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Characteristic Initial Data and Smoothness of Scri. I. Framework and Results

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Abstract. We analyze the Cauchy problem for the vacuum Einstein equations with data on a complete light-cone in an asymptotically Minkowskian space-time. We provide conditions on the free initial data which guarantee existence of global solutions of the characteristic constraint equations. We present necessary-and-sufficient conditions on characteristic initial data in $3 + 1$ dimensions to have no logarithmic terms in an asymptotic expansion at null infinity.

1. Introduction

An issue of central importance in general relativity is the understanding of gravitational radiation. This has direct implications for the soon-expected direct detection of gravitational waves. The current main effort in this topic appears to be a mixture of numerical modelling and approximation methods. From this perspective, there does not seem to be a need for a better understanding of the exact properties of the gravitational field in the radiation regime. However, as observations and numerics will have become routine, solid theoretical foundations for the problem will become necessary.

Now, a generally accepted framework for describing gravitational radiation seems to be the method of conformal completions of Penrose. Here, a key hypothesis is that a suitable conformal rescaling of the space-time metric becomes smooth on a new manifold with boundary $\mathscr{I}^+$. One then needs to face the question, if and how such space-times can be constructed. Ultimately, one would like to isolate the class of initial data, on a spacelike slice extending to spatial infinity, the evolution of which admits a Penrose-type conformal completion at infinity, and show that the class is large enough to model all physical

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processes at hand. Direct attempts to carry this out (see [17,22,23] and references therein) have not been successful so far. Similarly, the asymptotic behaviour of the gravitational field established in [3,10,19–21,24] is inconclusive as far as the smoothness of the conformally rescaled metric at $\mathscr{I}^+$ is concerned. The reader is referred to [18] for an extensive discussion of the issues arising.

On the other hand, clear-cut constructions have been carried out in less demanding settings, with data on characteristic surfaces as pioneered by Bondi et al. [5], or with initial data with hyperboloidal asymptotics. It has been found [1,2,12,30] that both generic Bondi data and generic hyperboloidal data, constructed out of conformally smooth seed data, will not lead to space-times with a smooth conformal completion. Instead, a *polyhomogeneous* asymptotics of solutions of the relevant constraint equations was obtained, with logarithmic terms appearing in asymptotic expansions of the fields.

The case for the necessity of a polyhomogeneous-at-best framework, as resulting from the above work, is not waterproof: In both cases, it is not clear whether initial data with logarithmic terms can arise from evolution of a physical system which is asymptotically flat in spacelike directions. There is a further issue with the Bondi expansions, because the framework of Bondi et al. [5,28] does not provide a well-posed system of evolution equations for the problem at hand.

The aim of this work is to rederive the existence of obstructions to smoothness of the metric at $\mathscr{I}^+$ in a framework in which the evolution problem for the Einstein vacuum equations is well-posed and where free initial data are given on a light-cone extending to null infinity, or on two characteristic hypersurfaces one of which extends to infinity, or in a mixed setting where part of the data are prescribed on a spacelike surface and part on a characteristic one extending to infinity. This can be viewed as a revisiting of the Bondi-type setting in a framework where an associated space-time is guaranteed to exist.

One of the attractive features of the characteristic Cauchy problem is that one can explicitly provide an exhaustive class of freely prescribable initial data. By "exhaustive class" we mean that the map from the space of free initial data to the set of solutions is surjective, where "solution" refers to that part of space-time which is covered by the domain of dependence of the smooth part of the light-cone, or of the smooth part of the null hypersurfaces issuing normally from a smooth submanifold of codimension two.[1] There is, moreover, considerable flexibility in prescribing characteristic initial data [13]. In this work, we will concentrate on the following approaches:

1. The free data are a triple $(\mathscr{N}, [\gamma], \kappa)$, where $\mathscr{N}$ is a n-dimensional manifold, $[\gamma]$ is a conformal class of symmetric two-covariant tensors on $\mathscr{N}$

[1] This should be contrasted with the spacelike Cauchy problem, where no exhaustive method for constructing non-CMC initial data sets is known. It should, however, be kept in mind that the spacelike Cauchy problem does not suffer from the serious problem of formation of caustics, inherent to the characteristic one.

of signature $(0,+,\dots,+)$, and κ is a field of connections on the bundles of tangents to the integral curves of the kernel of γ.[2]

2. Alternatively, the data are a triple $(\mathscr{N},\check g,\kappa)$, where $\check g$ is a field of symmetric two-covariant tensors on $\mathscr{N}$ of signature $(0,+,\dots,+)$, and κ is a field of connections on the bundles of tangents to the integral curves of the characteristic direction of $\check g$.[3] The pair $(\check g,\kappa)$ is further required to satisfy the constraint equation
$$\partial_r\tau-\kappa\tau+|\sigma|^2+\frac{\tau^2}{n-1}=0\,, \tag{1.1}$$
where τ is the divergence and σ is the shear (see Sect. 2.2 for details), which will be referred to as the *Raychaudhuri equation*.
3. Alternatively, the connection coefficient κ and all the components of the space-time metric are prescribed on $\mathscr{N}$, subject to the Raychaudhuri constraint equation. Here $\mathscr{N}$ is viewed as the hypersurface $\{u=0\}$ in the space-time to-be-constructed, and thus all metric components $g_{\mu\nu}$ are prescribed at $u=0$ in a coordinate system $(x^\mu)=(u,x^i)$, where (x^i) are local coordinates on $\mathscr{N}$.
4. Finally, schemes where tetrad components of the conformal Weyl tensor are used as free data are briefly discussed.

In the first two cases, to obtain a well-posed evolution problem, one needs to impose gauge conditions; in the third case, the initial data themselves determine the gauge, with the "gauge-source functions" determined from the initial data.

The aim of this work is to analyze the occurrence of log terms in the asymptotic expansions as r goes to infinity for initial data sets as above. The gauge choice $\kappa=O(r^{-3})$ below (in particular, the gauge choice $\kappa=\frac{r}{2}|\sigma|^2$, on which we focus in part II [26]), ensures that affine parameters along the generators of $\mathscr{N}$ diverge as r goes to infinity (cf. [26, Appendix B]), so that in the associated space-time the limit $r\to\infty$ will correspond to null geodesics approaching a (possibly non-smooth) null infinity.

It turns out that the simplest choice of gauge conditions, namely $\kappa=0$ and harmonic coordinates, is *not compatible* with smooth asymptotics at the conformal boundary at infinity: we prove that the *only* vacuum metric, constructed from characteristic Cauchy data on a light-cone, and which has a smooth conformal completion in this gauge, is Minkowski space-time.

It should be pointed out, that the observation that some sets of harmonic coordinates are problematic for an analysis of null infinity has already been made in [4,7]. Our contribution here is to make a precise *no-go* statement, without approximation procedures or supplementary assumptions.

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[2] Recall that a connection ∇ on each such bundle is uniquely described by writing $\nabla_r\partial_r=\kappa\partial_r$, in a coordinate system where ∂_r is in the kernel of γ. Once the associated space-time has been constructed, we will also have $\nabla_r\partial_r=\kappa\partial_r$, where ∇ now is the covariant derivative operator associated with the space-time metric.

[3] We will often write $(\check g,\kappa)$ instead of $(\mathscr{N},\check g,\kappa)$, with $\mathscr{N}$ being implicitly understood, when no precise description of $\mathscr{N}$ is required.

One way out of the problem is to replace the harmonic-coordinates condition by a wave-map gauge with non-vanishing gauge-source functions. This provides a useful tool to isolate those log terms which are gauge artefacts, in the sense that they can be removed from the solution by an appropriate choice of the gauge-source functions. There remain, however, some logarithmic coefficients which cannot be removed in this way. We identify those coefficients, and show that the requirement that these coefficients do not vanish is gauge independent. In part II of this work, we show that the logarithmic coefficients are non-zero for generic initial data. The equations which lead to vanishing logarithmic coefficients will be referred to as the *no-logs-condition*.

It is expected that for generic initial data sets, as considered here, the space-times obtained by solving the Cauchy problem will have a polyhomogeneous expansion at null infinity. There are, however, no theorems in the existing mathematical literature which guarantee existence of a polyhomogeneous $\mathscr{I}^+$ when the initial data have non-trivial log terms.

The situation is different when the no-logs-condition is satisfied. In part II of this work, we show that the resulting initial data lead to smooth initial data for Friedrich's conformal field equations [14] as considered in [6]. This implies that the no-logs-condition provides a necessary-and-sufficient condition for the evolved space-time to posses a smooth $\mathscr{I}^+$. For initial data close enough to Minkowskian ones, solutions global to the future are obtained.

It may still be the case that the logarithmic expansions are irrelevant as far as our understanding of gravitational radiation is concerned, either because they never arise from the evolution of isolated physical systems, or because their occurrence prevents existence of a sufficiently long evolution of the data, or because all essential physical issues are already satisfactorily described by smooth conformal completions. While we have not provided a definite answer to those questions, we hope that our results here will contribute to resolve the issue.

If not explicitly stated otherwise, all manifolds, fields, and expansion coefficients are assumed to be smooth.

2. The Characteristic Cauchy Problem on a Light-Cone

In this section, we will review some facts concerning the characteristic Cauchy problem. Most of the discussion applies to any characteristic surface. We concentrate on a light-cone, as in this case all the information needed is contained in the characteristic initial data together with the requirement of the smoothness of the metric at the vertex. The remaining Cauchy problems mentioned in the Introduction will be discussed in Sect. 7 below.

2.1. Gauge Freedom

2.1.1. Adapted Null Coordinates. Our starting point is a C^∞-manifold $\mathscr{M} \cong \mathbb{R}^{n+1}$ and a future light-cone $C_O \subset \mathscr{M}$ emanating from some point $O \in \mathscr{M}$. We make the assumption that the subset C_O can be *globally* represented in suitable coordinates (y^μ) by the equation of a Minkowskian cone, i.e.

$$C_O = \left\{ (y^\mu) : y^0 = \sqrt{\sum_{i=1}^{n} (y^i)^2} \right\} \subset \mathscr{M} \ .$$

Given a $C^{1,1}$-Lorentzian space-time, such a representation is always possible in some neighbourhood of the vertex. However, since caustics may develop along the null geodesics which generate the cone, it is a geometric restriction to assume the existence of a Minkowskian representation globally.

A treatment of the characteristic initial value problem at hand is easier in coordinates x^μ adapted to the geometry of the light-cone [9,27]. We consider space-time-dimensions $n+1 \geq 3$. It is standard to construct a set of coordinates $(x^\mu) \equiv (u, r, x^A)$, $A = 2, \ldots, n$, so that $C_O \setminus \{0\} = \{u = 0\}$. The x^A's denote local coordinates on the level sets $\Sigma_r := \{r = \text{const}, u = 0\} \cong S^{n-1}$, and are constant along the generators. The coordinate r induces, by restriction, a parameterization of the generators and is chosen so that the point O is approached when $r \to 0$. The general form of the trace $\overline{g}$ on the cone C_O of the space-time metric g reduces in these *adapted null coordinates* to

$$\overline{g} = \overline{g}_{00} \mathrm{d}u^2 + 2\nu_0 \mathrm{d}u \mathrm{d}r + 2\nu_A \mathrm{d}u \mathrm{d}x^A + \check{g}, \tag{2.1}$$

where

$$\nu_0 := \overline{g}_{01}, \quad \nu_A := \overline{g}_{0A},$$

and where

$$\check{g} = \check{g}_{AB} \mathrm{d}x^A \mathrm{d}x^B := \overline{g}_{AB} \mathrm{d}x^A \mathrm{d}x^B$$

is a degenerate quadratic form induced by g on C_O which induces on each slice Σ_r an r-dependent Riemannian metric $\check{g}_{\Sigma_r}$ (coinciding with $\check{g}(r, \cdot)$ in the coordinates above).[4]

The components $\overline{g}_{00}$, ν_0 and ν_A are gauge-dependent quantities. In particular, ν_0 changes sign when u is replaced by $-u$. Whenever useful and/or relevant, we will assume that ∂_r is future directed and ∂_u is past directed, which corresponds to requiring that $\nu_0 > 0$.

