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Dynamical Locality of the Free Maxwell Field

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Abstract. The extent to which the non-interacting and source-free Maxwell field obeys the condition of dynamical locality is determined in various formulations. Starting from contractible globally hyperbolic spacetimes, we extend the classical field theory to globally hyperbolic spacetimes of arbitrary topology in two ways, obtaining a 'universal' theory and a 'reduced' theory of the classical free Maxwell field and their corresponding quantisations. We show that the classical and the quantised universal theory fail local covariance and dynamical locality owing to the possibility of having non-trivial radicals in the classical pre-symplectic spaces and non-trivial centres in the quantised $*$-algebras. The classical and the quantised reduced theory are both locally covariant and dynamically local, thus closing a gap in the discussion of dynamical locality and providing new examples relevant to the question of how theories should be formulated so as to describe the same physics in all spacetimes.

1. Introduction

The purpose of this paper is to test various formulations of the free Maxwell field, both classical and quantised, for the property of *dynamical locality*. This property was introduced recently in connection with a discussion of a foundational problem for physics in curved spacetimes: namely, to understand how a theory should be formulated such that its physical content is preserved across the various spacetimes on which it is defined; i.e. so that it represents the same physics in all spacetimes (SPASs) [23]. This touches on what is actually meant by the physical content of a theory, which is not easy to make mathematically precise and it is conceivable there might be more than one satisfactory notion of SPASs, or possibly none at all.

A suitable framework to address such questions is the functorial framework of locally covariant quantum field theory set up in [12]. There, a quantum field theory is described as a functor between a category of curved spacetimes and a category of unital (C)*-algebras. Two quantum field theories are equivalent if and only if there is a natural isomorphism between the functors describing them. Due to the flexibility of the functorial framework, the ideas of locally covariant quantum field theory can be easily applied to other physical theories by a change of the target category, leading to the notion of locally covariant (physical) theories.

In [23], the issue of SPASs was addressed as follows. Any putative notion of SPASs can be represented by a class of locally covariant theories—those conforming to the notion in question. One can then assert axioms for what a good notion of SPASs should be as restrictions on such classes of theories. In particular, suppose one has two theories F, G, in a class T, each of which is supposed to represent the same physics in all spacetimes according to a common notion. If there is at least one spacetime in which the theories F and G coincide, then it seems natural to demand that they should coincide in all spacetimes. This idea was implemented mathematically for the case in which theory F is a subtheory of G: a class of theories T is said to have the *SPASs property* if and only if whenever F, G are locally covariant theories in T and $\eta : F \dot{\to} G$ is a partial natural isomorphism (i.e. at least one of its components is an isomorphism), then η is a natural isomorphism. It was pointed out in [23] that the collection of all locally covariant quantum field theories does not have the SPASs property, while the class of locally covariant theories which are furthermore *dynamically local* does. It was also noted that one might wish to consider other implementations of the underlying idea of SPASs, to which we will return in Sect. 8.4.

The condition of dynamical locality requires that two notions of the local physical content of a locally covariant theory should coincide: (a) the *kinematic* description provided by the functor applied to local regions considered as spacetimes in their own right, and (b) the *dynamical* description which singles out the elements that are invariant under changes to the metric in the causal complement of the region. The precise definition will be recalled in Sect. 5. Dynamical locality is also of interest its own right, regardless of SPASs, because of its consequences for locally covariant theories such as additivity, extended locality (see [36,43] for the original notion) and a no-go theorem concerning preferred states in locally covariant quantum field theories [23, § 6].

It is useful to summarise the current state of knowledge regarding dynamical locality. Klein–Gordon theory in spacetime dimension $n \geq 2$, with mass m and curvature coupling ξ, is known to be dynamically local provided at least one of m or ξ is non-zero [18,24]. The same is known to be true for the extended theory of Wick polynomials for $m > 0$ in the two cases of minimal and conformal coupling in dimensions $n \geq 2$ [18]; moreover, the Dirac field in $n = 4$ dimensions is dynamically local for $m \geq 0$ [17]. The massless minimally coupled scalar field fails to be dynamically local in all dimensions $n \geq 2$, which can be traced to the rigid gauge symmetry $\phi \mapsto \phi + \text{const}$ of the theory; as mentioned, dynamical locality is restored if either m or ξ become non-zero. Moreover, the free massless current is also dynamically local in dimensions $n \geq 3$, and also in $n = 2$ if we restrict to the category of connected spacetimes [24]. The inhomogeneous minimally coupled Klein–Gordon theory has recently been studied [22]; here, the category of spacetimes is replaced by a category of spacetimes with sources, and one modifies the definition of the relative Cauchy evolution and the dynamical net to take account of both metric and source perturbations. The result is that the inhomogeneous theory is dynamically local for all $n \geq 2$ and $m \geq 0$. Thus, we see that the failure of dynamical locality is lifted as soon interactions, in the form of curvature coupling or external sources (or, mass terms) are included. Note that, while the curvature and mass terms break the gauge symmetry, this is not the case for the inhomogeneous theory.[1] In this paper, we intend to close a gap by including the free (i.e. non-interacting and source-free) Maxwell field in the discussion of dynamical locality.

In fact, the formulation of the free Maxwell field and related models has attracted some interest recently, particularly in relation to local covariance. Results on the initial value problem and the quantisation in [16] were generalised to differential p-form fields in [40], Hadamard states were discussed in [15,21] and the Reeh–Schlieder property was analysed in [13]. However, these treatments have in common that they make some assumptions on the topology of the underlying spacetime. Approaches which do not make such assumptions are [14], which treats field strengths, [42], which treats the vector potential, and [25], which discusses the Gupta–Bleuler formalism in curved spacetimes. A consideration of electromagnetism in the spirit of Yang–Mills gauge theories is given in the series of papers [4–6]. Moving beyond electromagnetism, the renormalisability of quantum Yang–Mills theories in curved spacetimes was established in [33] and a general setting for linear quantised gauge field theories is given in [32]. One might also mention progress in linearised quantum gravity [20], which partly inspired some of the work just discussed.

There are various reasons for this interest in free electromagnetism. First, it is important as a model in which physical phenomena such as the Casimir effect can be described, and as a building block in the construction of the perturbative construction of interacting quantum field theories in curved spacetimes [11,34,35]. Second, it is a theory in which the effects of a non-trivial spacetime topology such as topological charges and their superselection rules [1,44] and Aharonov–Bohm-like effects [42] can be discussed. Related to this, topological effects result in a failure of electromagnetism and similar theories to obey the axioms of locally covariant physics—as emphasised by [42], it is locality, rather than covariance, which is lost, as the price for incorporating observables such as those related to Gauss' law. Finally, for our current purposes, the known failure of dynamical locality with the massless and minimally coupled free scalar field, as a result of a rigid gauge symmetry, evidently makes electromagnetism, as a local gauge field theory, an interesting test case for dynamical locality.

[1] There is a subtlety in [22]: not all generators of relative Cauchy evolutions correspond to observable (gauge-invariant) fields in the $m = 0$ case; if one excludes such relative Cauchy evolutions from the construction of the dynamical net, then dynamical locality fails. See [22, Remark 7.20 and §8].

A legitimate question is whether to use a field strength tensor [14] or a vector-potential description [4–6,16,42] of the free Maxwell field for the task of investigating dynamical locality. Our basic approach, following [14], will be to take the theory of the free Maxwell field on contractible curved spacetimes, where a field strength tensor description coincides with a gauge invariant vector potential description and leads to symplectic spaces and simple unital $(C)^*$-algebras, and to ask how it may be extended to curved spacetimes with arbitrary topologies in a functorial way. This differs from other, more global, approaches like [4–6,16,42] insofar as we are led to our global theory (on non-contractible curved spacetimes) by local reasoning. Such an extension was already achieved in [14] for the quantised free Maxwell field in terms of the field strength tensor using Fredenhagen's idea of the universal algebra [26–28]. The classical and the quantum field theory obtained in this way will be called 'universal'; as the field strength and vector potential formulations of electromagnetism coincide in contractible curved spacetimes, their corresponding universal theories are also equivalent: this is a generalisation of the "*natural algebraic relation*" described by Bongaarts [8] between the Borchers–Uhlmann algebras for the field strength description and the vector potential description of the quantum theory of the free Maxwell field in Minkowski space.

The classical and the quantised universal F-theories ("F" is to indicate the field strength tensor description) of the free Maxwell field do not obey local covariance since degenerate pre-symplectic spaces and non-simple unital $(C)^*$-algebras arise whenever the second de Rham cohomology group of the curved spacetime considered is non-trivial. However, they still satisfy the time-slice axiom and the dynamical net can be constructed, thus allowing one to test them for dynamical locality, which they will fail as well. It is therefore of interest to know whether these desirable properties can be restored in some way; moreover, it would also be closer to the original spirit of algebraic quantum field theory [30,31] to work with simple unital $(C)^*$-algebras (and thus with non-degenerate pre-symplectic spaces in the classical case. Hence, we will also consider 'reduced' theories of the free Maxwell field which quotient out non-trivial radicals or centres of the universal free F-theories—similar ideas have been mentioned in [42], where it is also stated that this cannot be done in a functorial way for the vector potential description. As we will show, the classical and the quantised reduced free F-theory are both locally covariant (by design) and, which is not so obvious, dynamically local.

The paper is structured as follows. We begin with some preliminary work, collecting notions of locally covariant quantum field theory in Sect. 2 and recalling some exterior calculus of differential forms in Sect. 3. Next, we review the classical and the quantum field theories of the free Maxwell field, which we wish to consider, in Sect. 4. In Sect. 5, we recap the general construction of the dynamical net. In Sect. 6, we will see that the classical and the quantised universal free F-theory obtained in Sect. 4 fail local covariance and dynamical locality due to the topological reasons already mentioned. This failure can be remedied, leading to the locally covariant and dynamically local, classical and quantised reduced F-theory of the free Maxwell field, which will be the topic of

Sect. 7. In Sect. 8, we discuss the status of dynamical locality, the categorical structure underlying some of our constructions, and the relation of our present work to the discussions of SPASs in [23,24].

2. Locally Covariant Physics

We briefly review the functorial framework of algebraic quantum field theory in curved spacetimes collected in [12], in which a quantum field theory is described as a functor between a category of spacetimes and a category of unital $(C)^*$-algebras, and its application to other physical systems.

2.1. Spacetimes and Physical Systems

The category of spacetimes, **Loc**, has as its objects all oriented globally hyperbolic spacetimes $\boldsymbol{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ of dimension 4 and signature $(+, -, -, -)$, where $\mathfrak{o}$ is the orientation and $\mathfrak{t}$ is the time-orientation. A **Loc**-morphism $\psi : \boldsymbol{M} \to \boldsymbol{N}$ is an isometric smooth embedding which preserves the orientation and the time-orientation and whose image $\psi(M)$ is causally convex[2] in $\boldsymbol{N}$ (preservation of the causal structure). For (algebraic) quantum field theory, the following two categories are of importance:

- $\mathbf{C^*Alg}_1^{\mathrm{m}}$: Objects are unital C^*-algebras; morphisms are unital $*$-monomorphisms.
- $\mathbf{^*Alg}_1^{\mathrm{m}}$: Objects are unital $*$-algebras (over $\mathbb{C}$); morphisms are unital $*$-monomorphisms.

Following standard notation, we denote the set of morphisms between objects A, B of $\mathbf{^*Alg}_1^{\mathrm{m}}$ by $\mathbf{^*Alg}_1^{\mathrm{m}}(A, B)$, and similarly for the other categories encountered.

A *locally covariant quantum field theory* is a functor $F : \mathbf{Loc} \to \mathbf{C^*Alg}_1^{\mathrm{m}}$ or $F : \mathbf{Loc} \to \mathbf{^*Alg}_1^{\mathrm{m}}$. This means that to each spacetime $\boldsymbol{M} \in \mathbf{Loc}$ the theory assigns an algebra $F\boldsymbol{M}$ (in $\mathbf{C^*Alg}_1^{\mathrm{m}}$ or $\mathbf{^*Alg}_1^{\mathrm{m}}$ as appropriate) and, importantly, to each embedding of spacetimes $\psi : \boldsymbol{M} \to \boldsymbol{N}$ in **Loc**, the theory assigns a morphism $F\psi : F\boldsymbol{M} \to F\boldsymbol{N}$; in the current algebraic setting, $F\psi$ is an injective unital $*$-homomorphism. The functorial axioms specify the action of F on identity morphisms and composite morphisms, namely, $F\,\mathrm{id}_{\boldsymbol{M}} = \mathrm{id}_{F\boldsymbol{M}}$ and $F(\psi \circ \varphi) = F\psi \circ F\varphi$. Note that it is important to insist on the injectivity of the unital *-homomorphisms in order to fully implement the principle of local covariance.

Due to the flexibility of the functorial framework, we can consider other physical situations by changing the target category. The physical systems under consideration shall form the objects of a category **Phys**, so that a morphisms of **Phys** represents an inclusion of one physical system as a physical subsystem of another. The category of **Phys** is subjected to further conditions [23, § 3.1]: to be specific it is required that all **Phys**-morphisms are monic and

[2] $\psi(M)$ is causally convex in $\boldsymbol{N}$ if and only if each causal smooth curve in $\boldsymbol{N}$ with endpoints in $\psi(M)$ is entirely contained in $\psi(M)$.

that **Phys** has equalisers, intersections, unions[3] and an initial object, which represents the trivial physical theory. A functor $F : \mathbf{Loc} \to \mathbf{Phys}$ is called a *locally covariant* (*physical*) *theory*. We will consider just a few candidates for **Phys** in this paper, namely, $^*\mathbf{Alg}_1^{\mathrm{m}}$ and

- $\mathbf{pSympl}_{\mathbb{K}}^{\mathrm{m}}$: Objects are (complexified if $\mathbb{K} = \mathbb{C}$) *pre-symplectic spaces*, (V, ω, C), where V is a $\mathbb{K}$-vector space, C a C-involution on V (which is omitted or set to be the identity on V if $\mathbb{K} = \mathbb{R}$)[4] and ω a (possibly degenerate) skew-symmetric $\mathbb{K}$-bilinear form satisfying $\omega \circ (C \times C) = {}^{-} \circ \omega$; the morphisms are symplectic C-monomorphisms, i.e. $f \in \mathbf{pSympl}_{\mathbb{K}}^{\mathrm{m}}((V, \omega, C), (V', \omega', C'))$ is an injective $\mathbb{K}$-linear map $f : V \to V'$ such that $\omega' \circ (f \times f) = \omega$ and $f \circ C = C' \circ f$.

We will also consider modifications of the categories mentioned so far as auxiliary structures. $\mathbf{Loc}_{©}$ is the full subcategory of **Loc** whose objects are contractible spacetimes. $^*\mathbf{Alg}_1$ is defined in the same way as its subcategory $^*\mathbf{Alg}_1^{\mathrm{m}}$, but dropping the restriction of injectivity and allowing general unital $*$-homomorphisms. Similarly, $\mathbf{pSympl}_{\mathbb{K}}$ is defined in the same way as its subcategory $\mathbf{pSympl}_{\mathbb{K}}^{\mathrm{m}}$, but dropping the restriction that the morphisms be injective maps.[5] Finally, $\mathbf{Sympl}_{\mathbb{K}}$ is the full subcategory of $\mathbf{pSympl}_{\mathbb{K}}^{\mathrm{m}}$, where the (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic form is now assumed to be weakly non-degenerate.

2.2. The Relative Cauchy Evolution

We call a **Loc**-morphism $\psi : \boldsymbol{M} \to \boldsymbol{N}$ *Cauchy* whenever the image $\psi(M)$ contains a Cauchy surface for $\boldsymbol{N}$; see [23, Appx.A.1] for some properties of Cauchy morphisms. A locally covariant theory $F : \mathbf{Loc} \to \mathbf{Phys}$ is said to obey the *time-slice axiom* if and only if $F\psi : F\boldsymbol{M} \to F\boldsymbol{N}$ is a **Phys**-isomorphism whenever $\psi \in \mathbf{Loc}(\boldsymbol{M}, \boldsymbol{N})$ is Cauchy.

For locally covariant theories obeying the time-slice axiom, it is possible to define the relative Cauchy evolution [12], which captures the dynamical reaction of the theory to a local perturbation of the background metric; its functional derivative with respect to the metric perturbation is closely related to the stress–energy tensor of the theory, see [12,23,24]. The relative Cauchy evolution can thus be regarded as the natural replacement of the action.

Let $\boldsymbol{M} = (M, g, \mathfrak{o}, \mathfrak{t}) \in \mathbf{Loc}$. A *globally hyperbolic perturbation* h of $\boldsymbol{M}$ is a compactly supported, symmetric and smooth tensor field such that the modification $\boldsymbol{M}[h] := (M, g + h, \mathfrak{o}, \mathfrak{t}_h)$ becomes a **Loc**-object, where $\mathfrak{t}_h$ is the unique choice for a time-orientation on $(M, g + h)$ that coincides with $\mathfrak{t}$ outside $\operatorname{supp} h$. We write $H(\boldsymbol{M})$ for all globally hyperbolic perturbations of $\boldsymbol{M}$, while $H(\boldsymbol{M}; K)$ denotes the subset of all globally hyperbolic perturbations

[3] For the categorical notions of equalisers, which are also known as *difference kernels*, intersections and unions see [39] or [23, Appx.B].

[4] A *C-involution* on a complex vector space V is a complex-conjugate linear map $C : V \to V$ satisfying $C \circ C = \mathrm{id}_V$.

[5] Note that in [24], $\mathbf{pSympl}_{\mathbb{K}}$ denotes the category we call $\mathbf{pSympl}_{\mathbb{K}}^{\mathrm{m}}$ here. As non-monic morphisms arise when considering the universal theory of the free Maxwell field, it is necessary to indicate unambiguously whether we only allow for monics or not.

whose support is contained in a subset $K \subseteq M$. For each $h \in H(\boldsymbol{M})$, we define open sets $M^{\pm}[h] := M \setminus J^{\mp}_{\boldsymbol{M}}(\operatorname{supp} h)$, which will become **Loc**-objects in their own right if endowed with the structures induced by $\boldsymbol{M}$ or $\boldsymbol{M}[h]$[6] by [23, Lem.3.2(a)]. We denote these **Loc**-objects by $\boldsymbol{M}^{\pm}[h] = \boldsymbol{M}|_{M^{\pm}[h]} = (M^{\pm}[h], g|_{M^{\pm}[h]}, \mathfrak{o}|_{M^{\pm}[h]}, \mathfrak{t}|_{M^{\pm}[h]})$. By [23, Lem.3.2(b)], the inclusion maps

$$\iota_{M^{\pm}[h]M} : M^{\pm}[h] \longrightarrow M \qquad \text{and} \qquad \iota_{M^{\pm}[h]M[h]} : M^{\pm}[h] \longrightarrow M[h]$$

become Cauchy morphisms, which we will denote by

$$\imath^{\pm}_{\boldsymbol{M}}[h] : \boldsymbol{M}^{\pm}[h] \longrightarrow \boldsymbol{M} \qquad \text{and} \qquad \jmath^{\pm}_{\boldsymbol{M}}[h] : \boldsymbol{M}^{\pm}[h] \longrightarrow \boldsymbol{M}[h].$$

Now, given a locally covariant theory $F : \mathbf{Loc} \to \mathbf{Phys}$ which obeys the time-slice axiom, the relative Cauchy evolution for F induced by $h \in H(\boldsymbol{M})$ is the **Phys**-automorphism $F\boldsymbol{M} \to F\boldsymbol{M}$ defined by

$$\operatorname{rce}^{F}_{\boldsymbol{M}}[h] := F\left(\imath^{-}_{\boldsymbol{M}}[h]\right) \circ \left(F\left(\jmath^{-}_{\boldsymbol{M}}[h]\right)\right)^{-1} \circ F\left(\jmath^{+}_{\boldsymbol{M}}[h]\right) \circ \left(F\left(\imath^{+}_{\boldsymbol{M}}[h]\right)\right)^{-1}. \quad (1)$$

3. Some Preliminaries on Differential Forms

Differential forms allow for an elegant geometrical description of electromagnetism, that extends to curved spacetimes and allows for a relatively easy quantisation. For $\boldsymbol{M} \in \mathbf{Loc}$, we denote the $\mathcal{C}^{\infty}(M, \mathbb{K})$-module of all smooth $\mathbb{K}$-valued differential p-forms ($p \geq 0$) by $\Omega^p(M; \mathbb{K})$. Adding the subscript "0", i.e. writing $\Omega^p_0(M; \mathbb{K})$, will denote the $\mathcal{C}^{\infty}(M, \mathbb{K})$-module of all smooth $\mathbb{K}$-valued differential p-forms of compact support. By convention, $\Omega^{-1}_{(0)}(M; \mathbb{K})$ is the trivial $\mathbb{K}$-vector space.

