Ann. Henri Poincaré 16 (2015), 1103–1153

1424-0637/15/051103-51
*published online* October 22, 2014
DOI 10.1007/s00023-014-0358-9

**Annales Henri Poincaré**

# On the Form Factors of Local Operators in the Bazhanov–Stroganov and Chiral Potts Models

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**Abstract.** We consider general cyclic representations of the six-vertex Yang–Baxter algebra and analyze the associated quantum integrable systems, the Bazhanov–Stroganov model and the corresponding chiral Potts model on finite size lattices. We first determine the *propagator operator* in terms of the chiral Potts transfer matrices and we compute the scalar product of *separate states* (including the transfer matrix eigenstates) as a single determinant formulae in the framework of Sklyanin's quantum separation of variables. Then, we solve the quantum inverse problem and reconstruct the local operators in terms of the separate variables. We also determine a basis of operators whose form factors are characterized by a single determinant formulae. This implies that the form factors of any local operator are expressed as finite sums of determinants. Among these form factors written in determinant form are in particular those which will reproduce the chiral Potts order parameters in the thermodynamic limit. The results presented here are the generalization to the present models associated to the most general cyclic representations of the six-vertex Yang–Baxter algebra of those we derived for the lattice sine–Gordon model.

## 1. Introduction

In the article [1], we developed an approach in the framework of the quantum inverse scattering method (QISM) [2–14] to achieve the complete solution of lattice integrable quantum models by the exact characterization of their spectrum and the computation of the matrix elements of local operators in the eigenstates basis. This approach is addressed to the large class of integrable quantum models whose spectrum (eigenvalues and eigenstates) can be determined by implementing Sklyanin's quantum separation of variables (SOV)

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method [15–17]. It can be considered as the generalization to this SOV framework of the Lyon group method[1] for the computation of matrix elements of local operators in the algebraic Bethe ansatz settings. In [1], the approach has been developed for the lattice quantum sine–Gordon model [5,14] associated by QISM to particular cyclic representations [53] of the six-vertex Yang–Baxter algebra. More in detail, in [54–56], the complete SOV spectrum characterization has been constructed for the lattice quantum sine–Gordon model, while in [1] the scalar product of separate states and the matrix elements of local operators have been computed. In the present article, we implement this approach for the quantum models associated by QISM to the most general cyclic representations of the six-vertex Yang–Baxter algebra, i.e. the inhomogeneous Bazhanov–Stroganov model and subsequently the chiral Potts (chP) model [57–76] by exploiting the well-known links between these two models [57]. We first build our two central tools for computing matrix elements of local operators, i.e. the expression of the scalar products of separate states in terms of a determinant formula and the local fields reconstruction in terms of quantum separate variables (by solving the so-called quantum inverse scattering problem). Then, we use these results to compute the form factors of local operators on the transfer matrix eigenstates and to express them as sums of determinants given by simple deformations of the ones giving the scalar product of separate states.

### 1.1. Literature Summary

Let us first summarize some known results concerning these quantum integrable models and that are relevant for our present work. In [57], the Bazhanov–Stroganov model was introduced from its Lax operator built as a general solution to the Yang–Baxter equation associated to the six-vertex $R$-matrix. For a specific subset of cyclic representations, in which the parameters lie on the algebraic curves associated to the chP-model, the construction of the Baxter $Q$-operator allowed for the analysis of the spectrum (eigenvalues). This $Q$-operator was shown to coincide with the transfer matrix of the integrable $Z_p$ chP-model [60–70]; in this way, a first remarkable connection between these two apparently very different models[2] was established. Additional functional

[1] This method has been introduced in [18] for the spin-1/2 $XXZ$ quantum chain [19–27] with periodic boundaries and further developed in[28–40]. Its generalization to the higher spin $XXX$ quantum chains and to the open spin-1/2 $XXZ$ quantum chains [46–52] with diagonal boundary conditions has been, respectively, implemented in [41–45].

[2] Note that in a two-dimensional statistical mechanics formulation, both models have Boltzmann weights which satisfy the star-triangle equations. However, while the weights of the Bazhanov–Stroganov model satisfy the difference property in the rapidities, those of the chP-model do not. In this respect, the link to classical integrable discrete models is quite illuminating [71–73]. It is worth recalling that the first solutions of the star-triangle equations with this non-difference property were obtained in [74,75], while in [64,65] the general solutions for the chP-model were derived.

equations of fusion hierarchy type[3] for commuting transfer matrices[4] were then exhibited in [58]. Bethe ansatz type equations play an important role in the special sub-variety of the super-integrable chP-model as it was first shown in [60–62]. The connection between the Bazhanov–Stroganov model and the chP-model allowed to introduce rigorously [76] the description of the super-integrable chP spectrum using algebraic Bethe ansatz. The Bethe ansatz construction was applied to the transfer matrix $\tau_2$ of the Bazhanov–Stroganov model, thus obtaining in a different way the Baxter results [68] on the subset of the translation-invariant eigenvectors of the super-integrable chP-model.[5] More recently, the extension of the eigenvalue analysis of the Bazhanov–Stroganov model to completely general cyclic representations was done by Baxter [59]. The main tool used there was the construction of a generalized $Q$-operator which satisfies the Baxter equation with the transfer matrix $\tau_2$ and the extension to these representations of the functional relations of the fused transfer matrices.

Another important feature of the chP-model which has been the subject of recent attention is the spontaneous magnetization. This order parameter was first described in [91] on the basis of perturbative calculations developed for the special class of super-integrable representations.[6] The first non-perturbative derivation of this order parameter was achieved only recently by Baxter [92, 93] under some natural analyticity assumptions and the use of a technique introduced by Jimbo et al. [94]. More classical techniques, such as the corner transfer matrix [95], could not be used, mainly because of the very nature of the chP-model [96]. The proof of the spontaneous magnetization formula [91] starting from direct computations on the finite lattice of matrix elements of the spin operators could only be achieved after the recent introduction by Baxter [97, 98] of a generalized version of the Onsager algebra for the special class of super-integrable representations of chP-model. The matrix elements used for this proof have been first analyzed by Au-Yang and Perk in a series of papers [79, 80, 99–101] for the case of the super-integrable chP-model. Their factorized form, first conjectured by Baxter [102], has been proven[7] by Iorgov et al. [103] and used to derive the spontaneous magnetization formula conjectured in [91]. Finally, it is worth recalling that, in the algebraic framework of generalized Onsager algebra, Baxter has also first conjectured [106] and successively proven

[3] The approach of fusion hierarchy of commuting transfer matrices was first introduced in [77, 78].

[4] The transfer matrix of the Bazhanov–Stroganov model is the second element in this hierarchy, this explains the name $\tau_2$ given some times to this model.

[5] For further analysis of the eigenstates of super-integrable chP-model, see also [79–82]. It is interesting to mention here also that in all these analysis the underlying Onsager algebra [83] and realizations of the sl2 loop algebra [84], which are symmetries for these super-integrable representations [64–66, 85–90], have played fundamental roles.

[6] This case both obeys Yang–Baxter integrability [64, 65] and has an underlying Onsager algebra [63].

[7] Note that factorized formulas for the spin matrix elements exist also for the 2D Ising model [104] and for the quantum $XY$-chain [105].

in [107] a determinant formula for the spontaneous magnetization of the super-integrable chP-model; this result is also used for a further derivation of the known formula of the order parameter in the thermodynamical limit.

### 1.2. Motivations for the Use of SOV

Let us comment that in the literature we just recalled, the spectral analysis has usually one or more of the following problems: there is no eigenstates construction for the functional methods based only on the Baxter $Q$-operator and the fusion of transfer matrices. The algebraic Bethe ansatz (ABA) applies only to very special representations of the Bazhanov–Stroganov model and similarly the algebraic framework of the generalized Onsager algebra is proven to exist only in the class of super-integrable representations of chiral Potts model. The proof of the completeness of eigenstates is not ensured by these methods and it was so far missing in the general $p$-state chP-model and Bazhanov–Stroganov model. Existing results about this issue are mainly restricted to the case of the 3-state super-integrable chP-model [108] and to the reduction of the 3-state Potts model to the trivial algebraic curve case [109], i.e. the Fateev–Zamolodchikov model [110], see also [111,112] for further applications of this method.

The circumstance interesting for us is that, in the case of the cyclic representations of the Bazhanov–Stroganov model for which the algebraic Bethe ansatz does not apply, Sklyanin's quantum SOV can be developed to analyze the system. This means that, for most[8] of the representations of this model, we have the opportunity to use the SOV method, which appears quite promising as it leads to both the eigenvalues and the eigenstates of the transfer matrix of the Bazhanov–Stroganov model with a complete spectrum construction if some simple conditions are satisfied. The SOV analysis of these representations was first introduced[9] in [113] and further developed in [118]. Here, we will use these SOV results as setup for the computation of the form factors of local operators. Let us recall that in [118], the functional equation characterization of the transfer matrix spectrum has been derived purely on the basis of the SOV spectrum characterization[10] together with a first proof of the completeness of the system of equations of Bethe ansatz type[11] for some

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[8] The values of the parameters of the representations for which ABA applies define a proper sub-variety in the full space of the parameters of the representations of the Bazhanov–Stroganov model.

[9] There the eigenvector analysis developed in [114] was used to obtain the SOV representations of the Bazhanov–Stroganov model. See also the series of works [115–117] where the form factors of local spin operators were computed by SOV for the special case ($p = 2$) of the generalized Ising model.

[10] Note that for cyclic representations, the SOV does not lead directly to the spectrum characterization by functional equations and so, in particular, it does not lead to Bethe equations.

[11] For Bethe ansatz methods, as the coordinate Bethe ansatz [20,95,119], the algebraic Bethe ansatz [3–5] and the analytic Bethe ansatz [120,121], a proof of the completeness was achieved only for few integrable quantum models, see as concrete examples [122] for the $XXX$ Heisenberg model, [123] for the infinite $XXZ$ spin chain with domain wall boundary conditions and [124] for the nonlinear quantum Schroedinger model.

classes of representations of Bazhanov–Stroganov model and chP-model and the simplicity of these transfer matrix spectra in the inhomogeneous models.

Beyond these motivations on the spectrum analysis, the summary presented in the previous subsection makes clear that the computations of matrix elements of local operators are so far mainly confined to the special class of super-integrable representations of chP-model as they were derived in the algebraic framework of the generalized Onsager algebra. This stresses the relevance of our approach using quantum SOV which leads to form factors of local operators and applies to generic representations of Bazhanov–Stroganov model and chiral Potts model to which the methods based on generalized Onsager algebra do not apply up to now.

### 1.3. Paper Organization

To make the paper self-contained, we dedicate Sects. 2 and 3 to review the material presented in [118] simultaneously integrating it with the presentation of new results needed for our purposes. In particular, Sect. 2 provides the definition of the Bazhanov–Stroganov model and the main results of [118] on SOV, while Sects. 2.3.1 and 2.4.2 contain new results on the SOV decomposition of the identity and the characterization of the transfer matrix eigenstates. Section 3 provides the definition of the chiral Potts model and the main results obtained by SOV method in [118]. The scalar products of separate states and the decomposition of the identity w.r.t. the transfer matrix eigenbasis are derived in Sect. 4. Section 5 contains the characterization of the propagator operator of the Bazhanov–Stroganov model in terms of the chiral Potts transfer matrices. The reconstruction of local operators in terms of separate variables is given in Sect. 6, while their form factors are expressed in terms of finite size determinants in Sect. 7. The last section addresses some comments on these results and a comparison with the existing literature.

## 2. The Bazhanov–Stroganov Model

We use this section to give our notations and to briefly recall the main results derived in [118] on the spectrum description by SOV of the Bazhanov–Stroganov model and chiral Potts model that are useful for our purposes.

### 2.1. The Bazhanov–Stroganov Model: Definitions and First Properties

We define in the $\mathsf{N}$ sites of the chain $\mathsf{N}$ local Weyl algebras $\mathcal{W}_n$ and denote by $\mathsf{u}_n$ and $\mathsf{v}_n$ their generators:

$$\mathsf{u}_n\mathsf{v}_m = q^{\delta_{n,m}}\mathsf{v}_m\mathsf{u}_n \quad \forall n, m \in \{1,\ldots,\mathsf{N}\}. \tag{2.1}$$

The Lax operator of the Bazhanov–Stroganov model reads:[12]

$$\mathsf{L}_n(\lambda) \equiv \begin{pmatrix} \lambda\alpha_n\mathsf{v}_n - \beta_n\lambda^{-1}\mathsf{v}_n^{-1} & \mathsf{u}_n\left(q^{-1/2}\mathbb{a}_n\mathsf{v}_n + q^{1/2}\mathbb{b}_n\mathsf{v}_n^{-1}\right) \\ \mathsf{u}_n^{-1}\left(q^{1/2}\mathbb{c}_n\mathsf{v}_n + q^{-1/2}\mathbb{d}_n\mathsf{v}_n^{-1}\right) & \gamma_n\mathsf{v}_n/\lambda - \delta_n\lambda/\mathsf{v}_n \end{pmatrix}, \tag{2.2}$$

[12] Up to different notations, this Lax operator coincides with the one introduced in [57].

where $\alpha_n$, $\beta_n$, $\gamma_n$, $\delta_n$, $\mathbb{a}_n$, $\mathbb{b}_n$, $\mathbb{c}_n$ and $\mathbb{d}_n$ are constants associated to the site $n$ of the chain subject to the relations:

$$\alpha_n\gamma_n = \mathbb{a}_n\mathbb{c}_n, \quad \beta_n\delta_n = \mathbb{b}_n\mathbb{d}_n. \tag{2.3}$$

The monodromy matrix of the model is defined in terms of the Lax operators by:

$$\mathsf{M}(\lambda) = \begin{pmatrix} \mathsf{A}(\lambda) & \mathsf{B}(\lambda) \\ \mathsf{C}(\lambda) & \mathsf{D}(\lambda) \end{pmatrix} \equiv \mathsf{L}_{\mathsf{N}}(\lambda)\ldots\mathsf{L}_1(\lambda). \tag{2.4}$$

It satisfies the quadratic Yang–Baxter relation:

$$R(\lambda/\mu)\,(\mathsf{M}(\lambda)\otimes 1)\,(1\otimes\mathsf{M}(\mu)) = (1\otimes\mathsf{M}(\mu))\,(\mathsf{M}(\lambda)\otimes 1)R(\lambda/\mu), \tag{2.5}$$

driven by the six-vertex (standard) $R$-matrix:

$$R(\lambda) = \begin{pmatrix} q\lambda - q^{-1}\lambda^{-1} & & & \\ & \lambda-\lambda^{-1} & q-q^{-1} & \\ & q-q^{-1} & \lambda-\lambda^{-1} & \\ & & & q\lambda - q^{-1}\lambda^{-1} \end{pmatrix}. \tag{2.6}$$

Then, the elements of $\mathsf{M}(\lambda)$ generate a representation $\mathcal{R}_{\mathsf{N}}$ of the so-called Yang–Baxter algebra. In particular, (2.5) yields the relation $[\mathsf{B}(\lambda), \mathsf{B}(\mu)] = 0$, for all $\lambda$ and $\mu$, and the mutual commutativity of the elements of the one parameter family of transfer matrix operators:

$$\tau_2(\lambda) \equiv \operatorname{tr}_{\mathbb{C}^2}\mathsf{M}(\lambda) = \mathsf{A}(\lambda) + \mathsf{D}(\lambda). \tag{2.7}$$

Let us introduce the operator:

$$\Theta = \prod_{n=1}^{\mathsf{N}} \mathsf{v}_n, \tag{2.8}$$

which plays the role of a *grading operator* in the Yang–Baxter algebra:[13]

**Lemma 2.1** (Lemma 1 of [118]). $\Theta$ *commutes with the transfer matrix* $\tau_2(\lambda)$. *More precisely, its commutation relations with the elements of the monodromy matrix are:*

$$\Theta\mathsf{C}(\lambda) = q\mathsf{C}(\lambda)\Theta, \quad [\mathsf{A}(\lambda), \Theta] = 0, \tag{2.9}$$
$$\mathsf{B}(\lambda)\Theta = q\Theta\mathsf{B}(\lambda), \quad [\mathsf{D}(\lambda), \Theta] = 0. \tag{2.10}$$

Besides, the $\Theta$-charge allows to express the following asymptotics in both $\lambda \to 0$ and $\lambda \to \infty$ of the leading operators of the Yang–Baxter algebras:

$$\mathsf{A}(\lambda) = \left(\lambda^{\mathsf{N}}\Theta\prod_{a=1}^{\mathsf{N}}\alpha_n + (-1)^{\mathsf{N}}\lambda^{-\mathsf{N}}\Theta^{-1}\prod_{a=1}^{\mathsf{N}}\beta_a\right) + \sum_{i=1}^{\mathsf{N}-1}\mathsf{A}_i\lambda^{\mathsf{N}-2i}, \tag{2.11}$$

$$\mathsf{D}(\lambda) = \left(\lambda^{-\mathsf{N}}\Theta\prod_{a=1}^{\mathsf{N}}\gamma_a + (-1)^{\mathsf{N}}\lambda^{\mathsf{N}}\Theta^{-1}\prod_{a=1}^{\mathsf{N}}\delta_a\right) + \sum_{i=1}^{\mathsf{N}-1}\mathsf{D}_i\lambda^{\mathsf{N}-2i}, \tag{2.12}$$

[13] The proof of the lemma is given following the same steps of that of Proposition 6 of [54].

with $\mathsf{A}_i$ and $\mathsf{D}_i$ being operators, and so

$$\lim_{\log\lambda\rightarrow\mp\infty}\lambda^{\pm\mathsf{N}}\tau_2(\lambda)=\left(\Theta^{\mp 1}a_{\mp}+\Theta^{\pm 1}d_{\mp}\right), \tag{2.13}$$

where $\lim_{\log\lambda\rightarrow-\infty}$ means $\lim_{\lambda\rightarrow 0}$, $\lim_{\log\lambda\rightarrow+\infty}$ means $\lim_{\lambda\rightarrow\infty}$ and:

$$a_+\equiv\prod_{a=1}^{\mathsf{N}}\alpha_a,\quad a_-\equiv(-1)^{\mathsf{N}}\prod_{a=1}^{\mathsf{N}}\beta_a,\quad d_+\equiv(-1)^{\mathsf{N}}\prod_{a=1}^{\mathsf{N}}\delta_a,\quad d_-\equiv\prod_{a=1}^{\mathsf{N}}\gamma_a. \tag{2.14}$$

We only consider here representations for which the Weyl algebra generators $\mathsf{u}_n$ and $\mathsf{v}_n$ are unitary operators; then, the following Hermitian conjugation properties of the generators of Yang–Baxter algebra hold:

**Lemma 2.2** (Lemma 2 of [118]). *Let $\epsilon\in\{+1,-1\}$, then under the following constrains on the parameters:*

$$\mathbb{c}_n=-\epsilon\mathbb{b}_n^*,\quad \mathbb{d}_n=-\epsilon\mathbb{a}_n^*,\quad \beta_n=\epsilon\left(\mathbb{a}_n^*\mathbb{b}_n\right)/\alpha_n^*, \tag{2.15}$$

*the generators of the Yang–Baxter algebra satisfy the following transformations under Hermitian conjugation:*

$$\mathsf{M}(\lambda)^\dagger\equiv\begin{pmatrix}\mathsf{A}^\dagger(\lambda) & \mathsf{B}^\dagger(\lambda)\\ \mathsf{C}^\dagger(\lambda) & \mathsf{D}^\dagger(\lambda)\end{pmatrix}=\begin{pmatrix}\mathsf{D}(\lambda^*) & -\epsilon\mathsf{C}(\lambda^*)\\ -\epsilon\mathsf{B}(\lambda^*) & \mathsf{A}(\lambda^*)\end{pmatrix}, \tag{2.16}$$

*which, in particular, imply the self-adjointness of the transfer matrix $\tau_2(\lambda)$ for real $\lambda$.*

### 2.2. General Cyclic Representations

Here, we will consider general cyclic representations for which $\mathsf{v}_n$ and $\mathsf{u}_n$ have discrete spectra, and we will restrict our study to the case where $q$ is a root of unity:

$$q=\mathrm{e}^{-i\pi\beta^2},\quad \beta^2=\frac{p'}{p},\quad p,p'\in\mathbb{Z}^{>0}, \tag{2.17}$$

with $p$ odd, $p=2l+1$, and $p'$ even being two co-prime numbers so that $q^p=1$. The condition (2.17) implies that the powers $p$ of the generators $\mathsf{u}_n$ and $\mathsf{v}_n$ are central elements of each Weyl algebra $\mathcal{W}_n$. In this case, we fix them to the identity:

$$\mathsf{v}_n^p=1,\quad \mathsf{u}_n^p=1. \tag{2.18}$$

We associate to any site $n$ of the chain a $p$-dimensional linear space $R_n$ ; we can define on it the following cyclic representation of $\mathcal{W}_n$:

$$\mathsf{v}_n|k_n\rangle\equiv q^{k_n}|k_n\rangle,\quad \mathsf{u}_n|k_n\rangle\equiv|k_n-1\rangle,\qquad\forall k_n\in\{0,\ldots,p-1\}, \tag{2.19}$$

with the following cyclic condition:

$$|k_n+p\rangle\equiv|k_n\rangle. \tag{2.20}$$

The vectors $|k_n\rangle$ give a $\mathsf{v}_n$-eigenbasis of the local space $R_n$. Let $\mathrm{L}_n$ be the linear space dual of $R_n$ and let $\langle k_n|$ be the vectors of the dual basis defined by:

$$\langle k_n|k_n'\rangle=(|k_n\rangle,|k_n'\rangle)\equiv\delta_{k_n,k_n'}\quad\forall k_n,k_n'\in\{0,\ldots,p-1\}. \tag{2.21}$$

The generators $\mathsf{u}_n$ and $\mathsf{v}_n$ being unitary, the covectors $\langle k_n|$ define a $\mathsf{v}_n$-eigenbasis in the dual space $\mathrm{L}_n$. This induces the following left representation of Weyl algebra $\mathcal{W}_n$:

$$\langle k_n|\mathsf{v}_n = q^{k_n}\langle k_n|, \quad \langle k_n|\mathsf{u}_n = \langle k_n+1|, \qquad \forall k_n \in \{0,\ldots,p-1\}, \tag{2.22}$$

with the cyclic condition:

$$\langle k_n| = \langle k_n + p|. \tag{2.23}$$

In the *left* and *right* linear spaces:

$$\mathcal{L}_{\mathsf{N}} \equiv \otimes_{n=1}^{\mathsf{N}}\mathrm{L}_n, \quad \mathcal{R}_{\mathsf{N}} \equiv \otimes_{n=1}^{\mathsf{N}}\mathrm{R}_n, \tag{2.24}$$

these representations of the Weyl algebras $\mathcal{W}_n$ determine left and right cyclic representations of dimension $p^{\mathsf{N}}$ of the monodromy matrix elements, and therefore of the Yang–Baxter algebra. In the following, we will denote with $\mathcal{R}_{\mathsf{N}}^{\text{S-adj}}$ the sub-variety of the space of representations $\mathcal{R}_{\mathsf{N}}$ defined by the condition (2.15).