The quadratic form $\check{g}$ is intrinsically defined on C_O, independently of the choice of the parameter r and of how the coordinates are extended off the cone.

Throughout this work, an overline denotes the restriction of space-time objects to C_O.

The restriction of the inverse metric to the light-cone takes the form

$$\overline{g}^\# \equiv \overline{g}^{\mu\nu} \partial_\mu \partial_\nu = 2\nu^0 \partial_u \partial_r + \overline{g}^{11} \partial_r \partial_r + 2\overline{g}^{1A} \partial_r \partial_A + \overline{g}^{AB} \partial_A \partial_B,$$

where

$$\nu^0 := \overline{g}^{01} = (\nu_0)^{-1}, \quad \nu^A := \overline{g}^{AB} \nu_B,$$
$$\overline{g}^{1A} = -\nu^0 \nu^A, \quad \overline{g}^{11} = (\nu^0)^2 (\nu^A \nu_A - \overline{g}_{00}),$$

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[4] The degenerate quadratic form denoted here by $\check{g}$ has been denoted by $\tilde{g}$ in [9,13]. However, here we will use $\tilde{g}$ to denote the conformally rescaled unphysical metric, as done in most of the literature on the subject.

and where $\overline{g}^{AB}$ is the inverse of $\overline{g}_{AB}$. The coordinate transformation relating the two coordinate systems (y^μ) and (x^μ) takes the form

$$u = \hat{r} - y^0, \quad r = \hat{r}, \quad x^A = \mu^A(y^i/\hat{r}), \quad \text{with} \quad \hat{r} := \sqrt{\sum_i (y^i)^2}.$$

The inverse transformation reads

$$y^0 = r - u, \quad y^i = r\Theta^i(x^A), \quad \text{with} \quad \sum_i (\Theta^i)^2 = 1.$$

Adapted null coordinates are singular at the vertex of the cone C_O and C^∞ elsewhere. They are convenient to analyze the initial data constraints satisfied by the trace $\overline{g}$ on the light-cone. Note that the space-time metric g will in general not be of the form (2.1) away from C_O. We further remark that adapted null coordinates are not uniquely fixed, for there remains the freedom to redefine the coordinate r (the only restriction being that r is strictly increasing on the generators and that $r = 0$ at the vertex; compare Sect. 2.2 below), and to choose local coordinates on S^{n-1}.

2.1.2. Generalized Wave-Map Gauge. Let us be given an auxiliary Lorentzian metric $\hat{g}$. A standard method to establish existence, and well-posedness, results for Einstein's vacuum field equations $R_{\mu\nu} = 0$ is a "hyperbolic reduction" where the Ricci tensor is replaced by the *reduced Ricci tensor in* $\hat{g}$*-wave-map gauge*,

$$R^{(H)}_{\mu\nu} := R_{\mu\nu} - g_{\sigma(\mu}\hat{\nabla}_{\nu)}H^\sigma. \tag{2.2}$$

Here,

$$H^\lambda := \Gamma^\lambda - \hat{\Gamma}^\lambda - W^\lambda, \quad \Gamma^\lambda := g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta}, \quad \hat{\Gamma}^\lambda := g^{\alpha\beta}\hat{\Gamma}^\lambda_{\alpha\beta}. \tag{2.3}$$

We use the hat symbol "ˆ" to indicate quantities associated with the *target metric* $\hat{g}$, while $W^\lambda = W^\lambda(x^\mu, g_{\mu\nu})$ denotes a vector field which is allowed to depend upon the coordinates and the metric g, but not upon derivatives of g.

The *wave-gauge vector* H^λ has been chosen of the above form [9,15,16] to remove some second-derivatives terms in the Ricci tensor, so that the *reduced vacuum Einstein equations*

$$R^{(H)}_{\mu\nu} = 0 \tag{2.4}$$

form a system of quasi-linear wave equations for g.

Any solution of (2.4) will provide a solution of the vacuum Einstein equations provided that the so-called $\hat{g}$*-generalized wave-map gauge condition*

$$H^\lambda = 0 \tag{2.5}$$

is satisfied. In the context of the characteristic initial value problem, the "gauge condition" (2.5) is satisfied by solutions of the reduced Einstein equations if it is satisfied on the initial characteristic hypersurfaces.

The vector field W^λ reflects the freedom to choose coordinates off the cone. Its components can be freely specified, or chosen to satisfy ad hoc equations. Indeed, by a suitable choice of coordinates, the gauge-source functions W^λ can locally be given any preassigned form, and conversely the W^λ's can be used to determine coordinates by solving wave equations, given appropriate initial data on the cone.

In most of this work, we will use a Minkowski target in adapted null coordinates, that is

$$\hat{g} = \eta \equiv -\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r + r^2 s_{AB}\mathrm{d}x^A\mathrm{d}x^B, \tag{2.6}$$

where s is the unit round metric on the sphere S^{n-1}.

2.2. The First Constraint Equation

Set $\ell \equiv \ell^\mu\partial_\mu \equiv \partial_r$. The Raychaudhuri equation $\overline{R}_{\mu\nu}\ell^\mu\ell^\nu \equiv \overline{R}_{11} = 0$ provides a constraining relation between the connection coefficient κ and other geometric objects on C_O, as follows: recall that the *null second fundamental form* of C_O is defined as

$$\chi_{ij} := \frac{1}{2}(\mathcal{L}_\ell \check{g})_{ij},$$

where $\mathcal{L}$ denotes the Lie derivative. In the adapted coordinates described above, we have

$$\chi_{AB} = -\overline{\Gamma}^0_{AB}\nu_0 = \frac{1}{2}\partial_r\overline{g}_{AB}, \quad \chi_{11} = 0, \quad \chi_{1A} = 0.$$

The null second fundamental form is sometimes called *null extrinsic curvature* of the initial surface C_O, which is misleading since only objects intrinsic to C_O are involved in its definition.

The *mean null extrinsic curvature* of C_O, or the *divergence* of C_O, which we denote by τ and which is often denoted by θ in the literature, is defined as the trace of χ:

$$\tau := \chi_A{}^A \equiv \overline{g}^{AB}\chi_{AB} \equiv \frac{1}{2}\overline{g}^{AB}\partial_r\overline{g}_{AB} \equiv \partial_r \log\sqrt{\det \check{g}_{\Sigma_r}}. \tag{2.7}$$

It measures the rate of change of area along the null geodesic generators of C_O. The traceless part of χ,

$$\sigma_A{}^B := \chi_A{}^B - \frac{1}{n-1}\delta_A{}^B\tau \equiv \overline{g}^{BC}\chi_{AC} - \frac{1}{n-1}\delta_A{}^B\tau \tag{2.8}$$

$$= \frac{1}{2}\gamma^{BC}(\partial_r\gamma_{AC})\breve{}, \tag{2.9}$$

is known as the *shear* of C_O. In (2.9), the field γ is any representative of the conformal class of $\check{g}_{\Sigma_r}$, which is sometimes regarded as the free initial data. The addition of the "$\breve{}$"-symbol to a tensor w_{AB} denotes "the trace-free part of":

$$\breve{w}_{AB} := w_{AB} - \frac{1}{n-1}\gamma_{AB}\gamma^{CD}w_{CD}. \tag{2.10}$$

We set

$$|\sigma|^2 := \sigma_A{}^B\sigma_B{}^A = -\frac{1}{4}(\partial_r\gamma^{AB})\breve{}(\partial_r\gamma_{AB})\breve{}. \tag{2.11}$$

We thus observe that the shear $\sigma_A{}^B$ depends merely on the conformal class of $\check{g}_{\Sigma_r}$. This is not true for τ, which is instead in one-to-one correspondence with the conformal factor relating $\check{g}_{\Sigma_r}$ and γ.

Imposing the generalized wave-map gauge condition $H^\lambda = 0$, the wave-gauge constraint equation induced by $\overline{R}_{11} = 0$ reads [9, equation (6.13)],

$$\partial_r\tau - \underbrace{\Big(\nu^0\partial_r\nu_0 - \frac{1}{2}\nu_0(\overline{W}^0 + \overline{\hat{\Gamma}}^0) - \frac{1}{2}\tau\Big)}_{=:\kappa}\tau + |\sigma|^2 + \frac{\tau^2}{n-1} = 0\,. \tag{2.12}$$

Under the allowed changes of the coordinate r, $r \mapsto \overline{r}(r, x^A)$, with $\partial\overline{r}/\partial r > 0$, $\overline{r}(0, x^A) = 0$, the tensor field g_{AB} transforms as a scalar,

$$\overline{g}_{AB}(\overline{r}, x^C) = g_{AB}(r(\overline{r}, x^C), x^C), \tag{2.13}$$

the field κ changes as a connection coefficient

$$\bar{\kappa} = \frac{\partial r}{\partial\overline{r}}\kappa + \frac{\partial\overline{r}}{\partial r}\frac{\partial^2 r}{\partial\overline{r}^2}, \tag{2.14}$$

while τ and σ_{AB} transform as one-forms:

$$\overline{\tau} = \frac{\partial r}{\partial\overline{r}}\tau, \quad \overline{\sigma}_{AB} = \frac{\partial r}{\partial\overline{r}}\sigma_{AB}. \tag{2.15}$$

The freedom to choose κ is thus directly related to the freedom to reparameterize the generators of C_O. Geometrically, κ describes the acceleration of the integral curves of ℓ, as seen from the identity $\nabla_\ell \ell^\mu = \kappa\ell^\mu$. The choice $\kappa = 0$ corresponds to the requirement that the coordinate r be an affine parameter along the rays. For a given κ, the first constraint equation splits into an equation for τ and, once this has been solved, an equation for ν_0.

Once a parameterization of generators has been chosen, we see that the metric function ν_0 is largely determined by the choice of the gauge-source function $\overline{W}^0$ and, in fact, the remaining gauge freedom in ν_0 can be encoded in $\overline{W}^0$.

2.3. The Wave-Map Gauge Characteristic Constraint Equations

Here, we present the whole hierarchical ODE-system of Einstein wave-map gauge constraints induced by the vacuum Einstein equations in a generalized wave-map gauge (cf. [9] for details) for given initial data $([\gamma], \kappa)$ and gauge-source functions $\overline{W}^\lambda$.

The equation (2.12) induced by $\overline{R}_{11} = 0$ leads to the equations

$$\partial_r\tau - \kappa\tau + |\sigma|^2 + \frac{\tau^2}{n-1} = 0, \tag{2.16}$$

$$\partial_r\nu^0 + \frac{1}{2}(\overline{W}^0 + \overline{\hat{\Gamma}}^0) + \nu^0(\frac{1}{2}\tau + \kappa) = 0. \tag{2.17}$$

Equation (2.16) is a Riccati differential equation for τ along each null ray, for $\kappa = 0$ it reduces to the standard form of the Raychaudhuri equation. Equation (2.17) is expressed in terms of

$$\nu^0 := \frac{1}{\nu_0}$$

rather than of ν_0, as then it becomes linear. Our aim is to analyze the asymptotic behaviour of solutions of the constraints, for this it turns out to be convenient to introduce an auxiliary positive function φ, defined as

$$\tau = (n-1)\partial_r \log \varphi, \tag{2.18}$$

which transforms (2.16) into a second-order *linear* ODE,

$$\partial_r^2 \varphi - \kappa \partial_r \varphi + \frac{|\sigma|^2}{n-1}\varphi = 0. \tag{2.19}$$

The function φ is essentially a rewriting of the conformal factor Ω relating $\breve{g}$ and the initial data γ, $\overline{g}_{AB} = \Omega^2 \gamma_{AB}$:

$$\Omega = \varphi \left(\frac{\det s}{\det \gamma} \right)^{1/(2n-2)} . \tag{2.20}$$

Here, $s = s_{AB}\mathrm{d}x^A \mathrm{d}x^B$ denotes the standard metric on S^{n-1}. The initial data symmetric tensor field $\gamma = \gamma_{AB} dx^A dx^B$ is assumed to form a one-parameter family of Riemannian metrics $r \mapsto \gamma(r, x^A)$ on S^{n-1}.