Several operators on smooth differential forms will be of importance to us. First, the *exterior derivative*[7] $d_{\boldsymbol{M}} : \Omega^p_{(0)}(M; \mathbb{K}) \to \Omega^{p+1}_{(0)}(M; \mathbb{K})$ is given, in abstract index notation, by

$$(d_{\boldsymbol{M}}\omega)_{a_1 \ldots a_{p+1}} = \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{a_i} \omega_{a_1 \ldots a_{i-1} a_{i+1} \ldots a_{p+1}}, \qquad \omega \in \Omega^p(M; \mathbb{K}),$$

where ∇ denotes the Levi–Civita connection on $\boldsymbol{M}$; by convention we set $d_{\boldsymbol{M}} : \Omega^{-1}_{(0)}(M; \mathbb{K}) \to \Omega^0_{(0)}(M; \mathbb{K})$ to be the zero map. The $\mathbb{K}$-vector space of all (compactly supported) smooth $\mathbb{K}$-valued differential p-forms $\omega \in \Omega^p_{(0)}(M; \mathbb{K})$ which are *closed*, that is, $d_{\boldsymbol{M}}\omega = 0$, is denoted by $\Omega^p_{(0),d}(M; \mathbb{K})$. We say that $\omega \in \Omega^p_d(M; \mathbb{K})$ is *exact* if and only if there is $\theta \in \Omega^{p-1}(M; \mathbb{K})$ such that $\omega = d_{\boldsymbol{M}}\theta$. The $\mathbb{K}$-vector spaces $H^p_{dR,(c)}(M; \mathbb{K}) := \Omega^p_{(0),d}(M; \mathbb{K}) / d_{\boldsymbol{M}}\Omega^{p-1}_{(0)}(M; \mathbb{K})$, called the *de Rham cohomology groups* (*with compact supports*), indicate to what extent the closed smooth differential forms (with compact support) of a smooth manifold fail to be exact (via compactly supported smooth differential forms)

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[6] It does not matter whether we use $\boldsymbol{M}$ or $\boldsymbol{M}[h]$ since $M^{\pm}[h] \cap \operatorname{supp} h = \emptyset$.

[7] The subscript '(0)' indicates that the map is well-defined for both with and without the subscript.

and are deeply connected to the topology of the manifold via singular homology. By Poincaré duality [29, § V.4], we have $H^p_{dR}(M;\mathbb{K}) \cong (H^{4-p}_{dR,c}(M;\mathbb{K}))^*$, where '$*$' denotes the vector space dual.

Next, the *Hodge-$*$-operator* $*_{\boldsymbol{M}} : \Omega^p_{(0)}(M;\mathbb{K}) \to \Omega^{4-p}_{(0)}(M;\mathbb{K})$ is the $\mathcal{C}^\infty(M,\mathbb{K})$-module isomorphism defined by

$$\omega \wedge *_{\boldsymbol{M}}\eta = \frac{1}{p!}\omega_{a_1\ldots a_p}\eta^{a_1\ldots a_p}\ \mathrm{vol}_{\boldsymbol{M}}, \qquad \omega,\eta \in \Omega^p\left(M;\mathbb{K}\right),$$

with inverse $*_{\boldsymbol{M}}^{-1} = (-1)^{p(4-p)+1}*_{\boldsymbol{M}}$. The Hodge-$*$-operator provides a weakly non-degenerate $\mathbb{K}$-bilinear pairing $\int_M (\cdot)\wedge *_{\boldsymbol{M}}(\cdot)$ of $\Omega^p(M;\mathbb{K})$ and $\Omega^p_0(M;\mathbb{K})$.

Using the exterior derivative and the Hodge-$*$-operator, we form the *exterior coderivative* $\delta_{\boldsymbol{M}} := (-1)^p *_{\boldsymbol{M}}^{-1} d_{\boldsymbol{M}} *_{\boldsymbol{M}} : \Omega^p_{(0)}(M;\mathbb{K}) \to \Omega^{p-1}_{(0)}(M;\mathbb{K})$, which is formally adjoint to $d_{\boldsymbol{M}}$ in the sense that

$$\int_M \omega \wedge *_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\eta = \int_M d_{\boldsymbol{M}}\omega \wedge *_{\boldsymbol{M}}\eta$$

whenever $\omega \in \Omega^p(M;\mathbb{K})$ and $\eta \in \Omega^{p+1}(M;\mathbb{K})$ such that $\operatorname{supp}\omega \cap \operatorname{supp}\eta$ is compact. In abstract index notation

$$\left(\delta_{\boldsymbol{M}}\omega\right)_{a_1\ldots a_{p-1}} = -\nabla_{a_0}\omega^{a_0}{}_{a_1\ldots a_{p-1}}, \qquad \omega \in \Omega^p\left(M;\mathbb{K}\right).$$

$\Omega^p_{(0),\delta}(M;\mathbb{K})$ will denote the $\mathbb{K}$-vector space of all (compactly supported) smooth $\mathbb{K}$-valued differential p-forms $\omega \in \Omega^p(M;\mathbb{K})$ which are *coclosed*, that is $\delta_{\boldsymbol{M}}\omega = 0$. $\omega \in \Omega^p_\delta(M;\mathbb{K})$ is called *coexact* if and only if there is $\eta \in \Omega^{p+1}(M;\mathbb{K})$ with $\omega = \delta_{\boldsymbol{M}}\eta$. Closed and coclosed as well as exact and coexact smooth differential forms are related to each other by the Hodge-$*$-operator.

The *d'Alembertian* or *wave operator* $\Box_{\boldsymbol{M}} : \Omega^p_{(0)}(M;\mathbb{K}) \to \Omega^p_{(0)}(M;\mathbb{K})$ is defined by $\Box_{\boldsymbol{M}} := -\delta_{\boldsymbol{M}}d_{\boldsymbol{M}} - d_{\boldsymbol{M}}\delta_{\boldsymbol{M}}$. In abstract index notation, we have

$$\left(\Box_{\boldsymbol{M}}\omega\right)_{a_1\ldots a_p} = g^{ab}\nabla_a\nabla_b\,\omega_{a_1\ldots a_p} + \sum_{i=1}^{p}(-1)^p g^{ab}\left[\nabla_a,\nabla_{a_i}\right]\omega_{ba_1\ldots a_{i-1}a_{i+1}\ldots a_p}, \qquad \omega \in \Omega^p\left(M;\mathbb{K}\right),$$

which establishes that $\Box_{\boldsymbol{M}}$ is a normally hyperbolic linear differential operator of metric type (see [3, § 1.5] for a definition but note that [3] employ the $(-,+,+,+)$-metric signature). Hence, [3] shows that $\Box_{\boldsymbol{M}}$ has a well-posed Cauchy problem and that there are unique retarded and advanced Green's operators $G^{\mathrm{ret/adv}}_{\boldsymbol{M}}$ such that $\operatorname{supp} G^{\mathrm{ret/adv}}_{\boldsymbol{M}}\omega \subseteq J^{+/-}_{\boldsymbol{M}}(\operatorname{supp}\omega)$ (usage of "advanced" and "retarded" is reversed in [3]). We will make extensive use of the difference $G_{\boldsymbol{M}} := G^{\mathrm{ret}}_{\boldsymbol{M}} - G^{\mathrm{adv}}_{\boldsymbol{M}}$,[8] and collect at this point some useful properties:

Lemma 3.1. *The following hold for any* $p \geq 0$: (a) *The identities* $G_{\boldsymbol{M}}d_{\boldsymbol{M}}\omega = d_{\boldsymbol{M}}G_{\boldsymbol{M}}\omega$ *and* $G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega = \delta_{\boldsymbol{M}}G_{\boldsymbol{M}}\omega$ *hold for all* $\omega \in \Omega^p_0(M;\mathbb{K})$. (b) *The kernel of* $\Box_{\boldsymbol{M}}$ *on* $\Omega^p_0(M;\mathbb{K})$ *is trivial, while the range of* $G_{\boldsymbol{M}}$ *on* $\Omega^p_0(M;\mathbb{K})$ *coincides with the space of* $\eta \in \Omega^p(M;\mathbb{K})$ *such that* $\Box_{\boldsymbol{M}}\eta = 0$ *and so that* η *has space-like*

[8] Note that the advanced-minus-retarded Green operator is often used in the literature, e.g., [21,24].

compact support (which is equivalent to having compact support on Cauchy surfaces [41]*). The kernel of* $G_{\boldsymbol{M}}$ *on* $\Omega_0^p(M;\mathbb{K})$ *is given by* $\Box_{\boldsymbol{M}}\Omega_0^p(M;\mathbb{K})$. (c) *The identity* $G_{\boldsymbol{M}}d_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega = -G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}d_{\boldsymbol{M}}\omega$ *holds for all* $\omega \in \Omega_0^p(M;\mathbb{K})$. (d) *The kernels of* $d_{\boldsymbol{M}}\Box_{\boldsymbol{M}}$ *and* $\delta_{\boldsymbol{M}}\Box_{\boldsymbol{M}}$ *on* $\Omega_0^p(M;\mathbb{K})$ *are* $\Omega_{0,d}^p(M;\mathbb{K})$ *and* $\Omega_{0,\delta}^p(M;\mathbb{K})$*, respectively.* (e) *The kernels of* $d_{\boldsymbol{M}}G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}$ *and* $\delta_{\boldsymbol{M}}G_{\boldsymbol{M}}d_{\boldsymbol{M}}$ *on* $\Omega_0^p(M;\mathbb{K})$ *are both equal to* $\Omega_{0,d}^p(M;\mathbb{K}) \oplus \Omega_{0,\delta}^p(M;\mathbb{K})$.

Proof. (a) is proved, e.g., in [40, Prop. 2.1]; (b) is standard for normally hyperbolic operators, e.g., [3, Thm. 3.4.7]; (c) is a special case of (b) using the definition of $\Box_{\boldsymbol{M}}$. For (d), we observe that $d_{\boldsymbol{M}}\Box_{\boldsymbol{M}}\alpha = 0$ for $\alpha \in \Omega_0^p(M;\mathbb{K})$ implies $\Box_{\boldsymbol{M}}d_{\boldsymbol{M}}\alpha = 0$ and hence that $d_{\boldsymbol{M}}\alpha = 0$ by (b); conversely, it is clear that $\alpha \in \Omega_{0,d}^p(M;\mathbb{K})$ implies $d_{\boldsymbol{M}}\Box_{\boldsymbol{M}}\alpha = 0$. Similarly, $\delta_{\boldsymbol{M}}\Box_{\boldsymbol{M}}\alpha = 0$ if and only if $\delta_{\boldsymbol{M}}\alpha = 0$. Finally, if $d_{\boldsymbol{M}}G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega = 0$ for $\omega \in \Omega_0^p(M;\mathbb{K})$ then we also have $G_{\boldsymbol{M}}d_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega = 0$ and hence $d_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega = \Box_{\boldsymbol{M}}\alpha$ for some $\alpha \in \Omega_0^p(M;\mathbb{K})$ by (b); as it is clear that $d_{\boldsymbol{M}}\Box_{\boldsymbol{M}}\alpha = 0$, (d) gives $\alpha \in \Omega_{0,d}^p(M;\mathbb{K})$. By (c), we also have $G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}d_{\boldsymbol{M}}\omega = 0$ and by similar arguments, $\delta_{\boldsymbol{M}}d_{\boldsymbol{M}}\omega = \Box_{\boldsymbol{M}}\beta$ for $\beta \in \Omega_{0,\delta}^p(M;\mathbb{K})$. We deduce that $\Box_{\boldsymbol{M}}(\omega + \alpha + \beta) = 0$ and hence $\omega \in \Omega_{0,d}^p(M;\mathbb{K}) + \Omega_{0,\delta}^p(M;\mathbb{K})$. This is actually a direct sum, because any $\omega \in \Omega_{0,d}^p(M;\mathbb{K}) \cap \Omega_{0,\delta}^p(M;\mathbb{K})$ obeys $\Box_{\boldsymbol{M}}\omega = 0$, so the intersection is trivial. The reverse inclusion is easily shown using (c). □

4. Classical and Quantum Maxwell Theories

4.1. The Initial Value Problem

For $\boldsymbol{M} \in \mathbf{Loc}$, the free Maxwell equations for the electromagnetic field strength tensor $F \in \Omega^2(M;\mathbb{K})$ are

$$d_{\boldsymbol{M}}F = 0 \qquad \text{and} \qquad \delta_{\boldsymbol{M}}F = 0. \tag{2}$$

Given the electric field $E \in \Omega_{0,\delta}^1(\Sigma;\mathbb{K})$ and the dualised magnetic field $B \in \Omega_{0,d}^2(\Sigma;\mathbb{K})$ on a smooth spacelike Cauchy surface Σ for $\boldsymbol{M}$ with inclusion map $\iota_\Sigma : \Sigma \to M$, we can formulate the well-posed initial value problem [14, Prop. 2.1]:

$$d_{\boldsymbol{M}}F = 0, \quad \delta_{\boldsymbol{M}}F = 0, \quad -\iota_\Sigma^* F = B \quad \text{and} \quad *_\Sigma\, \iota_\Sigma^*\, *_{\boldsymbol{M}}^{-1} F = E. \tag{3}$$

Borrowing terminology from [8], we will generally call this the *F-description* of the free Maxwell field.

As is well-known, on any $\boldsymbol{M} \in \mathbf{Loc}_{©}$, every solution of (2) can be expressed as $F = d_{\boldsymbol{M}}A$ (i.e. $F_{ab} = \nabla_a A_b - \nabla_b A_a$) because $H_{dR}^2(M;\mathbb{K}) = 0$, whereupon the free Maxwell equations (2) can be re-expressed as the single equation $\delta_{\boldsymbol{M}}d_{\boldsymbol{M}}A = 0$ for the electromagnetic vector potential $A \in \Omega^1(M;\mathbb{K})$. Owing to gauge freedom, however, the initial value problem

$$\delta_{\boldsymbol{M}}d_{\boldsymbol{M}}A = 0, \qquad -\iota_\Sigma^* A = \mathcal{A} \qquad \text{and} \qquad *_\Sigma\, \iota_\Sigma^*\, *_{\boldsymbol{M}}^{-1} d_{\boldsymbol{M}}A = E,$$

where Σ, ι_Σ and E as above and $\mathcal{A} \in \Omega_0^1(\Sigma;\mathbb{K})$ is the magnetic vector potential, i.e. $d_\Sigma\mathcal{A} = B$, is not well-posed. Instead, a well-posed initial value problem is obtained by passing to suitable equivalence classes of initial data and solutions

[16, 40, 42]. We will generally refer to the description in terms of the vector potential as the *A-description* of the free Maxwell field.

4.2. Classical Phase Space and Quantum Algebra: Contractible Spacetimes

As explained in the introduction, we start with the description of the classical and the quantised free Maxwell field on contractible curved spacetimes, where there is no dispute about the symplectic spaces and the unital *-algebras of the smeared quantum field and the F- and the A-description coincide.

Hence, we continue to assume that $\boldsymbol{M} \in \mathbf{Loc}_{\copyright}$. In the F- and the A-description of the free Maxwell field, there are three descriptions of the classical field theory in terms of a (possibly complexified) symplectic space: the phase space of the Cauchy data, the phase space of the solutions and the phase space of the test forms (cf. [16, § 3] for the case of the electromagnetic vector potential). However, these three choices are symplectomorphic and hence equivalent. In view of the unital *-algebras of the smeared quantum field and their relation to the classical phase space, we find it most convenient to work with the phase space of the test forms.

F-description. As shown in the proof of [14, Prop. 2.1], any solution of (3) with compact support on Cauchy surfaces is also a solution for the initial value problem of the wave equation $\Box_{\boldsymbol{M}} F = 0$ with compactly supported Cauchy data, and can be written as [14, Prop. 2.2]:

$$F = G_{\boldsymbol{M}} \left(d_{\boldsymbol{M}} \theta + \delta_{\boldsymbol{M}} \eta \right), \qquad \theta \in \Omega^1_{0,\delta} \left(M; \mathbb{K} \right), \eta \in \Omega^3_{0,d} \left(M; \mathbb{K} \right).$$

This general form may be simplified as $\boldsymbol{M}$ is contractible (so $H^1_{dR}(M;\mathbb{K})$ is trivial), and hence $\Omega^1_{0,\delta}(M;\mathbb{K}) = \delta_{\boldsymbol{M}} \Omega^2_0(M;\mathbb{K})$ and $\Omega^3_{0,d}(M;\mathbb{K}) = d_{\boldsymbol{M}} \Omega^2_0(M;\mathbb{K})$. Making use of Lemma 3.1, we see that any solution of (3) with compact support on Cauchy surfaces can be written as

$$F = d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} \omega, \qquad \omega \in \Omega^2_0 \left(M; \mathbb{K} \right).$$

By Lemma 3.1(e), $\omega, \eta \in \Omega^2_0(M;\mathbb{K})$ give rise to the same solution if and only if they differ by an element of $\Omega^2_{0,d}(M;\mathbb{K}) \oplus \Omega^2_{0,\delta}(M;\mathbb{K})$. As $\boldsymbol{M}$ is contractible, we have $\Omega^2_{0,d}(M;\mathbb{K}) = d_{\boldsymbol{M}} \Omega^1_0(M;\mathbb{K})$ and $\Omega^2_{0,\delta}(M;\mathbb{K}) = \delta_{\boldsymbol{M}} \Omega^3_0(M;\mathbb{K})$, so the space of the test forms may be described as a (complexified if $\mathbb{K} = \mathbb{C}$) symplectic space $\mathcal{F}\boldsymbol{M} := \left(\left[\Omega^2_0(M;\mathbb{K}) \right], \mathfrak{w}_{\boldsymbol{M}}, ^{-} \right)$,[9] where

$$\begin{aligned} \left[\Omega^2_0 \left(M; \mathbb{K} \right) \right] &:= \Omega^2_0 \left(M; \mathbb{K} \right) / \left(d_{\boldsymbol{M}} \Omega^1_0 \left(M; \mathbb{K} \right) \oplus \delta_{\boldsymbol{M}} \Omega^3_0 \left(M; \mathbb{K} \right) \right), \\ \mathfrak{w}_{\boldsymbol{M}} \left([\omega], [\eta] \right) &:= - \int_M G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} \omega \wedge *_{\boldsymbol{M}} \delta_{\boldsymbol{M}} \eta, \\ \overline{[\omega]} &:= [\overline{\omega}], \qquad [\omega], [\eta] \in \left[\Omega^2_0 \left(M; \mathbb{K} \right) \right], \end{aligned} \tag{4}$$

(the complex conjugation is to be omitted if $\mathbb{K} = \mathbb{R}$). The fact that $\mathfrak{w}_{\boldsymbol{M}}$ is a well-defined and non-degenerate follows immediately from the following result:

[9] The use of the same symbol $\mathcal{F}$ [and later $\mathcal{A}$] for both $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, should not give rise to any confusion.

Lemma 4.1. *Let $\boldsymbol{M} \in \mathbf{Loc}$ (contractibility is not assumed). Then*

$$(\omega, \eta) \mapsto -\int_M G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega \wedge *_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\eta,$$

is a skew-symmetric, $\mathbb{K}$-bilinear form on $\Omega^2_0(M;\mathbb{K})$, with radical $\Omega^2_{0,d}(M;\mathbb{K}) \oplus \Omega^2_{0,\delta}(M;\mathbb{K})$.