**2.2.1. Centrality of Operator Averages.** We define the average value $\mathcal{O}$ of any operator matrix element $\mathsf{O}$ of the monodromy matrix $\mathsf{M}(\lambda)$ by

$$\mathcal{O}(\Lambda) = \prod_{k=1}^{p}\mathsf{O}(q^k\lambda), \quad \Lambda = \lambda^p, \tag{2.25}$$

then the commutativity of each family of operators $\mathsf{A}(\lambda)$, $\mathsf{B}(\lambda)$, $\mathsf{C}(\lambda)$ and $\mathsf{D}(\lambda)$ implies that the corresponding average values are functions of $\Lambda$.

**Proposition 2.1** (Proposition 1 of [118]).

(a) *The average values of the monodromy matrix entries, $\mathcal{A}(\Lambda)$, $\mathcal{B}(\Lambda)$ , $\mathcal{C}(\Lambda)$, $\mathcal{D}(\Lambda)$, are central elements. They also satisfy, in the case of self-adjoint representations $\mathcal{R}_{\mathsf{N}}^{S\text{-}adj}$, the following relations under complex conjugation:*

$$(\mathcal{A}(\Lambda))^* \equiv \mathcal{D}(\Lambda^*), \quad (\mathcal{B}(\Lambda))^* \equiv -\epsilon\mathcal{C}(\Lambda^*), \tag{2.26}$$

(b) *Let*

$$\mathcal{M}(\Lambda) \equiv \begin{pmatrix} \mathcal{A}(\Lambda) & \mathcal{B}(\Lambda) \\ \mathcal{C}(\Lambda) & \mathcal{D}(\Lambda) \end{pmatrix} \tag{2.27}$$

*be the 2 × 2 matrix made of the average values of the elements of the monodromy matrix $\mathsf{M}(\lambda)$, then it holds:*

$$\mathcal{M}(\Lambda) = \mathcal{L}_{\mathsf{N}}(\Lambda)\,\mathcal{L}_{\mathsf{N}-1}(\Lambda)\,\ldots\,\mathcal{L}_1(\Lambda), \tag{2.28}$$

*where:*

$$\mathcal{L}_n(\Lambda) \equiv \begin{pmatrix} \Lambda\alpha_n^p - \beta_n^p/\Lambda & q^{p/2}(\mathbb{a}_n^p + \mathbb{b}_n^p) \\ q^{p/2}(\mathbb{c}_n^p + \mathbb{d}_n^p) & \gamma_n^p/\Lambda - \Lambda\delta_n^p \end{pmatrix}, \tag{2.29}$$

*is the 2 × 2 matrix made of the average values of the elements of the Lax matrix $\mathsf{L}_n(\lambda)$.*

**2.2.2. Quantum Determinant.** The following linear combination of products of the Yang–Baxter generators:

$$\det_{\mathrm{q}} \mathsf{M}(\lambda) \equiv \mathsf{A}(\lambda)\mathsf{D}(\lambda/q) - \mathsf{B}(\lambda)\mathsf{C}(\lambda/q), \tag{2.30}$$

is called quantum determinant and it is central[14] in this algebra. It admits the following factorized form:

$$\det_{\mathrm{q}} \mathsf{M}(\lambda) = \prod_{n=1}^{\mathsf{N}} \det_{\mathrm{q}} \mathsf{L}_n(\lambda), \tag{2.31}$$

in terms of the local quantum determinants:

$$\det_{\mathrm{q}} \mathsf{L}_n(\lambda) \equiv (\mathsf{L}_n(\lambda))_{11} (\mathsf{L}_n(\lambda/q))_{22} - (\mathsf{L}_n)_{12} (\mathsf{L}_n)_{21} . \tag{2.32}$$

In the Bazhanov–Stroganov model, it reads:

$$\begin{aligned}\det_{\mathrm{q}} \mathsf{M}(\lambda) &= \prod_{n=1}^{\mathsf{N}} k_n \left(\frac{\lambda}{\mu_{n,+}} - \frac{\mu_{n,+}}{\lambda}\right)\left(\frac{\lambda}{\mu_{n,-}} - \frac{\mu_{n,-}}{\lambda}\right) \\ &= (-q)^{\mathsf{N}} \prod_{n=1}^{\mathsf{N}} \frac{\beta_n \mathbb{a}_n \mathbb{c}_n}{\alpha_n} \left(\frac{1}{\lambda} + q^{-1} \frac{\mathbb{b}_n \alpha_n}{\mathbb{a}_n \beta_n}\lambda\right)\left(\frac{1}{\lambda} + q^{-1} \frac{\mathbb{d}_n \alpha_n}{\mathbb{c}_n \beta_n}\lambda\right), \end{aligned} \tag{2.33}$$

where:

$$k_n \equiv (\mathbb{a}_n \mathbb{b}_n \mathbb{c}_n \mathbb{d}_n)^{1/2}, \quad \mu_{n,h} \equiv \begin{cases} iq^{1/2} (\mathbb{a}_n \beta_n / \alpha_n \mathbb{b}_n)^{1/2} & h = +, \\ iq^{1/2} (\mathbb{c}_n \beta_n / \alpha_n \mathbb{d}_n)^{1/2} & h = -. \end{cases} \tag{2.34}$$

Moreover, for the representations that satisfy (2.15), the quantum determinant reads:[15]

$$\det_{\mathrm{q}} \mathsf{M}(\lambda) = q^{\mathsf{N}} \prod_{n=1}^{\mathsf{N}} \frac{|\mathbb{a}_n|^2 |\mathbb{b}_n|^2}{|\alpha_n|^2} \left(\frac{1}{\lambda} + \epsilon q^{-1} \frac{|\alpha_n|^2}{|\mathbb{a}_n|^2}\lambda\right)\left(\frac{1}{\lambda} + \epsilon q^{-1} \frac{|\alpha_n|^2}{|\mathbb{b}_n|^2}\lambda\right). \tag{2.35}$$

Let us define the following functions that will be crucial in the rest of the paper:

$$\bar{\mathrm{A}}(\lambda) \equiv \alpha(\lambda) \mathrm{A}(\lambda), \quad \bar{\mathrm{D}}(\lambda) \equiv \alpha^{-1}(q\lambda)\, \mathrm{D}(\lambda) \tag{2.36}$$

where:

$$\begin{aligned} \mathrm{A}(\lambda) &\equiv \prod_{n=1}^{\mathsf{N}} (\beta_n \alpha_n)^{1/2} \left(\frac{\lambda}{\mu_{n,+}} - \frac{\mu_{n,+}}{\lambda}\right), \\ \mathrm{D}(\lambda) &\equiv \prod_{n=1}^{\mathsf{N}} \left(\frac{\mathbb{a}_n \mathbb{b}_n \mathbb{c}_n \mathbb{d}_n}{\alpha_n \beta_n}\right)^{1/2} \left(\frac{q\lambda}{\mu_{n,-}} - \frac{\mu_{n,-}}{q\lambda}\right). \end{aligned} \tag{2.37}$$

They always satisfy the condition:

$$\det_{\mathrm{q}} \mathsf{M}(\lambda) = \bar{\mathrm{A}}(\lambda) \bar{\mathrm{D}}(\lambda/q), \tag{2.38}$$

[14] The centrality of the quantum determinant in the Yang–Baxter algebra was first discovered in [125].

[15] Remark that it depends on the parameters in Lax operators only through their modules.

while the function $\alpha(\lambda)$ is defined by the requirement:

$$\prod_{n=1}^{p} \bar{\mathrm{A}}(\lambda q^n) + \prod_{n=1}^{p} \bar{\mathrm{D}}(\lambda q^n) = \mathcal{A}(\Lambda) + \mathcal{D}(\Lambda). \tag{2.39}$$

Note that this last condition is a second-order equation in the average $\prod_{n=1}^{p} \alpha(q^n \lambda)$ and then we have only two possible choices for the averages of the functions $\bar{\mathrm{A}}(\lambda)$ and $\bar{\mathrm{D}}(\lambda)$:

$$\prod_{n=1}^{p} \bar{\mathrm{A}}(\lambda q^n) = \Omega_{\epsilon}(\Lambda), \quad \prod_{n=1}^{p} \bar{\mathrm{D}}(\lambda q^n) = \Omega_{-\epsilon}(\Lambda), \tag{2.40}$$

where $\epsilon = \mp$ and $\Omega_{\pm}$ are the two eigenvalues of the $2 \times 2$ matrix $\mathcal{M}(\Lambda)$ composed by the averages of the Yang–Baxter generators.

### 2.3. SOV-Representations and the Yang–Baxter Algebra

The spectral problem of the transfer matrix $\tau_2(\lambda)$ admits a separate variables representation in the basis which diagonalizes the commutative family of operators $\mathsf{B}(\lambda)$ as generally argued by Sklyanin [15–17]. In [118], it has been proven:

**Theorem 2.1** (Theorem 1 of [118]). *For almost all the values of the parameters of the representation, there exists a SOV representation for the Bazhanov–Stroganov model; in this case,* $\mathsf{B}(\lambda)$ *is diagonalizable and has simple spectrum.*

Let us recall here the left SOV-representations of the generators of the Yang–Baxter algebra for the Bazhanov–Stroganov model. Let $\langle\, \boldsymbol{\eta}_{\mathbf{k}} \,|$ be the generic element of a basis of eigenvectors of $\mathsf{B}(\lambda)$:

$$\langle\, \boldsymbol{\eta}_{\mathbf{k}} \,|\mathsf{B}(\lambda) = \eta_{\mathsf{N}}\, b_{\boldsymbol{\eta}_{\mathbf{k}}}(\lambda) \,\langle\, \boldsymbol{\eta}_{\mathbf{k}} \,|, \quad b_{\boldsymbol{\eta}_{\mathbf{k}}}(\lambda) \equiv \prod_{a=1}^{\mathsf{N}-1} \left(\lambda/\eta_a^{(k_a)} - \eta_a^{(k_a)}/\lambda\right), \tag{2.41}$$

and

$$\boldsymbol{\eta}_{\mathbf{k}} \in \mathsf{Z}_{\mathsf{B}} \equiv \left\{ (\eta_1^{(k_1)} \equiv q^{k_1}\eta_1^{(0)}, \ldots, \eta_{\mathsf{N}}^{(k_{\mathsf{N}})} \equiv q^{k_{\mathsf{N}}}\eta_{\mathsf{N}}^{(0)})\,;\, \mathbf{k} \equiv (k_1, \ldots, k_{\mathsf{N}}) \in \mathbb{Z}_p^{\mathsf{N}} \right\}, \tag{2.42}$$

where $\eta_a^{(0)}$ are fixed constants[16] of the representations. For simplicity, whenever possible we will omit the subscript $\mathbf{k}$ in $\langle \boldsymbol{\eta}_{\mathbf{k}} \,|$ as well as the superscript $k_a$ in $\eta_a^{(k_a)}$. The action of the remaining generators of the Yang–Baxter algebra on arbitrary states $\langle\, \boldsymbol{\eta}| \equiv \langle\, \eta_1, \ldots, \eta_{\mathsf{N}}|$ reads:

$$\begin{aligned} \langle\, \boldsymbol{\eta} \,|\mathsf{A}(\lambda) = {} & b_{\boldsymbol{\eta}}(\lambda) \left[\lambda \eta_{\mathsf{A}}^{(+)} \langle\, q^{-\delta_{\mathsf{N}}} \boldsymbol{\eta} \,| + \lambda^{-1} \eta_{\mathsf{A}}^{(-)} \langle\, q^{\delta_{\mathsf{N}}} \boldsymbol{\eta} \,|\right] \\ & + \sum_{a=1}^{\mathsf{N}-1} \prod_{b \neq a} \frac{\lambda/\eta_b - \eta_b/\lambda}{\eta_a/\eta_b - \eta_b/\eta_a}\, \mathsf{a}^{(\mathrm{SOV})}(\eta_a) \,\langle\, q^{-\delta_a} \boldsymbol{\eta} \,|, \end{aligned} \tag{2.43}$$

[16] Here, the simplicity of the spectrum of $\mathsf{B}(\lambda)$ is equivalent to the requirement $\left(\eta_a^{(0)}\right)^p \neq \left(\eta_b^{(0)}\right)^p$ for any $a \neq b \in \{1, \ldots, \mathsf{N}-1\}$.

$$\langle\, \boldsymbol{\eta}\,|\mathsf{D}(\lambda) = b_{\boldsymbol{\eta}}(\lambda)\left[\lambda\eta_{\mathsf{D}}^{(+)}\langle\, q^{\delta_{\mathsf{N}}}\boldsymbol{\eta}\,| + \lambda^{-1}\eta_{\mathsf{D}}^{(-)}\langle\, q^{-\delta_{\mathsf{N}}}\boldsymbol{\eta}\,|\right] + \sum_{a=1}^{\mathsf{N}-1}\prod_{b\neq a}\frac{\lambda/\eta_b-\eta_b/\lambda}{\eta_a/\eta_b-\eta_b/\eta_a}\,\mathtt{d}^{(\mathrm{SOV})}(\eta_a)\,\langle\, q^{\delta_a}\boldsymbol{\eta}\,|, \tag{2.44}$$

where:

$$\eta_{\mathsf{A}}^{(\pm)} = (\pm 1)^{\mathsf{N}-1}a_{\pm}\prod_{n=1}^{\mathsf{N}-1}\eta_n^{\pm 1}, \qquad \eta_{\mathsf{D}}^{(\pm)} = (\pm 1)^{\mathsf{N}-1}d_{\pm}\prod_{n=1}^{\mathsf{N}-1}\eta_n^{\pm 1}, \tag{2.45}$$

and the states $\langle\, q^{\pm\delta_a}\boldsymbol{\eta}\,|$ are defined by:

$$\langle\, q^{\pm\delta_a}\boldsymbol{\eta}\,| \equiv \langle\, \eta_1,\ldots,q^{\pm 1}\eta_a,\ldots,\eta_{\mathsf{N}}\,|. \tag{2.46}$$

Finally, the quantum determinant relation defines uniquely $\mathsf{C}(\lambda)$. The expressions (2.43) and (2.44) contain complex-valued coefficients $\mathtt{a}^{(\mathrm{SOV})}(\eta_a)$ and $\mathtt{d}^{(\mathrm{SOV})}(\eta_a)$ which completely characterize the SOV representation. These coefficients have to be solution of the quantum determinant conditions:

$$\det{}_{\mathrm{q}}\mathsf{M}(\eta_r) = \mathtt{a}^{(\mathrm{sov})}(\eta_r)\mathtt{d}^{(\mathrm{sov})}(q^{-1}\eta_r), \quad \forall r = 1,\ldots,\mathsf{N}-1, \tag{2.47}$$

and of the average conditions:

$$\begin{aligned}\mathcal{A}(Z_r) &\equiv \prod_{k=1}^{p}\mathtt{a}^{(\mathrm{sov})}(q^k\eta_r),\\ \mathcal{D}(Z_r) &\equiv \prod_{k=1}^{p}\mathtt{d}^{(\mathrm{sov})}(q^k\eta_r),\\ Z_r &\equiv \eta_r^p, \quad \forall r\in\{1,\ldots,\mathsf{N}-1\}.\end{aligned} \tag{2.48}$$

In a SOV representation, some freedom is left in the choice of $\mathtt{a}^{(\mathrm{sov})}(\eta_r)$ and $\mathtt{d}^{(\mathrm{sov})}(\eta_r)$. It can be parametrized by the gauge transformation written in terms of an arbitrary function $f$:

$$\tilde{\mathtt{a}}^{(\mathrm{sov})}(\eta_r) = \mathtt{a}^{(\mathrm{sov})}(\eta_r)\frac{f(\eta_r q^{-1})}{f(\eta_r)}, \quad \tilde{\mathtt{d}}^{(\mathrm{sov})}(\eta_r) = \mathtt{d}^{(\mathrm{sov})}(\eta_r)\frac{f(\eta_r q)}{f(\eta_r)}\,; \tag{2.49}$$

which just amounts to the following change of normalization for the states of the $\mathsf{B}$-eigenbasis:

$$\langle\, \boldsymbol{\eta}\,| \to \prod_{r=1}^{\mathsf{N}-1}f^{-1}(\eta_r)\langle\, \boldsymbol{\eta}\,|. \tag{2.50}$$

Similarly, we can construct a right SOV-representation of the Yang–Baxter generators by the following actions:

$$\mathsf{B}(\lambda)|\boldsymbol{\eta}\rangle = |\boldsymbol{\eta}\rangle\eta_{\mathsf{N}}b_{\boldsymbol{\eta}}(\lambda), \tag{2.51}$$

$$\mathsf{A}(\lambda)|\boldsymbol{\eta}\rangle = \left[|q^{\delta_{\mathsf{N}}}\boldsymbol{\eta}\rangle\eta_{\mathsf{A}}^{(+)}\lambda + |q^{-\delta_{\mathsf{N}}}\boldsymbol{\eta}\rangle\frac{\eta_{\mathsf{A}}^{(-)}}{\lambda}\right]b_{\boldsymbol{\eta}}(\lambda) + \sum_{a=1}^{\mathsf{N}-1}|q^{\delta_a}\boldsymbol{\eta}\rangle\prod_{b\neq a}\frac{(\lambda/\eta_b-\eta_b/\lambda)}{(\eta_a/\eta_b-\eta_b/\eta_a)}\bar{\mathtt{a}}^{(\mathrm{sov})}(\eta_a), \tag{2.52}$$

$$\mathsf{D}(\lambda)|\boldsymbol{\eta}\rangle = \left[|q^{-\delta_{\mathsf{N}}}\boldsymbol{\eta}\rangle\eta_{\mathsf{D}}^{(+)}\lambda + |q^{\delta_{\mathsf{N}}}\boldsymbol{\eta}\rangle\frac{\eta_{\mathsf{D}}^{(-)}}{\lambda}\right] b_{\boldsymbol{\eta}}(\lambda)$$
$$+\sum_{a=1}^{\mathsf{N}-1}|q^{-\delta_a}\boldsymbol{\eta}\rangle\prod_{b\neq a}\frac{(\lambda/\eta_b-\eta_b/\lambda)}{(\eta_a/\eta_b-\eta_b/\eta_a)}\bar{\mathsf{d}}^{(\mathrm{sov})}(\eta_a), \tag{2.53}$$

where $|\boldsymbol{\eta}\rangle \in \mathcal{R}_{\mathsf{N}}$ is the right $\mathsf{B}$-eigenstate corresponding to the generic $\boldsymbol{\eta} \in \mathsf{Z}_{\mathsf{B}}$. The coefficients $\bar{\mathsf{a}}^{(\mathrm{sov})}(\eta_a)$ and $\bar{\mathsf{d}}^{(\mathrm{sov})}(\eta_a)$ are solutions of the same average (2.48) and quantum determinant:

$$\det{}_{\mathrm{q}}\mathsf{M}(\eta_r) = \bar{\mathsf{d}}^{(\mathrm{sov})}(\eta_r)\bar{\mathsf{a}}^{(\mathrm{sov})}(q^{-1}\eta_r), \quad \forall r = 1,\ldots,\mathsf{N}-1 \tag{2.54}$$

conditions, while $\mathsf{C}(\lambda)$ is uniquely defined by the quantum determinant relation (2.30).

**2.3.1. SOV-Decomposition of the Identity.** The diagonalizability of the Yang–Baxter generator $\mathsf{B}(\lambda)$ and the simplicity of its spectrum imply the following spectral decomposition of the identity $\mathbb{I}$ in terms of the $\mathsf{B}$-eigenbasis:

$$\mathbb{I} \equiv \sum_{\mathbf{k}\in\mathbb{Z}_p^{\mathsf{N}}}\mu_{\mathbf{k}}|\boldsymbol{\eta}_{\mathbf{k}}\rangle\langle\boldsymbol{\eta}_{\mathbf{k}}|, \tag{2.55}$$

where:

$$\mu_{\mathbf{k}} \equiv \langle\boldsymbol{\eta}_{\mathbf{k}}|\boldsymbol{\eta}_{\mathbf{k}}\rangle^{-1} \quad \forall\,\mathbf{k}\in\mathbb{Z}_p^{\mathsf{N}}, \tag{2.56}$$

is the equivalent of the so-called Sklyanin's measure.[17] The non-Hermitian character of the operator family $\mathsf{B}(\lambda)$ clearly implies that, for generic $\mathbf{k}\in\mathbb{Z}_p^{\mathsf{N}}$, $(|\boldsymbol{\eta}_{\mathbf{k}}\rangle)^{\dagger}$ and $\langle\boldsymbol{\eta}_{\mathbf{k}}|$ are in general non-equal covectors in $\mathcal{L}_{\mathsf{N}}$; then, $\mu_{\mathbf{k}}$ is not a standard positive definite measure in our cyclic representations. Nevertheless, we will show that the above formula defines a proper orthogonal decomposition of the identity operator.

Now, we compute[18] this "measure" $\mu_{\mathbf{k}}$ and we show that up to an overall constant (i.e. a constant w.r.t. $\mathbf{k}\in\mathbb{Z}_p^{\mathsf{N}}$), it is completely fixed by the given left and right SOV-representations of the Yang–Baxter algebras when the gauges are fixed.

**Proposition 2.2.** *The following identities hold:*

$$\langle\boldsymbol{\eta}_{\mathbf{k}}|\boldsymbol{\eta}_{\mathbf{h}}\rangle = \langle\boldsymbol{\eta}_{\mathbf{h}}|\boldsymbol{\eta}_{\mathbf{h}}\rangle\prod_{j=1}^{\mathsf{N}}\delta_{k_i,h_i}, \quad \forall\,\mathbf{k},\mathbf{h}\in\mathbb{Z}_p^{\mathsf{N}}, \tag{2.57}$$

$$\mu_{\mathbf{h}} = \frac{\prod_{1\leq a<b\leq\mathsf{N}-1}\left(\left(\eta_a^{(h_a)}\right)^2-\left(\eta_b^{(h_b)}\right)^2\right)}{C_{\mathsf{N}}\prod_{a=1}^{\mathsf{N}-1}\omega_a\left(\eta_a^{(h_a)}\right)}, \quad \forall\,\mathbf{h}\in\mathbb{Z}_p^{\mathsf{N}}, \tag{2.58}$$

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[17] Sklyanin's measure has been first introduced by Sklyanin in his article [15] on quantum Toda chain [127–129]; see also [130,131] for further discussions on the measure in the quantum Toda chain and in the sinh–Gordon model, respectively.