The boundary conditions at the vertex O of the cone for the ODEs occurring in this work follow from the requirement of regularity of the metric there. When imposed, they guarantee that (2.17) and (2.19), as well as all the remaining constraint equations below, have unique solutions. The relevant conditions at the vertex have been computed in regular coordinates and then translated into adapted null coordinates in [9] for a natural family of gauges.

For ν^0 and φ, the boundary conditions read

$$\begin{cases} \lim_{r\to 0} \nu^0 = 1, \\ \lim_{r\to 0} \varphi = 0, \quad \lim_{r\to 0} \partial_r \varphi = 1. \end{cases}$$

The Einstein equations $\overline{R}_{1A} = 0$ imply the equations [9, Equation (9.2)] (compare [13, Equation (3.12)])

$$\frac{1}{2}(\partial_r + \tau)\xi_A - \check{\nabla}_B \sigma_A{}^B + \frac{n-2}{n-1}\partial_A \tau + \partial_A \kappa = 0, \tag{2.21}$$

where $\check{\nabla}$ denotes the Riemannian connection defined by $\breve{g}_{\Sigma_r}$, and

$$\xi_A := -2\overline{\Gamma}^1_{1A}.$$

When $\overline{H}^0 = 0$, one has $\overline{H}^A = 0$ if and only if

$$\begin{aligned} \xi_A = &- 2\nu^0 \partial_r \nu_A + 4\nu^0 \nu_B \chi_A{}^B + \nu_A(\overline{W}^0 + \overline{\hat{\Gamma}}{}^0) + \overline{g}_{AB}(\overline{W}^B + \overline{\hat{\Gamma}}{}^B) \\ &- \gamma_{AB}\gamma^{CD}\check{\Gamma}^B_{CD}. \end{aligned} \tag{2.22}$$

Here, $\check{\Gamma}^B_{CD}$ are the Christoffel symbols associated to the metric $\breve{g}_{\Sigma_r}$.

Given fields κ and $\overline{g}_{AB} = g_{AB}|_{u=0}$ satisfying the Raychaudhuri constraint equation, the equations (2.21) and (2.22) can be read as hierarchical linear first-order PDE-system which successively determines ξ_A and ν_A by solving ODEs. The boundary conditions at the vertex are

$$\lim_{r\to 0} \nu_A = 0 = \lim_{r\to 0} \xi_A.$$

The remaining constraint equation follows from the Einstein equation $\overline{g}^{AB}\overline{R}_{AB} = 0$ [9, Equations (10.33) & (10.36)],

$$(\partial_r + \tau + \kappa)\zeta + \check{R} - \frac{1}{2}\xi_A\xi^A + \check{\nabla}_A\xi^A = 0, \tag{2.23}$$

where we have set $\xi^A := \overline{g}^{AB}\xi_B$. The function $\check{R}$ is the curvature scalar associated to $\check{g}_{\Sigma_r}$. The auxiliary function ζ is defined as

$$\zeta := (2\partial_r + \tau + 2\kappa)\overline{g}^{11} + 2\overline{W}^1 + 2\overline{\hat{\Gamma}}^1, \tag{2.24}$$

and satisfies, if $\overline{H}^\lambda = 0$, the relation $\zeta = 2\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \tau\overline{g}^{11}$. The term $\overline{\hat{\Gamma}}^1$ depends upon the target metric chosen, and with our current Minkowski target $\hat{g} = \eta$ we have

$$\overline{\hat{\Gamma}}^1 = \overline{\hat{\Gamma}}^0 = -r\overline{g}^{AB}s_{AB}. \tag{2.25}$$

Taking the relation

$$\overline{g}^{11} = (\nu^0)^2(\nu^A\nu_A - \overline{g}_{00}) \tag{2.26}$$

into account, the definition (2.24) of ζ becomes an equation for $\overline{g}_{00}$ once ζ has been determined. The boundary conditions for (2.23) and (2.24) are

$$\lim_{r\to 0}\overline{g}^{11} = 1, \quad \lim_{r\to 0}(\zeta + 2r^{-1}) = 0.$$

2.4. Global Solutions

A prerequisite for obtaining asymptotic expansions is existence of solutions of the constraint equations defined for all r. The question of globally defined data becomes trivial when all metric components are prescribed on C_O: then, the only condition is that τ, as calculated from $\overline{g}_{AB}$, is strictly positive. Now, as is well-known, and will be rederived shortly in any case, negativity of τ implies formation of conjugate points in finite affine time, or geodesic incompleteness of the generators. In this work, we will only be interested in light-cones C_O which are globally smooth (except, of course, at the vertex), and extending all the way to conformal infinity. Such cones have complete generators without conjugate points, and so τ must remain positive. But then one can solve algebraically the Raychaudhuri equation to globally determine κ.

We note that the function τ depends upon the choice of parameterization of the generators, but its sign does not, hence the above discussion applies regardless of that choice. Recall that we assume that the tip of the cone corresponds to $r \to 0$ and that the condition that $\kappa = O(r^{-3})$ ensures that an affine parameter along the generators tends to infinity for $r \to \infty$, so that the parameterization of r covers the whole cone from O to null infinity.

In some situations, it might be convenient to request that κ vanishes, or takes some prescribed value. In this case, the Raychaudhuri equation becomes an equation for the function φ, and the question of its global positivity arises.

Recall that the initial conditions for φ at the vertex are $\varphi(0) = 0$ and $\partial_r\varphi(0) = 1$, and so both $\partial_r\varphi$ and φ are positive near zero. Now, (2.19) with $\kappa = 0$ shows that φ is concave as long as it is non-negative; equivalently, $\partial_r\varphi$ is non-increasing in the region where $\varphi > 0$. An immediate consequence of this is that if $\partial_r\varphi$ becomes negative at some $r_0 > 0$, then it stays so, with φ vanishing for some $r_0 < r_1 < \infty$, i.e. after some finite affine parameter time. We recover the result just mentioned, that negativity of $\partial_r\varphi$ indicates incompleteness, or occurrence of conjugate points, or both. In the first case, the solution will not be defined for all affine parameters r, in the second C_O will fail to be smooth for $r > r_1$ by standard results on conjugate points. Since the sign of $\partial_r\varphi$ is invariant under orientation-preserving reparameterizations, we conclude that:

Proposition 2.1. *Globally smooth and null-geodesically complete light-cones must have $\partial_r\varphi$ positive.*

A set of conditions guaranteeing global existence of positive solutions of the Raychaudhuri equation, viewed as an equation for φ, has been given in [9, Theorem 7.3]. Here, we shall give an alternative simpler criterion, as follows:

Suppose, first, that $\kappa = 0$. Integration of (2.19) gives

$$\partial_r\varphi(r,x^A) = 1 - \frac{1}{n-1}\int_0^r \big(\varphi|\sigma|^2\big)(\tilde r,x^A)\,\mathrm{d}\tilde r \;\le\; 1 \tag{2.27}$$

as long as φ remains positive. Since $\varphi(0)=0$, we see that we always have

$$\varphi(r,x^A)\le r$$

in the region where φ is positive, and in that region it holds

$$\begin{aligned}\partial_r\varphi(r,x^A) &\ge 1 - \frac{1}{n-1}\int_0^r \tilde r\,|\sigma(\tilde r,x^A)|^2\mathrm{d}\tilde r\\ &\ge 1 - \frac{1}{n-1}\int_0^\infty \tilde r\,|\sigma(\tilde r,x^A)|^2\mathrm{d}\tilde r.\end{aligned}$$

This implies that φ is strictly increasing if

$$\int_0^\infty r|\sigma|^2\,\mathrm{d}r \;<\; n-1. \tag{2.28}$$

Since φ is positive for small r, it remains positive as long as $\partial_r\varphi$ remains positive, and so global positivity of φ is guaranteed whenever (2.28) holds.

A rather similar analysis applies to the case $\kappa\neq 0$, in which we set

$$H(r,x^A) := \int_0^r \kappa(\tilde r,x^A)\mathrm{d}\tilde r. \tag{2.29}$$

Let

$$\varphi(r) = \mathring\varphi(s(r)),\quad \text{where}\quad s(r) := \int_0^r \mathrm{e}^{H(\hat r)}\mathrm{d}\hat r, \tag{2.30}$$

the x^A-dependence being implicit. The function $s(r)$ is strictly increasing with $s(0)=0$. If we assume that κ is continuous in r with $\kappa(0)=0$, defined for all r and, e.g.

$$\int_0^\infty \kappa > -\infty, \tag{2.31}$$

then $\lim_{r\to\infty} s(r) = +\infty$, and thus the function $r\mapsto s(r)$ defines a differentiable bijection from $\mathbb{R}^+$ to itself. Consequently, a differentiable inverse function $s\mapsto r(s)$ exists, and is smooth if κ is.

Expressed in terms of (2.30), (2.19) becomes

$$\partial_s^2 \mathring{\varphi}(s) + \mathrm{e}^{-2H(r(s))} \frac{|\sigma|^2(r(s))}{n-1} \mathring{\varphi}(s) = 0. \tag{2.32}$$

A global solution $\varphi > 0$ of (2.19) exists if and only if a global solution $\mathring{\varphi} > 0$ of (2.32) exists. It follows from the considerations above (note that $\mathring{\varphi}(s=0) = 0$ and $\partial_s \mathring{\varphi}(s=0) = 1$) that a sufficient condition for global existence of positive solutions of (2.32) is

$$\int_0^\infty s\mathrm{e}^{-2H(r(s))} |\sigma|^2(r(s))\, \mathrm{d}s < n-1$$
$$\Longleftrightarrow \int_0^\infty \Big(\int_0^r \mathrm{e}^{H(\hat r)} \mathrm{d}\hat r \Big) \mathrm{e}^{-H(r)} |\sigma|^2(r)\, \mathrm{d}r < n-1. \tag{2.33}$$

Consider now the question of positivity of ν^0. In the $\kappa = 0$-wave-map gauge with Minkowski metric as a target, we have (see [11, Equation (4.7)])

$$\nu^0(r, x^A) = \frac{\varphi^{-(n-1)/2}(r, x^A)}{2} \int_0^r \Big(\hat r \varphi^{(n-1)/2} \overline{g}^{AB} s_{AB} \Big)(\hat r, x^A)\, \mathrm{d}\hat r. \tag{2.34}$$

In an s-orthonormal coframe $\theta^{(A)}$, $\overline{g}^{AB} s_{AB}$ is the sum of the diagonal elements $\overline{g}^{(A)(A)} = \overline{g}^\sharp(\theta^{(A)}, \theta^{(A)})$, $A = 1, \ldots, n-1$, where $\overline{g}^\sharp$ is the scalar product on $T^*\Sigma_r$ associated to $\overline{g}_{AB} \mathrm{d}x^A \mathrm{d}x^B$, each of which is positive in Riemannian signature. Hence,

$$\overline{g}^{AB} s_{AB} > 0.$$

So, for globally positive φ, we obtain a globally defined strictly positive ν^0, hence also a globally defined strictly positive $\nu_0 \equiv 1/\nu^0$.

When $\kappa \neq 0$, and allowing further a non-vanishing W^0, we find instead

$$\nu^0(r, x^A) = \frac{\left(\mathrm{e}^{-H} \varphi^{-(n-1)/2}\right)(r, x^A)}{2} \times \int_0^r \Big(\mathrm{e}^H \varphi^{(n-1)/2} (\hat r \overline{g}^{AB} s_{AB} - \overline{W}^0) \Big)(\hat r, x^A)\, \mathrm{d}\hat r, \tag{2.35}$$

with H as in (2.29). If $\overline{W}^0 = 0$, we obtain positivity as before. More generally, we see that a necessary-and-sufficient condition for positivity of ν^0 is positivity of the integral in the last line of (2.35) for all r. This will certainly be the case if the gauge-source function $\overline{W}^0$ satisfies

$$\overline{W}^0 < r \overline{g}^{AB} s_{AB} = r \varphi^{-2} \Big(\frac{\det \gamma}{\det s} \Big)^{1/(n-1)} \gamma^{AB} s_{AB}. \tag{2.36}$$

Summarizing, we have proved:

Proposition 2.2. 1. *Solutions of the Raychaudhuri equation with prescribed κ and σ are global when (2.31) and (2.33) hold, and lead to globally positive functions φ and τ.*

2. *Any global solution of the Raychaudhuri equation with $\varphi > 0$ leads to a globally defined positive function ν_0 when the gauge-source function $\overline{W}^0$ satisfies (2.36). This condition will be satisfied for any $\overline{W}^0 \leq 0$.*

2.5. Positivity of φ_{-1} and $(\nu^0)_0$

For reasons that will become clear in Sect. 4, we are interested in fields φ and ν_0 which, for large r, take the form

$$\varphi(r, x^A) = \varphi_{-1}(x^A)r + o(r), \quad \nu^0(r, x^A) = (\nu^0)_0(x^A) + o(1), \tag{2.37}$$

with φ_{-1} and $(\nu^0)_0$ positive. The object of this section is to provide conditions which guarantee existence of such expansions, assuming a global positive solution φ.