Proof. Bilinearity is obvious and skew-symmetry follows from general properties of $G_{\boldsymbol{M}}$. Fixing $\omega \in \Omega^2_0(M;\mathbb{K})$ and noting that

$$\int_M G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega \wedge *_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\eta = \int_M d_{\boldsymbol{M}}G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega \wedge *_{\boldsymbol{M}}\eta \quad \forall\eta \in \Omega^2_0(M;\mathbb{K}), \quad (5)$$

the non-degeneracy of the pairing $\int_M(\cdot) \wedge *_{\boldsymbol{M}}(\cdot) : \Omega^2(M;\mathbb{K}) \times \Omega^2_0(M;\mathbb{K}) \to \mathbb{K}$ implies that the left-hand side of (5) vanishes for all $\eta \in \Omega^2_0(M;\mathbb{K})$ if and only if $d_{\boldsymbol{M}}G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega = 0$ and hence $\omega \in \Omega^2_{0,d}(M;\mathbb{K}) \oplus \Omega^2_{0,\delta}(M;\mathbb{K})$ by Lemma 3.1(e). □

The corresponding quantum version, that is, the unital *-algebra $\mathfrak{F}\boldsymbol{M}$ of the smeared quantum field for the free Maxwell field in terms of the field strength tensor is generated by the abstract elements $\mathbf{F}_{\boldsymbol{M}}(\omega)$, $\omega \in \Omega^2_0(M;\mathbb{C})$, which obey the following relations (cf. [14, Def. 3.1]):

- Linearity and Hermiticity:

$$\mathbf{F}_{\boldsymbol{M}}(\lambda\omega + \mu\eta) = \lambda\mathbf{F}_{\boldsymbol{M}}(\omega) + \mu\mathbf{F}_{\boldsymbol{M}}(\eta) \quad \text{and} \quad \mathbf{F}_{\boldsymbol{M}}(\omega)^* = \mathbf{F}_{\boldsymbol{M}}(\overline{\omega})$$
$$\forall\lambda, \mu \in \mathbb{C},\ \forall\omega, \eta \in \Omega^2_0(M;\mathbb{C}).$$

- Free Maxwell equations in the weak sense:

$$\mathbf{F}_{\boldsymbol{M}}(d_{\boldsymbol{M}}\theta) = 0 \quad \text{and} \quad \mathbf{F}_{\boldsymbol{M}}(\delta_{\boldsymbol{M}}\eta) = 0 \quad \forall\theta \in \Omega^1_0(M;\mathbb{C}),\ \forall\eta \in \Omega^3_0(M;\mathbb{C}).$$

- Commutation relations:[10]

$$[\mathbf{F}_{\boldsymbol{M}}(\omega), \mathbf{F}_{\boldsymbol{M}}(\eta)] = \left(-\mathrm{i}\int_M G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega \wedge *_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\eta\right) \cdot 1_{\mathfrak{F}\boldsymbol{M}} \quad \forall\omega, \eta \in \Omega^2_0(M;\mathbb{C}).$$

As will become clear from the discussion of the quantisation functor later, $\mathfrak{F}\boldsymbol{M}$ is simple.

A-description. In the A-description, the classical field theory can be described by the (complexified if $\mathbb{K} = \mathbb{C}$) symplectic space

$$\mathcal{A}\boldsymbol{M} = \left(\left[\delta_{\boldsymbol{M}}\Omega^2_0(M;\mathbb{K})\right], \mathfrak{v}_{\boldsymbol{M}}, {}^{-}\right),$$

where (omitting the complex conjugation if $\mathbb{K} = \mathbb{R}$)

$$\left[\delta_{\boldsymbol{M}}\Omega^2_0(M;\mathbb{K})\right] := \delta_{\boldsymbol{M}}\Omega^2_0(M;\mathbb{K}) / \delta_{\boldsymbol{M}}d_{\boldsymbol{M}}\Omega^1_0(M;\mathbb{K}),$$
$$\mathfrak{v}_{\boldsymbol{M}}([\theta], [\phi]) := -\int_M G_{\boldsymbol{M}}\theta \wedge *_{\boldsymbol{M}}\phi,$$
$$\overline{[\theta]} := \left[\overline{\theta}\right], \qquad [\theta], [\phi] \in \left[\delta_{\boldsymbol{M}}\Omega^2_0(M;\mathbb{K})\right],$$

[10] Also known as Lichnerowicz's commutation relations—see the remark in [16, § 4] and [37].

see [13,15,16,40]. Note, the first two references assume that $\boldsymbol{M}$ has compact Cauchy surfaces. This assumption is not necessary here (though we have contractibility at present). Also, recall the identity $\delta_{\boldsymbol{M}}\Omega^2_0(M;\mathbb{K}) = \Omega^1_{0,\delta}(M;\mathbb{K})$ due to the assumption $\boldsymbol{M} \in \mathbf{Loc}_{©}$. The corresponding simple unital *-algebra $\mathfrak{A}\boldsymbol{M}$ of the smeared quantum field for the free Maxwell field in terms of the vector potential is generated by the abstract symbols $[\mathbf{A}]_{\boldsymbol{M}}(\theta)$, $\theta \in \delta_{\boldsymbol{M}}\Omega^2_0(M;\mathbb{C})$, obeying the following relations [16,21,40,42]:

- Linearity and Hermiticity:
$$[\mathbf{A}]_{\boldsymbol{M}}(\lambda\theta + \mu\phi) = \lambda[\mathbf{A}]_{\boldsymbol{M}}(\theta) + \mu[\mathbf{A}]_{\boldsymbol{M}}(\phi) \quad \text{and} \quad [\mathbf{A}]_{\boldsymbol{M}}(\theta)^* = [\mathbf{A}]_{\boldsymbol{M}}(\overline{\theta}) \quad \forall\lambda,\mu \in \mathbb{C},\ \forall\theta,\phi \in \delta_{\boldsymbol{M}}\Omega^2_0(M;\mathbb{C}).$$
- Free Maxwell equations in the weak sense:
$$[\mathbf{A}]_{\boldsymbol{M}}(\delta_{\boldsymbol{M}} d_{\boldsymbol{M}}\theta) = 0 \qquad \forall\theta \in \Omega^1_0(M;\mathbb{C}).$$
- Commutation relations:
$$\big[[\mathbf{A}]_{\boldsymbol{M}}(\theta), [\mathbf{A}]_{\boldsymbol{M}}(\phi)\big] = \left(-\mathrm{i}\int_M G_{\boldsymbol{M}}\theta \wedge *_{\boldsymbol{M}}\phi\right) \cdot 1_{\mathfrak{A}\boldsymbol{M}} \ \ \forall\theta,\phi \in \delta_{\boldsymbol{M}}\Omega^2_0(M;\mathbb{C}).$$

Functorial properties. Let $\psi \in \mathbf{Loc}_{©}(\boldsymbol{M},\boldsymbol{N})$ be a morphism between contractible spacetimes $\boldsymbol{M}$ and $\boldsymbol{N}$. Then there is a natural pushforward of compactly supported smooth $\mathbb{K}$-valued differential forms, $\psi_* : \Omega^p_0(M;\mathbb{K}) \to \Omega^p_0(N;\mathbb{K})$ as well as the pullback $\psi^* : \Omega^p(N;\mathbb{K}) \to \Omega^p(M;\mathbb{K})$, and there is a well-known identity $\psi^* G_{\boldsymbol{N}}\psi_* = G_{\boldsymbol{M}}$ (cf. e.g., [24, Sec. 3]). Making use of these properties, we obtain $\mathbf{Sympl}_{\mathbb{K}}$-morphisms $\mathcal{F}\psi : \mathcal{F}\boldsymbol{M} \to \mathcal{F}\boldsymbol{N}$ and $\mathcal{A}\psi : \mathcal{A}\boldsymbol{M} \to \mathcal{A}\boldsymbol{N}$ by $\mathcal{F}\psi\,[\omega] := [\psi_*\omega]$ for $\omega \in \Omega^2_0(M;\mathbb{K})$ and $\mathcal{A}\psi\,[\theta] := [\psi_*\theta]$ for $\theta \in \delta_{\boldsymbol{M}}\Omega^2_0(M;\mathbb{K})$. Similarly, putting $\mathfrak{F}\psi(\mathbf{F}_{\boldsymbol{M}}(\omega)) := \mathbf{F}_{\boldsymbol{N}}(\psi_*\omega)$ for $\omega \in \Omega^2_0(M;\mathbb{K})$ and also $\mathfrak{A}\psi([\mathbf{A}]_{\boldsymbol{M}}(\theta)) := [\mathbf{A}]_{\boldsymbol{N}}(\psi_*\theta)$ for $\theta \in \delta_{\boldsymbol{M}}\Omega^2_0(M;\mathbb{K})$ well-defines ${}^*\mathbf{Alg}^{\mathrm{m}}_1$-morphisms $\mathfrak{F}\psi : \mathfrak{F}\boldsymbol{M} \to \mathfrak{F}\boldsymbol{N}$ and $\mathfrak{A}\psi : \mathfrak{A}\boldsymbol{M} \to \mathfrak{A}\boldsymbol{N}$. In this way, we obtain functors

$$\mathcal{F} : \mathbf{Loc}_{©} \longrightarrow \mathbf{Sympl}_{\mathbb{K}} \quad \text{and} \quad \mathcal{A} : \mathbf{Loc}_{©} \longrightarrow \mathbf{Sympl}_{\mathbb{K}},$$
$$\mathfrak{F} : \mathbf{Loc}_{©} \longrightarrow {}^*\mathbf{Alg}^{\mathrm{m}}_1 \quad \text{and} \quad \mathfrak{A} : \mathbf{Loc}_{©} \longrightarrow {}^*\mathbf{Alg}^{\mathrm{m}}_1.$$

It is straightforward to see that the map $\Omega^2_0(M;\mathbb{K}) \ni \omega \mapsto \delta_{\boldsymbol{M}}\omega \in \delta_{\boldsymbol{M}}\Omega^2_0(M;\mathbb{K})$ gives rise to a $\mathbf{Sympl}_{\mathbb{K}}$-isomorphism $\eta_{\boldsymbol{M}} : \mathcal{F}\boldsymbol{M} \to \mathcal{A}\boldsymbol{M}$ for each $\boldsymbol{M} \in \mathbf{Loc}_{©}$ and that the family $\{\eta_{\boldsymbol{M}}\}_{\boldsymbol{M}\in\mathbf{Loc}_{©}}$ thus obtained form the components of a natural isomorphism $\eta : \mathcal{F} \dot{\to} \mathcal{A}$. Thus, $\mathcal{F}$ and $\mathcal{A}$ are equivalent physical theories on $\mathbf{Loc}_{©}$. The quantum version of η is a natural isomorphism $\varepsilon : \mathfrak{F} \dot{\to} \mathfrak{A}$ determined by $\varepsilon_{\boldsymbol{M}}\mathbf{F}_{\boldsymbol{M}}(\omega) = [\mathbf{A}]_{\boldsymbol{M}}(\delta_{\boldsymbol{M}}\omega)$ for $\omega \in \Omega^2_0(M;\mathbb{K})$ and $\boldsymbol{M} \in \mathbf{Loc}_{©}$, and precisely generalises the "*natural algebraic relation*" between the Borchers–Uhlmann algebras for the F- and the A-descriptions discussed in [8] for Minkowski space.

4.3. Extensions to Non-contractible Spacetimes

The previous subsection described the classical and the quantised free Maxwell field on contractible curved spacetimes in terms of both the field strength tensor and the vector potential. However, there are physically relevant curved spacetimes with non-trivial topologies such that not every field strength tensor

F can be derived from a vector potential A via $F = dA$ ($F_{ab} = \nabla_a A_b - \nabla_b A_a$), which generally leads to interesting features such as topological (electric and magnetic) charges and superselection rules thereof [1,44]. An example of such a curved spacetime is the Schwarzschild–Kruskal spacetime, which has the topology of $M \cong \mathbb{R} \times \mathbb{R} \times S^2$, hence $H^2_{dR}(M;\mathbb{K}) \neq 0 \neq H^2_{dR,c}(M;\mathbb{K})$. Such features ultimately prevent one from having classical and quantised theories of the free Maxwell field in the usual, straightforward manner (in the F- as well as in the A-description).

Universal theories. To deal with non-trivial spacetime topologies and to analyse the impact they have on the (unital) $(C)^*$-algebras of the quantum field theory, Fredenhagen has suggested the use of (the analogue of) the universal algebra construction in [26–28], to obtain a 'minimal' description compatible with, and unifying, the local descriptions of the quantum field theory on contractible regions of the spacetime. This was addressed in [33, Appx. A] and carried out in detail in [14]. The final result of the analysis is very simple in the F-description: the universal algebra $\mathfrak{F}_u\boldsymbol{M}$ is precisely what would be obtained by removing the restriction to $\boldsymbol{M} \in \mathbf{Loc}_{\copyright}$ in the construction of $\mathfrak{F}\mathbf{M}$ and allowing general $\boldsymbol{M} \in \mathbf{Loc}$ instead (cf. [14, Prop. 3.1 + 3.2]). The difference lies in the fact that we now have $\Omega^2_{0,d}(M;\mathbb{K}) \neq d_{\boldsymbol{M}}\Omega^1_0(M;\mathbb{K})$ and $\Omega^2_{0,\delta}(M;\mathbb{K}) \neq \delta_{\boldsymbol{M}}\Omega^3_0(M;\mathbb{K})$ if $H^2_{dR}(M;\mathbb{K}) \neq 0$ (and hence $H^2_{dR,c}(M;\mathbb{K}) \neq 0$ by Poincaré duality). This implies that $\mathfrak{F}_u\boldsymbol{M}$ is non-simple and possesses a non-trivial centre whenever $\boldsymbol{M} \in \mathbf{Loc}$ such that $H^2_{dR}(M;\mathbb{K}) \neq 0$. Of course, $\mathfrak{F}_u\boldsymbol{M} = \mathfrak{F}\boldsymbol{M}$ for all $\boldsymbol{M} \in \mathbf{Loc}_{\copyright}$.

A similar construction can be carried out for the classical field theory, that is, there exists a 'universal' (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic space which can be constructed from the local (complexified if $\mathbb{K} = \mathbb{C}$) symplectic spaces of the contractible spacetime regions. For each $\boldsymbol{M} \in \mathbf{Loc}$, $\mathcal{F}_u\boldsymbol{M} := \left(\left[\Omega^2_0(M;\mathbb{K})\right], \mathfrak{w}_{u\boldsymbol{M}}, \overline{}\right)$ is the (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic space given by (omitting the complex conjugation if $\mathbb{K} = \mathbb{R}$)

$$\begin{aligned} \left[\Omega^2_0(M;\mathbb{K})\right] &:= \Omega^2_0(M;\mathbb{K}) \,/\, \left(d_{\boldsymbol{M}}\Omega^1_0(M;\mathbb{K}) \oplus \delta_{\boldsymbol{M}}\Omega^3_0(M;\mathbb{K})\right), \\ \mathfrak{w}_{u\boldsymbol{M}}([\omega],[\eta]) &:= -\int_M G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega \wedge *_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\eta, \\ \overline{[\omega]} &:= [\overline{\omega}], \qquad [\omega],[\eta] \in \left[\Omega^2_0(M;\mathbb{K})\right], \end{aligned} \tag{6}$$

which is well-defined as a consequence of Lemma 4.1. On contractible spacetimes $\boldsymbol{M} \in \mathbf{Loc}_{\copyright}$, $\mathcal{F}_u\boldsymbol{M}$ coincides precisely with $\mathcal{F}\boldsymbol{M}$ defined by (4). However, the skew-symmetric bilinear form $\mathfrak{w}_{u\boldsymbol{M}}$ is degenerate on spacetimes with non-trivial $H^2_{dR}(M;\mathbb{K})$ as can be seen from Lemma 4.1. Indeed, closed but non-exact $\omega \in \Omega^2_0(M;\mathbb{K})$ give rise to elements in the radical $\operatorname{rad}\mathfrak{w}_{u\boldsymbol{M}}$ that will be called *electric topological degeneracies*, while coclosed but non-coexact $\omega \in \Omega^2_0(M;\mathbb{K})$ give rise to *magnetic topological degeneracies* of $\mathfrak{w}_{u\boldsymbol{M}}$ (cf. [14, Prop.3.3]). Putting this another way, there is a linear isomorphism

$$\begin{aligned} H^2_{dR,c}(M) \oplus H^2_{dR,c}(M) &\longrightarrow \operatorname{rad}\mathfrak{w}_{u\boldsymbol{M}} \\ [\alpha] \oplus [\beta] &\longmapsto [\alpha + *_{\boldsymbol{M}}\beta], \end{aligned}$$

where the square brackets on the left are cohomology classes. The motivation for our nomenclature is that an electric topological degeneracy ω defines a classical observable $F \mapsto \int_M F \wedge *_{\boldsymbol{M}}\omega$. By means of Poincaré duality theory [10, § 1.5] the space of such observables is spanned by integrals of the form $\int_S *_{\boldsymbol{M}}F$ for some closed 2-surface S that can be chosen to lie in a space-like Cauchy surface, and so measures the topological electric charge enclosed by S. Likewise, magnetic degeneracies determine observables measuring magnetic fluxes.

Example 4.2. Let $\boldsymbol{M}$ be the Cauchy development of the exterior of a unit ball in the $t = 0$ hyperplane of Minkowski spacetime. Then $H^2_{dR}(M;\mathbb{K}) \cong \mathbb{K}$, and $H^2_{dR,c}(M;\mathbb{K}) \cong \mathbb{K}$ is generated by $\omega = f(t,r)\mathrm{d}t \wedge \mathrm{d}r$ in spherical polar coordinates, where $f \in C_0^\infty(\mathbb{R}\times(1,\infty))$. The observable $\int_M F \wedge *_{\boldsymbol{M}}\omega$ is proportional to the electric flux through any closed 2-surface in the $t = 0$ plane enclosing the excluded ball, while $\int_M F \wedge \omega$ is proportional to the magnetic flux. (It is interesting to compare this with [42, Example 3.7], in which context only the electric charges appear.)

For any **Loc**-morphism $\psi : \boldsymbol{M} \to \boldsymbol{N}$, we define $\mathcal{F}_u\psi : \mathcal{F}\boldsymbol{M} \to \mathcal{F}\boldsymbol{N}$ and $\mathfrak{F}_u\boldsymbol{M} \to \mathfrak{F}_u\boldsymbol{N}$ by extending the previous definitions, i.e. $\mathcal{F}_u\psi\,[\omega] := [\psi_*\omega]$ and $\mathfrak{F}_u\psi(\mathbf{F}_{\boldsymbol{M}}(\omega)) := \mathbf{F}_{\boldsymbol{N}}(\psi_*\omega)$ for $\omega \in \Omega^2_0(M;\mathbb{K})$, thus obtaining morphisms in $\mathbf{pSympl}_{\mathbb{K}}$ and $^*\mathbf{Alg}_1$, respectively. This yields functors $\mathcal{F}_u : \mathbf{Loc} \to \mathbf{pSympl}_{\mathbb{K}}$ and $\mathfrak{F}_u : \mathbf{Loc} \to {}^*\mathbf{Alg}_1$, which will be called the *classical universal F-theory* and *quantised universal F-theory* of the free Maxwell field. Note that on restriction to contractible spacetimes, we recover the previously constructed theories: $\mathcal{F}_u|_{\mathbf{Loc}_{\copyright}} = \mathcal{F}$ and $\mathfrak{F}_u|_{\mathbf{Loc}_{\copyright}} = \mathfrak{F}$.

Reduced theories. The degeneracies just mentioned ultimately turn out to obstruct desirable properties of local covariance and dynamical locality. To restore them, we may modify the theory by forming quotients by the larger direct sum of closed and coclosed forms, instead of quotienting by the direct sum of exact and coexact forms, thus obtaining a *classical reduced F-theory* of the free Maxwell field $\mathcal{R} : \mathbf{Loc} \to \mathbf{Sympl}_{\mathbb{K}}$. In more detail, for each $\boldsymbol{M} \in \mathbf{Loc}$, $\mathcal{R}\boldsymbol{M} := (\llbracket\Omega^2_0(M;\mathbb{K})\rrbracket, \mathfrak{r}_{\boldsymbol{M}}, \overline{})$, where

$$\begin{aligned} \llbracket\Omega^2_0(M;\mathbb{K})\rrbracket &:= \Omega^2_0(M;\mathbb{K}) \,/\, \left(\Omega^2_{0,d}(M;\mathbb{K}) \oplus \Omega^2_{0,\delta}(M;\mathbb{K})\right), \\ \mathfrak{r}_{\boldsymbol{M}}(\llbracket\omega\rrbracket, \llbracket\eta\rrbracket) &:= -\int_M G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega \wedge *_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\eta, \\ \overline{\llbracket\omega\rrbracket} &:= \llbracket\overline{\omega}\rrbracket, \qquad \llbracket\omega\rrbracket, \llbracket\eta\rrbracket \in \llbracket\Omega^2_0(M;\mathbb{K})\rrbracket, \end{aligned} \tag{7}$$

(omitting complex conjugation if $\mathbb{K} = \mathbb{R}$). For any **Loc**-morphism $\psi : \boldsymbol{M} \to \boldsymbol{N}$, $\mathcal{R}\psi : \mathcal{R}\boldsymbol{M} \to \mathcal{R}\boldsymbol{N}$ is given by $\llbracket\omega\rrbracket \longmapsto \llbracket\psi_*\omega\rrbracket$, $\omega \in \Omega^2_0(M;\mathbb{K})$. $\mathcal{R}\boldsymbol{M}$ is precisely obtained from $\mathcal{F}_u\boldsymbol{M}$ by quotienting out the radical of $\mathfrak{w}_{u\boldsymbol{M}}$. We also see that $\mathcal{R}\boldsymbol{M} = \mathcal{F}_u\boldsymbol{M}$ whenever $\boldsymbol{M} \in \mathbf{Loc}$ such that $H^2_{dR}(M;\mathbb{K}) = 0$ (which implies $H^2_{dR,c}(M;\mathbb{K}) = 0$ by Poincaré duality and thus, $\Omega^2_{0,d}(M;\mathbb{K}) = d_{\boldsymbol{M}}\Omega^1_0(M;\mathbb{K})$ and $\Omega^2_{0,\delta}(M;\mathbb{K}) = \delta_{\boldsymbol{M}}\Omega^3_0(M;\mathbb{K})$). Hence $\mathcal{R}|_{\mathbf{Loc}_{\copyright}} = \mathcal{F}$.