[18] Let us recall that this measure has been first derived in [115] for cyclic representations of Bazhanov–Stroganov model [57–59] through the recursion in the construction of left and right SOV-basis.

*where:*

$$\omega_a(\eta_a^{(h_a)}) \equiv \left(\eta_a^{(h_a)}\right)^{\mathsf{N}-1} \prod_{l_a=1}^{h_a} \mathsf{a}^{(\mathrm{sov})}\left(\eta_a^{(l_a)}\right)/\bar{\mathsf{a}}^{(\mathrm{sov})}\left(\eta_a^{(l_a-1)}\right) \tag{2.59}$$

*are gauge-dependent parameters and $C_{\mathsf{N}}$ in the formula for $\mu_{\mathbf{h}}$ is a constant w.r.t. $\mathbf{h} \in \mathbb{Z}_p^{\mathsf{N}}$. Then, the SOV-decomposition of the identity explicitly reads:*

$$\begin{aligned}\mathbb{I} \equiv \sum_{h_1,\ldots,h_{\mathsf{N}}=1}^{p} \prod_{1\le a<b\le \mathsf{N}-1} &\left(\left(\eta_a^{(h_a)}\right)^2 - \left(\eta_b^{(h_b)}\right)^2\right)\\ &\times \frac{\left|\eta_1^{(h_1)},\ldots,\eta_{\mathsf{N}}^{(h_{\mathsf{N}})}\rangle\langle\eta_1^{(h_1)},\ldots,\eta_{\mathsf{N}}^{(h_{\mathsf{N}})}\right|}{C_{\mathsf{N}}\prod_{b=1}^{\mathsf{N}-1}\omega_b(\eta_b^{(h_b)})},\end{aligned} \tag{2.60}$$

*Note that the constant $C_{\mathsf{N}}$ can be put equal to one by a trivial (constant) gauge transformation that does not affect the functions $\mathsf{a}^{(\mathrm{sov})}$ and $\bar{\mathsf{a}}^{(\mathrm{sov})}$.*

*Proof.* Computing in two different ways $\langle\boldsymbol{\eta}_{\mathbf{k}}|\mathsf{B}(\lambda)|\boldsymbol{\eta}_{\mathbf{h}}\rangle$, we get:

$$(b_{\boldsymbol{\eta}_{\mathbf{k}}}(\lambda) - b_{\boldsymbol{\eta}_{\mathbf{h}}}(\lambda))\langle\boldsymbol{\eta}_{\mathbf{k}}|\boldsymbol{\eta}_{\mathbf{h}}\rangle = 0 \quad \forall\lambda\in\mathbb{C},\ \forall\,\mathbf{k},\mathbf{h}\in\mathbb{Z}_p^{\mathsf{N}} \tag{2.61}$$

and then the simplicity of the spectrum of $\mathsf{B}(\lambda)$ implies (2.57). To compute $\mu_{\mathbf{h}}$, we compute the following matrix elements $\theta_a \equiv \langle\eta_1^{(h_1)},\ldots,\eta_a^{(h_a-1)},\ldots,\eta_{\mathsf{N}}^{(h_{\mathsf{N}})}|\mathsf{A}(\eta_a^{(h_a-1)})|\eta_1^{(h_1)},\ldots,\eta_a^{(h_a)},\ldots,\eta_{\mathsf{N}}^{(h_{\mathsf{N}})}\rangle$, using first the left action of $\mathsf{A}(\eta_a^{(h_a-1)})$, then the right action of $\mathsf{A}(\eta_a^{(h_a-1)})$ together with (2.57) and finally equating the two results we get:

$$\begin{aligned}&\frac{\left\langle\eta_1^{(h_1)},\ldots,\eta_a^{(h_a)},\ldots,\eta_{\mathsf{N}}^{(h_{\mathsf{N}})}|\eta_1^{(h_1)},\ldots,\eta_a^{(h_a)},\ldots,\eta_{\mathsf{N}}^{(h_{\mathsf{N}})}\right\rangle}{\left\langle\eta_1^{(h_1)},\ldots,\eta_a^{(h_a-1)},\ldots,\eta_{\mathsf{N}}^{(h_{\mathsf{N}})}|\eta_1^{(h_1)},\ldots,\eta_a^{(h_a-1)},\ldots,\eta_{\mathsf{N}}^{(h_{\mathsf{N}})}\right\rangle}\\ &\quad= \delta_{a,\mathsf{N}} + (1-\delta_{a,\mathsf{N}})\frac{\mathsf{a}^{(\mathrm{sov})}\left(\eta_a^{(h_a)}\right)}{\bar{\mathsf{a}}^{(\mathrm{sov})}\left(\eta_a^{(h_a-1)}\right)}\\ &\qquad\times\prod_{b\neq a,b=1}^{\mathsf{N}-1}\frac{\left(\eta_a^{(h_a-1)}/\eta_b^{(h_b)} - \eta_b^{(h_b)}/\eta_a^{(h_a-1)}\right)}{\left(\eta_a^{(h_a)}/\eta_b^{(h_b)} - \eta_b^{(h_b)}/\eta_a^{(h_a)}\right)},\end{aligned} \tag{2.62}$$

from which (2.58) simply follows. □

### 2.4. SOV-Characterization of the Spectrum

Let us denote with $\Sigma_{\tau_2}$ the set of eigenvalue functions $t(\lambda)$ of the transfer matrix $\tau_2(\lambda)$. We have then:

$$\Sigma_{\tau_2} \subset \mathbb{C}_{\mathrm{even}}[\lambda,\lambda^{-1}]_{\mathsf{N}} \text{ for } \mathsf{N} \text{ even}, \quad \Sigma_{\tau_2} \subset \mathbb{C}_{odd}[\lambda,\lambda^{-1}]_{\mathsf{N}} \text{ for } \mathsf{N} \text{ odd}, \tag{2.63}$$

where $\mathbb{C}_\epsilon[x,x^{-1}]_{\mathsf{M}}$ denotes the linear space in the field $\mathbb{C}$ of the Laurent polynomials of degree $\mathsf{M}$ in the variable $x$ which are even or odd as stated in the index $\epsilon$. The $\Theta$-charge naturally induces the grading $\Sigma_{\tau_2} = \bigcup_{k=0}^{2l} \Sigma_{\tau_2}^k$, where:

$$\Sigma_{\tau_2}^k \equiv \left\{ t(\lambda) \in \Sigma_{\tau_2} : \lim_{\log \lambda \to \mp\infty} \lambda^{\pm \mathsf{N}} t(\lambda) = \left( q^{\mp k} a_{\mp} + q^{\pm k} d_{\mp} \right) \right\}. \tag{2.64}$$

This simply follows from the commutativity of $\tau_2(\lambda)$ with $\Theta$ and from its asymptotics. In particular, any $t_k(\lambda) \in \Sigma_{\tau_2}^k$ is a $\tau_2$-eigenvalue corresponding to simultaneous eigenstates of $\tau_2(\lambda)$ and $\Theta$ with $\Theta$-eigenvalue $q^k$.

**2.4.1. Eigenvalues and Wave-Functions.** In the SOV representations, the spectral problem for $\tau_2(\lambda)$ is reduced to the following discrete system of Baxter-like equations in the wave-function $\Psi_t(\boldsymbol{\eta}) \equiv \langle \boldsymbol{\eta} \,|\, t \rangle$ of a $\tau_2$-eigenstate $|\, t \rangle$:

$$t(\eta_r)\Psi_t(\boldsymbol{\eta}) = \mathsf{a}^{(\mathrm{sov})}(\eta_r)\Psi_t(q^{-\delta_r}\boldsymbol{\eta}) + \mathsf{d}^{(\mathrm{sov})}(\eta_r)\Psi_t(q^{\delta_r}\boldsymbol{\eta}) \quad \forall r \in \{1, \ldots, \mathsf{N}-1\}, \tag{2.65}$$

plus the following equation in the variable $\eta_{\mathsf{N}}$:

$$\Psi_t(q^{\delta_{\mathsf{N}}}\boldsymbol{\eta}) = q^{-k}\Psi_t(\boldsymbol{\eta}), \quad \text{where} \quad q^{\pm\delta_r}\boldsymbol{\eta} \equiv (\eta_1, \ldots, q^{\pm 1}\eta_r, \ldots, \eta_{\mathsf{N}}), \tag{2.66}$$

for $t(\lambda) \in \Sigma_{\tau_2}^k$ with $k \in \{0, \ldots, 2l\}$. Let us introduce the one parameter family $D(\lambda)$ of $p \times p$ matrix:

$$D(\lambda) \equiv \begin{pmatrix} t(\lambda) & -\bar{\mathrm{D}}(\lambda) & 0 & \cdots & 0 & -\bar{\mathrm{A}}(\lambda) \\ -\bar{\mathrm{A}}(q\lambda) & t(q\lambda) & -\bar{\mathrm{D}}(q\lambda) & 0 & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ \vdots & & \cdots & & & \vdots \\ \vdots & & & \cdots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & 0 & -\bar{\mathrm{A}}(q^{2l-1}\lambda) & t(q^{2l-1}\lambda) & -\bar{\mathrm{D}}(q^{2l-1}\lambda) \\ -\bar{\mathrm{D}}(q^{2l}\lambda) & 0 & \cdots & 0 & -\bar{\mathrm{A}}(q^{2l}\lambda) & t(q^{2l}\lambda) \end{pmatrix} \tag{2.67}$$

then when we make the following choice of gauge for the left SOV-representation:

$$\mathsf{a}^{(\mathrm{sov})}(\lambda) \equiv \bar{\mathrm{A}}(\lambda), \quad \mathsf{d}^{(\mathrm{sov})}(\lambda) \equiv \bar{\mathrm{D}}(\lambda), \tag{2.68}$$

it holds:

**Theorem 2.2** (Theorems 2, 3 and 4 of [118]). *For almost all the values of the parameters of a Bazhanov–Stroganov representation, the spectrum of $\tau_2(\lambda)$ is simple. Moreover:*

(I) $\Sigma_{\tau_2}$ *coincides with the set of functions in (2.63) which are solutions of the functional equation:*

$$\det_p D(\Lambda) = 0, \quad \forall \Lambda \in \mathbb{C}. \tag{2.69}$$

*Then, up to an overall normalization, we can fix the $\tau_2$-eigenstate corresponding to $t_k(\lambda) \in \Sigma_{\tau_2}^k$ by:*

$$\Psi_{t_k}(\boldsymbol{\eta}) \equiv \langle \eta_1, \ldots, \eta_{\mathsf{N}} \,|\, t_k \rangle = \eta_{\mathsf{N}}^{-k} \prod_{r=1}^{\mathsf{N}-1} Q_{t_k}(\eta_r), \tag{2.70}$$

*where $Q_{t_k}(\lambda)$ is the only solution (up to quasi-constants) corresponding to $t_k(\lambda)$ of the Baxter equation:*

$$t_k(\lambda)Q_{t_k}(\lambda) = \bar{\text{A}}(\lambda)Q_{t_k}(\lambda/q) + \bar{\text{D}}(\lambda)Q_{t_k}(q\lambda). \tag{2.71}$$

(II) *In the self-adjoint representations of the Bazhanov–Stroganov model under the further constrains:*

$$\prod_{h=1}^{N} \frac{\alpha_h^*}{\alpha_h} = 1, \quad \frac{\mathbb{b}_n}{\mathbb{b}_n^*} = \frac{\mathbb{a}_n}{\mathbb{a}_n^*}, \quad \frac{\alpha_{n+1}^*\alpha_n^*}{\alpha_{n+1}\alpha_n} = \frac{\mathbb{b}_{n+1}^*\mathbb{b}_n}{\mathbb{b}_{n+1}\mathbb{b}_n^*}, \quad \forall n \in \{1, \ldots, \mathsf{N}\}, \tag{2.72}$$

*the functions $\bar{\text{A}}(\lambda)$ and $\bar{\text{D}}(\lambda)$ are gauge equivalent to the Laurent polynomials [$\epsilon$ being defined as in Eq. (2.15)]:*

$$\mathsf{a}(\lambda) \equiv i^{\mathsf{N}} \prod_{n=1}^{\mathsf{N}} \frac{\beta_n}{\lambda} \left(1 - i^{(1+\epsilon)/2} q^{-1/2} \frac{|\alpha_n|}{|\mathbb{a}_n|}\lambda\right)\left(1 - i^{(1+\epsilon)/2} q^{-1/2} \frac{|\alpha_n|}{|\mathbb{b}_n|}\lambda\right),$$
$$\mathsf{d}(\lambda) \equiv q^{\mathsf{N}}\mathsf{a}(-\lambda q), \tag{2.73}$$

*respectively, and for any $t_k(\lambda) \in \Sigma_{\tau_2}^k$, we can construct uniquely up to quasi-constants a $\epsilon$-real polynomial:*[19,20]

$$Q_{t_k}(\lambda) = \lambda^{a_{t_k}} \prod_{h=1}^{2l\mathsf{N}-(b_{t_k}+a_{t_k})} (\lambda_h - \lambda), \quad 0 \le a_{t_k} \le 2l, \; 0 \le b_{t_k} + a_{t_k} \le 2l\mathsf{N}, \tag{2.74}$$

*which is a solution of the Baxter functional equation (2.71) in the gauge (2.73) and:*

$$a_{t_k} = \pm k \bmod p, \quad b_{t_k} = \pm k \bmod p. \tag{2.75}$$

**2.4.2. Eigenvectors and Eigencovectors.** The SOV-decomposition of the identity (2.60) and the results of the previous subsections imply that the state:

$$|t_k\rangle = \sum_{h_1,\ldots,h_{\mathsf{N}}=1}^{p} \frac{q^{kh_{\mathsf{N}}}}{p^{1/2}} \prod_{a=1}^{\mathsf{N}-1} Q_{t_k}(\eta_a^{(h_a)}) \times \prod_{1\le a<b\le \mathsf{N}-1} \left(\left(\eta_a^{(h_a)}\right)^2 - (\eta_b^{(h_b)})^2\right) \frac{|\eta_1^{(h_1)}, \ldots, \eta_{\mathsf{N}}^{(h_{\mathsf{N}})}\rangle}{\prod_{b=1}^{\mathsf{N}-1} \omega_b\left(\eta_b^{(h_b)}\right)}, \tag{2.76}$$

is, up to an overall normalization, the only right $\tau_2$-eigenstate associated to $t_k(\lambda) \in \Sigma_{\mathsf{T}}^k$. Here, $Q_{t_k}(\lambda)$ is the only solution (up to quasi-constants) of the Baxter equation:

$$t_k(\lambda)Q_{t_k}(\lambda) = \bar{\text{A}}(\lambda)Q_{t_k}(\lambda q^{-1}) + \bar{\text{D}}(\lambda)Q_{t_k}(\lambda q), \tag{2.77}$$

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[19] I.e. it satisfies the following complex-conjugation conditions: $(Q_t(\lambda))^* \equiv Q_t(\epsilon\lambda^*)\ \ \forall\lambda \in \mathbb{C}$.
[20] Note that $Q_t(\lambda)$ has been constructed in terms of the cofactors of the matrix $D(\Lambda)$ in Theorem 3 of [118].

as defined in Theorem 2.2. Similarly, we can prove that the state:

$$\langle t_k| = \sum_{h_1,\dots,h_{\mathsf{N}}=1}^{p} \frac{q^{kh_{\mathsf{N}}}}{p^{1/2}} \prod_{a=1}^{\mathsf{N}-1} \bar{Q}_{t_k}\left(\eta_a^{(h_a)}\right) \times \prod_{1\le a<b\le \mathsf{N}-1} \left(\left(\eta_a^{(h_a)}\right)^2 - (\eta_b^{(h_b)})^2\right) \frac{\langle \eta_1^{(h_1)},\dots,\eta_{\mathsf{N}}^{(h_{\mathsf{N}})}|}{\prod_{b=1}^{\mathsf{N}-1}\omega_b(\eta_b^{(h_b)})}, \tag{2.78}$$

is, up to an overall normalization, the only left $\tau_2$-eigenstate associated to $t_k(\lambda) \in \Sigma_{\mathsf{T}}^k$. Here, $\bar{Q}_{t_k}(\lambda)$ is the only solution (up to quasi-constants) of the Baxter equation:

$$t_k(\lambda)\bar{Q}_{t_k}(\lambda) = \bar{\mathrm{D}}(\lambda/q)\bar{Q}_{t_k}(\lambda/q) + \bar{\mathrm{A}}(\lambda q)\bar{Q}_{t_k}(\lambda q), \tag{2.79}$$

when we make the following choice of gauge for the right SOV-representation:

$$\bar{\mathsf{a}}^{(\mathrm{sov})}(\lambda) \equiv \bar{\mathrm{A}}(\lambda q), \quad \bar{\mathsf{d}}^{(\mathrm{sov})}(\lambda) \equiv \bar{\mathrm{D}}(\lambda/q). \tag{2.80}$$

## 3. The Inhomogeneous Chiral Potts Model

### 3.1. Definitions and First Properties

The connections between the integrable chiral Potts model and the Bazhanov–Stroganov model restricted to parametrization by points on the algebraic curves $\mathcal{C}_k$ were first remarked in [57]. We can summarize them as follows:

(I) the fundamental $R$-matrix intertwining the Bazhanov–Stroganov Lax operator in the quantum space is given by the product of four chiral Potts Boltzmann weights;

(II) the transfer matrix of the chiral Potts model is a Baxter $\mathsf{Q}$-operator for the Bazhanov–Stroganov model.

Let us recall here how the spectrum of the inhomogeneous chiral Potts transfer matrix is characterized by SOV construction, thanks to the property (II). The algebraic curve $\mathcal{C}_k$ of modulus $k$ is by definition the locus of the points $\boldsymbol{f} \equiv (a_{\boldsymbol{f}}, b_{\boldsymbol{f}}, c_{\boldsymbol{f}}, d_{\boldsymbol{f}}) \in \mathbb{C}^4$ which satisfy the equations:

$$x_{\boldsymbol{f}}^p + y_{\boldsymbol{f}}^p = k(1 + x_{\boldsymbol{f}}^p y_{\boldsymbol{f}}^p), \quad k x_{\boldsymbol{f}}^p = 1 - k^{'} s_{\boldsymbol{f}}^{-p}, \quad k y_{\boldsymbol{f}}^p = 1 - k^{'} s_{\boldsymbol{f}}^{p}, \tag{3.1}$$

where:

$$x_{\boldsymbol{f}} \equiv a_{\boldsymbol{f}}/d_{\boldsymbol{f}}, \quad y_{\boldsymbol{f}} \equiv b_{\boldsymbol{f}}/c_{\boldsymbol{f}}, \quad s_{\boldsymbol{f}} \equiv d_{\boldsymbol{f}}/c_{\boldsymbol{f}}, \; t_{\boldsymbol{f}} \equiv x_{\boldsymbol{f}} y_{\boldsymbol{f}}, \quad k^2 + (k^{'})^2 = 1. \tag{3.2}$$

Let us introduce the following cyclic dilogarithm functions;[21] here, we use the notation:

$$\begin{aligned} \frac{W_{\boldsymbol{gf}}(z(n))}{W_{\boldsymbol{gf}}(z(0))} &= \left(\frac{s_{\boldsymbol{g}}}{s_{\boldsymbol{f}}}\right)^n \prod_{k=1}^{n} \frac{y_{\boldsymbol{f}} - q^{-2k} x_{\boldsymbol{g}}}{y_{\boldsymbol{g}} - q^{-2k} x_{\boldsymbol{f}}}, \\ \frac{\bar{W}_{\boldsymbol{gf}}(z(n))}{\bar{W}_{\boldsymbol{gf}}(z(0))} &= (s_{\boldsymbol{f}} s_{\boldsymbol{g}})^n \prod_{k=1}^{n} \frac{q^{-2} x_{\boldsymbol{g}} - q^{-2k} x_{\boldsymbol{f}}}{y_{\boldsymbol{f}} - q^{-2k} y_{\boldsymbol{g}}}, \end{aligned} \tag{3.3}$$

[21] They are the Boltzmann weights of the chiral Potts model [64,65], see also [131–142] for the study of the properties of dilogarithm functions.

where $z(n) = q^{-2n}$, $n \in \{0, \ldots, 2l\}$. They are solutions of the following recursion relations:

$$\begin{aligned} \frac{W_{\boldsymbol{gf}}(zq)}{W_{\boldsymbol{gf}}(zq^{-1})} &= -z\frac{s_{\boldsymbol{f}}}{s_{\boldsymbol{g}}}\frac{x_{\boldsymbol{f}}}{y_{\boldsymbol{f}}}q^{-1}\frac{1-\frac{y_{\boldsymbol{g}}}{x_{\boldsymbol{f}}}qz^{-1}}{1-\frac{x_{\boldsymbol{g}}}{y_{\boldsymbol{f}}}q^{-1}z}, \\ \frac{\bar{W}_{\boldsymbol{gf}}(zq)}{\bar{W}_{\boldsymbol{gf}}(zq^{-1})} &= -\frac{qz^{-1}}{s_{\boldsymbol{f}}s_{\boldsymbol{g}}}\frac{y_{\boldsymbol{f}}}{x_{\boldsymbol{f}}}\frac{1-\frac{y_{\boldsymbol{g}}}{y_{\boldsymbol{f}}}q^{-1}z}{1-\frac{x_{\boldsymbol{g}}}{x_{\boldsymbol{f}}}q^{-1}z^{-1}}. \end{aligned} \tag{3.4}$$

If the points $\boldsymbol{f}$ and $\boldsymbol{g}$ belong to the curves $\mathcal{C}_k$, they are well-defined functions of $z \in \mathbb{S}_p \equiv \{q^{2n}; n = 0, \ldots, 2l\}$ which satisfy the cyclicity condition:

$$\frac{\bar{W}_{\boldsymbol{gf}}(z(p))}{\bar{W}_{\boldsymbol{gf}}(z(0))} = 1, \quad \frac{W_{\boldsymbol{gf}}(z(p))}{W_{\boldsymbol{gf}}(z(0))} = 1. \tag{3.5}$$

Then, in the left and right $\mathsf{u}_n$-eigenbasis, the transfer matrix $\mathsf{T}^{\text{chP}}_{\lambda}$ of the inhomogeneous chiral Potts model[22] [57] is characterized by the following kernel:

$$\mathsf{T}^{\text{chP}}_{\lambda}(\mathrm{z}, \mathrm{z}') \equiv \langle \mathrm{z}|\mathsf{T}^{\text{chP}}_{\lambda}|\mathrm{z}\,'\rangle = \prod_{n=1}^{\mathsf{N}} W_{\boldsymbol{g}_n\boldsymbol{f}}(z_n/z'_n)\bar{W}_{\boldsymbol{r}_n\boldsymbol{f}}(z_n/z'_{n+1}), \tag{3.6}$$

where:

$$\lambda = t_{\boldsymbol{f}}^{-1/2}\mathsf{c}_0, \quad \boldsymbol{f}, \boldsymbol{g}_n, \boldsymbol{r}_n \in \mathcal{C}_k, \mathsf{c}_0 \in \mathbb{C}, \tag{3.7}$$

and z, z$'$ are the following multiple index $\mathrm{z} \equiv (z_1, \ldots, z_{\mathsf{N}})$ and $\mathrm{z}' \equiv (z'_1, \ldots, z'_{\mathsf{N}})$. Let us denote with $\mathcal{R}^{\text{chP}}_{\mathsf{N}}$ the sub-variety of the representations defined by the following parametrization of the Bazhanov–Stroganov Lax operator in terms of points of the curve:

$$\alpha_n = -b^2_{\boldsymbol{g}_n}/\mathsf{c}_0, \quad \mathbb{b}_n = -\mathbb{d}_n/q = -a_{\boldsymbol{g}_n}d_{\boldsymbol{g}_n}/q^{3/2}, \tag{3.8}$$

$$\beta_n = -\mathsf{c}_0 d^2_{\boldsymbol{g}_n}, \quad \mathbb{c}_n = -\mathbb{a}_n q = b_{\boldsymbol{g}_n}c_{\boldsymbol{g}_n}q^{1/2}, \tag{3.9}$$

and $\boldsymbol{g}_n \in \mathcal{C}_k$, $k \in \mathbb{C}$. $\mathsf{T}^{\text{chP}}_{\lambda}$ is then a Baxter $\mathsf{Q}$-operator[23] w.r.t. the transfer matrix of the Bazhanov–Stroganov model in $\mathcal{R}^{\text{chP}}_{\mathsf{N}}$:

$$\tau_2(\lambda)\mathsf{T}^{\text{chP}}_{\lambda} = a_{\text{BS}}(\lambda)\mathsf{T}^{\text{chP}}_{\lambda/q} + d_{\text{BS}}(\lambda)\mathsf{T}^{\text{chP}}_{q\lambda}, \tag{3.10}$$

$$[\tau_2(\lambda), \mathsf{T}^{\text{chP}}_{\lambda}] = 0, \quad [\Theta, \mathsf{T}^{\text{chP}}_{\lambda}] = 0, \quad [\mathsf{T}^{\text{chP}}_{\lambda}, \mathsf{T}^{\text{chP}}_{\mu}] = 0 \quad \forall\lambda, \mu \in \mathbb{C}, \tag{3.11}$$

with $a_{\text{BS}}$ and $d_{\text{BS}}$ defined in (5.8) and (5.9) of [118].