Let us further assume that $\mathrm{e}^{-2H}\varphi|\sigma|^2$ is continuous in r with

$$\int_0^\infty \left(\mathrm{e}^{-2H}\varphi|\sigma|^2\right)\big|_{r=r(s)}\mathrm{d}s = \int_0^\infty \mathrm{e}^{-H}\varphi|\sigma|^2\mathrm{d}r < \infty.$$

Integration of (2.32) and de l'Hospital rule at infinity give

$$\mathring{\varphi}_{-1} := \lim_{s\to\infty}\frac{\mathring{\varphi}(s)}{s} = \lim_{s\to\infty}\partial_s\mathring{\varphi}(s) = 1 - \frac{1}{n-1}\int_0^\infty \mathrm{e}^{-H}\varphi|\sigma|^2\mathrm{d}r. \tag{2.38}$$

This will be strictly positive if, for example, (2.33) holds, as

$$\begin{aligned}\int_0^r \mathrm{e}^{H(\tilde r)}\mathrm{d}\tilde r - \varphi(r) &= \int_0^r (\mathrm{e}^{H(\tilde r)} - \partial_{\tilde r}\varphi(\tilde r))\mathrm{d}\tilde r \\ &= \int_0^{r(s)} \underbrace{(1 - \partial_{\tilde s}\mathring{\varphi}(\tilde s))}_{\geq 0 \text{ by } (2.27)}\mathrm{d}\tilde s \geq 0,\end{aligned}$$

and thus by (2.38) and (2.33)

$$\mathring{\varphi}_{-1} \geq 1 - \frac{1}{n-1}\int_0^\infty \left(\int_0^r \mathrm{e}^{H(\hat r)}\mathrm{d}\hat r\right)\mathrm{e}^{-H(r)}|\sigma|^2(r)\,\mathrm{d}r > 0.$$

One can now use (2.38) to obtain (2.37), if we assume that the integral of κ over r converges:

$$\forall x^A \qquad -\infty < \beta(x^A) := \int_0^\infty \kappa(r, x^A)\mathrm{d}r < \infty, \tag{2.39}$$

so that

$$\int_0^r \kappa(s,\cdot)\mathrm{d}s = \beta(\cdot) + o(1). \tag{2.40}$$

Indeed, it follows from (2.39) that there exists a constant C such that the parameter s defined in (2.30) satisfies

$$C^{-1} \leq \frac{\partial s}{\partial r} \leq C, \quad C^{-1}r \leq s \leq Cr, \quad \lim_{r\to\infty}\frac{\partial s}{\partial r} = \mathrm{e}^\beta. \tag{2.41}$$

We then have

$$\begin{aligned}\varphi_{-1} &= \lim_{r\to\infty}\frac{\varphi(r)}{r} = \lim_{s\to\infty}\frac{\mathring{\varphi}(s)}{r(s)} = \lim_{s\to\infty}\frac{\partial_s\mathring{\varphi}(s)}{\partial_s r(s)} = \mathrm{e}^{-\beta}\mathring{\varphi}_{-1} \\ &= \mathrm{e}^{-\beta}\left(1 - \frac{1}{n-1}\int_0^\infty \mathrm{e}^{-H}\varphi|\sigma|^2\mathrm{d}r\right).\end{aligned} \tag{2.42}$$

We have proved:

Proposition 2.3. *Suppose that* (2.31), (2.33) *and* (2.39) *hold. Then, the function* φ *is globally positive with* $\varphi_{-1} > 0$.

Consider, next, the asymptotic behaviour of ν_0. In addition to (2.39), we assume now that $\varphi = \varphi_{-1} r + o(r)$, for some function of the angles φ_{-1}, and that there exists a bounded function of the angles α such that

$$r\overline{g}^{AB} s_{AB} - \overline{W}^0 = \frac{\alpha}{r} + o(r^{-1}). \tag{2.43}$$

Passing to the limit $r \to \infty$ in (2.35), one obtains

$$\nu^0(r, x^A) = \frac{\alpha(x^A)}{n-1} + o(1).$$

We see thus that

$$(\nu^0)_0 > 0 \quad \Longleftrightarrow \quad \alpha > 0, \qquad (\nu_0)_0 > 0 \quad \Longleftrightarrow \quad \alpha < \infty.$$

Remark 2.4. Note that (2.39) and (2.43) will hold with smooth functions α and β when the a priori restrictions (4.8)–(4.10), discussed below, are satisfied and when both φ and φ_{-1} are positive. Recall also that if $\overline{W}^0 \leq 0$ (in particular, if $\overline{W}^0 \equiv 0$), then the condition $\alpha \geq 0$ follows from the fact that both s_{AB} and $\overline{g}_{AB}$ are Riemannian.

So far we have justified the expansion (2.37). For the purposes of Sect. 3, we need to push the expansion one order further as follows:

Proposition 2.5. *Suppose that there exists a Riemannian metric* $(\gamma_{AB})_{-2} \equiv (\gamma_{AB})_{-2}(x^C)$ *and a tensor field* $(\gamma_{AB})_{-1} \equiv (\gamma_{AB})_{-1}(x^C)$ *on* S^{n-1} *such that for large* r, *we have*

$$\gamma_{AB} = r^2(\gamma_{AB})_{-2} + r(\gamma_{AB})_{-1} + o(r), \tag{2.44}$$

$$\partial_r\big(\gamma_{AB} - r^2(\gamma_{AB})_{-2} - r(\gamma_{AB})_{-1}\big) = o(1), \tag{2.45}$$

$$\int_0^r \kappa(s, x^A)\mathrm{d}s = \beta_0(x^A) + \beta_1(x^A)r^{-2} + o(r^{-2}). \tag{2.46}$$

Assume moreover that φ *exists for all* r, *with* $\varphi > 0$. *Then:*

1. *There exist bounded functions of the angles* $\varphi_{-1} \geq 0$ *and* φ_0 *such that*
$$\varphi(r) = \varphi_{-1} r + \varphi_0 + O(r^{-1}) \, . \tag{2.47}$$
2. *If, in addition,* ν_0 *exists for all* r, *if it holds that* $\varphi_{-1} > 0$ *and if* $\overline{W}^0$ *takes the form* $\overline{W}^0(r, x^A) = (\overline{W}^0)_1(x^A)r^{-1} + o(r^{-1})$ *with*
$$(\overline{W}^0)_1 < s_{AB}(\overline{g}^{AB})_2 = (\varphi_{-1})^{-2}\Big(\frac{\det\gamma_{-2}}{\det s}\Big)^{1/(n-1)} \gamma_{-2}^{AB} s_{AB}, \tag{2.48}$$
then
$$0 < (\nu_0)_0 < \infty.$$

Remark 2.6. If the space-time is not vacuum, then (2.32) becomes

$$\partial_s^2 \mathring{\varphi}(s) + \mathrm{e}^{-2H(r(s))} \frac{\big(|\sigma|^2 + \overline{R}_{rr}\big)(r(s))}{n-1} \mathring{\varphi}(s) = 0. \tag{2.49}$$

and the conclusions of Proposition 2.5 remain unchanged if we assume in addition that

$$\overline{R}_{rr} = O(r^{-4}). \tag{2.50}$$

Proof. From (2.11), one finds

$$|\sigma|^2 = O(r^{-4}) .$$

We have already seen that

$$\mathring{\varphi} = \mathring{\varphi}_{-1} s + o(s) .$$

Plugging this in the second term in (2.32) and integrating shows that

$$\partial_s \mathring{\varphi}(s) = \mathring{\varphi}_{-1} + O(s^{-2}), \quad \mathring{\varphi}(s) = \mathring{\varphi}_{-1} s + \mathring{\varphi}_0 + O(s^{-1}).$$

A simple analysis of the equation relating r with s gives now

$$\partial_r \varphi(r) = \varphi_{-1} + O(r^{-2}), \quad \varphi(r) = \varphi_{-1} r + \varphi_0 + O(r^{-1}).$$

This establishes point 1.

When φ_{-1} is positive, one finds that (2.43) holds, and from what has been said, the result follows. □

3. A no-go theorem for the ($\kappa = 0$, $\overline{W}^0 = 0$)-wave-map gauge

Rendall's proposal, to solve the characteristic Cauchy problem using the ($\kappa = 0$, $\overline{W}^\mu = 0$)-wave-map gauge, has been adopted by many authors. The object of this section is to show that, in $3+1$ dimensions, this approach will always lead to logarithmic terms in an asymptotic expansion of the metric *except for the Minkowski metric*. This makes clear the need to allow non-vanishing gauge-source functions $\overline{W}^\mu$.

More precisely, we prove (compare [7]):

Theorem 3.1. *Consider a four-dimensional vacuum space-time $(\mathscr{M}, g)$ which has a conformal completion at future null infinity $(\mathscr{M} \cup \mathscr{I}^+, \tilde{g})$ with a C^3 conformally rescaled metric, and suppose that there exists a point $O \in \mathscr{M}$ such that $\overline{C}_O \setminus \{O\}$, where $\overline{C}_O$ denotes the closure of C_O in $\mathscr{M} \cup \mathscr{I}^+$, is a smooth hypersurface in the conformally completed space-time. If the metric g has no logarithmic terms in its asymptotic expansion for large r in the $\overline{W}^0 = 0$ wave-map gauge, where r is an affine parameter on the generators of C_O, then $(\mathscr{M}, g)$ is the Minkowski space-time.*

Proof. Let $S \subset \mathscr{I}^+$ denote the intersection of $\overline{C}_O$ with $\mathscr{I}^+$. Elementary arguments show that $\overline{C}_O$ intersects $\mathscr{I}^+$ transversally and that S is diffeomorphic to S^2. Introduce near S coordinates so that S is given by the equation $\{u = 0 = x\}$, where x is an $\tilde{g}$-affine parameter along the generators of $\overline{C}_O$, with $x = 0$ at S, while the x^A's are coordinates on S in which the metric induced by $\breve{g}$ is manifestly conformal to the round-unit metric $s_{AB}\mathrm{d}x^A\mathrm{d}x^B$ on S^2. (Note that for finitely-differentiable metrics this construction might lead to the loss of one derivative of the metric.) The usual calculation shows that the g-affine parameter r along the generators of $\overline{C}_O$ equals $a(x^A)/x$ for some positive function of the angles $a(x^A)$. Discarding strictly positive conformal factors, we conclude that for large r the tensor field $\check{g}$ is conformal to a tensor field $\gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B$ satisfying

$$\gamma_{AB} = r^2\big(s_{AB} + (\gamma_{AB})_{-1}r^{-1} + o(r^{-1})\big), \tag{3.1}$$

$$\partial_r\big(\gamma_{AB} - r^2 s_{AB} - r(\gamma_{AB})_{-1}\big) = o(1). \tag{3.2}$$

The result follows now immediately from [8] and from our next Theorem 3.2. □

Theorem 3.2. *Suppose that the space-dimension n equals three. Let $r|\sigma|$, $r\overline{W}^0$ and $r^2\overline{R}_{\mu\nu}\ell^\mu\ell^\nu$ be bounded for small r. Suppose that $\gamma_{AB}(r,x^A)$ is positive definite for all $r>0$ and admits the expansion (3.1)–(3.2), for large r with the coefficients in the expansion depending only upon x^C. Assume that the first constraint equation (2.19) with $\kappa = 0$ and*

$$0 \le \overline{R}_{\mu\nu}\ell^\mu\ell^\nu = O(r^{-4})$$

has a globally defined positive solution satisfying $\varphi(0)=0$, $\partial_r\varphi(0)=1$, $\varphi>0$, and $\varphi_{-1}>0$. Then, there are no logarithmic terms in the asymptotic expansion of ν^0 in a gauge where $\kappa=0$ and $\overline{W}^0 = o(r^{-2})$ (for large r) if and only if

$$\sigma \equiv 0 \equiv \overline{R}_{\mu\nu}\ell^\mu\ell^\nu.$$

Proof of Theorem 3.2. At the heart of the proof lies the following observation:

Lemma 3.3. *In space-dimension n, suppose that $\kappa=0$ and set*

$$\Psi = r^2\exp\Big(\int_0^r \Big(\frac{\tau+\tau_1}{2} - \frac{n-1}{r}\Big)\,\mathrm{d}r\Big). \tag{3.3}$$

with $\tau_1 \equiv (n-1)/r$ We have $\tau = (n-1)r^{-1} + \tau_2 r^{-2} + o(r^{-2})$, where

$$\tau_2 := -\lim_{r\to\infty} r^2\Psi^{-1}\times\int_0^r (|\sigma|^2 + \overline{R}_{\mu\nu}\ell^\mu\ell^\nu)\Psi\,\mathrm{d}r, \tag{3.4}$$

provided that the limit exists.