The *quantised reduced F-theory* of the free Maxwell field, $\mathfrak{R} : \mathbf{Loc} \to {}^*\mathbf{Alg}_1^{\mathrm{m}}$, is given for each $\boldsymbol{M} \in \mathbf{Loc}$ by the simple unital *-algebra $\mathfrak{R}\boldsymbol{M}$ generated by the abstract elements $\mathbf{R}_{\boldsymbol{M}}(\omega)$, $\omega \in \Omega_0^2(M;\mathbb{C})$, subject to the following relations:

- Linearity and Hermiticity:

$$\mathbf{R}_{\boldsymbol{M}}\left(\lambda\omega + \mu\eta\right) = \lambda\mathbf{R}_{\boldsymbol{M}}\left(\omega\right) + \mu\mathbf{R}_{\boldsymbol{M}}\left(\eta\right) \quad \text{and} \quad \mathbf{R}_{\boldsymbol{M}}\left(\omega\right)^* = \mathbf{R}_{\boldsymbol{M}}\left(\overline{\omega}\right)$$
$$\forall \lambda, \mu \in \mathbb{C},\ \forall \omega, \eta \in \Omega_0^2\left(M;\mathbb{C}\right).$$

- Strengthened free Maxwell equations in the weak sense:

$$\mathbf{R}_{\boldsymbol{M}}\left(\omega\right) = 0 \qquad \forall \omega \in \Omega_{0,d}^2\left(M;\mathbb{C}\right) \oplus \Omega_{0,\delta}^2\left(M;\mathbb{C}\right).$$

- Commutation relations:

$$\left[\mathbf{R}_{\boldsymbol{M}}\left(\omega\right), \mathbf{R}_{\boldsymbol{M}}\left(\eta\right)\right] = \left(-\mathrm{i}\int_M G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega \wedge *_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\eta\right) \cdot 1_{\mathfrak{R}\boldsymbol{M}}$$
$$\forall \omega, \eta \in \Omega_0^2\left(M;\mathbb{C}\right).$$

The unital *-monomorphism $\mathfrak{R}\psi : \mathfrak{R}\boldsymbol{M} \to \mathfrak{R}\boldsymbol{N}$ is given by $\mathfrak{R}\psi(\mathbf{R}_{\boldsymbol{M}}(\omega)) := \mathbf{R}_{\boldsymbol{N}}(\psi_*\omega)$ for $\omega \in \Omega_0^2(M;\mathbb{K})$. It is readily seen that $\mathfrak{R}\boldsymbol{M} = \mathfrak{F}_u\boldsymbol{M}$ whenever $\boldsymbol{M} \in \mathbf{Loc}$ such that $H_{dR}^2(M;\mathbb{K}) = 0$ and thus, $\mathfrak{R}|_{\mathbf{Loc}_{©}} = \mathfrak{F}$.

We conclude this subsection with some remarks. First, our classical reduced theory of the free Maxwell field is closely related to the "*charge-zero phase space functor*" for electromagnetism given in [5, § 7]. The latter functor actually yields degenerate pre-symplectic spaces. However, as pointed out in [25], the treatment of affine theories used in [5] should be corrected; once this is done their approach would coincide with our reduced theory.

Second, it would also have been possible to start in the A-description of the free Maxwell field and then obtain a corresponding *classical* and *quantised universal A-theory* $\mathcal{A}_u : \mathbf{Loc} \to \mathbf{pSympl}_{\mathbb{K}}$ and $\mathfrak{A}_u : \mathbf{Loc} \to {}^*\mathbf{Alg}_1$ from $\mathcal{A} : \mathbf{Loc}_{©} \to \mathbf{pSympl}_{\mathbb{K}}$ and $\mathfrak{A} : \mathbf{Loc}_{©} \to {}^*\mathbf{Alg}_1^{\mathrm{m}}$, in the same way as $\mathcal{F}_u : \mathbf{Loc} \to \mathbf{pSympl}_{\mathbb{K}}$ and $\mathfrak{F}_u : \mathbf{Loc} \to {}^*\mathbf{Alg}_1$ were obtained from $\mathcal{F} : \mathbf{Loc}_{©} \to \mathbf{pSympl}_{\mathbb{K}}$ and $\mathfrak{F} : \mathbf{Loc}_{©} \to {}^*\mathbf{Alg}_1^{\mathrm{m}}$. As $\mathcal{A}$ and $\mathcal{F}$, and $\mathfrak{A}$ and $\mathfrak{F}$ are equivalent theories, that is, naturally isomorphic as functors, it follows on abstract categorical grounds that $\mathcal{A}_u$ is naturally isomorphic to $\mathcal{F}_u$ and that $\mathfrak{A}_u$ is naturally isomorphic to $\mathfrak{F}_u$. This means that the universal F-theory and the universal A-theory of the free Maxwell field are equivalent theories. Choosing the universal F-theory over the A-theory and vice versa has no physical significance and purely expresses a different point of view on the same theory. In the following sections, we will take the point of view of the F-description, which slightly simplifies some arguments as we are automatically working in a gauge invariant setting. In fact, there is a sense in which the universal algebra construction and its classical analogue favours the F-description. An explicit expression for $\mathfrak{A}_u\boldsymbol{M}$ and $\mathcal{A}_u\boldsymbol{M}$, where $\boldsymbol{M} \in \mathbf{Loc}$, purely in terms of (equivalence classes of) coclosed smooth $\mathbb{K}$-valued differential 1-forms appears only to be available when $H_{dR}^1(M;\mathbb{K})$ and $H_{dR}^2(M;\mathbb{K})$ are both trivial. The reason for this is that the universal algebra construction and its classical analogue allow not only for a 1-form potential for the field strength, but also for a description in terms of a 3-form potential, i.e. $V \in \Omega^3(M;\mathbb{K})$ such that $F = \delta V$.

Finally, we have described the various quantised theories $\mathfrak{F}$, $\mathfrak{F}_u$ and $\mathfrak{R}$, by constructing the algebras in each spacetime and giving the morphisms corresponding to spacetime embeddings explicitly. However, we can also describe them more abstractly as the result of composing the corresponding classical theories $\mathcal{F}$, $\mathcal{F}_u$ and $\mathcal{R}$ with a quantisation functor that implements the infinitesimal Weyl quantisation of (complexified) pre-symplectic spaces. Namely, given any complexified pre-symplectic space (V, ω, C),[11] the unital *-algebra $Q(V, \omega, C)$ is defined to be $*$-algebra generated by abstract elements $\mathbf{Q}_V(v)$ $(v \in V)$, subject to linearity of $v \mapsto \mathbf{Q}_V(v)$, the Hermiticity condition $\mathbf{Q}_V(v)^* = \mathbf{Q}_V(Cv)$, and commutation relations $[\mathbf{Q}_V(v), \mathbf{Q}_V(w)] = i\omega(v,w)1_{Q(V,\omega,C)}$. Further, if $f : (V, \omega_V, C_V) \to (W, \omega_W, C_W)$ is a symplectic C-homomorphism (again complexifying if needed), then $Qf : Q(V, \omega_V, C_V) \to Q(W, \omega_W, C_W)$ is the unique unital *-homomorphism such that $Qf(\mathbf{Q}_V(v)) = \mathbf{Q}_W(fv)$ for all $v \in V$. These descriptions give rise to functors $Q : \mathbf{pSympl}_{\mathbb{K}} \to {}^*\mathbf{Alg}_1$.

It turns out that $Q(V, \omega, C)$ is a simple unital *-algebra whenever ω is weakly non-degenerate [2, Scholium 7.1] and if f is injective, Qf will be injective.[12] Hence, Q also gives rise to functors $\mathbf{Sympl}_{\mathbb{K}} \to {}^*\mathbf{Alg}_1^{\mathrm{m}}$ and $\mathbf{pSympl}_{\mathbb{K}}^{\mathrm{m}} \to {}^*\mathbf{Alg}_1^{\mathrm{m}}$, all of which will be denoted by the same symbol Q and called the *quantisation functor*. For the various locally covariant theories introduced above it holds that $\mathfrak{R} = Q \circ \mathcal{R}$, $\mathfrak{F}_u = Q \circ \mathcal{F}_u$ and $\mathfrak{F} = Q \circ \mathcal{F}$ as well as $\mathfrak{A} = Q \circ \mathcal{A}$.

4.4. Electromagnetic Duality

As a slight digression, but with a view to later developments, we discuss the status of electromagnetic duality in our theories. In each $\boldsymbol{M} \in \mathbf{Loc}$, the Hodge-$*$ is a linear isomorphism of $\Omega_0^2(\boldsymbol{M};\mathbb{K})$ to itself. As $\delta_{\boldsymbol{M}} *_{\boldsymbol{M}} = (-1)^{p+1} *_{\boldsymbol{M}} d_{\boldsymbol{M}}$ and $d_{\boldsymbol{M}} *_{\boldsymbol{M}} = (-1)^p *_{\boldsymbol{M}} \delta_{\boldsymbol{M}}$ on $\Omega^p(\boldsymbol{M};\mathbb{K})$, it is easily seen that $*_{\boldsymbol{M}}$ induces an isomorphism of the quotient space $\left[\Omega_0^2(M;\mathbb{K})\right]$ given by $[\omega] \mapsto [*_{\boldsymbol{M}}\omega]$, and evidently obeys $\mathcal{F}\psi[*_{\boldsymbol{M}}\omega] = [*_{\boldsymbol{N}}\psi_*\omega]$ for every morphism $\psi : \boldsymbol{M} \to \boldsymbol{N}$ in $\mathbf{Loc}_{©}$. At the level of solutions, $d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} *_{\boldsymbol{M}} \omega = - *_{\boldsymbol{M}} d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} \omega$ for $\omega \in \Omega_0^2(\boldsymbol{M};\mathbb{K})$ and one easily derives from this that

$$\begin{aligned}\mathfrak{w}_{\boldsymbol{M}}\left([*_{\boldsymbol{M}}\omega], [*_{\boldsymbol{M}}\eta]\right) &= -\int_M d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} *_{\boldsymbol{M}} \omega \wedge (*_{\boldsymbol{M}} *_{\boldsymbol{M}} \eta) \\ &= -\int_M d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} \omega \wedge *_{\boldsymbol{M}} \eta \\ &= \mathfrak{w}_{\boldsymbol{M}}\left([\omega], [\eta]\right), \qquad [\omega], [\eta] \in \left[\Omega_0^2(M;\mathbb{K})\right].\end{aligned}$$

From these results, it follows that the electromagnetic duality rotations

$$\Theta_{\boldsymbol{M}}(\alpha)[\omega] = [\cos\alpha\,\omega + \sin\alpha *_{\boldsymbol{M}} \omega] \qquad \omega \in \Omega_0^2(\boldsymbol{M}),\ \boldsymbol{M} \in \mathbf{Loc}_{©} \tag{8}$$

yield automorphisms $\Theta(\alpha) \in \mathrm{Aut}(\mathcal{F})$ for $\alpha \in \mathbb{R}$; as $\Theta(\alpha)\Theta(\beta) = \Theta(\alpha + \beta)$ and $\Theta(\alpha + 2\pi) = \Theta(\alpha)$ for all $\alpha, \beta \in \mathbb{R}$, we see that there is a faithful homomorphism from $U(1)$ to $\mathrm{Aut}(\mathcal{F})$. In a similar way one may check that these

[11] Real pre-symplectic spaces are treated by first complexifying them.

[12] See [24] which discusses an equivalent description using the symmetric tensor product.

automorphisms lift to automorphisms of both the universal and reduced theories $\mathcal{F}_u$ and $\mathcal{R}$. Furthermore, they induce automorphisms of the quantised theories by $\hat{\Theta}_{\boldsymbol{M}}(\alpha)\mathbf{F}_{\boldsymbol{M}}(\omega) = \mathbf{F}_{\boldsymbol{M}}(\cos\alpha\,\omega + \sin\alpha *_{\boldsymbol{M}} \omega)$ and so forth.

Automorphisms of locally covariant theories have been identified as global gauge transformations [19]. This raises an interesting question, because the electromagnetic duality is not a symmetry of the Maxwell Lagrangian $\mathscr{L} = -\frac{1}{4}F \wedge *F$, which changes sign under $F \mapsto *F$. One might be concerned that the presence of these automorphisms is an indication that the theories under consideration are not true reflections of the original physics. Against this, we note that the Maxwell Lagrangian has other unusual properties, in particular, the field equations obtained by variation with respect to F are trivial. The Maxwell equations can be derived from the Lagrangian, however, by demanding conservation of the stress-energy tensor constructed by varying the action with respect to the metric. As electromagnetic duality rotations leave the stress-energy tensor invariant, there is good reason to accept them as *bona fide* symmetries.

5. Kinematical and Dynamical Nets, and Dynamical Locality

One of the virtues of locally covariant quantum field theory is that it generalises the framework of algebraic quantum field theory in a natural way [12, § 2.4]. Let $F : \mathbf{Loc} \to (\mathbf{C})^*\mathbf{Alg}_1^{\mathrm{m}}$ be a locally covariant quantum field theory, where we think of the algebra elements as local observables or as local smearings of the quantum field. For $\boldsymbol{M} \in \mathbf{Loc}$, we denote the set of all globally hyperbolic open subsets of $\boldsymbol{M}$ by $\mathscr{O}(\boldsymbol{M})$. Due to the functoriality of F, $\mathscr{O}(\boldsymbol{M}) \ni O \mapsto F\iota_O(F\boldsymbol{M}|_O)$ is a net of local unital $(C)^*$-algebras, where $\iota_O : O \to M$ denotes the inclusion map and $\boldsymbol{M}|_O$ denotes O endowed with the structures as an oriented globally hyperbolic spacetime induced by $\boldsymbol{M}$. We call this net of local unital $(C)^*$-algebras the *kinematical net* of F for $\mathbf{M}$ and also denote $\iota_O(F\boldsymbol{M}|_O)$ by $F^{\mathrm{kin}}(\boldsymbol{M}; O)$. The adjective "kinematical" is chosen because the construction only relies on the functoriality of F, which corresponds to isotony in algebraic quantum field theory, that is, a Haag–Araki–Kastler axiom referring to the kinematics of a quantum field theory.

By contrast, the *dynamical net* of F for $\boldsymbol{M} \in \mathbf{Loc}$ is a net of local unital $(C)^*$-algebras whose construction refers to the relative Cauchy evolution of F and hence to dynamical aspects of the locally covariant quantum field theory [23, § 5]. We thus assume that F obeys the time-slice axiom and take K compact in $\boldsymbol{M} \in \mathbf{Loc}$. Then, we consider all elements of $F\boldsymbol{M}$ which are insensitive to all globally hyperbolic perturbations $h \in H(\boldsymbol{M}; K^\perp)$ supported in the region $K^\perp := M \backslash J_{\boldsymbol{M}}(K)$ that is causally inaccessible to K,

$$F^\bullet(\boldsymbol{M}; K) = \left\{ a \in F\boldsymbol{M} \mid \mathrm{rce}_{\boldsymbol{M}}^F[h]\, a = a \;\; \forall h \in H\left(\boldsymbol{M}; K^\perp\right) \right\}.$$

This can be used to define what it means for an observable or smearing of the quantum field to be localised in K. Finally, to localise observables or smearings of the quantum field in globally hyperbolic open subsets of $\boldsymbol{M}$, we define for all $O \in \mathscr{O}(\boldsymbol{M})$,

$$F^{\mathrm{dyn}}(\boldsymbol{M};O) := \bigvee_{K\in\mathscr{K}(\boldsymbol{M};O)} F^{\bullet}(\boldsymbol{M};K),$$

that is, $F^{\mathrm{dyn}}(\boldsymbol{M};O)$ is defined as the unital $(C)^*$-algebra generated by the unital $(C)^*$-algebras $F^{\bullet}(\boldsymbol{M};K)$, where K ranges over a specific collection $\mathcal{K}(\boldsymbol{M};O)$ of compact subsets of O.

The definition of $\mathscr{K}(\boldsymbol{M};O)$ is slightly involved. First, we define a *Cauchy ball* to be an open set of a smooth spacelike Cauchy surface Σ for $\boldsymbol{M}$ diffeomorphic to an open ball of $\mathbb{R}^3$ under a smooth chart for Σ, with the chart image containing the ball's closure.[13] A finite union of causally disjoint Cauchy balls is called a *multi-diamond*, of which the Cauchy balls form the *base*. Finally, following [23, § 5], $\mathscr{K}(\boldsymbol{M};O)$ is the set of all compact subsets of $O \in \mathscr{O}(\boldsymbol{M})$ which have a multi-diamond open neighbourhood whose base is contained in O.

The assignment $\mathscr{O}(\boldsymbol{M}) \ni O \mapsto F^{\mathrm{dyn}}(\boldsymbol{M};O)$ is the *dynamical net* of F for $\boldsymbol{M}$ and a locally covariant quantum field theory is said to be *dynamically local* if and only if it obeys the time-slice axiom and the dynamical net coincides with the kinematical net. Note that $F^{\mathrm{kin}}(\boldsymbol{M};O)$ and $F^{\mathrm{dyn}}(\boldsymbol{M};O)$ are both subalgebras of $F\boldsymbol{M}$.

The kinematical and dynamical nets can be defined in much more general locally covariant theories, including categories **Phys** which do not have a notion of the image of a morphism. To do this, we redefine $F^{\mathrm{kin}}(\boldsymbol{M};O)$ as $F\boldsymbol{M}|_O$, and focus attention on the unital *-monomorphism $m^{\mathrm{kin}}_{\boldsymbol{M};O} : F^{\mathrm{kin}}(\boldsymbol{M};O) \to F\boldsymbol{M}$ given by $m^{\mathrm{kin}}_{\boldsymbol{M};O} = F\iota_{\boldsymbol{M};O}$, where $\iota_{\boldsymbol{M};O} : \boldsymbol{M}|_O \to \boldsymbol{M}$ is the **Loc**-morphism induced by the inclusion map of O in M. In categorical terminology, $m^{\mathrm{kin}}_{\boldsymbol{M};O}$ is *monic* and defines a *subobject* of $F\boldsymbol{M}$ (see [39] or [23, Appx. B]), thus allowing us to define the kinematical net for every locally covariant theory $F : \mathbf{Loc} \to \mathbf{Phys}$ by the rule assigning to each $O \in \mathscr{O}(\boldsymbol{M})$ the subobject $m^{\mathrm{kin}}_{\boldsymbol{M};O} : F^{\mathrm{kin}}(\boldsymbol{M};O) \to F\boldsymbol{M}$. Also the dynamical net can be characterised purely in terms of categorical notions, to be specific *equalisers*, *intersections* and *unions* of subobjects (see again [39] or [23, Appx. B]), and thus can be formulated for every locally covariant theory $F : \mathbf{Loc} \to \mathbf{Phys}$ obeying the time-slice axiom. The construction results in an assignment of $O \in \mathscr{O}(\boldsymbol{M})$ to a subobject $m^{\mathrm{dyn}}_{\boldsymbol{M};O} : F^{\mathrm{dyn}}(\boldsymbol{M};O) \to F\boldsymbol{M}$, details of which can be found in [23, § 5]. A locally covariant theory is dynamically local if and only if $m^{\mathrm{kin}}_{\boldsymbol{M};O}$ and $m^{\mathrm{dyn}}_{\boldsymbol{M};O}$ are equivalent subobjects (see yet again [39] or [23, Appx. B]) for every $O \in \mathscr{O}(\boldsymbol{M})$.

The net advantage of this abstract categorical viewpoint is an immense simplification in the argument that the quantised reduced free F-theory $\mathfrak{R} : \mathbf{Loc} \to {}^*\mathbf{Alg}^{\mathrm{m}}_1$ is dynamically local. Since $\mathfrak{R}$ is related to the classical reduced free F-theory $\mathcal{R} : \mathbf{Loc} \to \mathbf{Sympl}_{\mathbb{K}}$ via a functorial quantisation prescription, it

[13] Every point $x \in M$ is contained in a Cauchy ball: let Σ_x be any smooth spacelike Cauchy surface for $\boldsymbol{M}$ containing x, choose any smooth chart $\varphi : U \to W \subseteq \mathbb{R}^3$ for Σ_x with $x \in U$ and $\varepsilon > 0$ such that the ε-ball around $\varphi(x)$ is contained in W, and then take $B_x := \varphi^{-1}(B_\delta(\varphi(x)))$ with $\delta < \varepsilon$.

is enough to show that $\mathcal{R}$ is dynamically local and then prove a small number of additional properties listed as [24, ($\mathscr{L}1$–$\mathscr{L}4$)].