---

[22] For a direct comparison, see formula (4.12) of [97] with the following identifications:

$$z_j \equiv q^{2\sigma'_j}, \quad z'_j \equiv q^{2\sigma_j} \quad \forall j \in \{1, \ldots, \mathsf{N}\}.$$

Note that $\mathsf{T}^{\text{chP}}_{\lambda}$ is well defined, since the $W$-functions (3.3) are cyclic functions of their arguments.

[23] It is worth pointing out that while the Baxter equation (3.10) holds in the general inhomogeneous representations, the commutativity properties are proven only under the further restrictions $\boldsymbol{g}_n \equiv \boldsymbol{r}_n \quad \forall n\{1, \ldots, \mathsf{N}\}$ under which is characterized $\mathcal{R}^{\text{chP}}_{\mathsf{N}}$.

### 3.2. SOV-Spectrum Characterization

**Theorem 3.1** (Proposition 3, Theorem 5 and Lemma 13 of [118]). *For almost all the representations in $\mathcal{R}_{\mathsf{N}}^{\text{chP}}$, the spectrum of the chiral Potts transfer matrix $\mathsf{T}_{\lambda}^{\text{chP}}$ is simple. Moreover:*

*(I) All right and left eigenstates of the chiral Potts transfer matrix $\mathsf{T}_{\lambda}^{\text{chP}}$ are eigenstates of $\tau_2(\lambda)$ and they admit the SOV construction presented in point (I) of Theorem 2.2. The solution $Q_t(\lambda)$ of the functional Baxter equation (2.71) is gauge equivalent to the corresponding $\mathsf{T}_{\lambda}^{\text{chP}}$-eigenvalue $\mathsf{q}_{\lambda}^{\text{chP}}$ being the coefficients $a_{BS}(\lambda)$ and $d_{BS}(\lambda)$ of (3.10) gauge equivalent to the SOV-ones:*

$$a_{BS}(\lambda) = h_{BS}(\lambda)\bar{\mathrm{A}}(\lambda) \quad d_{BS}(\lambda) = h_{BS}^{-1}(\lambda q)\bar{\mathrm{D}}(\lambda). \tag{3.12}$$

*Here, $h_{BS}(\lambda)$ is a function whose average value is 1 for any $\lambda \in \mathbb{C}$.*

*(II) In the sub-variety $\mathcal{R}_{\mathsf{N}}^{\text{chP},S\text{-}adj} \equiv \mathcal{R}_{\mathsf{N}}^{\text{chP}} \cap \mathcal{R}_{\mathsf{N}}^{S\text{-}adj}$, characterized by (3.8)–(3.9) under the following constrains:*

$$\boldsymbol{g}_n = (a_{\boldsymbol{g}_n}, \epsilon q \epsilon_{0,n} a_{\boldsymbol{g}_n}^{*}, \epsilon_{0,n} d_{\boldsymbol{g}_n}^{*}, d_{\boldsymbol{g}_n}) \in \mathcal{C}_k, \quad \epsilon_{0,n} = \pm 1, \quad k^{*} = \epsilon k, \tag{3.13}$$

*the operator $\mathsf{T}_{\lambda}^{\text{chP}}$ is normal and $\tau_2(\lambda)$ is self-adjoint. Then, point (I) of Theorem 2.2 allows to construct the full simultaneous $(\mathsf{T}_{\lambda}^{\text{chP}}, \tau_2(\lambda), \Theta)$-eigenbasis associating to any $t(\lambda) \in \Sigma_{\tau_2}$ the corresponding eigenstate.*

## 4. Decomposition of the Identity in the Transfer Matrix Eigenbasis

### 4.1. Action of Left Separate States on Right Separate States

Here, we compute the action of covectors on vectors which in the left and right SOV-basis have a *separate form* similar to that of the transfer matrix eigenstates. To be more precise, let us give the following definition of a left $\langle \alpha_k |$ and a right $|\beta_k\rangle$ separate states characterized by the given arbitrary set of functions $\alpha_a$ and $\beta_a$:

$$\begin{aligned}\langle \alpha_k | = & \sum_{h_1,\ldots,h_{\mathsf{N}}=1}^{p} \frac{q^{k h_{\mathsf{N}}}}{p^{1/2}} \prod_{a=1}^{\mathsf{N}-1} \alpha_a\left(\eta_a^{(h_a)}\right) \\ & \times \prod_{1\le a<b\le \mathsf{N}-1} \left(\left(\eta_a^{(h_a)}\right)^2 - \left(\eta_b^{(h_b)}\right)^2\right) \frac{\langle \eta_1^{(h_1)}, \ldots, \eta_{\mathsf{N}}^{(h_{\mathsf{N}})} |}{\prod_{b=1}^{\mathsf{N}-1} \omega_b\left(\eta_b^{(h_b)}\right)},\end{aligned} \tag{4.1}$$

$$\begin{aligned}|\beta_k\rangle = & \sum_{h_1,\ldots,h_{\mathsf{N}}=1}^{p} \frac{q^{-k h_{\mathsf{N}}}}{p^{1/2}} \prod_{a=1}^{\mathsf{N}-1} \beta_a\left(\eta_a^{(h_a)}\right) \\ & \times \prod_{1\le a<b\le \mathsf{N}-1} \left(\left(\eta_a^{(h_a)}\right)^2 - \left(\eta_b^{(h_b)}\right)^2\right) \frac{| \eta_1^{(h_1)}, \ldots, \eta_{\mathsf{N}}^{(h_{\mathsf{N}})} \rangle}{\prod_{b=1}^{\mathsf{N}-1} \omega_b\left(\eta_b^{(h_b)}\right)}.\end{aligned} \tag{4.2}$$

**Proposition 4.1.** *The action of the left separate state $\langle\alpha_k|$ of form (4.1) on the right separate state $|\beta_h\rangle$ of form (4.2) reads:*

$$\begin{aligned}\langle\alpha_k|\beta_h\rangle &= \delta_{k,h}\det_{\mathsf{N}-1}\left\|\mathcal{M}^{(\alpha,\beta)}_{a,b}\right\| \\ \mathcal{M}^{(\alpha,\beta)}_{a,b} &\equiv \left(\eta_a^{(0)}\right)^{2(b-1)}\sum_{h=1}^{p}\frac{\alpha_a\left(\eta_a^{(h)}\right)\beta_a\left(\eta_a^{(h)}\right)}{\omega_a\left(\eta_a^{(h)}\right)}q^{2(b-1)h}.\end{aligned}\tag{4.3}$$

*Proof.* The SOV-decomposition of these states implies:

$$\begin{aligned}\langle\alpha_k|\beta_h\rangle = \sum_{h_{\mathsf{N}}=1}^{p}\frac{q^{(k-h)h_{\mathsf{N}}}}{p}\sum_{h_1,\ldots,h_{\mathsf{N}-1}=1}^{p}V\left(\left(\eta_1^{(h_1)}\right)^2,\ldots,\left(\eta_{\mathsf{N}-1}^{(h_{\mathsf{N}-1})}\right)^2\right)\\ \times\prod_{a=1}^{\mathsf{N}-1}\frac{\alpha_a\left(\eta_a^{(h_a)}\right)\beta_a\left(\eta_a^{(h_a)}\right)}{\omega_a\left(\eta_a^{(h_a)}\right)},\end{aligned}\tag{4.4}$$

where $V(x_1,\ldots,x_{\mathsf{N}})\equiv\prod_{1\le a<b\le\mathsf{N}-1}(x_a-x_b)$ is the Vandermonde determinant. Then, from the identity:

$$\delta_{k,h}=\sum_{h_{\mathsf{N}}=1}^{p}\frac{q^{(k-h)h_{\mathsf{N}}}}{p}\quad\text{when } q \text{ is a } p\text{-root of unit and } h,k\in\mathbb{Z}_p\tag{4.5}$$

and using the multilinearity of the determinant w.r.t. the rows, we prove the proposition. □

It is worth remarking that the previous determinant formulae define also scalar products for vectors in $\mathcal{R}_{\mathsf{N}}$ which have a separate form in the right $\mathsf{B}$-eigenbasis and in the dual of the left $\mathsf{B}$-eigenbasis. Indeed, $(\langle\alpha_k|)^\dagger\in\mathcal{R}_{\mathsf{N}}$ is a separate vector in the basis of $\mathcal{R}_{\mathsf{N}}$ formed out of the $(\langle\eta_{\mathbf{k}}|)^\dagger$ dual states of the left $\mathsf{B}$-eigenbasis. Then, these results represent the SOV analogue of the scalar product formulae [18,126] computed for Bethe states in the framework of the algebraic Bethe ansatz. Note that this formula is not restricted to the case in which one of the two states is an eigenstate of the transfer matrix. It is also interesting to remark that the previous scalar product formulae allow to prove directly, as in the case of the sine–Gordon model, that the action of a transfer matrix eigencovector on an eigenvector corresponding to different eigenvalue is zero.

**Corollary 4.1.** *Let $t_h(\lambda)$ and $t'_h(\lambda)\in\Sigma^h_{\tau_2}$ and $\langle t_h|$ and $|t'_h\rangle$ the $\tau_2$-eigenstates defined in Sect. 2.4.2, then for $t_h(\lambda)\neq t'_h(\lambda)$ the $(\mathsf{N}-1)\times(\mathsf{N}-1)$ matrix $\mathcal{M}^{(t_h,t'_h)}_{a,b}$ has rank equal or smaller than $\mathsf{N}-2$. Indeed, the non-zero $(\mathsf{N}-1)\times 1$ vector $V^{(t_h,t'_h)}$ defined by:*

$$V_b^{(t_h,t'_h)}\equiv c'_b-c_b\quad\forall b\in\{1,\ldots,\mathsf{N}-1\},\tag{4.6}$$

*where:*

$$t_h(\lambda) = \sum_{\epsilon=\pm1} \left(q^{\epsilon h} a_\epsilon + q^{-\epsilon h} d_\epsilon\right) \lambda^{\epsilon \mathsf{N}} + \sum_{b=1}^{\mathsf{N}-1} c_b \lambda^{-\mathsf{N}-2+2b}, \tag{4.7}$$

$$t'_h(\lambda) = \sum_{\epsilon=\pm1} \left(q^{\epsilon h} a_\epsilon + q^{-\epsilon h} d_\epsilon\right) \lambda^{\epsilon \mathsf{N}} + \sum_{b=1}^{\mathsf{N}-1} c'_b \lambda^{-\mathsf{N}-2+2b}, \tag{4.8}$$

*is an eigenvector of* $||\mathcal{M}_{a,b}^{(t_h,t'_h)}||$ *corresponding to the eigenvalue zero.*

*Proof.* Note that under the choice (2.68) for the left gauge and (2.80) for the right gauge, it holds:

$$\omega_a\left(\eta_a^{(h)}\right) = \left(\eta_a^{(h)}\right)^{\mathsf{N}-2}, \tag{4.9}$$

and then by the definitions (4.6), (4.7) and (4.8) it holds:

$$\sum_{b=1}^{\mathsf{N}-1} \mathcal{M}_{a,b}^{(t_h,t'_h)} \mathrm{V}_b^{(t_h,t'_h)} = \sum_{h=0}^{2s_a} Q_{t'_h}(\eta_a^{(h)}) \bar{Q}_{t_h}(\eta_a^{(h)})(t'_h(\eta_a^{(h)}) - t_h(\eta_a^{(h)})). \tag{4.10}$$

The desired result:

$$\sum_{b=1}^{\mathsf{N}-1} \mathcal{M}_{a,b}^{(t_h,t'_h)} \mathrm{V}_b^{(t_h,t'_h)} = 0 \quad \forall a \in \{1, \ldots, \mathsf{N}-1\}, \tag{4.11}$$

then follows as the Baxter equations (2.77) and (2.79) allow to write:

$$\begin{aligned} &Q_{t'_h}\left(\eta_a^{(k)}\right) \bar{Q}_{t_h}\left(\eta_a^{(k)}\right) (t'_h\left(\eta_a^{(k)}\right) - t_h\left(\eta_a^{(k)}\right)) \\ &\quad = (\bar{\mathrm{D}}\left(\eta_a^{(k+1)}\right) Q_{t'_h}\left(\eta_a^{(k+1)}\right) + \bar{\mathrm{A}}\left(\eta_a^{(k-1)}\right) Q_{t'_h}\left(\eta_a^{(k-1)}\right)) \bar{Q}_t\left(\eta_a^{(k)}\right) \\ &\qquad -(\bar{\mathrm{A}}\left(\eta_a^{(k)}\right) \bar{Q}_{t_h}\left(\eta_a^{(k+1)}\right) + \bar{\mathrm{D}}\left(\eta_a^{(k)}\right) \bar{Q}_{t_h}\left(\eta_a^{(k-1)}\right)) Q_{t'_h}\left(\eta_a^{(k)}\right), \end{aligned} \tag{4.12}$$

which substituted in (4.10) implies (4.11). □

### 4.2. Decomposition of the Identity in Transfer Matrix Eigenbasis

In the representations for which $\tau_2(\lambda)$ is diagonalizable, the simplicity of its spectrum plus the explicit characterizations of its left and right eigenstates allows to write the following decomposition of the identity:

$$\mathbb{I} = \sum_{k=0}^{p-1} \sum_{t(\lambda) \in \Sigma_{\tau_2}^k} \frac{|t_k\rangle\langle t_k|}{\langle t_k|t_k\rangle}, \tag{4.13}$$

where

$$\langle t_k|t_k\rangle = \det_{\mathsf{N}-1} \left\| \mathcal{M}_{a,b}^{(t_k,t_k)} \right\|$$

$$\text{with } \mathcal{M}_{a,b}^{(t_k,t_k)} \equiv \left(\eta_a^{(0)}\right)^{2(b-1)} \sum_{c=1}^{p} \frac{Q_{t_k}\left(\eta_a^{(c)}\right) \bar{Q}_{t_k}\left(\eta_a^{(c)}\right)}{\omega_a\left(\eta_a^{(c)}\right)} q^{2(b-1)c}, \tag{4.14}$$

is the action of the covector $\langle t_k|$ on the vector $|t_k\rangle$, both defined in Sect. 2.4.2. Note that in the representations which define a normal $\tau_2(\lambda)$, the simplicity of the spectrum implies the following identity:

$$(|t_k\rangle)^{\dagger} \equiv \alpha_{t_k}\langle t_k| \quad \text{where } \alpha_{t_k} = \frac{\||t_k\rangle\|^2}{\langle t_k|t_k\rangle} \in \mathbb{C} \tag{4.15}$$

for any eigenvector $|t_k\rangle$ of $\tau_2(\lambda)$. For these special representations, this stresses the interest in computing the norm $\||t_k\rangle\|$ as it allows to write left and right $\tau_2$-eigenstates as one which is the exact dual of the other. Let us mention that a similar decomposition of the identity was first proposed in the series of works [113–117] in particular for the case $p = 2$.

## 5. Propagator for the Bazhanov–Stroganov Model

In this section, we construct the propagator operator along the chain of the Bazhanov–Stroganov model for the representations parametrized by points on the chP curves.

### 5.1. Fundamental *R*-matrix of the Bazhanov–Stroganov Model

In the next proposition, we report adapting to our notations a fundamental result of the paper [57].

**Proposition 5.1** [57]. *Let $\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_2,\boldsymbol{r}_2)}$ be the operator defined on the tensor product of two p-dimensional spaces by:*

$$\begin{aligned}&\left\langle z_1, z_2|\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_2,\boldsymbol{r}_2)}|z_1', z_2'\right\rangle \\ &\quad\equiv \bar{W}_{\boldsymbol{g}_2\boldsymbol{g}_1}(z_1/z_2')\, W_{\boldsymbol{r}_2\boldsymbol{g}_1}(z_1'/z_2')\bar{W}_{\boldsymbol{r}_2\boldsymbol{r}_1}(z_2/z_1')W_{\boldsymbol{g}_2\boldsymbol{r}_1}(z_2/z_1),\end{aligned} \tag{5.1}$$

*Then, $\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_2,\boldsymbol{r}_2)}$ is the fundamental R-matrix intertwining the Bazhanov–Stroganov Lax operator in the quantum space, i.e. it holds:*

$$\begin{aligned}&\mathsf{L}_{02}(\lambda|\boldsymbol{g}_2,\boldsymbol{r}_2)\mathsf{L}_{01}(\lambda|\boldsymbol{g}_1,\boldsymbol{r}_1)\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_2,\boldsymbol{r}_2)} \\ &\quad= \mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_2,\boldsymbol{r}_2)}\mathsf{L}_{01}(\lambda|\boldsymbol{g}_1,\boldsymbol{r}_1)\mathsf{L}_{02}(\lambda|\boldsymbol{g}_2,\boldsymbol{r}_2).\end{aligned} \tag{5.2}$$

*Proof.* Let us just point out that the proof can be obtained by proving it for any matrix element $(i_1, i_2)\in \{1, 2\}\times\{1, 2\}$. Indeed, taking the matrix elements on the quantum states $\langle z_1, z_2|$ and $|z_1'', z_2''\rangle$, the proposition simply follows from the identities:

$$\begin{aligned}&\sum_{z_2',z_2'\in\mathbb{S}_p,j=1,2} (\mathsf{L}_{02})^{i_2,j}_{z_2 z_2'}(\lambda|\boldsymbol{g}_2,\boldsymbol{r}_2)\,(\mathsf{L}_{01})^{j,i_1}_{z_1 z_1'}(\lambda|\boldsymbol{g}_1,\boldsymbol{r}_1)\langle z_1', z_2'|\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\,\mathrm{q}_2,\boldsymbol{r}_2)}|z_1'', z_2''\rangle \\ &\quad= \sum_{z_2',z_2'\in\mathbb{S}_p,j=1,2}\left\langle z_1, z_2|\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_2,\boldsymbol{r}_2)}|z_1', z_2'\right\rangle \\ &\qquad\times (\mathsf{L}_{01})^{i_1,j}_{z_1' z_1''}(\lambda|\boldsymbol{g}_1,\boldsymbol{r}_1)\,(\mathsf{L}_{02})^{j,i_2}_{z_2' z_2''}(\lambda|\boldsymbol{g}_2,\boldsymbol{r}_2),\end{aligned} \tag{5.3}$$

once the elements of $\mathsf{L}_{0\mathrm{i}}$ are rewritten in terms of the points of $\mathcal{C}_k$ and we use the definition of the functions $W$ and $\bar{W}$. □

### 5.2. Propagator for the Bazhanov–Stroganov Model

The first transfer matrix of the chP-model has been defined in (3.6 ), while the second chP-transfer matrix reads:

$$\hat{\mathsf{T}}^{\mathrm{chP}}_{\lambda_{\boldsymbol{f}},(\boldsymbol{f}|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})}(\mathbf{z},\mathbf{z}') \equiv \left\langle \mathbf{z}|\hat{\mathsf{T}}^{\mathrm{chP}}_{\lambda_{\boldsymbol{f}},(\boldsymbol{f}|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})}|\mathbf{z}'\right\rangle = \prod_{n=1}^{N} W_{\boldsymbol{r}_n\boldsymbol{f}}(z_{n+1}/z'_n)\bar{W}_{\boldsymbol{g}_n\boldsymbol{f}}(z_n/z'_n). \tag{5.4}$$

Let us recall that the propagator operator $\mathsf{U}_n$ along the Bazhanov–Stroganov chain is defined by:

$$\mathsf{U}_n\mathsf{M}_{1,\dots,\mathsf{N}}(\lambda)\mathsf{U}_n^{-1}\equiv\mathsf{M}_{n,\dots,\mathsf{N},1,\dots,n-1}(\lambda)\equiv\mathsf{L}_{n-1}(\lambda)\dots\mathsf{L}_1(\lambda)\mathsf{L}_\mathsf{N}(\lambda)\dots\mathsf{L}_n(\lambda), \tag{5.5}$$

then, we can prove:

**Proposition 5.2.** *The propagator operator* $\mathsf{U}_m$ *has the following representation in terms of the chP-transfer matrices:*

$$\mathsf{U}_m^{-1} \equiv \mathsf{T}^{chP}_{\lambda_{\boldsymbol{r}_1},(\boldsymbol{r}_1|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})}\hat{\mathsf{T}}^{chP}_{\lambda_{\boldsymbol{g}_1},(\boldsymbol{g}_1|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})}\dots\mathsf{T}^{chP}_{\lambda_{\boldsymbol{r}_{m-1}},(\boldsymbol{r}_{m-1}|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})} \times\hat{\mathsf{T}}^{chP}_{\lambda_{\boldsymbol{g}_{m-1}},(\boldsymbol{g}_{m-1}|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})}. \tag{5.6}$$

*Proof.* The previous proposition implies that the operator $\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_2,\boldsymbol{r}_2)}$ satisfies the following equation

$$\left(\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_2,\boldsymbol{r}_2)}\right)^{-1}\mathsf{L}_{02}(\lambda|\boldsymbol{g}_2,\boldsymbol{r}_2)\mathsf{L}_{01}(\lambda|\boldsymbol{g}_1,\boldsymbol{r}_1)\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_2,\boldsymbol{r}_2)} = \mathsf{L}_{01}(\lambda|\boldsymbol{g}_1,\boldsymbol{r}_1)\mathsf{L}_{02}(\lambda|\boldsymbol{g}_2,\boldsymbol{r}_2), \tag{5.7}$$

then, it is simple to verify that:

$$\begin{aligned}&\left(\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_2,\boldsymbol{r}_2)}\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_3,\boldsymbol{r}_3)}\dots\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_\mathsf{N},\boldsymbol{r}_\mathsf{N})}\right)^{-1}\\ &\quad\mathsf{L}_{0\mathsf{N}}(\lambda|\boldsymbol{g}_\mathsf{N},\boldsymbol{r}_\mathsf{N})\dots\mathsf{L}_{02}(\lambda|\boldsymbol{g}_2,\boldsymbol{r}_2)\mathsf{L}_{01}(\lambda|\boldsymbol{g}_1,\boldsymbol{r}_1)\\ &\quad\times\left(\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\mathrm{q}_2,\boldsymbol{r}_2)}\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_3,\boldsymbol{r}_3)}\dots\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_\mathsf{N},\boldsymbol{r}_\mathsf{N})}\right)\\ &= \mathsf{L}_{01}(\lambda|\boldsymbol{g}_1,\boldsymbol{r}_1)\mathsf{L}_{0\mathsf{N}}(\lambda|\boldsymbol{g}_\mathsf{N},\boldsymbol{r}_\mathsf{N})\dots\mathsf{L}_{02}(\lambda|\boldsymbol{g}_2,\boldsymbol{r}_2)\end{aligned} \tag{5.8}$$

Let us compute the matrix elements:

$$\left\langle\mathbf{z}|\mathsf{T}^{\mathrm{chP}}_{\lambda_{\boldsymbol{r}_1},(\boldsymbol{r}_1|\{\mathrm{q}_n,\boldsymbol{r}_n\})}\hat{\mathsf{T}}^{\mathrm{chP}}_{\lambda_{\boldsymbol{g}_1},(\boldsymbol{g}_1|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})}|\mathbf{z}''\right\rangle = \sum_{\mathbf{z}'}\left\langle\mathbf{z}|\mathsf{T}^{\mathrm{chP}}_{\lambda_{\boldsymbol{r}_1},(\boldsymbol{r}_1|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})}|\mathbf{z}'\right\rangle\left\langle\mathbf{z}'|\hat{\mathsf{T}}^{\mathrm{chP}}_{\lambda_{\boldsymbol{g}_1},(\boldsymbol{g}_1|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})}|\mathbf{z}''\right\rangle \tag{5.9}$$

Using the relations $\bar{W}_{\boldsymbol{f}\,\boldsymbol{f}}(z/z') = \delta_{z,z'}$ and $W_{\boldsymbol{f}\,\boldsymbol{g}}(z)W_{\boldsymbol{g}\,\boldsymbol{f}}(z) = 1$, we get:

$$\left\langle\mathbf{z}|\mathsf{T}^{\mathrm{chP}}_{\lambda_{\boldsymbol{r}_1},(\boldsymbol{r}_1|\{\mathrm{q}_n,\boldsymbol{r}_n\})}\hat{\mathsf{T}}^{\mathrm{chP}}_{\lambda_{\boldsymbol{g}_1},(\boldsymbol{g}_1|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})}|\mathbf{z}''\right\rangle = \sum_{\mathbf{z}'}\delta_{z_1,z'_2}\delta_{z'_1,z''_1}\prod_{n\geq 2}\left\langle z'_n,z_n|\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_n,\boldsymbol{r}_n)}|z'_{n+1},z''_n\right\rangle \tag{5.10}$$

$$= \left\langle z_1,\dots,z_\mathsf{N}|\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_2,\boldsymbol{r}_2)}\dots\mathsf{S}_{(\boldsymbol{g}_1,\boldsymbol{r}_1|\boldsymbol{g}_\mathsf{N},\boldsymbol{r}_\mathsf{N})}|z''_1,\dots,z''_\mathsf{N}\right\rangle \tag{5.11}$$

Let us use the notation $\bar{\mathsf{S}}_i = \mathsf{T}^{\text{chP}}_{\lambda_{\boldsymbol{r}_i},(\boldsymbol{r}_i|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})}\hat{\mathsf{T}}^{\text{chP}}_{\lambda_{\boldsymbol{g}_i},(\boldsymbol{g}_i|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})}$, then (5.8) can be rewritten as it follows:

$$\begin{aligned}&\bar{\mathsf{S}}_1^{-1}\mathsf{L}_{0\mathsf{N}}(\lambda|\boldsymbol{g}_{\mathsf{N}},\boldsymbol{r}_{\mathsf{N}})\ldots\mathsf{L}_{02}(\lambda|\boldsymbol{g}_2,\boldsymbol{r}_2)\mathsf{L}_{01}(\lambda|\boldsymbol{g}_1,\boldsymbol{r}_1)\bar{\mathsf{S}}_1\\&\quad=\mathsf{L}_{01}(\lambda|\boldsymbol{g}_1,\boldsymbol{r}_1)\mathsf{L}_{0\mathsf{N}}(\lambda|\boldsymbol{g}_{\mathsf{N}},\boldsymbol{r}_{\mathsf{N}})\ldots\mathsf{L}_{02}(\lambda|\boldsymbol{g}_2,\boldsymbol{r}_2)\end{aligned}\tag{5.12}$$

and acting similarly with the others $\bar{\mathsf{S}}_n$ with $n>1$ it holds:

$$\begin{aligned}&\bar{\mathsf{S}}_n^{-1}\mathsf{L}_{0n-1}(\lambda|\boldsymbol{g}_{n-1},\boldsymbol{r}_{n-1})\ldots\mathsf{L}_{01}(\lambda|\boldsymbol{g}_1,\mathrm{r}_1)\mathsf{L}_{0\mathsf{N}}(\lambda|\boldsymbol{g}_{\mathsf{N}},\mathrm{r}_{\mathsf{N}})\\&\quad\ldots\mathsf{L}_{0n+1}(\lambda|\boldsymbol{g}_{n+1},\boldsymbol{r}_{n+1})\mathsf{L}_{0n}(\lambda|\boldsymbol{g}_n,\boldsymbol{r}_n)\bar{\mathsf{S}}_n\\&=\mathsf{L}_{0n}(\lambda|\boldsymbol{g}_n,\boldsymbol{r}_n)\ldots\mathsf{L}_{01}(\lambda|\boldsymbol{g}_1,\boldsymbol{r}_1)\mathsf{L}_{0\mathsf{N}}(\lambda|\boldsymbol{g}_{\mathsf{N}},\boldsymbol{r}_{\mathsf{N}})\\&\quad\ldots\mathsf{L}_{0n+2}(\lambda|\boldsymbol{g}_{n+2},\boldsymbol{r}_{n+2})\mathsf{L}_{0n+1}(\lambda|\boldsymbol{g}_{n+1},\boldsymbol{r}_{n+1}),\end{aligned}\tag{5.13}$$

from which defining:

$$\mathsf{U}_n^{-1}=\bar{\mathsf{S}}_1\bar{\mathsf{S}}_2\ldots\bar{\mathsf{S}}_{n-1}\tag{5.14}$$

$$\begin{aligned}&=\mathsf{T}^{\text{chP}}_{\lambda_{\boldsymbol{r}_1},(\boldsymbol{r}_1|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})}\hat{\mathsf{T}}^{\text{chP}}_{\lambda_{\boldsymbol{g}_1},(\boldsymbol{g}_1|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})}\\&\quad\ldots\mathsf{T}^{\text{chP}}_{\lambda_{\boldsymbol{r}_{n-1}},(\boldsymbol{r}_{n-1}|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})}\hat{\mathsf{T}}^{\text{chP}}_{\lambda_{\boldsymbol{g}_{n-1}},(\boldsymbol{g}_{n-1}|\{\boldsymbol{g}_n,\boldsymbol{r}_n\})},\end{aligned}\tag{5.15}$$

$\mathsf{U}_n$ surely satisfies the Eq. (5.5) which defines the propagator. □

It is worth noticing that the eigenvalues of the two chP-transfer matrices on the eigenstates of the $\tau_2$ transfer matrix are characterized according to the discussion made in Sect. 3.2, then the eigenvalues of $\mathsf{U}_m$ are also known. Moreover, let us point out that:

$$\lambda_{\boldsymbol{g}_n}=i\left(q\frac{\mathbb{a}_n\beta_n}{\alpha_n\mathbb{b}_n}\right)^{1/2},\quad \lambda_{\boldsymbol{r}_n}=i\left(q\frac{\mathbb{c}_n\beta_n}{\alpha_n\mathbb{d}_n}\right)^{1/2},\tag{5.16}$$

i.e. we are computing the $Q$-operators, $\mathsf{T}^{\text{chP}}_{\lambda_{\boldsymbol{r}_n}}\hat{\mathsf{T}}^{\text{chP}}_{\lambda_{\boldsymbol{g}_n}}$, in the zeros of the quantum determinant of the $\tau_2$-model. In the case of self-adjoint representations on trivial curves (like for sine–Gordon model), we have up to an overall constant:

$$\mathsf{U}_m^{-1}\equiv\mathsf{Q}_{\lambda_{\boldsymbol{r}_1}}\mathsf{Q}_{\lambda^*_{\boldsymbol{g}_1}}\ldots\mathsf{Q}_{\lambda_{\boldsymbol{r}_{m-1}}}\mathsf{Q}_{\lambda^*_{\boldsymbol{g}_{m-1}}}.\tag{5.17}$$

The case of Bethe ansatz representations corresponds to the case $\boldsymbol{g}_n=\boldsymbol{r}_n$, i.e. the two zeros of the quantum determinant coincide up to $p$-roots of units. In this case and in the homogeneous case, we reproduce the known result of [143] for the propagator.

## 6. Representation of Local Operators by Separate Variables

The results on the scalar product formulae define one of the main steps to compute matrix elements of local operators. The other one is to reconstruct local operators using the generators of the Yang–Baxter algebra, namely to invert the map from the local operators in the Lax matrices to the monodromy matrix elements. This inverse problem solution makes possible to compute the action of local operators on transfer matrix eigenstates in this way leading to the determination of form factors of local operators, once the scalar product formulae are used.

In [18], the first solution of this inverse problem has been obtained for the $XXZ$ spin 1/2 chain and then in [28] it has been generalized to all fundamental lattice models having isomorphic auxiliary and local quantum spaces characterized by a Lax operator matrix coinciding with the permutation operator for a special value of the spectral parameter. This reconstruction can be also used for non-fundamental lattice models, as derived in [28] for the higher spin $XXX$ chains using the fusion procedure [77]. For the Bazhanov–Stroganov model, we still do not know how to achieve this type of reconstruction and the known results reduce to those given by Oota [144]. However, Oota's results lead only to reconstruct some local operators of the Bazhanov–Stroganov model. We will explain in this section how to complete the Oota's reconstruction for all the local operators of the Bazhanov–Stroganov model associated to the most general cyclic representations of the six-vertex Yang–Baxter algebra. The procedure developed here is the natural generalization to these representations of the one for the special subclass presented in our previous paper [1]. The new technical tools required to handle these general representations will be also introduced in the next subsections.

### 6.1. Reconstruction of a Class of Local Operators

The results of Oota's paper [144] are reproduced here for the more general cyclic representations associated to the the Bazhanov–Stroganov model; this leads to the reconstruction of a subclass of local operators. In terms of quantum projectors, when computed in the zeros $\mu_{n,\pm}$ of the quantum determinant, the Lax operator $\mathsf{L}_n(\lambda)$ has the following factorization:

$$\mathsf{L}_n(\mu_{n,+}) \equiv \begin{pmatrix} (\mathsf{L}_n)_{12}\, \mathsf{u}_n^{-1/2} f_n \\ (\mathsf{L}_n)_{21}\, \mathsf{u}_n^{1/2} f_n^{-1} \end{pmatrix} \begin{pmatrix} \mathsf{u}_n^{-1/2} f_n & \mathsf{u}_n^{1/2} f_n^{-1} \end{pmatrix}, \tag{6.1}$$

$$\mathsf{L}_n(\mu_{n,-}) \equiv \begin{pmatrix} g_n \mathsf{u}_n^{1/2} \\ g_n^{-1} \mathsf{u}_n^{-1/2} \end{pmatrix} \begin{pmatrix} g_n \mathsf{u}_n^{1/2}\,(\mathsf{L}_n)_{21} & g_n^{-1} \mathsf{u}_n^{-1/2}\,(\mathsf{L}_n)_{12} \end{pmatrix}, \tag{6.2}$$

where $(\mathsf{L}_n)_{ij}$ stays for the matrix element $i, j$ of the Lax operator and:

$$f_n \equiv \left(-\frac{\alpha_n \beta_n}{\mathbb{a}_n \mathbb{b}_n}\right)^{1/4}, \quad g_n \equiv \left(-\frac{\alpha_n \beta_n}{\mathbb{c}_n \mathbb{d}_n}\right)^{1/4}. \tag{6.3}$$

These factorizations properties were used by Oota's to reconstruct local operators as it follows:

**Proposition 6.1.** *The following reconstructions of local operators hold:*

$$\begin{aligned} \mathsf{u}_n^{-1} &= \left(-\frac{\mathbb{a}_n \mathbb{b}_n}{\alpha_n \beta_n}\right)^{1/2} \mathsf{U}_n \mathsf{B}^{-1}(\mu_{n,+}) \mathsf{A}(\mu_{n,+}) \mathsf{U}_n^{-1} \\ &= \left(-\frac{\mathbb{a}_n \mathbb{b}_n}{\alpha_n \beta_n}\right)^{1/2} \mathsf{U}_n \mathsf{D}^{-1}(\mu_{n,+}) \mathsf{C}(\mu_{n,+}) \mathsf{U}_n^{-1}, \end{aligned} \tag{6.4}$$

$$\alpha_{0,n} = \mathsf{U}_n \mathsf{A}^{-1}(\mu_{n,-}) \mathsf{B}(\mu_{n,-}) \mathsf{U}_n^{-1} = \mathsf{U}_n \mathsf{C}^{-1}(\mu_{n,-}) \mathsf{D}(\mu_{n,-}) \mathsf{U}_n^{-1}. \tag{6.5}$$

*where we have defined:*

$$\alpha_{0,n} \equiv \left(\frac{-\mathbb{c}_n \mathbb{b}_n^2}{\alpha_n \beta_n \mathbb{d}_n}\right)^{1/2} \left(\frac{1+q^{-1}(\mathbb{a}_n/\mathbb{b}_n)\mathsf{v}_n^2}{1+q^{-1}(\mathbb{c}_n/\mathbb{d}_n)\mathsf{v}_n^2}\right) \mathsf{u}_n. \tag{6.6}$$

Reconstructions of local operators similar to (6.4)–(6.5) also appear in [145] and were used in [117]. Oota's formulae (6.4)–(6.5) clearly allow to reconstruct all the powers $\mathsf{u}_n^{-k} = \mathsf{U}_n \left(\mathsf{B}^{-1}(\mu_{n,+})\mathsf{A}(\mu_{n,+})\right)^k \mathsf{U}_n^{-1}$; however, the local operators $\mathsf{v}_n^k$ do not admit direct reconstructions as only rational functions like $\left(1+q^{-1}(\mathbb{a}_n/\mathbb{b}_n)\mathsf{v}_n^2\right) / \left(1+q^{-1}(\mathbb{c}_n/\mathbb{d}_n)\mathsf{v}_n^2\right)$ are reconstructed.

### 6.2. Reconstruction of all Local Operators

Here, we solve the inverse problem for the local operators $\mathsf{v}_n^k$ in this way completing the reconstruction of local operators. The cyclicity of the representations of the Bazhanov–Stroganov model will be the main property here used. Let us define the following local operators:

$$\beta_{k,n} \equiv \left(\mathsf{U}_n \mathsf{A}^{-1}(\mu_{n,+})\mathsf{B}(\mu_{n,+})\mathsf{U}_n^{-1}\right)^{-k-1} \alpha_{0,n} \left(\mathsf{U}_n \mathsf{A}^{-1}(\mu_{n,+})\mathsf{B}(\mu_{n,+})\mathsf{U}_n^{-1}\right)^k \tag{6.7}$$

then it holds:

**Proposition 6.2.** *For the cyclic representations of the Bazhanov–Stroganov model we consider, the local operators $\mathsf{v}_n^{2k}$ have the following reconstructions:*

$$\mathsf{v}_n^{2k} = \frac{1}{p}\left(-\frac{\mathbb{d}_n}{\mathbb{c}_n}\right)^k \frac{1+(\mathbb{c}_n/\mathbb{d}_n)^p}{(\mathbb{b}_n\mathbb{c}_n/\mathbb{a}_n\mathbb{d}_n)^{1/2} - (\mathbb{a}_n\mathbb{d}_n/\mathbb{b}_n\mathbb{c}_n)^{1/2}} \sum_{a=0}^{p-1} q^{k(2a+1)} \beta_{a,n}. \tag{6.8}$$

*Proof.* By definition in our cyclic representations, the powers $\mathsf{u}_n^p$ and $\mathsf{v}_n^p$ are central elements of the algebra coinciding with 1. Then, it holds:

$$\frac{1+(\mathbb{c}_n/\mathbb{d}_n)^p}{1+q^{-2k-1}(\mathbb{c}_n/\mathbb{d}_n)\mathsf{v}_n^2} = \sum_{i=0}^{p-1} \left(-q^{-2k-1}(\mathbb{c}_n/\mathbb{d}_n)\mathsf{v}_n^2\right)^i. \tag{6.9}$$

The previous formula and the reconstruction (6.4)–(6.5) allow to rewrite $\beta_{k,n}$ as the following finite sum in powers of $\mathsf{v}_n^2$:

$$\begin{aligned}\beta_{k,n} = &\frac{(\mathbb{b}_n\mathbb{c}_n/\mathbb{a}_n\mathbb{d}_n)^{1/2} + (\mathbb{a}_n\mathbb{d}_n/\mathbb{b}_n\mathbb{c}_n)^{1/2}(\mathbb{c}_n/\mathbb{d}_n)^p}{1+(\mathbb{c}_n/\mathbb{d}_n)^p} \\ &+ \frac{(\mathbb{b}_n\mathbb{c}_n/\mathbb{a}_n\mathbb{d}_n)^{1/2} - (\mathbb{a}_n\mathbb{d}_n/\mathbb{b}_n\mathbb{c}_n)^{1/2}}{1+(\mathbb{c}_n/\mathbb{d}_n)^p} \sum_{a=1}^{p-1} (-1)^a q^{-a(2k+1)} \left(\frac{\mathbb{c}_n}{\mathbb{d}_n}\right)^a \mathsf{v}_n^{2a},\end{aligned} \tag{6.10}$$

then, taking a discrete Fourier transformation, the reconstruction (6.8) is obtained together with the following sum rules

$$\sum_{a=0}^{p-1} \beta_{a,n} = p \frac{(\mathbb{b}_n \mathbb{c}_n/\mathbb{a}_n \mathbb{d}_n)^{1/2} + (\mathbb{a}_n \mathbb{d}_n/\mathbb{b}_n \mathbb{c}_n)^{1/2}(\mathbb{c}_n/\mathbb{d}_n)^p}{1+(\mathbb{c}_n/\mathbb{d}_n)^p}. \tag{6.11}$$

□

The formulae in (6.8) lead to the reconstruction of all the powers $\mathsf{v}_n^k$ for $k \in \{1, \ldots, p-1\}$ as it follows from the identities $\mathsf{v}_n^k = \mathsf{v}_n^{2h}$, for $k = 2h - p$ odd integer smaller than $p$. Hence, as desired, all the local operators of the cyclic representations of the Bazhanov–Stroganov model are reconstructed using the above proposition and the Oota's reconstructions.

### 6.3. Separate Variables Representations of all Local Operators

To compute the action of the local operators $\mathsf{v}_n^k$ and $\mathsf{u}_n^k$ on eigenstates of the transfer matrix and then their form factors, we need to determine their SOV-representations before. These SOV-representations are obtained from the above solution of the inverse problem. To this aim, we first prove two lemmas that are important to overcome the combinatorial problem associated to the computation of the SOV-representations of the local operators (6.4)–(6.5).