Proof. Let $\delta\tau = \tau - \tau_1$. It follows from the Raychaudhuri equation with $\kappa=0$ that $\delta\tau$ satisfies the equation

$$\frac{\mathrm{d}\delta\tau}{\mathrm{d}r} + \frac{\tau+\tau_1}{2}\delta\tau = -|\sigma|^2 - 8\pi\overline{T}_{rr}.$$

Solving, one finds

$$\delta\tau = -\Psi^{-1}\int_0^r (|\sigma|^2 + 8\pi\overline{T}_{rr})\Psi\,\mathrm{d}r$$
$$= \frac{\tau_2}{r^2} + o(r^{-2}),$$

as claimed. □

Let us return to the proof of Theorem 3.2. Proposition 2.5 and Remark 2.6 show that

$$\varphi(r,x^A) = \varphi_{-1}(x^A)r + \varphi_0(x^A) + o(r^{-1}), \tag{3.5}$$

$$\tau \equiv 2\partial_r\log\varphi = 2r^{-1} - 2\varphi_0(\varphi_{-1})^{-1}r^{-2} + o(r^{-2}). \tag{3.6}$$

Recall, next, the solution formula (2.34) for the constraint equation (2.17) with $\kappa = 0$ and $n = 3$:

$$\nu^0(r, x^A) = \frac{1}{2\varphi(r, x^A)} \int_0^r \varphi \left(s\overline{g}^{AB} s_{AB} - \overline{W}^0 \right) (s, x^A) \, \mathrm{d}s. \tag{3.7}$$

From (3.5)-(3.6), one finds

$$\overline{g}^{AB} = r^{-2}(\varphi_{-1})^{-2}[s^{AB} + r^{-1}(\tau_2 s^{AB} - \breve{\gamma}^{AB}_{-1}) + o(r^{-1})], \tag{3.8}$$

with

$$\breve{\gamma}^{AB}_{-1} := s^{AC} s^{BD}[(\gamma_{CD})_{-1} - \frac{1}{2} s_{CD} s^{EF} (\gamma_{EF})_{-1}].$$

Inserting this into (3.7), and assuming that $\overline{W}^0 = o(r^{-2})$, one finds for large r

$$\nu^0 = (\varphi_{-1})^{-2} + \frac{1}{2}\tau_2(\varphi_{-1})^{-2} \frac{\ln r}{r} + O(r^{-1}), \tag{3.9}$$

with the coefficient of the logarithmic term vanishing if and only if $\tau_2 = 0$ when a bounded positive coefficient φ_{-1} exists. One can check that the hypotheses of Lemma 3.3 are satisfied, and the result follows. □

4. Preliminaries to Solve the Constraints Asymptotically

4.1. Notation and Terminology

Consider a metric which has a smooth, or polyhomogeneous, conformal completion at infinity *à la Penrose*, and suppose that the closure (in the completed space-time) $\overline{\mathscr{N}}$ of a null hypersurface $\mathscr{N}$ of O meets $\mathscr{I}^+$ in a smooth sphere. One can then introduce Bondi coordinates (u, r, x^A) near $\mathscr{I}^+$, with $\overline{\mathscr{N}} \cap \mathscr{I}^+$ being the level set of a Bondi retarded coordinate u (see [29] in the smooth case, and [12, Appendix B] in the polyhomogeneous case). The resulting Bondi area coordinate r behaves as $1/\Omega$, where Ω is the compactifying factor. If one uses Ω as one of the coordinates near $\mathscr{I}^+$, say x, and chooses $1/x$ as a parameter along the generators of $\mathscr{N}$, one is led to an asymptotic behaviour of the metric which is captured by the following definition:

Definition 4.1. We say that a smooth metric tensor $\overline{g}_{\mu\nu}$ defined on a null hypersurface $\mathscr{N}$ given in adapted null coordinates has a *smooth conformal completion at infinity*, if the unphysical metric tensor field $\overline{\tilde{g}}_{\mu\nu}$ obtained via the coordinate transformation $r \mapsto 1/r =: x$ and the conformal rescaling $\overline{g} \mapsto \overline{\tilde{g}} \equiv x^2 \overline{g}$ is, as a Lorentzian metric, smoothly extendable across $\{x = 0\}$. We will say that $\overline{g}_{\mu\nu}$ is *polyhomogeneous*, if the conformal extension obtained as above is polyhomogeneous at $\{x = 0\}$, see Appendix A.

The components of a smooth tensor field on $\mathscr{N}$ will be said to be *smooth at infinity*, respectively, *polyhomogeneous at infinity*, whenever they admit, in the (x, x^A)-coordinates, a smooth, respectively, polyhomogeneous, extension in the conformally rescaled space-time across $\{x = 0\}$.

Remark 4.2. The reader is warned that the definition contains an implicit restriction, that $\mathscr{N}$ *is a smooth hypersurface in the conformally completed space-time*. In the case of a light-cone, this excludes existence of points which are null-conjugate to O both in $\mathscr{M}$ and on $\overline{C}_O \cap \mathscr{I}^+$.

We emphasize that Definition 4.1 concerns only fields on $\mathscr{N}$, and *no assumptions are made concerning existence, or properties, of an associated space-time*. In particular, there *might not* be an associated space-time; and if there is one, it *might or might not* have a smooth completion through a conformal boundary at null infinity.

The conditions of the definition are both conditions on the metric and on the coordinate system. While the definition restricts the class of parameters r, there remains considerable freedom, which will be exploited in what follows. It should be clear that the existence of a coordinate system as above on a globally smooth light-cone is a necessary condition for a space-time to admit a smooth conformal completion at null infinity, for points O such that $\overline{C}_O \cap \mathscr{I}^+$ forms a smooth hypersurface in the conformally completed space-time.

Consider a real-valued function

$$f : (0, \infty) \times S^{n-1} \longrightarrow \mathbb{R}, \quad (r, x^A) \longmapsto f(r, x^A).$$

If this function admits an asymptotic expansion in terms of powers of r (whether to finite or arbitrarily high order), we denote by f_n, or $(f)_n$, the coefficient of r^{-n} in the expansion.

We will write $f = \mathcal{O}(r^N)$ (or $f = \mathcal{O}(x^{-N})$, $x \equiv 1/r$), $N \in \mathbb{Z}$ if the function

$$F(x, \cdot) := x^N f(x^{-1}, \cdot) \tag{4.1}$$

is smooth at $x = 0$. We emphasize that this is a restriction on f for large r, and the condition does not say anything about the behaviour of f near the vertex of the cone (whenever relevant), where r approaches zero.

We write

$$f(r, x^A) \sim \sum_{k=-N}^{\infty} f_k(x^A) r^{-k}$$

if the right-hand side is the asymptotic expansion at $x = 0$ of the function $x \mapsto r^{-N} f(r, \cdot)|_{r=1/x}$, compare Appendix A.

The next lemma summarizes some useful properties of the symbol $\mathcal{O}$:

Lemma 4.3. *Let $f = \mathcal{O}(r^N)$ and $g = \mathcal{O}(r^M)$ with $N, M \in \mathbb{Z}$.*

1. *f can be asymptotically expanded as a power series starting from r^N,*

$$f(r, x^A) \sim \sum_{k=-N}^{\infty} f_k(x^A) r^{-k}$$

for some suitable smooth functions $f_k : S^{n-1} \rightarrow \mathbb{R}$.

2. *The n-th order derivative, $n \geq 0$, satisfies*

$$\partial_r^n f(r, x^A) = \begin{cases} \mathcal{O}(r^{N-n}), & \text{for } N < 0, \\ \mathcal{O}(r^{N-n}), & \text{for } N \geq 0 \text{ and } N - n \geq 0, \\ \mathcal{O}(r^{N-n-1}), & \text{for } N \geq 0 \text{ and } N - n \leq -1, \end{cases}$$

as well as

$$\partial_A^n f(r, x^B) = \mathcal{O}(r^N) \, .$$

3. $f^n g^m = \mathcal{O}(r^{nN+mM})$ *for all $n, m \in \mathbb{Z}$.*

4.2. Some a Priori Restrictions

To solve the constraint equations asymptotically and derive necessary-and-sufficient conditions concerning smoothness of the solutions at infinity in adapted coordinates, it is convenient to have some a priori knowledge regarding the lowest admissible orders of certain functions appearing in these equations, and to exclude the appearance of logarithmic terms in the expansions of fields such as ξ_A and $\overline{W}^\lambda$. Let us, therefore, derive the necessary restrictions on the metric, the gauge-source functions, etc. needed to end up with a trace of a metric on the light-cone which admits a smooth conformal completion at infinity.

4.2.1. Non-Vanishing of φ and ν^0. As described above, the Einstein wave-map gauge constraints can be represented as a system of *linear* ODEs for φ, ν^0, ν_A and $\overline{g}^{11}$, so that existence and uniqueness (with the described boundary conditions) of global solutions are guaranteed, if the coefficients in the relevant ODEs are globally defined. Indeed, we have to make sure that the resulting symmetric tensor field $\overline{g}_{\mu\nu}$ does not degenerate, so that it represents a regular Lorentzian metric in the respective adapted null coordinate system. In a setting where the starting point are conformal data $\gamma_{AB}(r, \cdot)\mathrm{d}x^A\mathrm{d}x^B$ which define a Riemannian metric for all $r > 0$, this will be the case if and only if φ and ν^0 are nowhere vanishing, in fact *strictly positive* in our conventions,

$$\varphi > 0, \quad \nu^0 > 0 \quad \forall\, r > 0. \tag{4.2}$$

4.2.2. A Priori Restrictions on $\overline{g}_{\mu\nu}$. Assume that $\overline{g}_{\mu\nu}$ admits a smooth conformal completion in the sense of Definition 4.1. Then, its conformally rescaled counterpart $\overline{\tilde{g}}_{\mu\nu} \equiv x^2 \overline{g}_{\mu\nu}$ satisfies

$$\overline{\tilde{g}}_{\mu\nu} = \mathcal{O}(1) \quad \text{with} \quad \overline{\tilde{g}}_{0x}|_{x=0} > 0, \quad \det \overline{\tilde{g}}_{AB}|_{x=0} > 0. \tag{4.3}$$

This imposes the following restrictions on the admissible asymptotic form of the components $g_{\mu\nu}$ in adapted null coordinates $(u, r \equiv 1/x, x^A)$:

$$\nu_0 = \mathcal{O}(1), \quad \nu_A = \mathcal{O}(r^2), \quad \overline{g}_{00} = \mathcal{O}(r^2), \quad \overline{g}_{AB} = \mathcal{O}(r^2), \tag{4.4}$$

with

$$(\nu_0)_0 > 0 \quad \text{and} \quad (\det \check{g}_{\Sigma_r})_{-4} > 0. \tag{4.5}$$

Moreover,

$$\tau \equiv \frac{1}{2}\overline{g}^{AB}\partial_r \overline{g}_{AB} = \frac{n-1}{r} + \mathcal{O}(r^{-2}), \tag{4.6}$$

and (recall that $\tau = (n-1)\partial_r \log\varphi$)

$$\varphi = \varphi_{-1} r + \mathcal{O}(1) \quad \text{for some positive function } \varphi_{-1} \text{ on } S^{n-1}. \tag{4.7}$$

Indeed assuming that φ_{-1} vanishes for some x^A, the function φ does not diverge as r goes to infinity along some null ray Υ emanating from O, i.e. $\varphi|_\Upsilon = \mathcal{O}(1)$ and $\det \breve{g}_{\Sigma_r}|_\Upsilon \equiv (\varphi^{2(n-1)} \det s)|_\Upsilon = \mathcal{O}(1)$, which is incompatible with (4.5).