6. Dynamical Locality of the Universal Theory

6.1. The Universal Theory Fails Local Covariance

It was already pointed out in [14, § 3.7] that the quantised universal free F-theory $\mathfrak{F}_u : \mathbf{Loc} \to {}^*\mathbf{Alg}_1$ is not a locally covariant quantum field theory according to [12] because algebra homomorphisms corresponding to spacetime embeddings are not always injective. The same is true for the classical universal free F-theory $\mathcal{F}_u : \mathbf{Loc} \to \mathbf{pSympl}_{\mathbb{K}}$. This problem arises whenever one has a $\mathbf{Loc}$-morphism $\psi : \boldsymbol{M} \to \boldsymbol{N}$ between objects obeying $H^2_{dR,c}(M;\mathbb{K}) \cong H^2_{dR}(M;\mathbb{K}) \neq 0$ and $H^2_{dR,c}(N;\mathbb{K}) \cong H^2_{dR}(N;\mathbb{K}) = 0$, where the isomorphisms are due to Poincaré duality. A specific instance may be given as follows:

Example 6.1. With $\boldsymbol{M}$ as in Example 4.2 and $\boldsymbol{N}$ taken to be Minkowski spacetime, let $\psi : \boldsymbol{M} \to \boldsymbol{N}$ be the inclusion morphism. Then $H^2_{dR}(M;\mathbb{K}) \cong \mathbb{K}$, while $H^2_{dR}(N;\mathbb{K}) = 0$.

Under such circumstances, there exists $\omega \in \Omega^2_{0,d}(M;\mathbb{K})\backslash d_{\boldsymbol{M}}\Omega^2_0(M;\mathbb{K})$, which corresponds to a nonzero element $[\omega] \in \left[\Omega^2_0(M;\mathbb{K})\right] = \mathcal{F}_u\boldsymbol{M}$, because ω cannot be written in the form $\omega = d_{\boldsymbol{M}}\theta + \delta_{\boldsymbol{M}}\eta$ for $\theta \in \Omega^1_0(M;\mathbb{K})$ and $\eta \in \Omega^3_0(M;\mathbb{K})$.[14] However, the push-forward $\psi_*\omega \in \Omega^2_0(N;\mathbb{K})$ obeys $d_{\boldsymbol{N}}\psi_*\omega = \psi_* d_{\boldsymbol{M}}\omega = 0$ and hence $\psi_*\omega \in d_{\boldsymbol{N}}\Omega^1_0(N;\mathbb{K}) \oplus \delta_{\boldsymbol{N}}\Omega^3_0(N;\mathbb{K})$ because $H^2_{dR,c}(N;\mathbb{K}) = 0$. Thus, $(\mathcal{F}_u\psi)[\omega] = [\psi_*\omega] = 0 \in \left[\Omega^2_0(N;\mathbb{K})\right]$ and, similarly, $(\mathfrak{F}_u\psi)(\mathbf{F}_{\boldsymbol{M}}(\omega)) = \mathbf{F}_{\boldsymbol{N}}(\psi_*\omega) = 0_{\mathfrak{F}_u\boldsymbol{N}}$, so neither $\mathcal{F}_u\psi$ nor $\mathfrak{F}_u\psi$ is injective.

A similar argument applies to $\omega \in \Omega^2_{0,\delta}(M;\mathbb{K})\backslash\delta_{\boldsymbol{M}}\Omega^2_0(M;\mathbb{K})$. The elements just described in this and the last paragraph precisely span the radical of $\mathfrak{w}_{u\boldsymbol{M}}$ and the centre of $\mathfrak{F}_u\boldsymbol{M}$, respectively, $\boldsymbol{M} \in \mathbf{Loc}$ (cf. [14, Prop.3.3]). Hence, local covariance of $\mathcal{F}_u$ and $\mathfrak{F}_u$ is precisely spoiled by the radical elements and the central elements, respectively.

Despite the failure of injectivity, the theories $\mathcal{F}_u$ and $\mathfrak{F}_u$ are well-behaved in other ways. For example, $\mathfrak{F}_u$ is still a causal functor—local algebras of causally disjoint regions commute—owing to the form of Lichnerowicz's commutator. As we will see shortly, both $\mathcal{F}_u$ and $\mathfrak{F}_u$ obey the time-slice axiom, i.e. $\mathcal{F}_u\psi$ is a $\mathbf{pSympl}_{\mathbb{K}}$-isomorphism and $\mathfrak{F}_u\psi$ is an ${}^*\mathbf{Alg}_1$-isomorphism whenever $\psi \in \mathbf{Loc}(\boldsymbol{M},\boldsymbol{N})$ is Cauchy.

6.2. The Universal Theory Obeys the Time-Slice Axiom

We start with some helpful, more general statements, which will allow us to show the validity of the time-slice axiom and to compute inverses. The specification of inverses usually involves certain choices (of representatives of equivalences classes and of smooth spacelike Cauchy surfaces) and *time-slice maps*

[14] Otherwise, $\delta_{\boldsymbol{M}}\eta = G^{\mathrm{ret}}_{\boldsymbol{M}}\Box_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\eta = -G^{\mathrm{ret}}_{\boldsymbol{M}}\delta_{\boldsymbol{M}}d_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\eta = -G^{\mathrm{ret}}_{\boldsymbol{M}}\delta_{\boldsymbol{M}}d_{\boldsymbol{M}}(d_{\boldsymbol{M}}\theta - \omega) = 0$, so $\omega = d_{\boldsymbol{M}}\theta$, a contradiction.

will help us to efficiently deal with these choices. For the rest of this subsection, let $\psi \in \mathbf{Loc}(\boldsymbol{M}, \boldsymbol{N})$ be Cauchy, $\xi = (E, N, \pi, V)$ a smooth $\mathbb{K}$-vector bundle over N and $P : \Gamma^\infty(\xi) \to \Gamma^\infty(\xi)$ a normally hyperbolic differential operator of metric type.

Definition 6.2. A *time-slice map* for (ψ, ξ, P) is a $\mathbb{K}$-linear map $L : \Gamma_0^\infty(\xi) \to \Gamma_0^\infty(\xi)$ such that

$$\operatorname{supp}\left(\left(\mathrm{id}_{\Gamma_0^\infty(\xi)} - PL\right)\sigma\right) \subseteq \psi(M) \qquad \forall \sigma \in \Gamma_0^\infty(\xi).$$

If a particular time-slice map is understood, we will write

$$\sigma = \sigma_{€} + P\sigma_{£}$$

for the corresponding decomposition $\sigma_{£} := L\sigma$, $\sigma_{€} := \sigma - P\sigma_{£}$.

Time-slice maps exist by slight modification of a standard construction: fix any two smooth spacelike Cauchy surfaces Σ^f and Σ_p for $\boldsymbol{N}$ such that $\Sigma^f, \Sigma_p \subseteq \psi(M)$ and Σ^f lies strictly in the future of Σ_p. This can be achieved using [23, Lem.A.2] and the splitting theorem of Bernal and Sánchez [7, Prop.2.4]. Further, let $\{\chi^+, \chi^-\}$ be a smooth partition of unity subordinated to the open cover $\left\{I^+_{\boldsymbol{N}}(\Sigma_p), I^-_{\boldsymbol{N}}(\Sigma^f)\right\}$ of N. Define for each $\sigma \in \Gamma_0^\infty(\xi)$

$$\sigma_{€} := \sigma - P\chi^+ G^{\mathrm{adv}}\sigma - P\chi^- G^{\mathrm{ret}}\sigma, \tag{9}$$

where G^{adv} and G^{ret} are the advanced and the retarded Green's operator for P, which exist and are unique [3, Cor.3.4.3]. By the properties of $\chi^\pm$ and $G^{\mathrm{ret/adv}}$, $\operatorname{supp}\sigma_{€}$ is compactly supported in $\psi(M)$. Finally, $\sigma_{£} \in \Gamma_0^\infty(\xi)$ is defined by $\sigma_{£} := \chi^+ G^{\mathrm{adv}}\sigma + \chi^- G^{\mathrm{ret}}\sigma$. However, many properties of time-slice maps can be proved without using a specific formula. The main technical point is that any compactly supported solution ϕ to the inhomogeneous equation $P\phi = \sigma$, where $\sigma \in \Gamma_0^\infty(\xi)$, must be supported in the intersection $J^+_{\boldsymbol{N}}(\operatorname{supp}\sigma) \cap J^-_{\boldsymbol{N}}(\operatorname{supp}\sigma)$ because $\phi = G^{\mathrm{ret/adv}}\sigma$. Let us observe

Lemma 6.3. *If L is any time-slice map for (ψ, ξ, P), we have*

$$\operatorname{supp} L\sigma \subseteq \psi(M)$$

whenever $\sigma \in \Gamma_0^\infty(\xi)$ with $\operatorname{supp}\sigma \subseteq \psi(M)$; if K is another time-slice map for (ψ, ξ, P), then

$$\operatorname{supp}(K - L)\sigma \subseteq \psi(M) \qquad \forall \sigma \in \Gamma_0^\infty(\xi).$$

Hence,

$$\sigma_{€_K} - \sigma_{€_L} = P\tau,$$

where $\tau \in \Gamma_0^\infty(\xi)$ with $\operatorname{supp}\tau \subseteq \psi(M)$; moreover,

$$L\sigma\Big|_{N \setminus J^{-/+}_{\boldsymbol{N}}(\psi(M))} = G^{adv/ret}_{\boldsymbol{N}}\sigma\Big|_{N \setminus J^{-/+}_{\boldsymbol{N}}(\psi(M))}.$$

Proof. Taking any $\sigma \in \Gamma_0^\infty(\xi)$ with $\operatorname{supp}\sigma \subseteq \psi(M)$, $PL\sigma = \sigma - (\mathrm{id}_{\Gamma_0^\infty(\xi)} - PL)\sigma$ is (compactly) supported in $\psi(M)$. As $L\sigma$ is compactly supported, it follows that $L\sigma$ is supported in $J^+_{\boldsymbol{N}}(\psi(M)) \cap J^-_{\boldsymbol{N}}(\psi(M)) = \psi(M)$ as required. Next, let $\sigma \in \Gamma_0^\infty(\xi)$. Then by definition of time-slice maps, $P(K - L)\sigma$ has support in $\psi(M)$, while $(K-L)\sigma$ has compact support. Thus, $(K-L)\sigma$ is (compactly) supported in $J^+_{\boldsymbol{N}}(\psi(M)) \cap J^-_{\boldsymbol{N}}(\psi(M)) = \psi(M)$. The penultimate formula follows from this and the definition $\sigma_{€} := \sigma - P\sigma_{£} = \sigma - PL\sigma$ for $\sigma \in \Gamma^\infty_0(\xi)$. Finally, our result shows that the action of any timeslice map on σ is fixed modulo terms compactly supported in $\psi(M)$. Outside this set, all timeslice maps agree, so we may use the formula implicit in (9) to obtain the final result. ☐

As a digression, the existence of a time-slice map for (ψ, ξ, P) implies that the following is a short exact sequence of $\mathbb{K}$-linear maps

$$0 \longrightarrow P\Gamma^\infty_0(\xi, \psi(M)) \xrightarrow{\alpha} \Gamma^\infty_0(\xi) \oplus P\Gamma^\infty_0(\xi, \psi(M)) \xrightarrow{\beta} \Gamma^\infty_0(\xi) \longrightarrow 0$$

where $\alpha : \sigma \longmapsto (\sigma, -\sigma)$ and $\beta : (\sigma, \tau) \longmapsto \sigma + \tau$, and we denote smooth sections of ξ with compact support in $O \subset N$ by $\Gamma^\infty_0(\xi, O)$. Exactness at $P\Gamma^\infty_0(\xi, \psi(M))$ is immediate because α is injective; moreover, its image is precisely the kernel of β, so, we have exactness at $\Gamma^\infty_0(\xi) \oplus P\Gamma^\infty_0(\xi, \psi(M))$. Any time-slice map L for (ψ, ξ, P) induces $\gamma : \Gamma^\infty_0(\xi) \to \Gamma^\infty_0(\xi) \oplus P\Gamma^\infty_0(\xi, \psi(M))$ by $\gamma : \sigma \longmapsto (\sigma - PL\sigma, PL\sigma)$, and as $\beta \circ \gamma = \mathrm{id}_{\Gamma^\infty_0(\xi)}$, it is clear that β is surjective and we have a split short exact sequence.

Lemma 6.4. *Let $\eta = (D, N, \varrho, W)$ be a smooth $\mathbb{K}$-vector bundle with a normally hyperbolic differential operator $Q : \Gamma^\infty(\eta) \to \Gamma^\infty(\eta)$ such that P and Q are intertwined by a linear differential operator $\partial : \Gamma^\infty(\xi) \to \Gamma^\infty(\eta)$, i.e. $\partial \circ P = Q \circ \partial$. Suppose L and K are time-slice maps for (ψ, ξ, P) and (ψ, η, Q), then for any $\sigma \in \Gamma^\infty_0(\xi)$,*

$$\mathrm{supp}\,(\partial L\sigma - K\partial\sigma) \subseteq \psi(M)$$

and accordingly

$$(\partial\sigma)_{€_K} - \partial\sigma_{€_L} = Q\,(\partial L\sigma - K\partial\sigma) = Q\tau$$

with $\tau \in \Gamma^\infty_0(\eta)$, $\mathrm{supp}\,\tau \subseteq \psi(M)$.

Proof. We calculate for $\sigma \in \Gamma^\infty_0(\xi)$

$$\begin{aligned} Q\,(\partial L\sigma - K\partial\sigma) &= \partial PL\sigma - QK\partial\sigma = \partial\,(\sigma - \sigma_{€_L}) - \big(\partial\sigma - (\partial\sigma)_{€_K}\big) \\ &= (\partial\sigma)_{€_K} - \partial\sigma_{€_L}, \end{aligned}$$

where $\mathrm{supp}((\partial\sigma)_{€_K} - \partial\sigma_{€_L}) \subseteq \psi(M)$. Hence, $\partial L\sigma - K\partial\sigma$ is compactly supported in $\psi(M)$ and the remaining assertion follows. ☐

Finally, let us apply this to smooth differential forms with a view to the description of electromagnetism. Let our smooth $\mathbb{K}$-vector bundles be the (complexified if $\mathbb{K} = \mathbb{C}$) p-th exterior power λ^p_N of the cotangent bundle τ^*_N of N for $p \geq 0$ and let $\psi \in \mathbf{Loc}(\boldsymbol{M}, \mathbf{N})$ be Cauchy. Then taking the appropriate wave operators as the normally hyperbolic differential operators acting on smooth differential p-forms, the exterior derivative and the exterior coderivative provide intertwining operators. The previous lemma now gives the following.

Lemma 6.5. *For any time-slice map $L : \Omega_0^p(N;\mathbb{K}) \to \Omega_0^p(N;\mathbb{K})$, we have for $\omega \in \Omega_0^p(N;\mathbb{K})$,*

$$(d_{\boldsymbol{N}}\omega)_{\in} - d_{\boldsymbol{N}}\omega_{\in} = \Box_{\boldsymbol{N}}\eta \quad \text{and} \quad (\delta_{\boldsymbol{N}}\omega)_{\in} - \delta_{\boldsymbol{N}}\omega_{\in} = \Box_{\boldsymbol{N}}\theta,$$

where $\eta \in \Omega_0^{p+1}(N;\mathbb{K})$ with $\operatorname{supp}\eta \subseteq \psi(M)$ and $\theta \in \Omega_0^{p-1}(N;\mathbb{K})$ with $\operatorname{supp}\theta \subseteq \psi(M)$. Further, if $\omega \in \Omega_{0,d}^p(N;\mathbb{K}) \oplus \Omega_{0,\delta}^p(N;\mathbb{K})$, then

$$\omega_{\in} = \alpha + \beta, \tag{10}$$

where $\alpha \in \Omega_{0,d}^p(N;\mathbb{K})$ with $\operatorname{supp}\alpha \subseteq \psi(M)$ and $\beta \in \Omega_{0,\delta}^p(N;\mathbb{K})$ with $\operatorname{supp}\beta \subseteq \psi(M)$.

Proof. The first part is a direct consequence of Lemma 6.4. Now, suppose that $d_{\mathbf{N}}\omega = 0$, then $\operatorname{supp}(d_{\mathbf{N}}L\omega) \subseteq \psi(M)$. Using $\Box_{\boldsymbol{N}} = -(d_{\boldsymbol{N}}\delta_{\boldsymbol{N}} + \delta_{\boldsymbol{N}}d_{\boldsymbol{N}})$, we have

$$\omega = \omega_{\in} + \Box_{\mathbf{N}}L\omega \qquad \text{or equivalently} \qquad \omega + d_{\boldsymbol{N}}\delta_{\boldsymbol{N}}L\omega = \omega_{\in} - \delta_{\boldsymbol{N}}d_{\boldsymbol{N}}L\omega$$

the right-hand side of which is obviously supported in $\psi(M)$. Hence, the left-hand side of the second equation must have the same support and is in the kernel of $d_{\boldsymbol{N}}$. Thus, (10) holds for closed ω, and as the same argument applies to coexact ω, the result is proved. □

We will now apply these general statements to show that $\mathcal{F}_u$ and $\mathfrak{F}_u$ obey the time-slice axiom. In the proof, we will explicitly construct the inverses of $\mathcal{F}_u\psi$ and $\mathfrak{F}_u\psi$, where $\psi \in \mathbf{Loc}(\boldsymbol{M},\boldsymbol{N})$ is Cauchy, which will be helpful when computing a concrete expression for the relative Cauchy evolution for $\mathcal{F}_u$ and $\mathfrak{F}_u$. Since functors preserve isomorphisms and $\mathfrak{F}_u = Q \circ \mathcal{F}_u$ (where $Q : \mathbf{pSympl}_{\mathbb{K}} \to {}^*\mathbf{Alg}_1$ is the quantisation functor) it is enough to concentrate on the classical universal free F-theory.

Proposition 6.6. *For $\psi \in \mathbf{Loc}(\boldsymbol{M},\boldsymbol{N})$ Cauchy, $\mathcal{F}_u\psi$ is a $\mathbf{pSympl}_{\mathbb{K}}$-isomorphism whose inverse is explicitly given by*

$$(\mathcal{F}_u\psi)^{-1} : \mathcal{F}_u\boldsymbol{N} \to \mathcal{F}_u\boldsymbol{M}, \qquad [\omega] \longmapsto [\psi^*\omega_{\in}],$$

for any time-slice map of $(\psi, \lambda_N^2, \Box_{\boldsymbol{N}})$ and any representative ω of the equivalence class $[\omega] \in \left[\Omega_0^2(N;\mathbb{K})\right]$. Thus, both $\mathcal{F}_u$ and $\mathfrak{F}_u$ obey the time-slice axiom.