Let us introduce, the coordinate operators $\hat{\boldsymbol{\eta}}_i$ for $i \in \{1, \ldots, \mathsf{N}\}$, $\hat{\boldsymbol{\eta}}_{\mathsf{A}}^{(\pm)}$ and $\hat{\boldsymbol{\eta}}_{\mathsf{D}}^{(\pm)}$ such that:

$$\langle \boldsymbol{\eta} | \hat{\boldsymbol{\eta}}_i \equiv \eta_i \langle \boldsymbol{\eta}|, \quad \langle \boldsymbol{\eta} | \hat{\boldsymbol{\eta}}_{\mathsf{A}}^{(\pm)} \equiv \eta_{\mathsf{A}}^{(\pm)} \langle \boldsymbol{\eta}|, \quad \langle \boldsymbol{\eta} | \hat{\boldsymbol{\eta}}_{\mathsf{D}}^{(\pm)} \equiv \eta_{\mathsf{D}}^{(\pm)} \langle \boldsymbol{\eta}|, \tag{6.12}$$

and the operator $\mathsf{T}_i^{\pm}$ are defined on the left and right SOV-representations by:[24]

$$\langle \boldsymbol{\eta} | \mathsf{T}_i^{\pm} \equiv \langle q^{\pm \delta_i} \boldsymbol{\eta}|, \quad \mathsf{T}_i^{\pm} | \boldsymbol{\eta} \rangle \equiv | q^{\mp \delta_i} \boldsymbol{\eta} \rangle \tag{6.13}$$

and clearly the commutation relations hold:

$$\mathsf{T}_i^{\pm} \hat{\boldsymbol{\eta}}_j = q^{\pm \delta_{i,j}} \hat{\boldsymbol{\eta}}_j \mathsf{T}_i^{\pm}. \tag{6.14}$$

**Lemma 6.1.** *We have the expansion*

$$\begin{aligned}\left(\hat{\boldsymbol{\Omega}}(f)\right)^k = & \sum_{\substack{\vec{\alpha}=\{\alpha_1 \ldots \alpha_{N-1}\} \\ \sum \alpha_i = k}} \begin{bmatrix} k \\ \vec{\alpha} \end{bmatrix} \prod_{i=1}^{\mathsf{N}-1} \\ & \times \left( \prod_{h=0}^{\alpha_i - 1} f(q^{-h} \hat{\boldsymbol{\eta}}_i) \prod_{j \neq i} \frac{1}{q^{\alpha_j - h} \hat{\boldsymbol{\eta}}_i / \hat{\boldsymbol{\eta}}_j - q^{-\alpha_j + h} \hat{\boldsymbol{\eta}}_j / \hat{\boldsymbol{\eta}}_i} \right) \prod_{i=1}^{\mathsf{N}-1} (\mathsf{T}_i^-)^{\alpha_i}\end{aligned} \tag{6.15}$$

*for the operator*

$$\hat{\boldsymbol{\Omega}}(f) = \sum_{a=1}^{\mathsf{N}-1} \prod_{b \neq a} \frac{1}{\hat{\boldsymbol{\eta}}_a / \hat{\boldsymbol{\eta}}_b - \hat{\boldsymbol{\eta}}_b / \hat{\boldsymbol{\eta}}_a} f(\hat{\boldsymbol{\eta}}_a) \mathsf{T}_a^-, \tag{6.16}$$

---

[24] It is worth remarking that from the definition of the SOV-representations of the generators of the Yang–Baxter algebra, given in Sect. 2.3, and the definitions in (6.13), it follows that the SOV-representation of the charge $\Theta$ coincides with the operator $\mathsf{T}_{\mathsf{N}}^-$.

*with*

$$\begin{bmatrix} k \\ \vec{\alpha} \end{bmatrix} \equiv \frac{[k]!}{\prod_{j=1}^{\mathsf{N}-1}[\alpha_j]!}, \ [k]! \equiv [k][k-1]\dots[1], \ [a] \equiv \frac{q^a - q^{-a}}{q - q^{-1}}. \tag{6.17}$$

*Proof.* The lemma holds for $k = 1$ and we prove it by induction for $k > 1$. Let us take $\mathsf{N} - 1$ integers $\alpha_i$:

$$\sum_{i=1}^{\mathsf{N}-1} \alpha_i = k, \tag{6.18}$$

from which we define the set of integers $I = \{i \in \{1, \dots, \mathsf{N}-1\} : \alpha_i \neq 0\}$ and $\hat{\mathbf{C}}^{(k)}_{\vec{\alpha}}$ as the operator coefficient of $\prod \mathsf{T}_i^{-\alpha_i}$ (put to the left) in the expansion of the $k$-th power of $\hat{\mathbf{\Omega}}(f)$. By writing $(\hat{\mathbf{\Omega}}(f))^k = (\hat{\mathbf{\Omega}}(f))^{k-1}\hat{\mathbf{\Omega}}(f)$ and using the induction hypothesis for the power $k-1$ of $\hat{\mathbf{\Omega}}(f)$, we have:

$$\begin{aligned} \hat{\mathbf{C}}^{(k)}_{\vec{\alpha}} &= \sum_{a\in I} \begin{bmatrix} k-1 \\ \vec{\alpha} - \vec{\delta}_a \end{bmatrix} \prod_{j=1}^{\mathsf{N}-1} \prod_{h=0}^{\alpha_j - \delta_{a,j} - 1} \\ &\quad \times \left( f(q^{-h}\hat{\boldsymbol{\eta}}_j) \times \prod_{i\neq j, i=1}^{\mathsf{N}-1} \frac{1}{q^{\alpha_i - \delta_{a,i} - h}\hat{\boldsymbol{\eta}}_j/\hat{\boldsymbol{\eta}}_i - \hat{\boldsymbol{\eta}}_i/q^{\alpha_i - \delta_{a,i} - h}\hat{\boldsymbol{\eta}}_j} \right) \\ &\quad \times f(\hat{\boldsymbol{\eta}}_a q^{-\alpha_a + 1}) \prod_{i\in I\setminus\{a\}} \frac{1}{q^{\alpha_a - \alpha_i - 1}\hat{\boldsymbol{\eta}}_i/\hat{\boldsymbol{\eta}}_a - \hat{\boldsymbol{\eta}}_a/q^{\alpha_a - \alpha_i - 1}\hat{\boldsymbol{\eta}}_i}, \end{aligned} \tag{6.19}$$

with $\vec{\delta}_a \equiv (\delta_{1,a}, \dots, \delta_{\mathsf{N},a})$. The first term in r.h.s. is the coefficient of $\prod \mathsf{T}_i^{-\alpha_i + \delta_{a,i}}$ in $(\hat{\mathbf{\Omega}}(f))^{k-1}$ and the second is the coefficient of $\mathsf{T}_a^{-1}$ in $\hat{\mathbf{\Omega}}(f)$ once the commutations between $\prod \mathsf{T}_i^{-\alpha_i + \delta_{a,i}}$ and the $\hat{\boldsymbol{\eta}}_i$ have been performed. Hence, we get:

$$\begin{aligned} \hat{\mathbf{C}}^{(k)}_{\vec{\alpha}} &= \frac{[k-1]!}{\prod[\alpha_i]!} \left( \prod_{j=1}^{\mathsf{N}-1} \prod_{h=0}^{\alpha_j - 1} ( \prod_{i\neq j, i=1}^{\mathsf{N}-1} \frac{1}{q^{\alpha_i - h}\hat{\boldsymbol{\eta}}_j/\hat{\boldsymbol{\eta}}_i - \hat{\boldsymbol{\eta}}_i/q^{\alpha_i - h}\hat{\boldsymbol{\eta}}_j} ) f(q^{-h}\hat{\boldsymbol{\eta}}_j) \right) \\ &\quad \times \sum_{a\in I} \left( [\alpha_a] \prod_{i\in I\setminus\{a\}} \frac{q^{\alpha_a}\hat{\boldsymbol{\eta}}_i/\hat{\boldsymbol{\eta}}_a - \hat{\boldsymbol{\eta}}_a/q^{\alpha_a}\hat{\boldsymbol{\eta}}_i}{q^{\alpha_a - \alpha_i}\hat{\boldsymbol{\eta}}_i/\hat{\boldsymbol{\eta}}_a - \hat{\boldsymbol{\eta}}_a/q^{\alpha_a - \alpha_i}\hat{\boldsymbol{\eta}}_i} \right), \end{aligned} \tag{6.20}$$

which leads to our result using the relation:

$$\sum_{a=1}^{n} [\alpha_a] \prod_{i\neq a} \frac{q^{\alpha_a}\eta_i/\eta_a - \eta_a/q^{\alpha_a}\eta_i}{q^{\alpha_a - \alpha_i}\eta_i/\eta_a - \eta_a/q^{\alpha_a - \alpha_i}\eta_i} = \left[ \sum_{a=1}^{n} \alpha_a \right]. \tag{6.21}$$

Note that the above formula holds for any $n$, for any set of numbers $\eta_i$ and for any non-negative integers $\alpha_i$. This is proven by studying the analytical properties of the function

$$g(z) = \frac{1}{z} \prod \frac{z - \eta_i^2}{z - q^{-2\alpha_i}\eta_i^2}. \tag{6.22}$$

$\square$

**Lemma 6.2.** *The SOV-representation of the powers of $B^{-1}(\lambda)A(\lambda)$ is given by*

$$\left(B^{-1}(\lambda)A(\lambda)\right)^m = \sum_{i+j+k=m} \frac{(-1)^j}{\hat{\boldsymbol{\eta}}_{\mathsf{N}}^m} \left(\lambda \prod_{a=1}^{\mathsf{N}-1} \hat{\boldsymbol{\eta}}_a\right)^{i-j} a_+^i a_-^j q^{\frac{i(i-1)-j(j-1)}{2}} \begin{bmatrix} m \\ i,j,k \end{bmatrix} \hat{\boldsymbol{\sigma}}(\lambda)^k \mathsf{T}_{\mathsf{N}}^{j-i} \tag{6.23}$$

*with*

$$\hat{\boldsymbol{\sigma}}(\lambda) = \sum_{a=1}^{\mathsf{N}-1} \prod_{b\neq a} \frac{1}{\hat{\boldsymbol{\eta}}_a/\hat{\boldsymbol{\eta}}_b - \hat{\boldsymbol{\eta}}_b/\hat{\boldsymbol{\eta}}_a} \frac{\mathsf{a}^{(\mathrm{sov})}(\hat{\boldsymbol{\eta}}_a)}{\lambda/\hat{\boldsymbol{\eta}}_a - \hat{\boldsymbol{\eta}}_a/\lambda} \mathsf{T}_a^-, \tag{6.24}$$

*where the powers of $\hat{\boldsymbol{\sigma}}(\lambda)$ are given by the previous lemma.*

*Proof.* Let $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ be three operators satisfying the relations

$$\hat{\mathbf{b}}\hat{\mathbf{a}} = q^{-2}\hat{\mathbf{a}}\hat{\mathbf{b}}, \hat{\mathbf{c}}\hat{\mathbf{b}} = q^2\hat{\mathbf{b}}\hat{\mathbf{c}}, \hat{\mathbf{c}}\hat{\mathbf{a}} = q^{-2}\hat{\mathbf{a}}\hat{\mathbf{c}} \tag{6.25}$$

It is easy to prove by induction that

$$\left(\hat{\mathbf{a}} + \hat{\mathbf{b}} + \hat{\mathbf{c}}\right)^m = \sum_{i+j+k=m} q^{k(j-i)-ij} \begin{bmatrix} m \\ i,j,k \end{bmatrix} \hat{\mathbf{a}}^i \hat{\mathbf{b}}^j \hat{\mathbf{c}}^k \tag{6.26}$$

The SOV-representation of $B^{-1}(\lambda)A(\lambda)$ is the sum of three main terms,

$$\hat{\mathbf{a}} = \frac{\prod_{i=1}^{\mathsf{N}-1} \hat{\boldsymbol{\eta}}_i}{\hat{\boldsymbol{\eta}}_{\mathsf{N}}} \lambda a_+ \mathsf{T}_{\mathsf{N}}^- \tag{6.27}$$

$$\hat{\mathbf{b}} = -\frac{\prod_{i=1}^{\mathsf{N}-1} \hat{\boldsymbol{\eta}}_i^{-1}}{\hat{\boldsymbol{\eta}}_{\mathsf{N}}} \lambda^{-1} a_- \mathsf{T}_{\mathsf{N}}^+ \tag{6.28}$$

$$\hat{\mathbf{c}} = \frac{1}{\hat{\boldsymbol{\eta}}_{\mathsf{N}}} \sum_{a=1}^{\mathsf{N}-1} \prod_{b\neq a} \frac{1}{\hat{\boldsymbol{\eta}}_a/\hat{\boldsymbol{\eta}}_b - \hat{\boldsymbol{\eta}}_b/\hat{\boldsymbol{\eta}}_a} \frac{\mathsf{a}^{(\mathrm{sov})}(\hat{\boldsymbol{\eta}}_a)}{\lambda/\hat{\boldsymbol{\eta}}_a - \hat{\boldsymbol{\eta}}_a/\lambda} \mathsf{T}_a^- \tag{6.29}$$

Since they satisfy the commutation relations (6.25), the power of $B^{-1}(\lambda)A(\lambda)$ can be computed using the formula ( 6.26), which ends the proof. □

*Remark* 1. The quantum multinomials have the property

$$\begin{bmatrix} p \\ \vec{\alpha} \end{bmatrix} = \begin{cases} 1 & \text{if } \exists i \in \{1,\ldots,\mathsf{N}-1\} : \alpha_i = p\delta_{a,i} \ \forall a \in \{1,\ldots,\mathsf{N}-1\}, \\ 0 & \text{otherwise,} \end{cases} \tag{6.30}$$

This property yields that the power $p$ of $\mathsf{B}^{-1}(\lambda)\mathsf{A}(\lambda)$ is a central element of the Yang–Baxter algebra and it reads:

$$(\mathsf{B}^{-1}(\lambda)\mathsf{A}(\lambda))^p = \mathcal{B}(\Lambda)^{-1}\mathcal{A}(\Lambda), \tag{6.31}$$

result which is consistent with the commutation relations:

$$\mathsf{B}^{-1}(q\lambda)\mathsf{A}(q\lambda) = \mathsf{A}(\lambda)\mathsf{B}^{-1}(\lambda). \tag{6.32}$$

The two previous lemmas allow to expand the SOV-representation of the operators $\mathsf{u}_n^k$. However, they do not apply directly to the expansion of $\mathsf{v}_n$. The aim of the following lemma is to transform the operators $\beta_{k,n}$, whose linear combination gives the powers of $\mathsf{v}_n$.

**Lemma 6.3.** *The operator $\beta_{k,n}$ has the following expansion:*

$$\beta_{k,n} = \frac{\mathcal{B}(\mu_{n,-}^p)}{\mathcal{A}(\mu_{n,-}^p)\mathcal{B}(\mu_{n,+}^p)} \frac{\mu_{n,+}/\mu_{n,-} - \mu_{n,-}/\mu_{n,+}}{q^k\mu_{n,+}/\mu_{n,-} - q^{-k}\mu_{n,-}/\mu_{n,+}} B^{-1}(\mu_{n,+})A(\mu_{n,+})$$
$$\times \prod_{i=1}^{p-k} B(q^{-i}\mu_{n,+}) \left(B^{-1}(\mu_{n,-})A(\mu_{n,-})\right)^{p-1} \prod_{i=p-k+1}^{p} B(q^{-i}\mu_{n,+})$$
$$+ \frac{q^k - q^{-k}}{q^k\mu_{n,+}/\mu_{n,-} - q^{-k}\mu_{n,-}/\mu_{n,+}} \tag{6.33}$$

*Proof.* A simple induction on the Yang–Baxter relation $B(\lambda)A(q^{-1}\lambda) = A(\lambda)B(q^{-1}\lambda)$ shows that

$$\left(A^{-1}(\lambda)B(\lambda)\right)^k = \prod_{i=1}^{k} B(q^{-i}\lambda) \prod_{i=1}^{k} A^{-1}(q^{-i}\lambda) = \prod_{i=0}^{k-1} A^{-1}(q^{i}\lambda) \prod_{i=0}^{k-1} B(q^{i}\lambda). \tag{6.34}$$

From the definition of the average values of operators, we get

$$\left(A^{-1}(\lambda)B(\lambda)\right)^k = \mathcal{A}(\Lambda)^{-1} \prod_{i=1}^{p-k} A(q^{-i}\lambda) \prod_{i=p-k+1}^{p} B(q^{-i}\lambda), \tag{6.35}$$

$$\left(A^{-1}(\lambda)B(\lambda)\right)^{-k} = \mathcal{B}(\Lambda)^{-1} \prod_{i=1}^{p-k} B(q^{-i}\lambda) \prod_{i=p-k+1}^{p} A(q^{-i}\lambda). \tag{6.36}$$

It also yields

$$\left(A^{-1}(\lambda)B(\lambda)\right)^p = \mathcal{A}^{-1}(\Lambda)\mathcal{B}(\Lambda) \tag{6.37}$$

and

$$A^{-1}(\lambda)B(\lambda) = \mathcal{A}^{-1}(\Lambda)\mathcal{B}(\Lambda)\left(B^{-1}(\lambda)A(\lambda)\right)^{p-1}. \tag{6.38}$$

Standard arguments give the relation

$$B(\mu_{n,-}) \prod_{i=1}^{p-k} A(q^{-i}\mu_{n,+})$$
$$= \frac{q^k - q^{-k}}{q^k\mu_{n,+}/\mu_{n,-} - q^{-k}\mu_{n,-}/\mu_{n,+}} A(\mu_{n,-}) \prod_{i=1}^{p-k-1} A(q^{-i}\mu_{n,+})B(q^k\mu_{n,+})$$
$$+ \frac{\mu_{n,+}/\mu_{n,-} - \mu_{n,-}/\mu_{n,+}}{q^k\mu_{n,+}/\mu_{n,-} - q^{-k}\mu_{n,-}/\mu_{n,+}} \prod_{i=1}^{p-k} A(q^{-i}\mu_{n,+})B(\mu_{n,-}). \tag{6.39}$$

Eventually, the use of these relations proves the lemma. □

## 7. Form Factors of Local Operators

In this section, we present the main results of our paper on the form factors of the local operators. One of the main peculiarities emerging in quantum separate variables is a feature of universality in the representation of these dynamical observables. In fact, the comparison between the results presented here for the most general cyclic representations of the six-vertex Yang–Baxter algebra and those previously derived in our paper [1] defines one peculiar and evident instance of this universality.

### 7.1. Form Factors of $\mathbf{u}_n^{-1}$ and $\boldsymbol{\alpha}_{0,n}^{-1}$

The form factors of some local operators written as single determinants are here provided.

**Proposition 7.1.** *Let us denote with* $\varphi_n^{(t_k)}$ *and* $\varphi_n^{(t'_{k'})}$ *the eigenvalues of the shift operator* $\mathsf{U}_n$*, respectively, on the left* $\langle t_k|$ *and right* $|t'_{k'}\rangle$ *eigenstates of the transfer matrix* $\tau_2(\lambda)$*, then the following determinant formula is verified:*

$$\langle t_k|\mathsf{u}_n^{-1}|t'_{k'}\rangle = \left(-\frac{\mathbb{a}_n\mathbb{b}_n}{\alpha_n\beta_n}\right)^{1/2}\frac{\varphi_n^{(t_k)}}{\varphi_n^{(t'_{k'})}}\delta_{k,k'-1}\det_{\mathsf{N}-1}(||\mathcal{U}_{a,b}^{(t_k,t'_{k'})}(\mu_{n,+})||). \tag{7.1}$$

*Here,* $||\mathcal{U}_{a,b}^{(t_k,t'_{k'})}(\lambda)||$ *is the* $(\mathsf{N}-1)\times(\mathsf{N}-1)$ *matrix defined by:*

$$\mathcal{U}_{a,b}^{(t_k,t'_{k'})}(\lambda) \equiv \mathcal{M}_{a,b+1/2}^{(t_k,t'_{k'})} \quad \textit{for } b\in\{1,\ldots,\mathsf{N}-2\}, \tag{7.2}$$

$$\begin{aligned}
&\mathcal{U}_{a,\mathsf{N}-1}^{(t_k,t'_{k'})}(\lambda)\\
&\equiv \frac{1}{\eta_{\mathsf{N}}^{(0)}}\sum_{h=1}^{p}\frac{\left(\eta_a^{(h)}\right)^{\mathsf{N}-2}Q_{t'_{k'}}\left(\eta_a^{(h)}\right)}{\omega_a\left(\eta_a^{(h)}\right)}\left[\frac{\bar{Q}_{t_k}(\eta_a^{(h+1)})}{(\lambda/\eta_a^{(h+1)}-\eta_a^{(h+1)}/\lambda)}\bar{\mathsf{a}}^{(\mathrm{sov})}\left(\eta_a^{(h)}\right)\right.\\
&\quad\left.+\bar{Q}_{t_k}\left(\eta_a^{(h)}\right)\left(a_+\lambda\left(\eta_a^{(h)}\right)^{\mathsf{N}-1}q^{k'}-\frac{a_-}{\lambda}\left(\eta_a^{(h)}\right)^{-(\mathsf{N}-1)}q^{-k'}\right)\right].
\end{aligned} \tag{7.3}$$

*Proof.* The operator $\mathsf{B}^{-1}(\lambda)\mathsf{A}(\lambda)$ admits the following SOV-representation:

$$\begin{aligned}
\mathsf{B}^{-1}(\lambda)\mathsf{A}(\lambda) = \frac{1}{\hat{\boldsymbol{\eta}}_{\mathsf{N}}}&\left(\lambda\hat{\boldsymbol{\eta}}_{\mathsf{A}}^{(+)}\mathsf{T}_{\mathsf{N}}^{-}+\frac{\hat{\boldsymbol{\eta}}_{\mathsf{A}}^{(-)}}{\lambda}\mathsf{T}_{\mathsf{N}}^{+}\right)\\
&+\sum_{a=1}^{\mathsf{N}-1}\mathsf{T}_a^{-}\frac{\bar{\mathsf{a}}^{(\mathrm{sov})}(\hat{\boldsymbol{\eta}}_a)}{\hat{\boldsymbol{\eta}}_{\mathsf{N}}(\lambda/\hat{\boldsymbol{\eta}}_a q-\hat{\boldsymbol{\eta}}_a q/\lambda)}\prod_{b\neq a}\frac{1}{(\hat{\boldsymbol{\eta}}_a/\hat{\boldsymbol{\eta}}_b-\hat{\boldsymbol{\eta}}_b/\hat{\boldsymbol{\eta}}_a)}.
\end{aligned} \tag{7.4}$$