The assumptions $\varphi(r, x^A) > 0$ and $\varphi_{-1}(x^A) > 0$ imply the non-existence of conjugate points on the light-cone up-to-and-including conformal infinity.

4.2.3. A Priori Restrictions on Gauge-Source Functions. Assume that there exists a smooth conformal completion of the metric, as in Definition 4.1. We wish to find the class of gauge functions κ and $\overline{W}^\mu$ which are compatible with this asymptotic behaviour.

The relation $\overline{g}_{AB} = \mathcal{O}(r^2)$ together with $\partial_r = -x^2\partial_x$ and the definition (2.8) implies

$$\sigma_A{}^B = \mathcal{O}(r^{-2}), \quad |\sigma|^2 = \mathcal{O}(r^{-4}). \tag{4.8}$$

Using the estimate (4.7) for τ and the Raychaudhuri equation (2.16), we find

$$\kappa = \mathcal{O}(r^{-3}) \,, \tag{4.9}$$

where cancellations in both the leading and the next-to-leading terms in (2.16) have been used. Then, (2.17), (4.4), (4.7) and (4.9) imply

$$\overline{W}^0 = \mathcal{O}(r^{-1}). \tag{4.10}$$

Similar to $\kappa = \overline{\Gamma}^r_{rr}$, ξ_A corresponds to the restriction to C_O of certain connection coefficients (cf. [9,13])

$$\xi_A = -2\overline{\Gamma}^r_{rA}.$$

We will use this equation to determine the asymptotic behaviour of ξ_A; the main point is to show that there needs to exist a gauge in which ξ_A has no logarithmic terms. We note that the argument here requires assumptions about the whole space-time metric and some of its derivatives transverse to the characteristic initial surface, rather than on $\overline{g}_{AB}$.

A necessary condition for the space-time metric to be smoothly extendable across $\mathscr{I}^+$ is that the Christoffel symbols of the unphysical metric $\tilde{g}$ in coordinates $(u, x \equiv 1/r, x^A)$ are smooth at $\mathscr{I}^+$, in particular

$$\overline{\tilde{\Gamma}}{}^x_{xA} = \mathcal{O}(1). \tag{4.11}$$

The formula for the transformation of Christoffel symbols under conformal rescalings of the metric, $\tilde{g} = \Theta^2 g$, reads

$$\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + \frac{1}{\Theta}\left(\delta_\nu{}^\rho\partial_\mu\Theta + \delta_\mu{}^\rho\partial_\nu\Theta - g_{\mu\nu}g^{\rho\sigma}\partial_\sigma\Theta\right),$$

and shows that (4.11) is equivalent to

$$\overline{\Gamma}^x_{xA} = \mathcal{O}(1), \quad \text{or} \quad \overline{\Gamma}^r_{rA} = \mathcal{O}(1); \tag{4.12}$$

the second equation is obtained from the first one using the transformation law of the Christoffel symbols under the coordinate transformation $x \mapsto r \equiv 1/x$.

Hence, $\xi_A = \mathcal{O}(1)$. Inspection of the leading-order terms in (2.21) leads now to

$$\xi_A = \mathcal{O}(r^{-1}). \tag{4.13}$$

One can insert all this into (2.22), viewed as an equation for $\overline{W}^A$, to obtain

$$\overline{W}^A = \mathcal{O}(r^{-1}).$$

We note the formula

$$\zeta = \overline{2g^{AB}\Gamma^r_{AB} + \tau g^{rr}}$$

which allows one to relate ζ to the Christoffel symbols of g, and hence also to those of $\tilde{g}$. However, when relating $\overline{\tilde{\Gamma}}{}^x_{AB}$ and $\overline{\Gamma}{}^r_{AB}$, derivatives of the conformal factor Θ appear which are transverse to the light-cone and whose expansion is a priori not clear. Therefore, this formula cannot be used to obtain information about ζ in a direct way, and one has to proceed differently. Assuming, from now on, that we are in space-dimension three, it will be shown in part II of this work that the above a priori restrictions and the constraint equation (2.23) *imply* that the auxiliary function ζ has the asymptotic behaviour

$$\zeta = \mathcal{O}(r^{-1}). \tag{4.14}$$

It then follows from (2.24) and (4.4) that

$$\overline{W}^1 = \mathcal{O}(r). \tag{4.15}$$

This is our final condition on the gauge functions. To summarize, necessary conditions for existence of both a smooth conformal completion of the metric $\overline{g}$ and of smooth extensions of the connection coefficients $\overline{\Gamma}{}^1_{1A}$ are

$$\xi_A = \mathcal{O}(r^{-1}), \quad \overline{W}^0 = \mathcal{O}(r^{-1}), \quad \overline{W}^A = \mathcal{O}(r^{-1}). \tag{4.16}$$

Moreover,

$$\text{if } \zeta = \mathcal{O}(r^{-1}) \text{ then } \overline{W}^1 = \mathcal{O}(r)\ . \tag{4.17}$$

5. Asymptotic Expansions

We have seen in Sect. 4.2.2 that existence of a smooth completion at null infinity requires $g_{AB} = \mathcal{O}(r^2)$ with $(\det \overline{g}_{AB})_{-4} > 0$, and thus $\varphi = \mathcal{O}(r)$ with $\varphi_{-1} > 0$. But then

$$\frac{1}{\sqrt{\det \gamma}}\gamma_{AB} = \varphi^{-2}\frac{1}{\sqrt{\det s}}g_{AB} = \mathcal{O}(1).$$

Since only the conformal class of γ_{AB} matters, we see that there is no loss of generality to assume that $\gamma_{AB} = \mathcal{O}(r^2)$, with $(\det \gamma_{AB})_{-4} \neq 0$; this is convenient because then γ_{AB} and $\overline{g}_{AB}$ will display similar asymptotic behaviour. Moreover, since any Riemannian metric on the 2-sphere is conformal to the standard metric $s = s_{AB}\mathrm{d}x^A\mathrm{d}x^B$, in the case of smooth conformal completions we may without loss of generality require the initial data γ to be of the form, for large r,

$$\gamma_{AB} \sim r^2\left(s_{AB} + \sum_{n=1}^{\infty} h^{(n)}_{AB} r^{-n}\right), \tag{5.1}$$

for some smooth tensor fields $h^{(n)}_{AB}$ on S^2. (Recall that the symbol "$\sim$" has been defined in Sect. 4.1.) If the initial data γ_{AB} are not directly of the form (5.1), they can either be brought to (5.1) via an appropriate choice of coordinates and conformal rescaling, or lead to a metric $\overline{g}_{\mu\nu}$ which is not smoothly extendable across $\mathscr{I}^+$.

In the second part of this work [26], the following theorem will be proved:

Theorem 5.1. *Consider the characteristic initial value problem for Einstein's vacuum field equations in four space-time dimensions with smooth conformal data $\gamma = \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B$ and gauge functions κ and $\overline{W}^\lambda$ on a cone C_O which has smooth closure in the conformally completed space-time. The following conditions are necessary-and-sufficient for the trace of the metric $g = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ on C_O, obtained as solution to Einstein's wave-map characteristic vacuum constraint equations (2.19) and (2.21)–(2.24), to admit a smooth conformal completion at infinity and for the connection coefficients $\overline{\Gamma}^r_{rA}$ to be smooth at $\mathscr{I}^+$, in the sense of Definition 4.1, when imposing a generalized wave-map gauge condition $H^\lambda = 0$:*

(i) *There exists a conformal factor so that the conformally rescaled γ satisfies (5.1).*

(ii) *The functions φ, ν^0, φ_{-1} and $(\nu_0)_0$ have no zeros on $C_O\backslash\{0\}$ and S^2, respectively, with the non-vanishing of $(\nu^0)_0$ being equivalent to*

$$(\overline{W}^0)_1 < 2(\varphi_{-1})^{-2} \,. \tag{5.2}$$

(iii) *The gauge functions satisfy $\kappa = \mathcal{O}(r^{-3})$, $\overline{W}^0 = \mathcal{O}(r^{-1})$, $\overline{W}^A = \mathcal{O}(r^{-1})$, $\overline{W}^1 = \mathcal{O}(r)$ and, setting $\overline{W}_A := \overline{g}_{AB}\overline{W}^A$,*

$$(\overline{W}^0)_2 = \Big[\frac{1}{2}(\overline{W}^0)_1 + (\varphi_{-1})^{-2}\Big]\tau_2 \,, \tag{5.3}$$

$$\begin{aligned}(\overline{W}_A)_1 = {} & 4(\sigma_A{}^B)_2\mathring{\nabla}_A \log\varphi_{-1} - (\check{\varphi}_{-1})^{-2}[(\nu_0)_2(\overline{W}_A)_{-1} + (\nu_0)_1(\overline{W}_A)_0] \\ & - \mathring{\nabla}_A\tau_2 - \frac{1}{2}(w_A{}^B)_1(w_B{}^C)_1(\overline{W}_C)_{-1} - \frac{1}{2}(w_A{}^B)_2(\overline{W}_B)_{-1} \\ & - (w_A{}^B)_1\Big[(\overline{W}_B)_0 + (\check{\varphi}_{-1})^2(\nu_0)_1(\overline{W}_B)_{-1}\Big] \,,\end{aligned} \tag{5.4}$$

$$\begin{aligned}(\overline{W}^1)_2 = {} & \frac{\zeta_2}{2} + (\varphi_{-1})^{-2}\tau_2 + \frac{\tau_2}{4}\check{R}_2 + \frac{\tau_2}{2}(\overline{W}^1)_1 + \Big[\frac{\tau_3}{4} + \frac{\kappa_3}{2} - \frac{(\tau_2)^2}{8}\Big](\overline{W}^1)_0 \\ & \Big[\frac{1}{48}(\tau_2)^3 - \frac{1}{8}\tau_2\tau_3 - \frac{1}{4}\tau_2\kappa_3 + \frac{1}{6}\tau_4 + \frac{1}{3}\kappa_4\Big](\overline{W}^1)_{-1} \,,\end{aligned} \tag{5.5}$$

where $\mathring{\nabla}$ is the covariant derivative operator of the unit round metric on the sphere $s_{AB}\mathrm{d}x^A\mathrm{d}x^B$, $\check{R}_2$ is the r^{-2}-coefficient of the scalar curvature $\check{R}$ of the metric $\check{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B$, $\check{\varphi}_{-1} := [(\varphi_{-1})^{-2} - \frac{1}{2}(\overline{W}^0)_1]^{-1/2}$, and the expansion coefficients $(w_A{}^B)_n$ are defined using

$$w_A{}^B := \Big[\frac{r}{2}\nu_0(\overline{W}^0 + \overline{\Gamma}^0) - 1\Big]\delta_A{}^B + 2r\chi_A{}^B.$$

(iv) *The* no-logs-condition *is satisfied:*

$$(\sigma_A{}^B)_3 = \tau_2(\sigma_A{}^B)_2. \tag{5.6}$$

Remark 5.2. If any of the Eqs. (5.3)–(5.6) fail to hold, the resulting characteristic initial data sets will have a *polyhomogeneous* expansion in terms of powers of r.

