pi\sqrt{2})^{n-1}\sqrt{n-1}} \times \left(\int_{\mathbb{R}_+^{n-1}} \frac{|x| \sin^2((x_1-1)\pi/2) \dots \sin^2((x_{n-1}-1)\pi/2)}{(x_1^2-1)^2 \dots (x_{n-1}^2-1)^2} dx_1 \dots dx_{n-1} \right)^{\frac{1}{2}}. \quad (11.5)$$

Writing $|x| \leq |x_1| + \cdots + |x_{n-1}|$ and making permutations on the variables, we obtain,

$$J_2 \leq \frac{4^{n-1}}{(\pi\sqrt{2})^{n-1}} \left(\int_0^{+\infty} \frac{t \sin^2((t-1)\pi/2)}{(t^2-1)^2} dt \right)^{\frac{1}{2}} \times \left(\int_0^{+\infty} \frac{\sin^2((t-1)\pi/2)}{(t^2-1)^2} dt \right)^{\frac{n-2}{2}}$$

Setting

$$L_1 := \int_0^{+\infty} \frac{t \sin^2((t-1)\pi/2)}{(t^2-1)^2} dt; \quad L_2 := \int_0^{+\infty} \frac{\sin^2((t-1)\pi/2)}{(t^2-1)^2} dt,$$

it becomes,

$$J_2 \leq \left(\frac{L_1}{L_2} \right)^{\frac{1}{2}} \left(\frac{4\sqrt{L_2}}{\pi\sqrt{2}} \right)^{n-1}.$$

The integrals L_1 and L_2 can be computed exactly, and one finds,

$$L_1 = -\frac{1}{2} + \frac{\pi}{4}\mathrm{Si}(\pi) \approx 0.9545; \quad L_2 = \frac{\pi^2}{8}.$$

In particular, for ε small enough, we have

$$\tilde{\tau}_1^2 + \tilde{\tau}_2^2 < \frac{8}{10} + \frac{1}{\sqrt{n-1}} \left(\left(\frac{\pi^2}{8} \right)^{n-1} - 1 \right), \quad (11.6)$$

and one can check that this quantity is strictly less than 4 when $2 \leq n \leq 12$.

At this point, we can complete the proof as in the two-dimensional case. $\square$

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Appendix A.

We prove Lemma 4.4. For $k \geq 0$, we can represent $h_k(R) = H_k(R\sqrt{\rho})$ by the formula (see, e.g., [20]),

$$h_k(R) = \frac{1}{i\pi} \int_{-\infty}^{+\infty+i\pi} e^{R\sqrt{\rho}\sinh t - kt} dt,$$

that we split into,

$$\begin{aligned} h_k(R) &= \frac{1}{i\pi} \int_{-\infty}^{0} e^{R\sqrt{\rho}\sinh t - kt} dt + \frac{1}{\pi} \int_{0}^{\pi} e^{i(R\sqrt{\rho}\sin\theta - k\theta)} d\theta \\ &\quad + \frac{1}{i\pi} \int_{0}^{+\infty} e^{-R\sqrt{\rho}\sinh t - kt - ik\pi} dt \\ &= \frac{1}{i\pi} \int_{-\infty}^{0} e^{R\sqrt{\rho}\sinh t - kt} dt + \mathcal{O}(1). \end{aligned}$$

In the latter integral, we make the change of variable: $t \mapsto -t - \ln k$, and we obtain,

$$h_k(R) = \frac{k^k}{i\pi} \int_{-\ln k}^{+\infty} e^{k\psi(t)} a_k(t) dt + \mathcal{O}(1),$$

with,

$$\psi(t) := t - R\sqrt{\rho}e^t/2; \quad a_k(t) := e^{R\sqrt{\rho}e^{-t}/2k}.$$

Here, we observe that, for any $j \geq 0$, we have $a_{k,R}^{(j)}(t) = \mathcal{O}(1)$ uniformly on $[-\ln k, +\infty)$. Moreover, the phase function ψ admits a unique critical point at $t_c := \ln(2/R\sqrt{\rho})$, and $\psi''(t_c) = -1$. In particular, since also $\operatorname{Re} t_c > 0$ and $\operatorname{Im} t_c \to 0$ as $\varepsilon \to 0$, we can apply the method of steepest descent to estimate this integral, and we obtain,

$$h_k(R) = -i\sqrt{\frac{2}{\pi}}\, k^{k-\frac{1}{2}} e^{k\psi(t_c)} \left(a_k(t_c) + \mathcal{O}(k^{-1})\right) + \mathcal{O}(1),$$

that is,

$$h_k(R) = -i\sqrt{\frac{2}{\pi}}\, k^{k-\frac{1}{2}} \left(\frac{2}{eR\sqrt{\rho}}\right)^k \left(a_k(t_c) + \mathcal{O}(k^{-1})\right) + \mathcal{O}(1).$$

Since $a_k(t_c) = 1 + \mathcal{O}(k^{-1})$, the result follows.

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