For brevity, we denote with $[\mathsf{B}^{-1}(\lambda)\mathsf{A}(\lambda)]$ the sum on the r.h.s. of (7.4). Then, from the SOV-decomposition of the $\tau_2$-eigenstates, it holds:

$$\begin{aligned}
&\langle t_k|[\mathsf{B}^{-1}(\lambda)\mathsf{A}(\lambda)]|t'_{k'}\rangle\\
&\quad=\frac{\sum_{h_{\mathsf{N}}=1}^{p}q^{(k+1-k')h_{\mathsf{N}}}}{p\eta_{\mathsf{N}}^{(0)}}\sum_{a=1}^{\mathsf{N}-1}\sum_{h_1,\ldots,h_{\mathsf{N}-1}=1}^{p}V\left(\left(\eta_1^{(h_1)}\right)^2,\ldots,\left(\eta_{\mathsf{N}-1}^{(h_{\mathsf{N}-1})}\right)^2\right)
\end{aligned}$$

$$\times \prod_{b\neq a,b=1}^{\mathsf{N}-1} \frac{\eta_b^{(h_b)} Q_{t'_{k'}}\left(\eta_b^{(h_b)}\right) \bar{Q}_{t_k}\left(\eta_b^{(h_b)}\right)}{\omega_b\left(\eta_b^{(h_b)}\right)\left(\left(\eta_a^{(h_a)}\right)^2-\left(\eta_b^{(h_b)}\right)^2\right)}$$
$$\times \frac{\bar{Q}_{t_k}\left(\eta_a^{(h+1)}\right) Q_{t'_{k'}}\left(\eta_a^{(h_a)}\right)}{\omega_a\left(\eta_a^{(h_a)}\right)} \frac{\left(\eta_a^{(h_a)}\right)^{(\mathsf{N}-2)} \bar{\mathsf{a}}^{(\mathrm{sov})}\left(\eta_a^{(h_a)}\right)}{(\lambda/\eta_a^{(0)} q^{h_a+1}-\eta_a^{(0)} q^{h_a+1}/\lambda)}, \tag{7.5}$$

and so:

$$\begin{aligned}
&\langle t_k|[\mathsf{B}^{-1}(\lambda)\mathsf{A}(\lambda)]|t'_{k'}\rangle \\
&\quad = \frac{\delta_{k,k'-1}}{\eta_{\mathsf{N}}^{(0)}} \sum_{a=1}^{\mathsf{N}-1} \underbrace{\sum_{h_1,\ldots,h_{\mathsf{N}}=1}^{p}}_{h_a \text{ is missing.}} \underbrace{\hat{V}_a(\left(\eta_1^{(h_1)}\right)^2,\ldots,\left(\eta_{\mathsf{N}-1}^{(h_{\mathsf{N}-1})}\right)^2)}_{\text{(The row } a \text{ is removed.)}} \\
&\quad\times \prod_{b\neq a,b=1}^{\mathsf{N}-1} \frac{\eta_b^{(h_b)} Q_{t'_{k'}}\left(\eta_b^{(h_b)}\right) \bar{Q}_{t_k}\left(\eta_b^{(h_b)}\right)}{\omega_b\left(\eta_b^{(h_b)}\right)} \\
&\quad\times(-1)^{(\mathsf{N}-1+a)} \sum_{h_a=1}^{p} \frac{\bar{Q}_{t_k}(\eta_a^{(0)} q^{h_a+1}) Q_{t'_{k'}}\left(\eta_a^{(h_a)}\right)\left(\eta_a^{(h_a)}\right)^{(\mathsf{N}-2)} \bar{\mathsf{a}}^{(\mathrm{sov})}\left(\eta_a^{(h_a)}\right)}{\omega_a\left(\eta_a^{(h_a)}\right)(\lambda/\eta_a^{(h_a+1)}-\eta_a^{(h_a+1)}/\lambda)},
\end{aligned} \tag{7.6}$$

inserting the sum over $(h_1,\ldots,\widehat{h_a},\ldots,h_{\mathsf{N}-1})$ in the Vandermonde determinant $\hat{V}_a$, the above expression reduces to the expansion of the following determinant:

$$\langle t_k|[\mathsf{B}^{-1}(\lambda)\mathsf{A}(\lambda)]|t'_{k'}\rangle = \delta_{k,k'-1} \det_{\mathsf{N}-1}\left(\left\|\left[\mathcal{U}_{a,b}^{(t_k,t'_{k'})}(\lambda)\right]\right\|\right), \tag{7.7}$$

where $\left[\mathcal{U}_{a,b}^{(t_k,t'_{k'})}(\lambda)\right]$ is just $\mathcal{M}_{a,b+1/2}^{(t_k,t'_{k'})}$ for $b\in\{1,\ldots,\mathsf{N}-2\}$, while:

$$\left[\mathcal{U}_{a,\mathsf{N}-1}^{(t_k,t'_{k'})}(\lambda)\right] \equiv \frac{\left(\eta_a^{(0)}\right)^{\mathsf{N}-2}}{\eta_{\mathsf{N}}^{(0)}} \sum_{h=1}^{p} \frac{q^{(\mathsf{N}-2)h} Q_{t'_{k'}}\left(\eta_a^{(h)}\right) \bar{Q}_{t_k}\left(\eta_a^{(h_a+1)}\right)}{\omega_a\left(\eta_a^{(h)}\right)(\lambda/\eta_a^{(h_a+1)}-\eta_a^{(h_a+1)}/\lambda)} \bar{\mathsf{a}}^{(\mathrm{sov})}(\eta_a^{(h)}). \tag{7.8}$$

We compute now the matrix elements:

$$\begin{aligned}
&\langle t_k|\hat{\boldsymbol{\eta}}_{\mathsf{N}}^{-1}\hat{\boldsymbol{\eta}}_{\mathsf{A}}^{(\pm)}\mathsf{T}_{\mathsf{N}}^{\mp}|t'_{k'}\rangle \\
&\quad = \frac{\pm a_{\pm} q^{\pm k'} \sum_{h_{\mathsf{N}}=1}^{p} q^{(k+1-k')h_{\mathsf{N}}}}{p\eta_{\mathsf{N}}^{(0)}} \sum_{h_1,\ldots,h_{\mathsf{N}-1}=1}^{p} V(\left(\eta_1^{(h_1)}\right)^2,\ldots,\left(\eta_{\mathsf{N}-1}^{(h_{\mathsf{N}-1})}\right)^2) \\
&\quad\times \prod_{b=1}^{\mathsf{N}-1} \frac{\left(\eta_b^{(h_b)}\right)^{\pm 1} Q_{t'_{k'}}\left(\eta_b^{(h_b)}\right) \bar{Q}_{t_k}(\eta_b^{(h_b)})}{\omega_b\left(\eta_b^{(h_b)}\right)},
\end{aligned} \tag{7.9}$$

hence leading to:

$$\left\langle t_k|\hat{\boldsymbol{\eta}}_{\mathsf{N}}^{-1}\hat{\boldsymbol{\eta}}_{\mathsf{A}}^{(\pm)}\mathsf{T}_{\mathsf{N}}^{\mp}|t'_{k'}\right\rangle = \frac{\pm a_{\pm}q^{\pm k'}\delta_{k,k'-1}}{\eta_{\mathsf{N}}^{(0)}}\det_{\mathsf{N}-1}\left(\left\|\mathcal{M}_{a,b\pm 1/2}^{(t_k,t'_{k'})}\right\|\right). \tag{7.10}$$

Then, our result follows as the matrices of formula (7.7) and (7.10) have $\mathsf{N}-2$ common columns. Let us note that the above formula holds for any value of $\lambda$. □

*Remark* 2. (I) The matrix elements $\langle t_k|\alpha_{0,n}^{-1}|t'_{k'}\rangle$ of the local operators $\alpha_{0,n}^{-1}$ are given by:

$$\left\langle t_k|\alpha_{0,n}^{-1}|t'_{k'}\right\rangle = \frac{\varphi_n^{(t_k)}}{\varphi_n^{(t'_{k'})}}\delta_{k,k'-1}\det_{\mathsf{N}-1}\left(\left\|\mathcal{U}_{a,b}^{(t_k,t'_{k'})}(\mu_{n,-})\right\|\right). \tag{7.11}$$

(II) In the case of general representations $\mathcal{R}_{\mathsf{N}}$, the matrix elements $\langle t_k|\mathsf{u}_n|t'_{k'}\rangle$ can be computed using the reconstruction:

$$\mathsf{u}_n = \left(-\frac{\alpha_n\beta_n}{\mathbb{a}_n\mathbb{b}_n}\right)^{1/2}\mathsf{U}_n\mathsf{C}^{-1}(\mu_{n,+})\mathsf{D}(\mu_{n,+})\mathsf{U}_n^{-1}, \tag{7.12}$$

in the SOV $\mathsf{C}$-representation. Here, we do not make this explicitly as the result will have the same type of form presented for $\langle t_k|\mathsf{u}_n^{-1}|t'_{k'}\rangle$; the difference will be that all the quantities will be written in the SOV $\mathsf{C}$-representation.

### 7.2. Determinant Representations of Form Factors for a Suitable Basis of Operators

In this section, we construct an operator basis for which the form factors of any operator in this basis are written by a one determinant formula. For this reason, we will refer to it as the basis of elementary operators. The idea of the construction goes back to the sine–Gordon case [1].

#### 7.2.1. Introduction of the Basis of Elementary Operators.

**Lemma 7.1.** *Let us define the operators:*

$$\mathcal{O}_{a,k} \equiv \frac{\mathsf{B}\left(\eta_a^{(p+k-1)}\right)\mathsf{B}(\eta_a^{(p+k-2)})\ldots\mathsf{B}\left(\eta_a^{(k+1)}\right)\mathsf{A}\left(\eta_a^{(k)}\right)}{p\hat{\boldsymbol{\eta}}_{\mathsf{N}}^{p-1}\prod_{b\neq a,b=1}^{\mathsf{N}-1}(Z_a/Z_b - Z_b/Z_a)} \quad \text{with } k\in\{0,\ldots,p-1\}, \tag{7.13}$$

*with* $Z_r \equiv \eta_r^p$ *as in* (2.48), *then they satisfy the following properties:*

$$\mathcal{O}_{a,k}\mathcal{O}_{a,h} \text{ is non-zero if and only if } h = k-1, \tag{7.14}$$

*and*

$$\mathcal{O}_{a,k}\mathcal{O}_{a,k-1}\ldots\mathcal{O}_{a,k+1-p}\mathcal{O}_{a,k-p} = \frac{\mathcal{A}(Z_a)}{\prod_{b\neq a,b=1}^{\mathsf{N}-1}(Z_a/Z_b - Z_b/Z_a)}\mathcal{O}_{a,k}. \tag{7.15}$$

*The following commutation relations are furthermore satisfied:*

$$\hat{\boldsymbol{\eta}}_{\mathsf{A}}^{(\pm)}\mathcal{O}_{a,k} = q^{\mp 1}\mathcal{O}_{a,k}\hat{\boldsymbol{\eta}}_{\mathsf{A}}^{(\pm)}, \quad [\hat{\boldsymbol{\eta}}_{\mathsf{N}},\mathcal{O}_{a,k}] = [\mathsf{T}_{\mathsf{N}}^{-},\mathcal{O}_{a,k}] = 0, \tag{7.16}$$

*and*

$$\mathcal{O}_{a,k}\mathcal{O}_{b,h}=\frac{\left(\eta_a^{(k-h+1)}/\eta_b^{(0)}-\eta_b^{(0)}/\eta_a^{(k-h+1)}\right)}{\left(\eta_a^{(k-h-1)}/\eta_b^{(0)}-\eta_b^{(0)}/\eta_a^{(k-h-1)}\right)}\,\mathcal{O}_{b,h}\mathcal{O}_{a,k} \tag{7.17}$$

*for* $a\neq b\in\{1,\ldots,\mathsf{N}-1\}$.

*Proof.* Since $\mathcal{B}(Z_a)=0$ with $\mathcal{B}(\Lambda)$, the average value of the operator $\mathsf{B}(\lambda)$, the first is quite immediate. Moreover, the following identity:

$$\langle\eta_1,\ldots,\eta_a^{(h)},\ldots,\eta_{\mathsf{N}}|\mathcal{O}_{a,k}=\frac{a\left(\eta_a^{(k)}\right)\delta_{h,k}}{\prod_{b\neq a,b=1}^{\mathsf{N}}\left(\eta_a^{(k)}/\eta_b-\eta_b/\eta_a^{(k)}\right)}\times\langle\eta_1,\ldots,\eta_a^{(k-1)},\ldots,\eta_{\mathsf{N}}|, \tag{7.18}$$

is a direct consequence of the definition of the operators $\mathcal{O}_{a,k}$ so that the second identity of the lemma follows. Now, using the following Yang–Baxter commutation relation:

$$(\lambda/\mu-\mu/\lambda)\mathsf{A}(\lambda)\mathsf{B}(\mu)=(\lambda/q\mu-\mu q/\lambda)\mathsf{B}(\mu)\mathsf{A}(\lambda)+(q-q^{-1})\mathsf{B}(\lambda)\mathsf{A}(\mu) \tag{7.19}$$

and moving the $\mathsf{A}(\eta_a^{(k)})$ to the right through all the $\mathsf{B}(\eta_b^{(j)})$, for $j\neq h$, remarking that only the first term of the r.h.s of (7.19) survives, and after moving the $\mathsf{A}(\eta_b^{(h)})$ to the left, we get the last identity of the lemma. □

Now, we define *elementary operators* by the following monomials:

$$\mathcal{E}_{k,k_0,(a_1,k_1),\ldots,(a_r,k_r)}^{(\alpha_1,\ldots,\alpha_r)}\equiv\hat{\boldsymbol{\eta}}_{\mathsf{N}}^{-k}\left(\hat{\boldsymbol{\eta}}_{\mathsf{A}}^{(+)}\mathsf{T}_{\mathsf{N}}^{-}\right)^{k_0}\mathcal{O}_{a_1,k_1}^{(\alpha_1)}\ldots\mathcal{O}_{a_r,k_r}^{(\alpha_r)}, \tag{7.20}$$

where $\sum_{h=1}^{r}\alpha_h\leq p$, $k,k_i\in\{0,\ldots,p-1\}$, $a_i<a_j\in\{1,\ldots,\mathsf{N}-1\}$ for $i<j\in\{1,\ldots,\mathsf{N}-1\}$ and:

$$\mathcal{O}_{a,k}^{(\alpha)}\equiv\mathcal{O}_{a,k}\mathcal{O}_{a,k-1}\ldots\mathcal{O}_{a,k+1-\alpha},\quad\text{with }\alpha\in\{1,\ldots,p\}. \tag{7.21}$$

**Lemma 7.2.** *Once the set of the elementary operators is dressed by the shift operator* $\mathsf{U}_n$ *as it follows:*

$$\mathsf{U}_n\mathcal{E}_{k,k_0,(a_1,k_1),\ldots,(a_r,k_r)}^{(\alpha_1,\ldots,\alpha_r)}\mathsf{U}_n^{-1}, \tag{7.22}$$

*a basis is defined in the space of the local operators at the quantum site* $n$, $\forall n\in\{1,\ldots,\mathsf{N}\}$.

*Proof.* To prove the lemma, the local operators in site $n$ generated by $\mathsf{u}_n^k$ and $\mathsf{v}_n^k$ for $k\in\{1,\ldots,p-1\}$ have to be written as linear combinations of the dressed elementary operators (7.20) and thanks to Proposition 6.2 this is equivalent to prove the same statement for the following basis of local operators:

$$\mathsf{u}_n^{-k}=\mathsf{U}_n\left(\mathsf{B}^{-1}(\mu_{n,+})\mathsf{A}(\mu_{n,+})\right)^k\mathsf{U}_n^{-1}, \tag{7.23}$$

$$\tilde{\beta}_{k,n}=\mathsf{U}_n\left(\mathsf{B}^{-1}(\mu_{n,+})\mathsf{A}(\mu_{n,+})\right)^k\mathsf{B}^{-1}(\mu_{n,-})\mathsf{A}(\mu_{n,-})\times\left(\mathsf{B}^{-1}(\mu_{n,+})\mathsf{A}(\mu_{n,+})\right)^{p-1-k}\mathsf{U}_n^{-1} \tag{7.24}$$

The operator $\mathsf{B}^{-1}(\lambda)$ is invertible for $\lambda^p \neq Z_a$ with $a \in \{1,\ldots,\mathsf{N}-1\}$ so that the centrality of the average values implies:

$$\mathsf{B}^{-1}(\lambda)\mathsf{A}(\lambda) = \frac{\mathsf{B}(\lambda q^{p-1})\mathsf{B}(\lambda q^{p-2})\ldots\mathsf{B}(\lambda q)\mathsf{A}(\lambda)}{\mathcal{B}(\Lambda)}. \tag{7.25}$$

The monomial $\mathsf{B}(\lambda q^{p-1})\mathsf{B}(\lambda q^{p-2})\ldots\mathsf{B}(\lambda q)\mathsf{A}(\lambda)$ is an even Laurent polynomial of degree $p(\mathsf{N}-1)+1$ in $\lambda$ and so we can write:

$$\mathsf{B}^{-1}(\lambda)\mathsf{A}(\lambda) = \frac{1}{\hat{\boldsymbol{\eta}}_{\mathsf{N}}}\left(\lambda\hat{\boldsymbol{\eta}}_{\mathsf{A}}^{(+)}\mathsf{T}_{\mathsf{N}}^{-} + \frac{\hat{\boldsymbol{\eta}}_{\mathsf{A}}^{(-)}}{\lambda}\mathsf{T}_{\mathsf{N}}^{+}\right) + \frac{1}{\hat{\boldsymbol{\eta}}_{\mathsf{N}}}\sum_{a=1}^{\mathsf{N}-1}\sum_{k=0}^{p-1}\frac{\mathcal{O}_{a,k}}{(\lambda/\eta_a^{(k)} - \eta_a^{(k)}/\lambda)}. \tag{7.26}$$

It is then clear that the local operators $\mathsf{u}_n^{-k}$ and $\tilde{\beta}_{k,n}$ are linear combinations of the monomials:

$$\mathsf{U}_n\hat{\boldsymbol{\eta}}_{\mathsf{N}}^{-h}\left(\hat{\boldsymbol{\eta}}_{\mathsf{A}}^{(+)}\mathsf{T}_{\mathsf{N}}^{-}\right)^{h_0}\mathcal{O}_{a_1,h_1}\ldots\mathcal{O}_{a_s,h_s}\mathsf{U}_n^{-1} \tag{7.27}$$

for $s \leq p$, $a_i \in \{1,\ldots,\mathsf{N}-1\}$ and $h, h_i \in \{0,\ldots,p-1\}$. The commutation rules (7.17) allow to rewrite any monomial $\mathcal{O}_{a_1,h_1}\ldots\mathcal{O}_{a_s,h_s}$ in a way that operators with the same index $a$ are adjacent and those with different $a$ are ordered in a way $a_i < a_j$ for $i < j \in \{1,\ldots,\mathsf{N}-1\}$. Then, the rule (7.14) tells us if the monomial is zero or non-zero. The property (7.15) finally implies:

$$\mathcal{O}_{a,k}^{(p+\alpha)} = \frac{\mathcal{A}(Z_a)}{\prod_{b\neq a,b=1}^{\mathsf{N}}(Z_a/Z_b - Z_b/Z_a)}\mathcal{O}_{a,k}^{(\alpha)}, \tag{7.28}$$

and so that all the non-zero monomials $\mathcal{O}_{a_1,h_1}\ldots\mathcal{O}_{a_s,h_s}$ are rewritable in the form (7.20). □

#### 7.2.2. Determinant Representation of Elementary Operator Form Factors.

**Lemma 7.3.** *The elementary operators admit the following simple characterizations for their form factors:*

$$\left\langle t_k|\mathcal{E}_{(h,h_0,(a_1,h_1),\ldots,(a_r,h_r))}^{(\alpha_1,\ldots,\alpha_r)}|t'_{k'}\right\rangle = \frac{\delta_{k,k'+h}a_+^{h_0}q^{h_0k'}}{\left(\eta_{\mathsf{N}}^{(0)}\right)^h}\mathsf{f}_{(h_0,\{\alpha\},\{a\})}\det_{\mathsf{N}-1+rp-g}\left(\left\|\mathrm{O}_{a,b}^{(h_0,\{\alpha\},\{a\})}\right\|\right). \tag{7.29}$$

*Here, $\langle t_k|$ and $|t'_{k'}\rangle$ are two eigenstates of the transfer matrix $\tau_2(\lambda)$ and $||\mathrm{O}_{a,b}^{(h_0,\{\alpha\},\{a\})}||$ is the $(\mathsf{N}-1+rp-g)\times(\mathsf{N}-1+rp-g)$ matrix of elements:*

$$\mathrm{O}_{a,\sum_{h=1}^{m-1}(p-\alpha_h+1)+j_m}^{(h_0,\{\alpha\},\{a\})} \equiv \left(\eta_{a_m}^2 q^{2j_m}\right)^{2(a-1)} \quad \textit{for } j_m \in \{0,\ldots,p-\alpha_m\}, \quad m \in \{1,\ldots,r\}, \tag{7.30}$$

$$\mathrm{O}_{a,\sum_{h=1}^{r}(p-\alpha_h+1)+i}^{(h_0,\{\alpha\},\{a\})} \equiv \mathcal{M}_{b_i,a+(h_0+g)/2}^{(t,t')}, \quad \textit{for } i \in \{1,\ldots,\mathsf{N}-1-r\}, \quad g \equiv \sum_{h=1}^{r}\alpha_h, \tag{7.31}$$

*for any* $a \in \{1,\dots,\mathsf{N}-1+rp-g\}$. *Moreover, we have used the following notations* $\{b_1,\dots,b_{\mathsf{N}-1-r}\} \equiv \{1,\dots,\mathsf{N}-1\}\backslash\{a_1,\dots,a_r\}$ *where the elements are ordered by* $b_i < b_j$ *for* $i < j$,

$$\begin{aligned}
&\mathsf{f}_{(h_0,\{\alpha\},\{a\})}\\
&\equiv \frac{\prod_{i=1}^{r} Q_{t'}\left(\eta_{a_i} q^{-\alpha_i}\right)\bar{Q}_t\left(\eta_{a_i}\right)\left(\eta_{a_i}^{h_0+\alpha_i(\mathsf{N}-1-r)}/\omega_{a_i}\left(\eta_{a_i}\right)\right)\prod_{h=0}^{\alpha_i-1} a\left(\eta_i q^{-h}\right)}{\prod_{i=1}^{r}\prod_{h=0}^{\alpha_i-1}\prod_{j=1}^{i-1}(q^{\alpha_j-h}\eta_{a_i}/\eta_{a_j}-\eta_{a_j}/q^{\alpha_j-h}\eta_{a_i})\prod_{j=i+1}^{r}\left(\eta_{a_i}/q^h\eta_{a_j}-\eta_{a_j}q^h/\eta_{a_i}\right)}\\
&\times\frac{(-1)^{\sum_{i=1}^{r}(a_i-i)}\prod_{i=1}^{r} q^{-(\mathsf{N}-1-r)\alpha_i(\alpha_i-1)/2}V\left(\eta_{a_1}^2,\dots,\eta_{a_r}^2\right)}{\prod_{i=1}^{r}\prod_{j=1}^{\mathsf{N}-1-r}(Z_{a_i}^2-Z_{b_j}^2)V\left(\eta_{a_1}^2,\eta_{a_1}^2q^2,\dots,\eta_{a_1}^2q^{2(p-\alpha_1)},\dots,\eta_{a_r}^2,\eta_{a_r}^2q^2,\dots,\eta_{a_r}^2q^{2(p-\alpha_r)}\right)},
\end{aligned}\tag{7.32}$$