Remark 5.3. Theorem 5.1 is independent of the particular setting used (and remains also valid when the light-cone is replaced by one of two transversally intersecting null hypersurfaces meeting $\mathscr{I}^+$ in a sphere), cf. Sect. 7: As long as the generalized wave-map gauge condition is imposed, one can always compute $\overline{W}^\lambda$, τ, σ etc. and check the validity of (5.3)–(5.6), whatever the prescribed initial data sets are. Some care is needed when the Minkowski target is replaced by some other target metric, cf. [26].

All the conditions in (ii) and (iii) which involve κ or $\overline{W}^\lambda$ can always be satisfied by an appropriate choice of coordinates. Equivalently, those logarithmic terms which appear if these conditions are not satisfied are pure gauge artefacts.

Recall that to solve the equation for ξ_A both κ and φ need to be known. This requires a choice of the κ-gauge. Since the choice of $\overline{W}^0$ does not affect the ξ_A-equation, there is no gauge freedom left in that equation and if the no-logs-condition (5.6) does not hold there is no possibility to get rid of the log terms that arise in this equation. (In Sect. 6, we will return to the question, whether (5.6) can be satisfied by a choice of κ.) Similarly, there is no gauge freedom left when the equation for ζ is integrated but, due to the special structure of the asymptotic expansion of its source term, no new log terms arise in the expansion of ζ.

The no-logs-condition involves two functions, φ_{-1} and φ_0, which are globally determined by the gauge function κ and the initial data γ, cf. (2.19). The dependence of these functions on the gauge and on the initial data is rather intricate. Thus, the question arises for which class of initial data one can find a function $\kappa = \mathcal{O}(r^{-3})$, such that the no-logs-condition holds, and accordingly what the geometric restrictions are for this to be possible. This issue will be analyzed in part II of this work, using a gauge scheme adjusted to the initial data so that all relevant globally defined integration functions can be computed explicitly.

6. The No-Logs-Condition

6.1. Gauge Independence

In this section, we show gauge independence of (5.6). It is shown in paper II [26] that (5.6) arises from integration of the equation for ξ_A, which is independent of the gauge functions W^μ. Equation (5.6) is, therefore, independent of those functions, as well. So the only relevant freedom is that of rescaling the r-coordinate parameterizing the null rays. We, therefore, need to compute how (5.6) transforms under rescalings of r. For this, we consider a smooth coordinate transformation

$$r \mapsto \tilde{r} = \tilde{r}(r, x^A) \ . \tag{6.1}$$

Under (6.1), the function φ transforms as a scalar. We have seen above that a necessary condition for the metric to be smoothly extendable across $\mathscr{I}^+$ is that φ has the asymptotic behaviour

$$\varphi(r,x^A)=\varphi_{-1}(x^A)r+\varphi_0+\mathcal{O}(r^{-1})\ ,\quad \text{with}\quad \varphi_{-1}>0. \tag{6.2}$$

The transformed φ thus takes the form

$$\tilde\varphi(\tilde r,x^A):=\varphi(r(\tilde r),x^A)=\varphi_{-1}(x^A)r(\tilde r)+\varphi_0+O(r(\tilde r)^{-1})\ ,$$
$$\partial_{\tilde r}\tilde\varphi(\tilde r,x^A)=\frac{\partial r}{\partial\tilde r}\partial_r\varphi(r(\tilde r),x^A)=\frac{\partial r}{\partial\tilde r}\varphi_{-1}(x^A)r(\tilde r)+\frac{\partial r}{\partial\tilde r}O(r(\tilde r)^{-2}).$$

If we require $\tilde\varphi$ to be of the form (6.2) as well, it is easy to check that we must have

$$r(\tilde r,x^A)=r_{-1}(x^A)\tilde r+r_0+O(\tilde r^{-1})\quad\text{and} \tag{6.3}$$
$$\partial_{\tilde r}r(\tilde r,x^A)=r_{-1}(x^A)+O(\tilde r^{-2}),\quad\text{with}\quad r_{-1}>0. \tag{6.4}$$

We have:

Proposition 6.1. *The no-logs-condition* (5.6) *is invariant under the coordinate transformations* (6.3)–(6.4).

Proof. For the transformation behaviour of the expansion coefficients, we obtain

$$\varphi_{-1}=(r_{-1})^{-1}\tilde\varphi_{-1},\quad \varphi_0=\tilde\varphi_0-r_0(r_{-1})^{-1}\tilde\varphi_{-1}$$
$$\Longrightarrow \tau_2=-2(\varphi_{-1})^{-1}\varphi_0=r_{-1}\tilde\tau_2+2r_0.$$

Moreover, with (6.3)–(6.4), we find

$$\begin{aligned}
\tilde\sigma_A{}^B&=\frac{\partial r}{\partial\tilde r}\sigma_A{}^B\\
&=\left[r_{-1}+O(\tilde r^{-2})\right]\left[(\sigma_A{}^B)_2r(\tilde r)^{-2}+(\sigma_A{}^B)_3r(\tilde r)^{-3}+\mathcal{O}(r(\tilde r)^{-4})\right]\\
&=(r_{-1})^{-1}(\sigma_A{}^B)_2\tilde r^{-2}+\left[(r_{-1})^{-2}(\sigma_A{}^B)_3-2r_0(r_{-1})^{-2}(\sigma_A{}^B)_2\right]\tilde r^{-3}\\
&\quad+O(\tilde r^{-4})\\
&\Longrightarrow\ (\sigma_A{}^B)_2=r_{-1}(\tilde\sigma_A{}^B)_2\ ,\\
&\qquad (\sigma_A{}^B)_3=(r_{-1})^2(\tilde\sigma_A{}^B)_3+2r_0r_{-1}(\tilde\sigma_A{}^B)_2.
\end{aligned}$$

Hence,

$$(\sigma_A{}^B)_3-\tau_2(\sigma_A{}^B)_2=(r_{-1})^2[(\tilde\sigma_A{}^B)_3-\tilde\tau_2(\tilde\sigma_A{}^B)_2]\ .$$

□

Although the No-Go Theorem 3.1 shows that the ($\kappa=0$, $\overline{W}^\lambda=0$)-wave-map gauge invariably produces logarithmic terms except in the flat case, one can decide whether the logarithmic terms can be transformed away by checking (5.6) using this gauge, or in fact any other. In the $([\gamma],\kappa)$ scheme, this requires to determine τ_2 by solving the Raychaudhuri equation, which makes this scheme not practical for the purpose. In particular, it is not a priori clear within this scheme whether *any* initial data satisfying this condition exist unless both $(\sigma^A{}_B)_2$ and $(\sigma^A{}_B)_3$ vanish. On the other hand, in any gauge scheme where $\breve{g}$ is prescribed on the cone, the no-logs-condition (5.6) is a straightforward condition on the asymptotic behaviour of the metric.

Let us assume that (5.6) is violated for say $\kappa = 0$. We know that the metric cannot have a smooth conformal completion at infinity in an adapted null coordinate system arising from the $\kappa = 0$-gauge via a transformation which *is not* of the asymptotic form (6.3)–(6.4). On the other hand, if the transformation *is* of the form (6.3), then the no-logs-condition will also be violated in the new coordinates. We conclude that we cannot have a smooth conformal completion in *any* adapted null coordinate system. That yields

Theorem 6.2. *Consider initial data γ on a light-cone C_O in a $\kappa = 0$-gauge with asymptotic behaviour $\gamma_{AB} \sim r^2(s_{AB} + \sum_{n=1}^{\infty} h^{(n)}_{AB} r^{-n})$. Assume that φ, ν^0 and φ_{-1} are strictly positive on $C_O \setminus \{O\}$ and S^2, respectively. Then, there exists a gauge w.r.t. which the trace $\overline{g}$ of the metric on the cone admits a smooth conformal completion at infinity and where the connection coefficients $\overline{\Gamma}^r_{rA}$ are smooth at $\mathscr{I}^+$ (in the sense of Definition 4.1) if and only if the no-logs-condition (5.6) holds in one (and then any) coordinate system related to the original one by a coordinate transformation of the form (6.3)–(6.4).*

6.2. Geometric Interpretation

Here, we provide a geometric interpretation of the no-logs-condition (5.6) in terms of the conformal Weyl tensor. This ties our results with the analysis in [1] (compare also Sect. 7.4).

For this purpose, let us consider the components of the conformal Weyl tensor, ${C_{rAr}}^B$, on the cone. To end up with smooth initial data for the conformal fields equations, we need to require its rescaled counterpart ${\overline{\tilde{d}}_{rAr}}{}^B = \overline{\Theta}^{-1}{\overline{\tilde{C}}_{rAr}}{}^B = \overline{\Theta}^{-1}{\overline{C}_{rAr}}{}^B$ to be smooth at $\mathscr{I}^+$, which is equivalent to

$$
{\overline{C}_{rAr}}{}^B = \mathcal{O}(r^{-5}). \tag{6.5}
$$

In particular, the ${\overline{C}_{rAr}}{}^B$-components of the Weyl tensor need to vanish one order faster than naively expected from the asymptotic behaviour of the metric. In adapted null coordinates and in vacuum we have, using the formulae of [9, Appendix A],

$$
\begin{aligned}
{\overline{C}_{rAr}}{}^B &= {\overline{R}_{rAr}}{}^B = -\partial_r \overline{\Gamma}^B_{rA} + \overline{\Gamma}^B_{rA}\overline{\Gamma}^r_{rr} - \overline{\Gamma}^B_{rC}\overline{\Gamma}^C_{rA} \\
&= -(\partial_r - \kappa){\chi_A}^B - {\chi_A}^C{\chi_C}^B \\
&= -\frac{1}{2}(\partial_r\tau - \kappa\tau + \frac{1}{2}\tau^2){\delta_A}^B - (\partial_r + \tau - \kappa){\sigma_A}^B - {\sigma_A}^C{\sigma_C}^B \\
&= \frac{1}{2}|\sigma|^2{\delta_A}^B - (\partial_r + \tau - \kappa){\sigma_A}^B - {\sigma_A}^C{\sigma_C}^B.
\end{aligned}
$$

Assuming, for definiteness, that $\kappa = \mathcal{O}(r^{-3})$ and $\overline{g}_{AB} = \mathcal{O}(r^2)$ with $(\det \overline{g}_{AB})_{-4} > 0$, we have

$$
\begin{aligned}
{\overline{C}_{rAr}}{}^B = \Big(&({\sigma_A}^B)_3 - \tau_2({\sigma_A}^B)_2 + \frac{1}{2}({\sigma_C}^D)_2({\sigma_D}^C)_2{\delta_A}^B - ({\sigma_A}^C)_2({\sigma_C}^B)_2 \Big) r^{-4} \\
&+ \mathcal{O}(r^{-5}).
\end{aligned}
$$

As an s-symmetric, trace-free tensor $(\sigma_A{}^C)_2$ has the property

$$(\sigma_A{}^C)_2(\sigma_C{}^B)_2 = \frac{1}{2}(\sigma_D{}^C)_2(\sigma_C{}^D)_2\delta_A{}^B,$$

i.e.

$$\overline{C}_{rAr}{}^B = \big[(\sigma_A{}^B)_3 - \tau_2(\sigma_A{}^B)_2\big]r^{-4} + \mathcal{O}(r^{-5}),$$

and (6.5) holds if and only if the no-logs-condition is satisfied.

7. Other Settings

We pass now to the discussion, how to modify the above when other data sets are given, or Cauchy problems other than a light-cone are considered.

7.1. Prescribed $(\check{g}_{AB}, \kappa)$

In this setting, the initial data are a symmetric degenerate twice-covariant tensor field $\check{g}$, and a connection κ on the family of bundles tangent to the integral curves of the kernel of $\check{g}$, satisfying the Raychaudhuri constraint (2.16).