Proof. By Lemmas 6.3 and 6.4, the map $\Xi : \mathcal{F}_u\boldsymbol{N} \to \mathcal{F}_u\boldsymbol{M}$, $[\omega] \longmapsto [\psi^*\omega_{\in}]$, is well-defined, i.e. independent of the representative of $[\omega] \in \left[\Omega_0^2(N;\mathbb{K})\right]$ and the time-slice map chosen (cf. the paragraph after Lemma 6.4). It is not difficult to check that Ξ is $\mathbb{K}$-linear, symplectic and intertwines with the C-involution in the case $\mathbb{K} = \mathbb{C}$. The computations

$$(\Xi \circ (\mathcal{F}_u\psi))[\omega] = \Xi[\psi_*\omega] = \left[\psi^*(\psi_*\omega)_{\in}\right] = [\psi^*\psi_*\omega] = [\omega] \qquad \forall [\omega] \in \left[\Omega_0^2(M;\mathbb{K})\right],$$

where we have used Lemma 6.3, and

$$((\mathcal{F}_u\psi) \circ \Xi)[\omega] = (\mathcal{F}_u\psi)[\psi^*\omega_{\in}] = [\psi_*\psi^*\omega_{\in}] = [\omega_{\in}] = [\omega] \qquad \forall [\omega] \in \left[\Omega_0^2(N;\mathbb{K})\right]$$

show the rest. □

6.3. The Relative Cauchy Evolution of the Universal Theory

The explicit inverse computed in Proposition 6.6 allows us to compute the relative Cauchy evolution for $\mathcal{F}_u$ and $\mathfrak{F}_u$ induced by $h \in H(\boldsymbol{M})$. To this end, let $L^{\pm} : \Omega^2_0(M;\mathbb{K}) \to \Omega^2_0(M;\mathbb{K})$ be time-slice maps for $(\imath^{+}_{\boldsymbol{M}}[h] : \boldsymbol{M}^{+}[h] \to \boldsymbol{M}, \lambda^2_M, \Box_{\boldsymbol{M}})$ and $(\imath^{-}_{\boldsymbol{M}}[h] : \boldsymbol{M}^{-}[h] \to \boldsymbol{M}[h], \lambda^2_M, \Box_{\boldsymbol{M}[h]})$ respectively and use the symbols '$€^{\pm}$' to correspond to $L^{\pm}$. Then we have, for any $[\omega] \in \left[\Omega^2_0(M;\mathbb{K})\right]$,

$$\begin{aligned}\mathrm{rce}^{\mathcal{F}_u}_{\boldsymbol{M}}[h]\,[\omega] &= [(\omega_{€^+})_{€^-}] = [\omega_{€^+}] - \left[\Box_{\boldsymbol{M}[h]} L^{-}\omega_{€^+}\right] \\ &= [\omega] + \left[(\Box_{\boldsymbol{M}} - \Box_{\boldsymbol{M}[h]})L^{-}\omega_{€^+}\right],\end{aligned}$$

where we have used the fact that $L^{-}\omega_{€^+}$ is compactly supported and hence $[\Box_{\boldsymbol{M}}L^{-}\omega_{€^+}] = 0$. Now, $\Box_{\boldsymbol{M}}$ and $\Box_{\boldsymbol{M}[h]}$ differ only on the support of h, which lies outside and to the future of the range of $\imath^{-}_{\boldsymbol{M}}[h]$, allowing us to replace L^{-} by $G^{\mathrm{adv}}_{\boldsymbol{M}[h]}$ (by the last part of Lemma 6.3). Hence

$$\begin{aligned}\mathrm{rce}^{\mathcal{F}_u}_{\boldsymbol{M}}[h]\,[\omega] &= [\omega] + \left[(\Box_{\boldsymbol{M}} - \Box_{\boldsymbol{M}[h]})G^{\mathrm{adv}}_{\boldsymbol{M}[h]}\omega_{€^+}\right] \\ &= [\omega] - \left[(\Box_{\boldsymbol{M}} - \Box_{\boldsymbol{M}[h]})G_{\boldsymbol{M}[h]}\omega_{€^+}\right],\end{aligned}$$

where we have used the fact that $G^{\mathrm{ret}}_{\boldsymbol{M}[h]}\omega_{€^+}$ vanishes on the support of h. This expression is independent of the time-slice map L^{+}, because $\omega_{€^+}$ is fixed modulo the image of $\Box_{\boldsymbol{M}}$ on smooth differential 2-forms compactly supported in the image of $\imath^{+}_{\boldsymbol{M}}[h]$, on which $\Box_{\boldsymbol{M}}$ and $\Box_{\boldsymbol{M}[h]}$ agree. Standard manipulations with smooth differential forms and the equivalence relation give

$$\mathrm{rce}^{\mathcal{F}_u}_{\boldsymbol{M}}[h]\,[\omega] = [\omega] - \left[\left(\delta_{\boldsymbol{M}[h]} - \delta_{\boldsymbol{M}}\right) G_{\boldsymbol{M}[h]} d_{\boldsymbol{M}}\omega_{€^+}\right], \tag{11}$$

for any $[\omega] \in \left[\Omega^2_0(M;\mathbb{K})\right]$. Finally, the relative Cauchy evolution of $\mathfrak{F}_u$ is given by the application of the quantisation functor:

$$\mathrm{rce}^{\mathfrak{F}_u}_{\boldsymbol{M}}[h] = Q\left(\mathrm{rce}^{\mathcal{F}_u}_{\boldsymbol{M}}[h]\right).$$

6.4. The Failure of Dynamical Locality for the Universal Theory

The failure of injectivity demonstrated in Sect. 6.3, already shows that $\mathcal{F}_u$ and $\mathfrak{F}_u$ cannot possibly be dynamically local in the original sense of this definition [23]. For if $\psi \in \mathbf{Loc}(\boldsymbol{M},\boldsymbol{N})$ is such that $\mathcal{F}_u\psi$ is non-injective (e.g., as in Example 6.1) then the same holds for $\mathrm{f}^{\mathrm{kin}}_{\boldsymbol{N};M}$, which is thereby inequivalent to the (necessarily injective/monic) subobject $\mathrm{f}^{\mathrm{dyn}}_{\boldsymbol{N};M} : \mathcal{F}^{\mathrm{dyn}}_u(\boldsymbol{N};M) \to \mathcal{F}_u\boldsymbol{N}$. Similarly, in the quantised case, the subobject $\varphi^{\mathrm{dyn}}_{\boldsymbol{N};M} : \mathfrak{F}^{\mathrm{dyn}}_u(\boldsymbol{N};M) \to \mathfrak{F}_u\boldsymbol{N}$ cannot be equivalent to the non-monic $\varphi^{\mathrm{kin}}_{\boldsymbol{N};M} = \mathfrak{F}_u\psi : \mathfrak{F}^{\mathrm{kin}}_u(\boldsymbol{N};M) = \mathfrak{F}_u\boldsymbol{M} \to \mathfrak{F}_u\boldsymbol{N}$.

In this subsection, we show that the failure of dynamical locality for these theories is even more severe and cannot be achieved even if we restrict to *contractible* globally hyperbolic open subsets. There is no harm now in shifting our focus from the abstract categorical subobjects $\mathrm{f}^{\mathrm{kin}}_{\boldsymbol{M};O}$, $\mathrm{f}^{\mathrm{dyn}}_{\boldsymbol{M};O}$, $\varphi^{\mathrm{kin}}_{\boldsymbol{M};O}$

and $\varphi^{\rm dyn}_{\boldsymbol{M};O}$ to the concrete (complexified if $\mathbb{K}=\mathbb{C}$) pre-symplectic spaces and unital *-algebras $\mathcal{F}^{\rm kin}_u(\boldsymbol{M};O)$, $\mathcal{F}^{\rm dyn}_u(\boldsymbol{M};O)$, $\mathfrak{F}^{\rm kin}_u(\boldsymbol{M};O)$ and $\mathfrak{F}^{\rm dyn}_u(\boldsymbol{M};O)$.

Let $\boldsymbol{M} \in \mathbf{Loc}$ be such that $H^2_{dR}(M;\mathbb{K}) \neq 0$. By arguments given in Sect. 6.1, there exists $\omega \in \Omega^2_0(M;\mathbb{K})$ satisfying $d_{\boldsymbol{M}}\omega = 0$ but $[\omega] \neq 0 \in \left[\Omega^2_0\left(M;\mathbb{K}\right)\right]$ (and hence $\mathbf{F}_{\boldsymbol{M}}(\omega) \neq 0 \in \mathfrak{F}_u\boldsymbol{M}$). In other words, $[\omega]$ corresponds to a electric topological degeneracy. Lemma 6.5 and (11) give

$$\mathrm{rce}^{\mathcal{F}_u}_{\boldsymbol{M}}\left[h\right]\left[\omega\right] = \left[\omega\right] \qquad \forall h \in H(\boldsymbol{M}),$$

and hence $(\mathrm{rce}^{\mathfrak{F}_u}_{\boldsymbol{M}}\left[h\right])(\mathbf{F}_{\boldsymbol{M}}(\omega)) = \mathbf{F}_{\boldsymbol{M}}(\omega)$ for all $h \in H(\boldsymbol{M})$. Consequently, $[\omega] \in \mathcal{F}^\bullet_u(\boldsymbol{M};K)$ and $\mathbf{F}_{\boldsymbol{M}}(\omega) \in \mathfrak{F}^\bullet_u(\boldsymbol{M};K)$ for all $K \in \mathscr{K}(\boldsymbol{M};O)$ and for all contractible $O \in \mathscr{O}(\boldsymbol{M})$. This implies $[\omega] \in \mathcal{F}^{\rm dyn}_u(\boldsymbol{M};O)$ and $\mathbf{F}_{\boldsymbol{M}}(\omega) \in \mathfrak{F}^{\rm dyn}_u(\boldsymbol{M};O)$ for all contractible $O \in \mathscr{O}(\boldsymbol{M})$. As $[\omega]$ is in the radical of the (complexified if $\mathbb{K}=\mathbb{C}$) pre-symplectic form on $\mathcal{F}_u\boldsymbol{M}$, it follows that $\mathcal{F}^{\rm dyn}_u(\boldsymbol{M};O)$ has a degenerate (complexified if $\mathbb{K}=\mathbb{C}$) pre-symplectic form, while $\mathcal{F}^{\rm kin}_u(\boldsymbol{M};O)$ is weakly non-degenerate; thus, these two (complexified if $\mathbb{K}=\mathbb{C}$) pre-symplectic spaces cannot possibly be symplectomorphic for any contractible $O \in \mathscr{O}(\boldsymbol{M})$; i.e. dynamical locality fails.

The same is true for the quantised universal free F-theory because, for every contractible $O \in \mathscr{O}(\boldsymbol{M})$, $\mathfrak{F}^{\rm dyn}_u(\boldsymbol{M};O)$ is not simple, while $\mathfrak{F}^{\rm kin}_u(\boldsymbol{M};O)$ is simple; hence these two unital *-algebras are not unital *-isomorphic. As far as the dynamical net is concerned, the elements $[\omega]$ resp. $\mathbf{F}_{\boldsymbol{M}}(\omega)$ are local to all regions.

We have shown that the electric topological degeneracies spoil dynamical locality. This is also true for magnetic topological degeneracies, i.e. $\omega \in \Omega^2_0(M;\mathbb{K})$ satisfying $\delta_{\boldsymbol{M}}\omega = 0$ but $[\omega] \neq 0 \in \left[\Omega^2_0(M;\mathbb{K})\right]$ and $\mathbf{F}_{\boldsymbol{M}}(\omega) \neq 0 \in \mathfrak{F}_u\boldsymbol{M}$. These are also fixed points under relative Cauchy evolution, which is not obvious from (11), but can be shown abstractly because electric and magnetic topological degeneracies are exchanged by the electromagnetic duality rotation $\Theta(\pi/2)$ defined by (8), which is an automorphism of $\mathcal{F}_u$ and therefore, commutes with the relative Cauchy evolution [19, Prop. 2.1]. The application of the quantisation functor yields the analogous result for the quantised universal free F-theory.

We summarise:

Theorem 6.7. *The classical and the quantised universal free F-theory (and hence also the A-theory) are not dynamically local (even in the weakened sense obtained by restricting to contractible open globally hyperbolic subsets).*

7. Dynamical Locality of the Reduced Theory

In the last section, we saw that the classical and the quantised universal free F-theory (and hence A-theory) fail local covariance and dynamical locality. However, we were also able to clearly identify what causes this failure, namely the possibility of having non-trivial radicals in the classical case and non-trivial centres in the quantum case. The reduced theories are free of these features and, as we will show, they are dynamically local. We work in the F-description, but all our statements have analogues in the equivalent A-description.

7.1. The Relative Cauchy Evolution of the Reduced Theory

Having established local covariance, we will now show that the classical and the quantised reduced free F-theories obey the time-slice axiom. We will compute their respective relative Cauchy evolutions and differentiate them with respect to the metric perturbation, thus obtaining the stress-energy tensor for the classical reduced free F-theory. Since $\mathfrak{R} = Q \circ \mathcal{R}$, we can concentrate on the classical case.

The only difference to Sects. 6.2 and 6.3 is so far the use of a different equivalence relation and hence different equivalence classes, i.e. $[\![\cdot]\!]$ instead of $[\cdot]$. Assume $\psi \in \mathbf{Loc}(\boldsymbol{M}, \mathbf{N})$ is Cauchy and $L : \Omega^p_0(N;\mathbb{K}) \to \Omega^p_0(N;\mathbb{K})$ is a time-slice map for $(\psi, \lambda^p_N, \Box_{\mathbf{N}})$. By Lemma 6.5, $\omega_{\Subset} = \alpha + \beta$ with $\alpha \in \Omega^p_{0,d}(N;\mathbb{K})$ such that $\operatorname{supp}\alpha \subseteq \psi(M)$ and $\beta \in \Omega^p_{0,\delta}(N;\mathbb{K})$ such that $\operatorname{supp}\beta \subseteq \psi(M)$ for $\omega \in \Omega^p_0(M;\mathbb{K})$ such that $d_{\boldsymbol{M}}\omega = 0$ or $\delta_{\boldsymbol{M}}\omega = 0$. Thus, we can adapt the results of Sects. 6.2 and 6.3 by just replacing $[\cdot]$ with $[\![\cdot]\!]$. In particular, $\mathcal{R}$ and $\mathfrak{R}$ obey the time-slice axiom and their respective relative Cauchy evolutions induced by $h \in H(\boldsymbol{M})$ are given (in the same conventions as in Sect. 6.3; in particular,'$\Subset^+$' refers to an arbitrary time-slice map $L^+ : \Omega^2_0(M;\mathbb{K}) \to \Omega^2_0(M;\mathbb{K})$ for $(\imath^+_{\boldsymbol{M}}[h] : \boldsymbol{M}^+[h] \to \boldsymbol{M}, \lambda^2_M, \Box_{\boldsymbol{M}}))$ by

$$\begin{aligned}\operatorname{rce}^{\mathcal{R}}_{\boldsymbol{M}}[h]\,[\![\omega]\!] &= [\![\omega]\!] + \left[\!\!\left[\left(\delta_{\boldsymbol{M}[h]} - \delta_{\boldsymbol{M}}\right) G^{\mathrm{adv}}_{\boldsymbol{M}[h]} d_{\boldsymbol{M}} \omega_{\Subset^+}\right]\!\!\right] \\ &= [\![\omega]\!] - \left[\!\!\left[\left(\delta_{\boldsymbol{M}[h]} - \delta_{\boldsymbol{M}}\right) G_{\boldsymbol{M}[h]} d_{\boldsymbol{M}} \omega_{\Subset^+}\right]\!\!\right], \quad [\![\omega]\!] \in \left[\!\!\left[\Omega^2_0(M;\mathbb{K})\right]\!\!\right],\end{aligned} \tag{12}$$

and also

$$\operatorname{rce}^{\mathfrak{R}}_{\boldsymbol{M}}[h] = Q\left(\operatorname{rce}^{\mathcal{R}}_{\boldsymbol{M}}[h]\right). \tag{13}$$

The intermediate expression in (12) allows us to employ a Born expansion as in [24, (B.2)],

$$G^{\mathrm{adv}}_{\boldsymbol{M}[h]}\omega = G^{\mathrm{adv}}_{\boldsymbol{M}}\omega - G^{\mathrm{adv}}_{\boldsymbol{M}}\left(\Box_{\boldsymbol{M}[h]} - \Box_{\boldsymbol{M}}\right) G^{\mathrm{adv}}_{\boldsymbol{M}[h]}\omega \qquad \forall \omega \in \Omega^2_0(M;\mathbb{K}),$$

to further compute:

$$\begin{aligned}\operatorname{rce}^{\mathcal{R}}_{\boldsymbol{M}}[h]\,[\![\omega]\!] &= [\![\omega]\!] + \left[\!\!\left[\left(\delta_{\boldsymbol{M}[h]} - \delta_{\boldsymbol{M}}\right) G^{\mathrm{adv}}_{\boldsymbol{M}} d_{\boldsymbol{M}} \omega_{\Subset^+}\right]\!\!\right] \\ &= [\omega] + - \left[\!\!\left[\left(\delta_{\boldsymbol{M}[h]} - \delta_{\boldsymbol{M}}\right) G^{\mathrm{adv}}_{\boldsymbol{M}}\left(\Box_{\boldsymbol{M}[h]} - \Box_{\boldsymbol{M}}\right) G^{\mathrm{adv}}_{\boldsymbol{M}[h]} d_{\boldsymbol{M}} \omega_{\Subset^+}\right]\!\!\right], \\ &\qquad [\![\omega]\!] \in \left[\!\!\left[\Omega^2_0(M;\mathbb{K})\right]\!\!\right].\end{aligned}$$

Now, $\operatorname{supp} G^{\mathrm{ret}}_{\boldsymbol{M}}\omega_{\Subset^+} \cap \operatorname{supp} h = \emptyset$ by construction for any $\omega \in \Omega^2_0(M;\mathbb{K})$ and, as $\delta_{\boldsymbol{M}[h]} - \delta_{\boldsymbol{M}}$ vanishes outside $\operatorname{supp} h$, we can replace $G^{\mathrm{adv}}_{\boldsymbol{M}} d_{\boldsymbol{M}} \omega_{\Subset^+}$ by $-G_{\boldsymbol{M}} d_{\boldsymbol{M}} \omega_{\Subset^+} = -G_{\boldsymbol{M}} d_{\boldsymbol{M}} \omega$ to obtain

$$\begin{aligned}\operatorname{rce}^{\mathcal{R}}_{\boldsymbol{M}}[h]\,[\![\omega]\!] = [\![\omega]\!] - \Bigg[\!\!\Bigg[\left(\delta_{\boldsymbol{M}[h]} - \delta_{\boldsymbol{M}}\right)\Big(G_{\boldsymbol{M}} d_{\boldsymbol{M}} \omega \\ + G^{\mathrm{adv}}_{\boldsymbol{M}}\left(\Box_{\boldsymbol{M}[h]} - \Box_{\boldsymbol{M}}\right) G^{\mathrm{adv}}_{\boldsymbol{M}[h]} d_{\boldsymbol{M}} \omega_{\Subset^+}\Big)\Bigg]\!\!\Bigg], \quad [\![\omega]\!] \in \left[\!\!\left[\Omega^2_0(M;\mathbb{K})\right]\!\!\right].\end{aligned} \tag{14}$$

For $\boldsymbol{M} \in \mathbf{Loc}$, we can associate to each $[\![\omega]\!] \in [\![\Omega_0^2(M;\mathbb{K})]\!]$ a solution of the free Maxwell equations (2) with compact support on smooth space-like Cauchy surfaces for $\boldsymbol{M}$ by setting $F_{[\![\omega]\!]} := d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} \omega$ for any representative $\omega \in \Omega_0^2(M;\mathbb{K})$. Clearly, all representatives will give rise to the same solution and if $d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} \eta = F_{[\![\omega]\!]}$ for $[\![\eta]\!] \in [\![\Omega_0^2(M;\mathbb{K})]\!]$, $[\![\eta]\!] = [\![\omega]\!]$ necessarily. Thus, in the classical reduced free F-theory, we are only dealing with solutions of (3) which are of the form $d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} \omega$ for $\omega \in \Omega_0^2(M;\mathbb{K})$. Note that all solutions of the Cauchy problem (3) take this form if $\boldsymbol{M} \in \mathbf{Loc}_{©}$ (cf. Subsection 4.2). This provides a nice interpretation of the relative Cauchy evolution: namely,

$$d_{\boldsymbol{M}[h]} G_{\boldsymbol{M}[h]} \delta_{\boldsymbol{M}[h]} \left(\mathcal{R}\jmath_{\boldsymbol{M}}^{+}[h]\right) \left(\left(\mathcal{R}\imath_{\boldsymbol{M}}^{+}[h]\right)^{-1} [\![\omega]\!]\right) = d_{\boldsymbol{M}[h]} G_{\boldsymbol{M}[h]} \delta_{\boldsymbol{M}[h]} \omega_{\Subset^{+}}$$

is the unique solution of the free Maxwell equations on $\boldsymbol{M}[h]$ which coincides with $F_{[\![\omega]\!]}$ on $M^{+}[h]$ (cf. [24]). The agreement is not difficult to see, the uniqueness follows from the well-posedness of the Cauchy problem. Then, if $\eta \in \Omega_0^2(M;\mathbb{K})$ is a representative of $\mathrm{rce}_{\boldsymbol{M}}^{\mathcal{R}}[h]\,[\![\omega]\!]$, then $d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} \eta$ is the unique solution of the free Maxwell equations for the field strength on $\boldsymbol{M}$ agreeing with $d_{\boldsymbol{M}[h]} G_{\boldsymbol{M}[h]} \delta_{\boldsymbol{M}[h]} \omega_{\Subset^{+}}$ on $M^{-}[h]$. This interpretation of the relative Cauchy evolution will become very helpful in the proof of Lemma 7.1.

7.2. The Stress–Energy Tensor of the Classical Modified Theory

To show that $\mathcal{R}$ and $\mathfrak{R}$ are dynamically local, it will be helpful to relate the relative Cauchy evolution to the stress-energy tensor for the classical reduced free F-theory. This can be done as follows: taking any compactly supported, symmetric and smooth tensor field[15] $h \in \Gamma_0^\infty(\tau_M^* \odot \tau_M^*)$, there exists $\varepsilon > 0$ such that $th \in H(\boldsymbol{M})$ for all $t \in (-\varepsilon, \varepsilon)$ (cf. [24, §§ 2 and 3]). The relative Cauchy evolution for $\mathcal{R}$ induced by $th \in H(\boldsymbol{M})$ for $\boldsymbol{M} \in \mathbf{Loc}$ is differentiable in the weak symplectic topology (cf. [24, § 3 and Appx.B]), i.e. there is a $\mathbb{K}$-linear map $H_{\boldsymbol{M}}[h] : \mathcal{R}\boldsymbol{M} \to \mathcal{R}\boldsymbol{M}$ such that

$$\mathfrak{r}_{\boldsymbol{M}}\left(H_{\boldsymbol{M}}[h]\,[\![\omega]\!], [\![\eta]\!]\right) = \frac{\mathrm{d}}{\mathrm{d}t} \mathfrak{r}_{\boldsymbol{M}}\left(\mathrm{rce}_{\boldsymbol{M}}^{\mathcal{R}}[th]\,[\![\omega]\!], [\![\eta]\!]\right)\Big|_{t=0}, \qquad [\![\omega]\!], [\![\eta]\!] \in [\![\Omega_0^2(M;\mathbb{K})]\!] \tag{15}$$

and the derivative on the right-hand side exists for all such $[\![\omega]\!], [\![\eta]\!]$. Note, $H_{\boldsymbol{M}}[h]$ is called $F_{\boldsymbol{M}}[h]$ in [24], a notation we avoid for obvious reasons. Inserting (14) and already dropping some terms of order t^2 and higher, we need to compute (up to first order in t)

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \mathfrak{r}_{\boldsymbol{M}}\left(\mathrm{rce}_{\boldsymbol{M}}^{\mathcal{R}}[th]\,[\![\omega]\!], [\![\eta]\!]\right)\Big|_{t=0} \\ &\quad = \lim_{t\to 0} \mathfrak{r}_{\boldsymbol{M}}\left(\left[\!\left[-t^{-1}\left(\delta_{\boldsymbol{M}[th]} - \delta_{\boldsymbol{M}}\right) d_{\boldsymbol{M}} G_{\boldsymbol{M}} \omega\right]\!\right], [\![\eta]\!]\right), \\ &\quad = -\lim_{t\to 0} \int_M t^{-1}\left(\delta_{\boldsymbol{M}[th]} - \delta_{\boldsymbol{M}}\right) d_{\boldsymbol{M}} G_{\boldsymbol{M}} \omega \wedge *_{\boldsymbol{M}} d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} \eta, \\ &\qquad\qquad [\![\omega]\!], [\![\eta]\!] \in [\![\Omega_0^2(M;\mathbb{K})]\!]. \end{aligned} \tag{16}$$

[15] Recall, τ_M^* denotes the cotangent bundle of the smooth manifold M.

The coderivative $\delta_{\boldsymbol{M}[th]}$ may be expanded by a lengthy but straightforward computation (being careful to recall that the inverse metric to $g+th$ is $(g+th)^{-1} = g^{-1} - th^{\sharp\sharp} + O(t^2)$, which reads in abstract index notation $g^{ab} - th^{ab} + O(t^2)$):

$$t^{-1}\left(\left(\delta_{\boldsymbol{M}[th]} - \delta_{\boldsymbol{M}}\right)\varpi\right)_{cd} = \nabla_a\left(h^{ab}\varpi_{bcd}\right) - \frac{1}{2}\left(\nabla_b h^a_a\right)\varpi^b{}_{cd} + \left(\nabla_a h_{bc}\right)\varpi^{ab}{}_d - \left(\nabla_a h_{bd}\right)\varpi^{ab}{}_c + O(t), \quad \varpi \in \Omega^3(M;\mathbb{K}),$$

where ∇ stands for the Levi–Civita connection with respect to g. This yields

$$H_{\boldsymbol{M}}[h]\,[\![\omega_{cd}]\!] = \Big[\!\!\Big[-\nabla_a\left(h^{ab}\left(G_{\boldsymbol{M}} d_{\boldsymbol{M}}\omega\right)_{bcd}\right) + \frac{1}{2}\left(\nabla_b h^a_a\right)\left(G_{\boldsymbol{M}} d_{\boldsymbol{M}}\omega\right)^b{}_{cd} - \left(\nabla_a h_{bc}\right)\left(G_{\boldsymbol{M}} d_{\boldsymbol{M}}\omega\right)^{ab}{}_d + \left(\nabla_a h_{bd}\right)\left(G_{\boldsymbol{M}} d_{\boldsymbol{M}}\omega\right)^{ab}{}_c\Big]\!\!\Big], \quad [\![\omega]\!] \in \left[\!\left[\Omega^2_0(M;\mathbb{K})\right]\!\right], \tag{17}$$

whose well-definedness can be seen using the weak non-degeneracy of $\mathfrak{r}_{\boldsymbol{M}}$. To bring (16) into a nicer form, we define $\varpi := d_{\boldsymbol{M}} G_{\boldsymbol{M}}\omega \in \Omega^3(M;\mathbb{K})$ and $F_{[\![\eta]\!]} := d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}}\eta$. The divergence theorem entails the following identities

$$\int_M \nabla_a\left(h^{ab}\varpi_{bcd}\right) F^{cd}_{[\![\eta]\!]}\,\mathrm{vol}_{\boldsymbol{M}} = -\int_M h^{ab}\varpi_{bcd}\nabla_a F^{cd}_{[\![\eta]\!]}\,\mathrm{vol}_{\boldsymbol{M}},$$

$$\int_M \left(\nabla_b h^a_a\right)\varpi^b{}_{cd} F^{cd}_{[\![\eta]\!]}\,\mathrm{vol}_{\boldsymbol{M}} = -\int_M \left(h^a_a\nabla_b\left(\varpi^b{}_{cd}\right)F^{cd}_{[\![\eta]\!]} + h^a_a\varpi^b{}_{cd}\nabla_b F^{cd}_{[\![\eta]\!]}\right)\mathrm{vol}_{\boldsymbol{M}}$$

and

$$\int_M \left(\nabla_a h_{bc}\right)\varpi^{ab}{}_d F^{cd}_{[\![\eta]\!]}\,\mathrm{vol}_{\boldsymbol{M}} = -\int_M \left(h^b_c\left(\nabla_a\varpi^a{}_{bd}\right)F^{cd}_{[\![\eta]\!]}) + h_{bc}\varpi^{ab}{}_d\nabla_a F^{cd}_{[\![\eta]\!]}\right)\mathrm{vol}_{\boldsymbol{M}},$$

where $\nabla_b(\varpi^b{}_{cd}) = -(\delta_{\boldsymbol{M}}\varpi)_{cd} = +(d_{\boldsymbol{M}} G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega)_{cd} =: F_{[\![\omega]\!]cd}$; together with $d_{\boldsymbol{M}} F_{[\![\eta]\!]} = 0$ and

$$\varpi^b{}_{cd}\nabla_b F^{cd}_{[\![\eta]\!]}\,\mathrm{vol}_{\boldsymbol{M}} = \varpi_{bcd}\nabla^{[b}F^{cd]}_{[\![\eta]\!]}\,\mathrm{vol}_{\boldsymbol{M}} = 3!\,\varpi\wedge *_{\boldsymbol{M}} d_{\boldsymbol{M}} F_{[\![\eta]\!]} = 0,$$

they yield overall

$$\begin{aligned}\mathfrak{r}_{\boldsymbol{M}}\left(H_{\boldsymbol{M}}[h]\,[\![\omega]\!],[\![\eta]\!]\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{r}_{\boldsymbol{M}}\left(\mathrm{rce}^{\mathcal{R}}_{\boldsymbol{M}}[h]\,[\![\omega]\!],[\![\eta]\!]\right)\Big|_{t=0}\\ &= -\int_M h_{ab}\left(\frac{1}{4}g^{ab}F_{[\![\omega]\!]mn}F^{mn}_{[\![\eta]\!]} - g_{mn}F^{am}_{[\![\omega]\!]}F^{bn}_{[\![\eta]\!]}\right)\mathrm{vol}_{\boldsymbol{M}}\\ &= -\int_M h_{ab}T^{ab}_{\boldsymbol{M}}\left([\![\omega]\!],[\![\eta]\!]\right)\mathrm{vol}_{\boldsymbol{M}},\\ &\qquad [\![\omega]\!],[\![\eta]\!] \in \left[\!\left[\Omega^2_0(M;\mathbb{K})\right]\!\right].\end{aligned}$$

(There is a sign error in the analogous formula [24, Eq. (3.7)], which however does not alter the main results of that reference.) Here $T_{\boldsymbol{M}}([\![\omega]\!],[\![\eta]\!])$ is the polarised form of the stress-energy tensor for the classical reduced free F-theory on $\boldsymbol{M} \in \mathbf{Loc}$

$$T^{ab}_{\boldsymbol{M}}\left(\llbracket\omega\rrbracket, \llbracket\eta\rrbracket\right) = \frac{1}{4}\, g^{ab} F_{\llbracket\omega\rrbracket mn} F^{mn}_{\llbracket\eta\rrbracket} - g_{mn} F^{am}_{\llbracket\omega\rrbracket} F^{bn}_{\llbracket\eta\rrbracket}, \quad \llbracket\omega\rrbracket, \llbracket\eta\rrbracket \in \left\llbracket\Omega^2_0\left(M;\mathbb{K}\right)\right\rrbracket, \tag{18}$$

where $F_{\llbracket\omega\rrbracket} := d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} \omega$ with a representative $\omega \in \Omega^2_0(M;\mathbb{K})$ for $\llbracket\omega\rrbracket \in \left\llbracket\Omega^2_0(M;\mathbb{K})\right\rrbracket$. The same expression (18) is obtained for the stress-energy tensor of the classical universal free F-theory if $\llbracket\cdot\rrbracket$ is replaced with $[\cdot]$.

7.3. Verification of Dynamical Locality for the Reduced Theories

We will now prove that the reduced free F-theory $\mathcal{R} : \mathbf{Loc} \to \mathbf{Sympl}_{\mathbb{K}}$ obeys dynamical locality (hence the same is true for the corresponding reduced A-theory). To do so, we can work with concrete (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic spaces and avoid referring to underlying categorical notions such as subobjects. We will follow the reasoning of [24] using the stress-energy tensor of $\mathcal{R}$ to characterise the dynamical net. The main technical point of difference is that the field strength tensor satisfies not only the wave equation but also the free Maxwell equations.

Lemma 7.1. *Let K be any compact subset of $\boldsymbol{M} \in \mathbf{Loc}$. Then,*

$$\begin{aligned} \mathcal{R}^{\bullet}\left(\boldsymbol{M};K\right) &= \left\{\llbracket\omega\rrbracket \in \mathcal{R}\boldsymbol{M} \mid \operatorname{supp} T_{\boldsymbol{M}}\left(\llbracket\omega\rrbracket, \overline{\llbracket\omega\rrbracket}\right) \subseteq J_{\boldsymbol{M}}\left(K\right)\right\} \\ &= \bigcap_{\substack{h \in \Gamma^\infty_0\left(\tau^*_{\boldsymbol{M}} \odot \tau^*_{\boldsymbol{M}}\right) \\ \operatorname{supp} h \subseteq K^\perp}} \ker H_{\boldsymbol{M}}\left[h\right], \end{aligned} \tag{19}$$

and also $\mathcal{R}^{\bullet}(\boldsymbol{M};K) = \left\{\llbracket\omega\rrbracket \in \mathcal{R}\boldsymbol{M} \mid \operatorname{supp} F_{\llbracket\omega\rrbracket} \subseteq J_{\boldsymbol{M}}(K)\right\}$.

Proof. Labelling the members of (19) as I, II and III respectively, we will prove that I $\subseteq$ III $\subseteq$ II $\subseteq$ I. Starting with I $\subseteq$ III, suppose $\llbracket\omega\rrbracket \in \mathcal{R}^{\bullet}(\boldsymbol{M};K)$. For $h \in \Gamma^\infty_0(\tau^*_{\boldsymbol{M}} \odot \tau^*_{\boldsymbol{M}})$ with support $\operatorname{supp} h \subseteq K^\perp$, there is $\varepsilon > 0$ such that $th \in H(\boldsymbol{M};K^\perp)$ for all $t \in (-\varepsilon,\varepsilon)$. As $\operatorname{rce}^{\mathcal{R}}_{\boldsymbol{M}}[th]\llbracket\omega\rrbracket = \llbracket\omega\rrbracket$ for all $t \in (-\varepsilon,\varepsilon)$, we have $\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{r}_{\boldsymbol{M}}(\operatorname{rce}^{\mathcal{R}}_{\boldsymbol{M}}[th]\llbracket\omega\rrbracket, \llbracket\eta\rrbracket)\big|_{t=0} = 0$ for all $\llbracket\eta\rrbracket \in \mathcal{R}\boldsymbol{M}$. Hence also $\mathfrak{r}_{\boldsymbol{M}}(H_{\boldsymbol{M}}[h]\llbracket\omega\rrbracket, \llbracket\eta\rrbracket) = 0$ for all $\llbracket\eta\rrbracket \in \mathcal{R}\boldsymbol{M}$ and so by weak non-degeneracy, $\llbracket\omega\rrbracket \in \ker H_{\boldsymbol{M}}[h]$; as h was arbitrary, we have I $\subseteq$ III. For III $\subseteq$ II, if

$$\llbracket\omega\rrbracket \in \bigcap_{\substack{h \in \Gamma^\infty_0\left(\tau^*_{\boldsymbol{M}} \odot \tau^*_{\boldsymbol{M}}\right) \\ \operatorname{supp} h \subseteq K^\perp}} \ker H_{\boldsymbol{M}}\left[h\right],$$

then $\mathfrak{r}_{\boldsymbol{M}}(H_{\boldsymbol{M}}[h]\llbracket\omega\rrbracket, \overline{\llbracket\omega\rrbracket}) = -\int_M h_{ab} T^{ab}_{\boldsymbol{M}}(\llbracket\omega\rrbracket, \overline{\llbracket\omega\rrbracket})\,\mathrm{vol}_{\boldsymbol{M}} = 0$ for all $h \in \Gamma^\infty_0(\tau^*_{\boldsymbol{M}} \odot \tau^*_{\boldsymbol{M}})$ with support $\operatorname{supp} h \subseteq K^\perp$, so $\operatorname{supp} T_{\boldsymbol{M}}(\llbracket\omega\rrbracket, \overline{\llbracket\omega\rrbracket}) \subseteq J_{\boldsymbol{M}}(K)$ as required. Finally, to prove II $\subseteq$ I, we note that $\operatorname{supp} T_{\boldsymbol{M}}(\llbracket\omega\rrbracket, \overline{\llbracket\omega\rrbracket}) \subseteq J_{\boldsymbol{M}}(K)$ implies that $\operatorname{supp}(F_{\llbracket\omega\rrbracket}) \subseteq J_{\boldsymbol{M}}(K)$ because the energy density, which is the sum of the squares of the off-diagonal components of $F_{\llbracket\omega\rrbracket}$ (in some local framing), must vanish at each point $p \notin J_{\boldsymbol{M}}(K)$. Accordingly, $F_{\llbracket\omega\rrbracket}$ is a solution of Maxwell's equations in the perturbed spacetime $\boldsymbol{M}[h]$ for every $h \in H(\boldsymbol{M};K^\perp)$. Hence, it is the unique solution on $\boldsymbol{M}[h]$ that coincides with

$F_{[\![\omega]\!]}$ on $M^+[h]$ and also the unique solution on $\boldsymbol{M}$ that coincides with $F_{[\![\omega]\!]}$ on $M^-[h]$. Thus, $[\![\omega]\!]$ and $\mathrm{rce}^{\mathcal{R}}_{\boldsymbol{M}}[h][\![\omega]\!]$ give rise to the same solution of the free Maxwell equations on $\boldsymbol{M}$ which implies $\mathrm{rce}^{\mathcal{R}}_{\boldsymbol{M}}[h][\![\omega]\!] = [\![\omega]\!]$ and consequently, $[\![\omega]\!] \in \mathcal{R}^\bullet(\boldsymbol{M}; K)$. The final statement is immediate from the argument just given. □

Lemma 7.2. *For all $O \in \mathscr{O}(\boldsymbol{M})$, we have $\mathcal{R}^{\mathrm{kin}}(\boldsymbol{M}; O) \subseteq \mathcal{R}^{\mathrm{dyn}}(\boldsymbol{M}; O)$.*

Proof. Let $[\![\omega]\!] \in \mathcal{R}^{\mathrm{kin}}(\boldsymbol{M}; O)$ and $\omega \in \Omega^2_0(M; \mathbb{K})$, $\operatorname{supp}\omega \subseteq O$ a representative of $[\![\omega]\!]$. Choosing for each $x \in \operatorname{supp}\omega$ a Cauchy ball B_x containing x and taking the Cauchy developments, we have found an open cover $\{D_{\boldsymbol{M}}(B_x)\}_{x\in\operatorname{supp}\omega}$ of $\operatorname{supp}\omega$ in $\boldsymbol{M}$. Since $\operatorname{supp}\omega$ is compact, finitely many of these sets are enough to cover $\operatorname{supp}\omega$, say $\operatorname{supp}\omega \subseteq \bigcup_{i=0}^n D_{\boldsymbol{M}}(B_i)$ with $n \geq 0$.

Let $\{\chi, \chi^i \mid i = 0, \ldots, n\}$ be a smooth partition of unity subordinated to the open cover $\{M \backslash \operatorname{supp}\omega, D_{\boldsymbol{M}}(B_i) \mid i = 0, \ldots, n\}$ of M. Defining for all $i \in I$ $\omega_i := \chi^i\omega \in \Omega^2_0(M; \mathbb{K})$ with $\operatorname{supp}\omega_i \subseteq D_{\boldsymbol{M}}(B_i) \cap O$, we can write $\omega = \sum_{i=0}^n \omega_i$. By construction, $\operatorname{supp}\omega_i \in \mathscr{K}(\boldsymbol{M}; O)$. As $\operatorname{supp} T_{\boldsymbol{M}}([\![\omega_i]\!], \overline{[\![\omega_i]\!]}) \subseteq J_{\boldsymbol{M}}(\operatorname{supp}\omega_i)$, Lemma 7.1 yields $[\![\omega_i]\!] \in \mathcal{R}^\bullet(\boldsymbol{M}; \operatorname{supp}\omega_i)$ and hence, $[\![\omega]\!] = \sum_{i=0}^n [\![\omega_i]\!] \in \mathcal{R}^{\mathrm{dyn}}(\boldsymbol{M}; O)$ because $\mathcal{R}^{\mathrm{dyn}}(\boldsymbol{M}; O)$ is the smallest (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic subspace of $\mathcal{R}\boldsymbol{M}$ containing $\mathcal{R}^\bullet(\boldsymbol{M}; K)$ for all $K \in \mathscr{K}(\boldsymbol{M}; O)$. □

The following lemma can be considered as an analogue to [24, Lem. 3.1] and is integral to the proof that the kinematical and the dynamical nets coincide.

Lemma 7.3. *Let $\boldsymbol{M} \in$ **Loc** and $K \subseteq O \in \mathscr{O}(\boldsymbol{M})$ compact. There exists $\chi \in \mathcal{C}^\infty(M)$ such that every solution $F \in \Omega^2(M, \mathbb{K})$ of Maxwell's equations with $\operatorname{supp} F \subseteq J_{\boldsymbol{M}}(K)$ can be written as $F = G_{\boldsymbol{M}}\Box_{\boldsymbol{M}}\chi F$, where $\Box_{\boldsymbol{M}}\chi F \in \Omega^2_0(M; \mathbb{K})$, $\delta_{\boldsymbol{M}}\chi F \in \Omega^1_0(M, \mathbb{K})$ and $d_{\boldsymbol{M}}\chi F \in \Omega^3_0(M, \mathbb{K})$ are supported in O.*

Proof. The proof works in exactly the same way as that of [24, Lem. 3.1(i)]. The additional point is that due to $d_{\boldsymbol{M}}F = 0$ and $\delta_{\boldsymbol{M}}F = 0$, the Leibniz rule gives $d_{\boldsymbol{M}}\chi F = 0$ and $\delta_{\boldsymbol{M}}\chi F = 0$ outside of the compact set $K_0 \subseteq O$ defined in [24, Lem. 3.1(i)], and are thereby compactly supported in O. □

Recall from Sect. 5 that for $\boldsymbol{M} \in$ **Loc** and $O \in \mathscr{O}(\boldsymbol{M})$, $\mathcal{R}^{\mathrm{dyn}}(\boldsymbol{M}; O)$ is the (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic subspace of $\mathcal{R}\boldsymbol{M}$ generated by $\bigcup_{K\in\mathscr{K}(\boldsymbol{M};O)} \mathcal{R}^\bullet(\boldsymbol{M}; K)$.