Recall that so far we have mainly been considering a characteristic Cauchy problem where $([\gamma], \kappa)$ is given. There (2.19) was used to solve for the conformal factor relating $\check{g}$ and γ:

$$\check{g} \equiv \overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B = \varphi^2\left(\frac{\det s_{CD}}{\det \gamma_{EF}}\right)^{\frac{1}{n-1}}\gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B. \tag{7.1}$$

But then a pair $(\check{g}, \kappa)$ satisfying (2.16) is obtained. So, in fact, prescribing the pair $(\check{g}, \kappa)$ satisfying (2.16) can be viewed as a special case of the $([\gamma], \kappa)$-prescription, where one sets $\gamma := \check{g}$. Indeed, when $\check{g}$ and κ are suitably regular at the vertex, uniqueness of solutions of (2.19) with the boundary conditions $\varphi(0) = 0$ and $\partial_r\varphi(0) = 1$ shows that

$$\varphi = \left(\frac{\det \overline{g}_{EF}}{\det s_{CD}}\right)^{\frac{1}{2(n-1)}} \quad\Longleftrightarrow\quad \overline{g}_{AB} \equiv \gamma_{AB} \quad\Longleftrightarrow\quad \check{g} \equiv \gamma. \tag{7.2}$$

In particular, all the results so far apply to this case.

If τ is nowhere vanishing, as necessary for a smooth null-geodesically complete light-cone extending to null infinity, then (2.16) can be algebraically solved for κ, so that the constraint becomes trivial.

7.2. Prescribed $(\overline{g}_{\mu\nu}, \kappa)$

In this approach, one prescribes all metric functions $\overline{g}_{\mu\nu}$ on the initial characteristic hypersurface, together with the connection coefficient κ, subject to the Raychaudhuri equation (2.16). Equation (2.12) relating κ and ν_0 becomes an algebraic equation for the gauge-source function $\overline{W}^0$, while the equations $\overline{R}_{rA} = 0 = \overline{g}^{AB}\overline{R}_{AB}$ become algebraic equations for $\overline{W}^A$ and $\overline{W}^r$.

In four space-time dimensions, a smooth conformal completion at null infinity will exist if and only if $r^{-2}\overline{g}_{\mu\nu}$ can be smoothly extended as a Lorentzian metric across $\mathscr{I}^+$ and no logarithmic terms appear in the asymptotic expansion of $\overline{\Gamma}^r_{rA}$; this last fact is equivalent to (5.6). To see this, note that since the equations for $\overline{W}^\mu$ are algebraic, no log terms arise in these fields as long as no log terms appear in the remaining fields appearing in the constraint equation. Similarly, no log terms arise in the ζ-equation. The only possible source of log terms is thus the ξ_A-equation, and the appearance of log terms there is excluded precisely by the no-logs-condition. The existence of an associated space-time with a "piece of smooth $\mathscr{I}^+$" follows then from the analysis of the initial data for Friedrich's conformal equations in part II of this work, together with the analysis in [6].

We conclude that (5.6) is again a necessary-and-sufficient condition for existence of a smooth $\mathscr{I}^+$ for the current scheme in space-time dimension four.

7.3. Frame Components of σ as Free Data

In this section, we consider as free data the components χ_{ab} in an adapted parallel-propagated frame as in [13, Section 5.6]. We will assume that

$$\chi^a{}_b = \frac{1}{r}\delta^a{}_b + \mathcal{O}(r^{-2}), \quad a,b \in \{2,3\}\ . \tag{7.3}$$

There are actually at least two schemes which would lead to this form of $\chi^a{}_b$: One can, for example, prescribe any $\chi^a{}_b$ satisfying (7.3) such that $\chi^2{}_2 + \chi^3{}_3 = \chi_{22} + \chi_{33}$ has no zeros, define $\sigma_{ab} = \chi_{ab} - \frac{1}{2}(\chi^2{}_2 + \chi^3{}_3)\delta_{ab}$, and solve algebraically the Raychaudhuri equation for κ. Another possibility is to prescribe directly a symmetric trace-free tensor σ_{ab} in the $\kappa = 0$ gauge, use the Raychaudhuri equation to determine τ, and construct χ_{ab} using

$$\chi^a{}_b = \frac{\tau}{2}\delta^a{}_b + \sigma^a{}_b, \quad a,b \in \{2,3\}. \tag{7.4}$$

The asymptotics (7.3) will then hold if $\sigma^a{}_b$ is taken to be $\mathcal{O}(r^{-2})$.

Given χ_{ab}, the tensor field $\check{g}$ is obtained by setting

$$\check{g} = \left(\theta^2{}_A\theta^2{}_B + \theta^3{}_A\theta^3{}_B\right)\mathrm{d}x^A\mathrm{d}x^B, \tag{7.5}$$

where the co-frame coefficients $\theta^a{}_A$ are solutions of the equation [13]

$$\partial_r\theta^a{}_A = \chi^a{}_b\theta^b{}_A, \quad a,b \in \{2,3\}. \tag{7.6}$$

Assuming (7.3), one finds that solutions of (7.6) have an asymptotic expansion for $\theta^a{}_A$ without log terms:

$$\theta^a{}_A = r\varphi^a{}_A + \mathcal{O}(1), \quad a,b \in \{2,3\} \tag{7.7}$$

for some globally determined functions $\varphi^a{}_A$. If the determinant of the two-by-two matrix $(\varphi^a{}_A)$ does not vanish, one obtains a tensor field $\check{g}$ to which our previous considerations apply. This leads again to the no-logs-condition (5.6).

Writing, as usual,

$$\sigma_{ab} = (\sigma_{ab})_2 r^{-2} + (\sigma_{ab})_3 r^{-3} + \mathcal{O}(r^{-4}), \quad a,b \in \{2,3\}, \tag{7.8}$$

the no-logs-condition rewritten in terms of σ_{ab} reads

$$(\sigma_{ab})_3 = \tau_2(\sigma_{ab})_2, \quad a,b \in \{2,3\}. \tag{7.9}$$

7.4. Frame Components of the Weyl Tensor as Free Data

Let $C_{\alpha\beta\gamma\delta}$ denote the space-time Weyl tensor. For $a, b \geq 2$ let

$$\psi_{ab} := e_a{}^A e_b{}^B \overline{C}_{ArBr}$$

represent the components of $\overline{C}_{ArBr}$ in a parallelly transported adapted frame, as in Sect. 7.3. The tensor field ψ_{ab} is symmetric, with vanishing η-trace, and we have in space-time dimension four (cf. e.g. [13, Section 5.7])

$$(\partial_r - \kappa)\chi_{ab} = -\sum_{c=2}^{3} \chi_{ac}\chi_{cb} - \psi_{ab} - \frac{1}{2}\eta_{ab}\overline{T}_{rr}. \tag{7.10}$$

Given (κ, ψ_{ab}), we can integrate this equation in vacuum to obtain the tensor field χ_{ab} needed in Sect. 7.3. However, this approach leads to at least two difficulties: First, it is not clear under which conditions on ψ_{ab}, the solutions will exist for all values of r. Next, it is not clear that the global solutions will have the desired asymptotics. We will not address these questions but, taking into account the behaviour of the Weyl tensor under conformal transformations, we will assume that

$$\kappa = \mathcal{O}(r^{-3}), \quad \psi_{ab} = \mathcal{O}(r^{-4}), \tag{7.11}$$

and that the associated tensor field χ_{ab} exists globally and satisfies (7.3). The no-logs-condition will then hold if and only if

$$\psi_{ab} = \mathcal{O}(r^{-5}) \qquad \Longleftrightarrow \qquad (\psi_{ab})_4 = 0. \tag{7.12}$$

Note that one can reverse the procedure just described: given χ_{ab} we can use (7.10) to determine ψ_{ab}. Assuming (7.3), the no-logs-condition will hold if and only if the ψ_{ab}-components of the Weyl tensor vanish one order faster than naively expected from the asymptotic behaviour of the metric (cf. Sect. 6.2).

Equation (7.12) is the well-known starting point of the analysis in [25], and has also been obtained previously as a necessary condition for existence of a smooth $\mathscr{I}$ in the analysis of the hyperboloidal Cauchy problem [1]. It is, therefore, not surprising that it reappears in the analysis of the characteristic Cauchy problem. However, as pointed out above, a satisfactory treatment of the problem using ψ_{ab} as initial data requires further work.

7.5. Characteristic Surfaces Intersecting Transversally

Consider two characteristic surfaces, say $\mathscr{N}_1$ and $\mathscr{N}_2$, intersecting transversally along a smooth submanifold S diffeomorphic to S^2. Assume, moreover, that the initial data on $\mathscr{N}_1$ (in any of the versions just discussed) are such that the metric $\overline{g}_{\mu\nu}$ admits a smooth conformal completion across the sphere $\{x = 0\}$, as in Definition 4.1. The no-logs-condition (5.6) remains unchanged. Indeed, the only difference is the integration procedure for the constraint equations: while on the light-cone we have been integrating from the tip of the light-cone, on $\mathscr{N}_1$ we integrate from the intersection surface S. This leads to the need to provide supplementary data at S which render the solutions unique. Hence, the asymptotic values of the fields, which arise from the integration of the constraints, will also depend on the supplementary data at S.

7.6. Mixed Spacelike-Characteristic Initial Value Problem

Consider a mixed initial value problem, where the initial data set consists of:

1. A spacelike initial data set $(\mathscr{S}, {}^3g, K)$, where ${}^3g$ is a Riemannian metric on $\mathscr{S}$ and K is a symmetric two-covariant tensor field on $\mathscr{S}$. The three-dimensional manifold $\mathscr{S}$ is supposed to have a compact smooth boundary S diffeomorphic to S^2, and the fields $({}^3g, K)$ are assumed to satisfy the usual vacuum Einstein constraint equations.
2. A hypersurface $\mathscr{N}_1$ with boundary S equipped with a characteristic initial data set, in any of the configurations discussed so far. Here, $\mathscr{N}_1$ should be thought of as a characteristic initial data surface emanating from S in the outgoing direction.
3. The data on $\mathscr{S}$ and $\mathscr{N}_1$ satisfy a set of "corner conditions" at S, to be defined shortly.

The usual evolution theorems for the spacelike general relativistic initial value problem provide a unique future maximal globally hyperbolic vacuum development $\mathscr{D}^+$ of $(\mathscr{S}, {}^3g, K)$. Since $\mathscr{S}$ has a boundary, $\mathscr{D}^+$ will also have a boundary. Near S, the null part of the boundary of $\partial\mathscr{D}^+$ will be a smooth null hypersurface emanating from S, say $\mathscr{N}_2$, generated by null geodesics normal to S and "pointing towards $\mathscr{S}$" at S. In particular, the space-time metric on $\mathscr{D}^+$ induces characteristic initial data on $\mathscr{N}_2$. In fact, all derivatives of the metric, both in directions tangent and transverse to $\mathscr{N}_2$, will be determined on $\mathscr{N}_2$ by the initial data set $(\mathscr{S}, {}^3g, K)$. This implies that the characteristic initial data needed on $\mathscr{N}_1$, as well as their derivatives in directions tangent to $\mathscr{N}_1$, are determined on S by $(\mathscr{S}, {}^3g, K)$. These are the "corner conditions" which have to be satisfied by the data on $\mathscr{N}_1$ at S, with these data being arbitrary otherwise. The corner conditions can be calculated algebraically in terms of the fields $({}^3g, K)$, the gauge-source functions W^μ, and the derivatives of those fields, at S, using the vacuum Einstein equations.

One can use now the Cauchy problem discussed in Section 7.5 to obtain the metric to the future of $\mathscr{N}_1 \cup \mathscr{N}_2$, and the discussion of the no-logs-condition given in Sect. 7.5 applies.

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Appendix A. Polyhomogeneous Functions

A function f defined on an open set $\mathscr{U}$ with smooth boundary $\partial\mathscr{U} = \{x = 0\}$ is said to be *polyhomogeneous* at $x = 0$ if $f \in C^\infty(\mathscr{U})$ and if there exist integers N_i, real numbers n_i, and functions $f_{ij} \in C^\infty(\overline{\mathscr{U}})$ such that

$$\forall m \in \mathbb{N}, \quad \exists N(m) \in \mathbb{N}, \quad f - \sum_{i=0}^{N(m)} \sum_{j=0}^{N_i} f_{ij} x^{n_i} \ln^j x \in C^m(\overline{\mathscr{U}}). \tag{A.1}$$

We will then write

$$f \sim \sum_{i,j} f_{ij} x^{n_i} \ln^j x.$$

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