Lemma 7.4. *For all $O \in \mathscr{O}(\boldsymbol{M})$, we have $\mathcal{R}^{\mathrm{dyn}}(\boldsymbol{M}; O) \subseteq \mathcal{R}^{\mathrm{kin}}(\boldsymbol{M}; O)$.*

Proof. We have to show that for each $K \in \mathscr{K}(\boldsymbol{M}; O)$, $[\![\omega]\!] \in \mathcal{R}^\bullet(\boldsymbol{M}; K)$ has a representative $\eta \in \Omega^2_0(M; \mathbb{K})$ with $\operatorname{supp}\eta \subseteq O$. By Lemma 7.1, we have $\operatorname{supp} d_{\boldsymbol{M}}G_{\boldsymbol{M}}\delta_{\boldsymbol{M}}\omega \subseteq J_{\boldsymbol{M}}(K)$ for any representative $\omega \in \Omega^2_0(M; \mathbb{K})$ of $[\![\omega]\!]$. Now, by definition of $\mathscr{K}(\boldsymbol{M}; O)$, K has a neighbourhood comprising finitely many causally disjoint diamonds $\{D_{\boldsymbol{M}}(B_i)\}_{i=0,\ldots,n}$, $n \geq 0$, based in smooth spacelike Cauchy surfaces for $\boldsymbol{M}$ such that the bases $\{B_i\}_{i=0,\ldots,n}$ are contained in O. Note that these diamonds might not be entirely contained in O. Hence,

$\{D_{\boldsymbol{M}|_O}(B_i)\}_{i=0,\ldots,n}$ are globally hyperbolic open subsets of both $\boldsymbol{M}|_O$ and $\boldsymbol{M}$, which are furthermore contractible. Because of the causal disjointness, their (disjoint) union $U := \bigsqcup_{i=0}^{n} D_{\boldsymbol{M}|_O}(B_i)$ is a globally hyperbolic open subset of $\boldsymbol{M}|_O$ and $\boldsymbol{M}$, contains[16] K and each connected component is contractible. We apply Lemma 7.3 to U and find that $F := d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} \omega = G_{\boldsymbol{M}} \Box_{\boldsymbol{M}} \chi F = -G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} d_{\boldsymbol{M}} \chi F - G_{\boldsymbol{M}} d_{\boldsymbol{M}} \delta_{\boldsymbol{M}} \chi F$, where $d_{\boldsymbol{M}} \chi F \in \Omega_0^3(M;\mathbb{K})$ and $\delta_{\boldsymbol{M}} \chi F \in \Omega_0^1(M;\mathbb{K})$ are compactly supported in U. Since each connected component of U is contractible, there are $\eta_1, \eta_2 \in \Omega_0^2(M;\mathbb{K})$ with $\operatorname{supp}\eta_1, \operatorname{supp}\eta_2 \subseteq U$ satisfying the equalities $d_{\boldsymbol{M}} \chi F = d_{\boldsymbol{M}} \eta_1$ and $\delta_{\boldsymbol{M}} \chi F = \delta_{\boldsymbol{M}} \eta_2$. Thus, $d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} \omega = d_{\boldsymbol{M}} G_{\boldsymbol{M}} \delta_{\boldsymbol{M}} (\eta_1 - \eta_2)$, which shows $[\![\omega]\!] = [\![\eta_1 - \eta_2]\!]$. Accordingly, $\eta := \eta_1 - \eta_2 \in \Omega_0^2(M;\mathbb{K})$ is a representative of $[\![\omega]\!]$ that is compactly supported in O (because η is compactly supported in $U \subseteq O$). □

Combining Lemmas 7.2 and 7.4, the main statement of this subsection follows:

Theorem 7.5. *The classical reduced theory of the free Maxwell field is dynamically local.*

From Theorem 7.5, we may deduce that the quantised reduced free F-theory $\mathfrak{R} : \mathbf{Loc} \to {}^*\mathbf{Alg}_1^{\mathrm{m}}$ (and hence the quantised reduced free A-theory) is dynamically local:

Corollary 7.6. *The quantised reduced theory of the free Maxwell field obeys dynamical locality.*

Proof. $\mathfrak{R} = Q \circ \mathcal{R}$ with the quantisation functor $Q : \mathbf{pSympl}_{\mathbb{K}}^{\mathrm{m}} \to {}^*\mathbf{Alg}_1^{\mathrm{m}}$ and as a result of that we need to check $(\mathscr{L}1 - \mathscr{L}4)$ of [24, p.1688]:

($\mathscr{L}$1) The relative Cauchy evolution of $\mathcal{R}$ is differentiable in the weak symplectic topology as in (15), and the resulting maps obey (the sign appears incorrectly in [24])

$$\mathfrak{r}_{\boldsymbol{M}} \left(H_{\boldsymbol{M}}[h] [\![\omega]\!], \overline{[\![\omega]\!]} \right) = -\int_M h_{ab} T_{\boldsymbol{M}}^{ab} \left([\![\omega]\!], \overline{[\![\omega]\!]} \right) \mathrm{vol}_{\boldsymbol{M}},$$
$$[\![\omega]\!] \in [\![\Omega_0^2(M;\mathbb{K})]\!],\ h \in H(\boldsymbol{M};O),\ O \in \mathscr{O}(\boldsymbol{M}),\ \boldsymbol{M} \in \mathbf{Loc},$$

where $T_{\boldsymbol{M}}([\![\omega]\!], \overline{[\![\omega]\!]}) \in \Gamma^\infty(\tau_{\boldsymbol{M}} \odot \tau_{\boldsymbol{M}})$ for each $[\![\omega]\!] \in [\![\Omega_0^2(M;\mathbb{K})]\!]$ and $\boldsymbol{M} \in \mathbf{Loc}$.

($\mathscr{L}$2) For each $O \in \mathscr{O}(\boldsymbol{M})$ containing $\operatorname{supp} h$ of $h \in \Gamma_0^\infty(\tau_{\boldsymbol{M}}^* \odot \tau_{\boldsymbol{M}}^*)$, $\operatorname{img} H_{\boldsymbol{M}}[h]$ can be identified with a subset of $\mathcal{R}^{\mathrm{kin}}(\boldsymbol{M};O)$.

($\mathscr{L}$3) $\mathcal{R}$ obeys extended locality, i.e. $\mathcal{R}^{\mathrm{kin}}(\boldsymbol{M};O_1) \cap \mathcal{R}^{\mathrm{kin}}(\boldsymbol{M};O_2) = 0 \in \mathcal{R}\boldsymbol{M}$ for spacelike separated $O_1, O_2 \in \mathscr{O}(\boldsymbol{M})$, $\boldsymbol{M} \in \mathbf{Loc}$.

($\mathscr{L}$4) $\mathcal{R}^\bullet(\boldsymbol{M};K) = \bigcap_{\substack{h \in \Gamma_0^\infty(\tau_{\boldsymbol{M}}^* \odot \tau_{\boldsymbol{M}}^*) \\ \operatorname{supp} h \subseteq K^\perp}} \ker H_{\boldsymbol{M}}[h]$ for K compact in $\boldsymbol{M} \in \mathbf{Loc}$.

($\mathscr{L}$1) is obvious from what was done in Sect. 7.2. For $\boldsymbol{M} \in \mathbf{Loc}$, the image of $H_{\boldsymbol{M}}[h]$, where $h \in \Gamma_0^\infty(\tau_{\boldsymbol{M}}^* \odot \tau_{\boldsymbol{M}}^*)$, can be identified with a subset of

[16] $D_{\boldsymbol{M}|_O}(B_i) = D_{\boldsymbol{M}}(B_i) \cap O$ for $i = 0, \ldots, n$ because O is causally convex in $\boldsymbol{M}$.

$\mathcal{R}^{\mathrm{kin}}(\boldsymbol{M};O)$ for each $O \in \mathscr{O}(\boldsymbol{M})$ with $\operatorname{supp} h \subseteq O \in \mathscr{O}(\boldsymbol{M})$ by (17). ($\mathscr{L}3$) is obvious and ($\mathscr{L}4$) is proven by Lemma 7.1. Hence [24, Thm.5.3][17] applies and proves the result. □

8. Discussion

8.1. Summary

In this paper, we have discussed the notion of dynamical locality for the free Maxwell field. Describing the quantum field theory in terms of the universal algebra of the unital $*$-algebras of smeared quantum field (cf. [14]), and describing the classical field theory by the equivalent for (complexified if $\mathbb{K}=\mathbb{C}$) presymplectic spaces, we showed that the classical and the quantised universal theories, given by functors $\mathcal{F}_u, \mathcal{A}_u : \mathbf{Loc} \to \mathbf{pSympl}_{\mathbb{K}}$ and $\mathfrak{F}_u, \mathfrak{A}_u : \mathbf{Loc} \to {}^*\mathbf{Alg}_1^{\mathrm{m}}$, fail dynamical locality due to **Loc**-objects $\boldsymbol{M}$ with $H^2_{dR}(M;\mathbb{K}) \neq 0$. However, we were able to modify the classical and the quantised universal F-theory to obtain locally covariant and dynamically local theories $\mathcal{R} : \mathbf{Loc} \to \mathbf{Sympl}_{\mathbb{K}}$ and $\mathfrak{R} : \mathbf{Loc} \to {}^*\mathbf{Alg}_1^{\mathrm{m}}$. In establishing this, we have used the same chain of arguments as [24] for the free real scalar field.

To conclude, we discuss three aspects in more detail, namely the status of dynamical locality, the categorical structure underlying some of our constructions, and the relation of our present work to the discussions of SPASs in [23, 24].

8.2. Dynamical Locality

Our present results on the free Maxwell field contribute to the emerging picture of dynamical locality as follows. The failure of dynamical locality for the universal free F-theory can be traced to the existence of topological charges present whenever the second de Rham cohomology is non-trivial. These observables are invariant under all relative Cauchy evolutions and so are common to every element of the dynamical net, which does not distinguish between observables that are local to every region and "*observables that are localised at infinity*". Actually, these observables can have unusual spatial localisation as well: it is possible for such an element to be common to spacelike separated elements of the kinematic net, giving a failure of extended locality [36, 43]. In the quantum field theory, the topological charges are central elements which parameterise different superselection sectors of the theory [1, 44], again underlining their global nature. By contrast, the reduced F-theory of the free Maxwell field in $n = 4$ dimensions provides a well-behaved locally covariant and dynamically local theory (at the cost of giving up topological observables labelled by the first and the second de Rham cohomology group with compact supports). Overall, dynamical locality appears to be a reasonable expectation for theories of local observables, but to fail where theories admit observables of an essentially global nature that are stabilised by topological or other constraints.

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[17] The sign error in [24] does not affect the validity of this result because the focus is on solutions with vanishing stress-energy tensor.

8.3. Categorical Structures

A number of ideas concerning the ‘universal’ and the ‘reduced’ theory for the classical and the quantised free Maxwell field can be put in a broader categorical context. The details of the following discussion have been worked out and will appear in B.L.’s forthcoming Ph.D. thesis.

For each $\boldsymbol{M} = (M, g, \mathfrak{o}, \mathfrak{t}) \in \mathbf{Loc}$, we can consider the category $\mathscr{J}_{\mathbf{M}}$ whose objects are those $\boldsymbol{N} = (N, g_N, \mathfrak{o}_N, \mathfrak{t}_N) \in \mathbf{Loc}_{©}$ such that $N \subseteq M$ is a globally hyperbolic open subsets (excluding $N = M$ if $\boldsymbol{M} \in \mathbf{Loc}_{©}$), $g_N = g|_N$, $\mathfrak{o}_N = \mathfrak{o}|_N$ and $\mathfrak{t}_N = \mathfrak{t}|_N$; the morphisms in $\mathscr{J}_{\boldsymbol{M}}$ are the inclusion maps. We can thus restrict each of the functors $\mathfrak{F}, \mathfrak{A} : \mathbf{Loc}_{©} \to {}^*\mathbf{Alg}_1^{\mathrm{m}}$ to $\mathscr{J}_{\boldsymbol{M}}$ and obtain functors $\mathfrak{F}_{\boldsymbol{M}}, \mathfrak{A}_{\boldsymbol{M}} : \mathscr{J}_{\boldsymbol{M}} \to {}^*\mathbf{Alg}_1^{\mathrm{m}}$. The universal algebras $\mathfrak{F}_u\boldsymbol{M}$ and $\mathfrak{A}_u\boldsymbol{M}$ are now precisely the universal objects of the *colimits* (see [39, Sec.2.5], [9, Sec.2.6] or [38, Sec.III.3] for this categorical notion) for the functors $\mathfrak{F}_{\boldsymbol{M}}$ and $\mathfrak{A}_{\boldsymbol{M}}$ but viewed as functors $\mathfrak{F}_{\boldsymbol{M}}, \mathfrak{A}_{\boldsymbol{N}} : \mathscr{J}_{\boldsymbol{M}} \to {}^*\mathbf{Alg}_1$. Here, it is crucial to drop the restriction to injective unital *-homomorphisms, because ${}^*\mathbf{Alg}_1$ is *cocomplete*, i.e. the colimit for any functor from any small category to ${}^*\mathbf{Alg}_1$ always exists, while ${}^*\mathbf{Alg}_1^{\mathrm{m}}$ is not; in fact, the colimits for $\mathfrak{F}_{\mathbf{M}}$ and $\mathfrak{A}_{\mathbf{M}}$ do not exist in ${}^*\mathbf{Alg}_1^{\mathrm{m}}$ for general $\boldsymbol{M}$. This justifies the use of the term ‘universal’. At this point, we get the functorial property of $\mathfrak{F}_u, \mathfrak{A}_u : \mathbf{Loc} \to {}^*\mathbf{Alg}_1$ for free because they necessarily turn out to be the *left Kan extensions* (see [9, Sec.3.7] or [38, Chap.X]) of $\mathfrak{F}, \mathfrak{A} : \mathbf{Loc}_{©} \to {}^*\mathbf{Alg}_1$ (again, one must work in ${}^*\mathbf{Alg}_1$ rather than ${}^*\mathbf{Alg}_1^{\mathrm{m}}$). Hence, from this categorical point of view, the universal theories of the quantised free Maxwell field are highly distinguished extensions of the quantum field theories on contractible curved spacetimes.

The notion of a colimit and a left Kan extension also make sense for the categories $\mathbf{pSympl}_{\mathbb{K}}$, $\mathbf{pSympl}_{\mathbb{K}}^{\mathrm{m}}$ and $\mathbf{Sympl}_{\mathbb{K}}$, but none of these three categories is cocomplete. However, it can be shown that the functors $\mathcal{F}_{\boldsymbol{M}}, \mathcal{A}_{\boldsymbol{M}} : \mathscr{J}_{\boldsymbol{M}} \to \mathbf{pSympl}_{\mathbb{K}}$ have colimits whose universal objects are precisely $\mathcal{F}_u\boldsymbol{M}$ and $\mathcal{A}_u\boldsymbol{M}$ respectively, and that $\mathcal{F}_u, \mathcal{A}_u : \mathbf{Loc} \to \mathbf{pSympl}_{\mathbb{K}}$ are the left Kan extensions of $\mathcal{F}, \mathcal{A} : \mathbf{Loc}_{©} \to \mathbf{pSympl}_{\mathbb{K}}$. Moreover, the relations $Q(\mathcal{F}_u\boldsymbol{M}) = \mathfrak{F}_u\boldsymbol{M}$ and $Q(\mathcal{A}_u\boldsymbol{M}) = \mathfrak{A}_u\boldsymbol{M}$ can be understood as special cases of a general result. Although the colimits for $\mathcal{F}_{\boldsymbol{M}}, \mathcal{A}_{\boldsymbol{M}} : \mathscr{J}_{\boldsymbol{M}} \to \mathbf{pSympl}_{\mathbb{K}}^{\mathrm{m}}$ (or $\mathbf{Sympl}_{\mathbb{K}}$) do not exist, the non-existence of colimits does not rule out the existence of left Kan extensions and it would be indeed interesting to know if $\mathfrak{F}, \mathfrak{A} : \mathbf{Loc}_{©} \to {}^*\mathbf{Alg}_1^{\mathrm{m}}$ and $\mathcal{F}, \mathcal{A} : \mathbf{Loc}_{©} \to \mathbf{pSympl}_{\mathbb{K}}$ (or $\mathbf{Sympl}_{\mathbb{K}}$) have left Kan extensions in ${}^*\mathbf{Alg}_1^{\mathrm{m}}$ and $\mathbf{pSympl}_{\mathbb{K}}$ (or $\mathbf{Sympl}_{\mathbb{K}}$). If they do exist, the resulting theories would be distinguished as the minimal locally covariant extensions of the theory on contractible curved spacetimes; while we have not reached a conclusion on the question of existence, it can however be shown that *if* these extensions exist, they would coincide with the reduced theories.

8.4. Theories of the Free Maxwell Field and SPASs

The models for the free Maxwell field in curved spacetimes studied in this paper provide a new viewpoint on the issue of the same physics in all spacetimes (SPASs), in relation to locally covariant (quantum field) theories that can be regarded as extensions of others. The locally covariant theories $\mathcal{F}_u$ and

$\mathcal{R}$ (resp., $\mathfrak{F}_u$ and $\mathfrak{R}$) coincide on all spacetimes (of dimension $n = 4$) with trivial second de Rham cohomology group. To be specific, let $\mathbf{Loc}_2$ be the full subcategory of $\mathbf{Loc}$ formed by the spacetimes $\boldsymbol{M}$ with $H^2_{dR}(M;\mathbb{K}) = 0$, and let $K : \mathbf{Loc}_2 \to \mathbf{Loc}$ be the inclusion functor. Then there are natural isomorphisms $\mathcal{F}_u \circ K \dot{\to} \mathcal{R} \circ K$ and $\mathfrak{F}_u \circ K \dot{\to} \mathfrak{R} \circ K$. However, the locally covariant theories are not equivalent on $\mathbf{Loc}$ and it is evidently not tenable to regard both the universal and reduced F-theory of the free Maxwell field as each representing the same physics in all spacetimes according to a common notion.

As far as we are aware, there is no way of embedding the reduced free F-theory as a subtheory of its universal cousin.[18] However, it would be natural to regard the universal-free F-theory as an *extension* of its reduced counterpart. On the classical level, we have a short left exact sequence

$$0 \stackrel{\cdot}{\longrightarrow} \operatorname{rad} \mathfrak{w}_u \underset{m}{\stackrel{\cdot}{\longrightarrow}} \mathcal{F}_u \underset{e}{\stackrel{\cdot}{\longrightarrow}} \mathcal{R},$$

of functors from $\mathbf{Loc}$ to $\mathbf{pSympl}_{\mathbb{K}}$, where all components of e are epic. Here, 0 denotes the constant functor returning the zero (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic space and $\operatorname{rad} \mathfrak{w}_u$ is the functor assigning the radical $\operatorname{rad} \mathfrak{w}_{u\,\boldsymbol{M}}$ (equipped with the zero (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic form) to each $\boldsymbol{M} \in \mathbf{Loc}$, and with morphisms obtained by restriction from $\mathcal{F}_u$. The components of m, which are given by the inclusion maps of $\operatorname{rad} \mathfrak{w}_{u\,\boldsymbol{M}}$ into $\mathcal{F}_u$, are necessarily monic. As $\mathbf{pSympl}_{\mathbb{K}}$ lacks a zero object,[19] it is not possible to write a short exact sequence, and we have to insist on e being epic separately. Applying the quantisation functor, we obtain a similar short left exact sequence in the quantum case. In general, we could consider any sequence $\mathcal{C} \underset{m}{\stackrel{\cdot}{\longrightarrow}} \mathcal{B} \underset{e}{\stackrel{\cdot}{\longrightarrow}} \mathcal{A}$ with monic m and epic e as indicating that $\mathcal{B}$ is an extension of $\mathcal{A}$ (by $\mathcal{C}$), where $\mathcal{A}, \mathcal{B}, \mathcal{C} : \mathbf{Loc} \to \mathbf{Phys}$ (for these purposes, we would allow $\mathbf{Phys}$ to admit non-monic morphisms). One may then formulate a version of the SPASs property to cover extensions: a class $\mathfrak{T}$ of theories $\mathbf{Loc} \to \mathbf{Phys}$ has the *SPASs property for extensions* if, whenever $\mathcal{A}, \mathcal{B} \in \mathfrak{T}$ and $\mathcal{B}$ is an extension of $\mathcal{A}$ so that e is a partial natural isomorphism, then e is a natural isomorphism. It would be very interesting to know whether the class of dynamically local theories satisfies this version of SPASs in addition to the subtheory version studied in [23]. Our results on the free Maxwell field studied here are certainly consistent with a positive answer to that question.

[18] In any spacetime $\boldsymbol{M}$ one can find a symplectic C-monomorphism (resp. symplectic $\mathbb{R}$-linear injection if $\mathbb{K} = \mathbb{R}$) from $\mathcal{R}\boldsymbol{M}$ to $\mathcal{F}_u\boldsymbol{M}$, e.g., $[\![\omega]\!] \mapsto \sum_\alpha [\chi_\alpha \omega]$, where χ_α is a partition of unity subordinate to a covering by contractible globally hyperbolic open subsets; the problem is that such maps are not generally unique and (lacking a global Hodge theory for $\mathbf{Loc}$) there is no natural choice.

[19] The *zero space* 0 is an initial object but not a terminal object in $\mathbf{pSympl}_{\mathbb{K}}$, and hence is not a *zero object*.

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