$V(x_1,\dots,x_{\mathsf{N}}) \equiv \prod_{1\le a<b\le\mathsf{N}}(x_a-x_b)$ *is the Vandermonde determinant and for brevity:*

$$\eta_{a_m} \equiv \eta_{a_m}^{(h_m)}.\tag{7.33}$$

*Proof.* The following actions hold:

$$\begin{aligned}
\langle t_k|\hat{\boldsymbol{\eta}}_{\mathsf{N}}^{-h}\left(\hat{\boldsymbol{\eta}}_{\mathsf{A}}^{(+)}\mathsf{T}_{\mathsf{N}}^{-}\right)^{h_0} &= \frac{a_+^{h_0}q^{h_0(k-h)}}{\left(\eta_{\mathsf{N}}^{(0)}\right)^h}\sum_{h_1,\dots,h_{\mathsf{N}}=1}^{p}\frac{q^{(k-h)h_{\mathsf{N}}}}{p^{1/2}}\prod_{a=1}^{\mathsf{N}-1}\left(\eta_a^{(h_a)}\right)^{h_0}\bar{Q}_t\left(\eta_a^{(h_a)}\right)\\
&\times\prod_{1\le a<b\le\mathsf{N}-1}\left(\left(\eta_a^{(h_a)}\right)^2-(\eta_b^{(h_b)})^2\right)\frac{\langle\eta_1^{(h_1)},\dots,\eta_{\mathsf{N}}^{(h_{\mathsf{N}})}|}{\prod_{b=1}^{\mathsf{N}-1}\omega_b\left(\eta_b^{(h_b)}\right)}.
\end{aligned}\tag{7.34}$$

From the formula (7.18), it follows:

$$\langle\eta_1,\dots,\eta_{a_i}^{(f)},\dots,\eta_{\mathsf{N}}|\mathcal{O}_{a_i,h_i}^{(\alpha_i)} = \frac{\prod_{h=0}^{\alpha_i-1}a\left(\eta_{a_i}q^{-h}\right)\delta_{f,h_i}\langle\eta_1,\dots,\eta_{a_i}q^{-\alpha_i},\dots,\eta_{\mathsf{N}}|}{\prod_{b\neq a_i,b=1}^{\mathsf{N}-1}\prod_{h=0}^{\alpha_i-1}\left(\eta_{a_i}q^{-h}/\eta_b-\eta_b/\eta_{a_i}q^{-h}\right)},\tag{7.35}$$

where $\eta_{a_i}$ is defined in (7.33). The action of $\mathcal{O}_{a_1,h_1}^{(\alpha_1)}\dots\mathcal{O}_{a_r,h_r}^{(\alpha_r)}$ can be computed now taking into account the order of the operators which appear in the monomial, then using the scalar product formula we get:

$$\begin{aligned}
&\left\langle t_k|\mathcal{E}_{(h,h_0,(a_1,h_1),\dots,(a_r,h_r)}^{(\alpha_1,\dots,\alpha_r)}|t'_{k'}\right\rangle\\
&\quad= \frac{a_+^{h_0}q^{h_0(k-h)}}{\left(\eta_{\mathsf{N}}^{(0)}\right)^h}\sum_{k_1,\dots,k_{\mathsf{N}}=1}^{p}\frac{q^{\left[(k-h)-k'\right]k_{\mathsf{N}}}}{p}\prod_{a=1}^{\mathsf{N}-1}\left(\eta_a^{(h_a)}\right)^{h_0}\\
&\qquad\times\prod_{i=1}^{r}\frac{\prod_{h=0}^{\alpha_i-1}a\left(\eta_{a_i}q^{-h}\right)\delta_{k_{a_i},h_i}}{\prod_{j=1}^{\mathsf{N}-1-r}\prod_{h=0}^{\alpha_i-1}\left(\eta_{a_i}q^{-h}/\eta_{b_j}^{(k_{b_j})}-\eta_{b_j}^{(k_{b_j})}/\eta_{a_i}q^{-h}\right)}\\
&\qquad\times\prod_{i=1}^{r}\prod_{h=0}^{\alpha_i-1}\frac{\prod_{j=i+1}^{r}(\eta_{a_i}q^{-h}/\eta_{a_j}-\eta_{a_j}/\eta_{a_i}q^{-h})^{-1}}{\prod_{j=1}^{i-1}\left(\eta_{a_i}q^{\alpha_j-h}/\eta_{a_j}-\eta_{a_j}/\eta_{a_i}q^{\alpha_j-h}\right)}
\end{aligned}$$

$$\times \prod_{j=1}^{\mathsf{N}-1-r} \frac{Q_{t'}\left(\eta_{b_j}^{(k_{b_j})}\right)\bar{Q}_t\left(\eta_{b_j}^{(k_{b_j})}\right)}{\omega_{b_j}\left(\eta_{b_j}^{(k_{b_j})}\right)} \prod_{i=1}^{r} \frac{Q_{t'}\left(\eta_{a_i}q^{-\alpha_i}\right)\bar{Q}_t\left(\eta_{a_i}\right)}{\omega_{a_i}\left(\eta_{a_i}\right)} V$$
$$\times \left(\eta_1^2,\ldots,\eta_{\mathsf{N}-1}^2\right). \tag{7.36}$$

The presence of the $\prod_{i=1}^{r}\delta_{k_{a_i},h_i}$ reduces the sum $\sum_{k_1,\ldots,k_{\mathsf{N}}=1}^{p}$ to $\delta_{k,k'+h}$ times the sum $\sum_{k_{b_1},\ldots,k_{b_{\mathsf{N}-(r+1)}}=1}^{p}$ where:

$$\{a_1,\ldots,a_r\}\cup\{b_1,\ldots,b_{\mathsf{N}-(r+1)}\}=\{1,\ldots,\mathsf{N}-1\}. \tag{7.37}$$

We get our formula (7.29) multiplying each term of the sum by:

$$1=\prod_{\epsilon=\pm 1}\prod_{i=1}^{r}\prod_{j=1}^{\mathsf{N}-1-r}\prod_{h=-p+\alpha_i}^{-1}\left(\eta_{a_i}^2 q^{-2h}-(\eta_{b_j}^{(k_{b_j})})^2\right)^{\epsilon}$$
$$\times\left(\frac{V\left(\eta_{a_1}^2,\eta_{a_1}^2q^2,\ldots,\eta_{a_1}^2q^{2(p-\alpha_1)},\ldots,\eta_{a_r}^2,\eta_{a_r}^2q^2,\ldots,\eta_{a_r}^2q^{2(p-\alpha_r)}\right)}{V\left(\eta_{a_1}^2,\ldots,\eta_{a_r}^2\right)}\right)^{\epsilon}. \tag{7.38}$$

Indeed, the power $+1$ leads to the construction of the Vandermonde determinant:

$$V(\underbrace{\eta_{a_1}^2,\ldots,\eta_{a_1}^2q^{2(p-\alpha_1)}}_{p-\alpha_1+1\text{ columns}},\ldots,\underbrace{\eta_{a_r}^2,\ldots,\eta_{a_r}^2q^{2(p-\alpha_r)}}_{p-\alpha_r+1\text{ columns}},\underbrace{\left(\eta_{b_1}^{(k_{b_1})}\right)^2,\ldots,\left(\eta_{b_{(\mathsf{N}-1)-r}}^{(k_{b_{(\mathsf{N}-1)-r}})}\right)^2}_{(\mathsf{N}-1)-r\text{ columns}}), \tag{7.39}$$

and the sum $\sum_{k_{b_1},\ldots,k_{b_{\mathsf{N}-(r+1)}}=1}^{p}$ becomes sum over columns which can be brought inside the determinant. □

### 7.3. The Chiral Potts Model Order Parameters

The results presented in the previous subsections are as well results for the matrix elements of local operators in the inhomogeneous chiral Potts model. In particular, let $|t_k\rangle$ and $|t'_{k'}\rangle$ be two eigenstates of the chiral Potts transfer matrix, then the matrix elements:

$$\langle t_k|\mathsf{u}_n^{-1}|t'_{k'}\rangle,\quad \langle t_k|\alpha_{0,n}^{-1}|t'_{k'}\rangle\quad\text{and}\quad\left\langle t_k|\mathcal{E}_{(h,h_0,(a_1,h_1),\ldots,(a_r,h_r))}^{(\alpha_1,\ldots,\alpha_r)}|t'_{k'}\right\rangle$$

are given, respectively, by the formulae (7.1), (7.11) and (7.29). Furthermore, in the representations $\mathcal{R}_{\mathsf{N}}^{\text{chP,S-adj}}$, the formulae (7.1), (7.11) and (7.29 ) are always matrix elements of the corresponding local operators on chiral Potts eigenstates. As clarified below, some of these matrix elements generate the chiral Potts order parameters under the homogeneous and thermodynamic limits.

**7.3.1. Local Hamiltonians and Order Parameters.** It is worth recalling that the following local quantum Hamiltonians:

$$H \equiv H_0 + kH_1, \; H_0 \equiv \sum_{n=1}^{\mathsf{N}} \left[ \sum_{r=1}^{p-1} f_r(\theta) \mathsf{u}_n^r \mathsf{u}_{n+1}^{-r} \right], \quad H_1 \equiv \sum_{n=1}^{\mathsf{N}} \left[ \sum_{r=1}^{p-1} f_r(\bar{\theta}) \mathsf{v}_n^r \right], \tag{7.40}$$

$$f_r(\theta) \equiv \frac{\mathrm{e}^{i(2r-p)\theta/p}}{\sin \pi r/p}, \quad \cos\bar{\theta} = \frac{\cos\theta}{k}, \quad \mathrm{e}^{i(2\theta-\pi)/p} \equiv \frac{x_{\boldsymbol{g}_n}}{y_{\boldsymbol{g}_n}} = \frac{x_{\boldsymbol{r}_n}}{y_{\boldsymbol{r}_n}}, \tag{7.41}$$

first constructed by von Gehlen and Rittenberg [66], commute with the homogeneous $\mathrm{Z}_p$ chP transfer matrices. Indeed, they are generated by derivative of these transfer matrices w.r.t. the spectral parameter, see for example [60] for a derivation. Then, the order parameters associated to the homogeneous $\mathrm{Z}_p$ chP models:

$$\mathcal{M}_r \equiv \frac{\langle g.s.|\mathsf{u}_1^r|g.s.\rangle}{\langle g.s.|g.s.\rangle}, \quad \forall r \in \{1, \ldots, p-1\} \tag{7.42}$$

admit a natural interpretation as spontaneous magnetizations in terms of the spin chain formulation associated to these local Hamiltonians. They have been mainly analyzed in the special representations associated to the super-integrable $\mathrm{Z}_p$ chP model, characterized by the following constrains:

$$x_{\boldsymbol{g}_n}^p = y_{\boldsymbol{g}_n}^p = x_{\boldsymbol{r}_n}^p = y_{\boldsymbol{r}_n}^p = \frac{1+k'}{k}, \quad \forall n \in \{1, \ldots, \mathsf{N}\} \to \bar{\theta} = \theta = \pi/2. \tag{7.43}$$

In these special representations, the $\mathrm{Z}_p$ chP model also has an underlying Onsager algebra [63] generated by the two components $H_0$ and $H_1$ of the local quantum Hamiltonians. The following thermodynamic limits:

$$\mathcal{M}_r = (1-k^2)^{\frac{r(p-r)}{2p^2}}, \quad \forall r \in \{1, \ldots, p-1\} \tag{7.44}$$

have been first argued by perturbative computations in [91] and then proven with techniques[25] which apply only starting from finite lattice computations in the super-integrable case. Nevertheless, as argued in [100], the formulae (7.44) should hold true for the general homogeneous $\mathrm{Z}_p$ chP models. It is then relevant pointing out that our approach should give us the possibility to prove this statement for general representations without the need to be restricted to the super-integrable case and our SOV results already provide simple determinant formulae for the matrix elements associated to $\mathcal{M}_{p-1}$ in the finite size and inhomogeneous regime.

## 8. Conclusion and Outlook

### 8.1. Conclusions

In this article, we have considered general cyclic representations of the six-vertex Yang–Baxter algebra on $\mathsf{N}$-sites finite lattices and analyzed the associated Bazhanov–Stroganov model and consequently the chiral Potts model.

[25] See Sect. 1.1 for an historical recall.

We have derived a reconstruction for all local operators in terms of standard Sklyanin's quantum separate variables and characterized by one determinant formulae of $\mathsf{N} \times \mathsf{N}$ matrices the scalar products of separate states. These findings imply that the action of any local operator on transfer matrix eigenstates reduces to a finite sum of separate states which allows to characterize matrix elements of any local operator as finite sum of determinants of the scalar product type. Moreover, we have obtained: form factors of the local operators $\mathsf{u}_n^{-1}$ and $\alpha_{0,n}^{-1}$ expressed by one determinant formulae obtained by modifying a single row in the scalar product matrices; form factors of a basis of operators expressed by one determinant formulae obtained by modifying the scalar product matrices by introducing rows which coincide with those of Vandermonde's matrix computed in the spectrum of the separate variables.

Let us comment that it would be desirable to get also for the generators $\mathsf{v}_n$ of the local Weyl algebras simple one determinant formulae as for the generators $\mathsf{u}_n$ (at this moment we have expressed its form factors as finite sums of determinants); this interesting issue is currently under investigation. One important motivation to derive form factors of local operators by simple determinant formulae is for their use as efficient tools for the computations of correlation functions. The decomposition of the identity (4.13) allows to write correlation functions in spectral series of form factors and so it allows to analyze them numerically mainly by the same tools developed in [146] in the ABA framework and used in the series of works[26] [146–152]. Indeed, in our SOV framework, we have determinant representations of the form factors and eventually complete characterization of the transfer matrix spectrum in terms of the solutions of a system of Bethe equations type. Let us mention that in a recent series of papers [160–170], the problem to compute the asymptotic behavior of correlation functions has been successfully addressed[27] with a method which is, in principle, susceptible to be extended to any (integrable) quantum model possessing determinant representations for the form factors of local operators [169] and so also to the models analyzed by our approach in the SOV framework.

To make this program operative, one important step to address is a stringent analysis of the similarities and differences which appear in the characterization of the form factors obtained by us in the SOV approach and those derived in the framework of the ABA. Indeed, these last characterizations were the starting point for the asymptotic analysis of [160–170]. In particular, it is natural to compare the determinant formulae for the scalar products appearing in the SOV and ABA frameworks. This should help understanding the large size behavior of the determinant representations we obtained in the present article. One important feature of the representation of scalar products

[26] By this numerical approach, relevant physical observables (like the so-called dynamical structure factors) were evaluated and successfully compared with the measurements accessible by neutron scattering experiments [153–159].

[27] These results have been also successfully compared with those obtained previously with a method relying mainly on the Riemann–Hilbert analysis of related Fredholm determinants [171–173].

and form factors in the SOV framework is that they are written in a rather uniform and universal way in terms of the $Q$ operator eigenvalues. We believe this property to make the corresponding determinants suitable for their thermodynamic limit analysis.

Finally, let us remark that the originality and interest of our current results are also due to the fact that so far the exact determination of matrix elements was achieved only for some local operators and mainly confined to the special class of super-integrable representations of $\mathrm{Z}_p$ chiral Potts model. As these representations can be obtained by taking well-defined limits on the parameters of a generic (non-super-integrable) representation to which SOV applies, it is then an interesting issue to investigate how from our form factor results one can reproduce also those known in the super-integrable case. About this point it is worth mentioning that in the special case ($p = 2$) of the generalized Ising model, it was already remarked in [103] that the matrix elements of the local spin operators obtained in the SOV framework in [116] admit factorized forms similar to those conjectured in [98] and proven in [103] for the super-integrable $\mathrm{Z}_p$ cases for general $p \geq 2$.

In a future paper, we will analyze the homogeneous and thermodynamic limits focusing the attention on the derivation of the order parameter formulae for the general homogeneous $\mathrm{Z}_p$ chiral Potts models. These formulae were proven with techniques working only in the super-integrable case but they are expected to be true [100] for the general homogeneous $\mathrm{Z}_p$ chiral Potts models. Our approach should give access to a proof of this statement from the finite lattice in general representations and we find encouraging the fact that the matrix element describing the order parameter:

$$\mathcal{M}_{p-1} \equiv \frac{\langle g.s.|\mathsf{u}_1^{-1}|g.s.\rangle}{\langle g.s.|g.s.\rangle} \tag{8.1}$$

admits simple determinant formula in our approach.

Anyhow, it is worth admitting that in fact the novelty of the results here derived can be also at the origin of some technical difficulties. Indeed, in our SOV framework, we are obliged to start mainly from zero the analysis of problems like the computation of thermodynamic limit of matrix elements of local operators; problems which instead in the ABA framework have been already largely analyzed in the literature and for which exact results are known [30].

### 8.2. Outlook

It is worth recalling that in the literature of quantum integrable models, there exist some results on form factors derived by different applications of separation of variable methods. For a more detailed analysis of the most relevant preexisting results and an explicit comparison with those obtained by our method in SOV, we address the reader to [1]. Here, we want to just recall the Smirnov's results [130], in the case of the integrable quantum Toda chain [15,127–129] and those of Babelon et al. [174,175], in the case of the restricted sine–Gordon at the reflectionless points. In both these cases, form factors of local operators were argued[28] to have a determinant form. A strong similarity in the form of the results appears: the elements of the matrices whose determinants give the form factors are expressed as "convolutions", over the spectrum of each separate variable, of the product of the corresponding separate components of the wave functions times contributions associated to the action of local operators. It is then remarkable that also our results fall in this general form. This observation and the potential generality of the SOV method lead to the expectation of an universality in the SOV characterization of form factors.

A natural project is then to develop explicitly our method for a set of fundamental integrable quantum models providing determinant representations for form factors. This SOV method is not restricted to the case of cyclic representation and applies to a large class of integrable quantum models which were not tractable with other methods and in particular by algebraic Bethe ansatz. There exist already several key integrable quantum models associated by QISM to highest weight representations of the Yang–Baxter algebras and generalization of it for which this program has been developed. In [180–185] our approach has been, respectively, implemented for the spin-1/2 $XXZ$ and the spin-s $XXX$ inhomogeneous quantum chains with antiperiodic boundary conditions, for the spin-1/2 $XXZ$ and $XYZ$ open quantum chains with general non-diagonal integrable boundary conditions [46–52] and finally for the spin-1/2 representations of highest weight type of the dynamical six-vertex Yang–Baxter algebra. In all these models, the universality we just discussed in the structure of the matrix elements of local operator has been verified.

## Acknowledgements

N. G. is supported by the University of Cergy-Pontoise. He acknowledges the support of ENS Lyon and ANR grant ANR-10-BLAN-0120-04-DIADEMS during his Ph.D. thesis when most of this work was done. N. G. would also like to thank the YITP Institute of Stony Brook for hospitality. J. M. M. is supported by CNRS, ENS Lyon and by the grant ANR-10-BLAN-0120-04-DIADEMS. G. N. gratefully thanks Barry McCoy for his teachings on pre-existing results on the Bazhanov–Stroganov model, in particular about the order parameters. G. N. is supported by National Science Foundation grants PHY-0969739. G. N. gratefully acknowledges the YITP Institute of Stony Brook for the opportunity to develop his research programs. G. N. would like to thank the Theoretical

[28] The absence of a direct reconstruction of the local operators in terms of the Sklyanin's quantum separate variables was the motivation in [130, 175] to use some well-educated guess relying on counting arguments for the characterization of local operators basis and to use semi-classical arguments relying on the classical SOV-reconstruction for the identification of primary fields [174, 179]. Note that a reconstruction of local operators in the lattice Toda model has been achieved in [176] in terms of a set of quantum separate variables defined by a change of variables in terms of the original Sklyanin's quantum separate variables. Recent analysis of this reconstruction problem for the lattice Toda model appears also in [177, 178].

Physics Group of the Laboratory of Physics at ENS-Lyon for hospitality, under support of ANR-10-BLAN- 0120-04-DIADEMS, which made possible this collaboration.

## References

[1] Grosjean, N., Maillet, J.M., Niccoli, G.: On the form factors of local operators in the lattice sine-Gordon model. J. Stat. Mech. P10006 (2012)

[2] Sklyanin, E.K., Faddeev, L.D.: Quantum mechanical approach to completely integrable field theory models. Sov. Phys. Dokl. **23**, 902 (1978)

[3] Faddeev, L.D., Takhtajan, L.A.: The quantum method of the inverse problem and the Heisenberg $XYZ$ model. Russ. Math. Surv. **34**(5), 11 (1979)

[4] Kulish, P.P., Sklyanin, E.K.: Quantum inverse scattering method and the Heisenberg ferromagnet. Phys. Lett. A **70**, 461 (1979)

[5] Faddeev, L.D., Sklyanin, E.K., Takhtajan, L.A.: Quantum inverse problem method. I. Theor. Math. Phys. **40**, 688 (1979)

[6] Faddeev, L.D.: Quantum completely integrable models in field theory. Sov. Sci. Rev. C Math. Phys. **1**, 107–155 (1980)

[7] Sklyanin, E.K.: Quantum version of the method of inverse scattering problem. J. Sov. Math. **19**, 1546–1596 (1982)

[8] Kulish, P.P., Sklyanin, E.K.: Quantum spectral transform method recent developments. Lect. Notes Phys. **151**, 61 (1982)

[9] Fadeev, L.D.: Integrable models in 1 + 1 dimensional quantum field theory. In: Zuber, J.-B., Stora, R. (eds.) Recent Advances in Field Theory and Statistical Mechanics, Les Houches, Session XXXIX, pp. 561–608. North Holland Publishing Company, Amsterdam (1984). ISBN: 0444866752

[10] Fadeev, L.D.: How Algebraic Bethe Ansatz works for integrable model. hep-th/9605187v1

[11] Jimbo, M.: Yang-Baxter Equation in Integrable Systems. Advanced series in mathematical physics, vol. 10. Scientific, Singapore (1990). ISBN: 978-981-02-0120-3

[12] Shastry, B.S., Jha, S.S., Singh, V.: Exactly solvable problems in condensed matter and relativistic field theory. Lecture Notes in Physics, vol. 242. Springer, Berlin, Heidelberg (1985)

[13] Thacker, H.B.: Exact integrability in quantum field theory and statistical systems. Rev. Mod. Phys. **53**, 253 (1981)

[14] Izergin, A.G., Korepin, V.E.: Lattice versions of quantum field theory models in two dimensions. Nucl. Phys. B **205**, 401–413 (1982)

[15] Sklyanin, E.K.: The quantum Toda chain. Lect. Notes Phys. **226**, 196–233 (1985)

[16] Sklyanin, E.K.: Quantum inverse scattering method. Selected topics. In: Ge, M.-L. (ed.) Quantum Group and Quantum Integrable Systems: Nankai Lectures on Mathematical Physics. World Academic, Singapore (1992). ISBN: 978-9810207458. hep-th/9211111

[17] Sklyanin, E.K.: Separation of variables, new trends. Prog. Theor. Phys. Suppl. **118**, 35–60 (1995)

[18] Kitanine, N., Maillet, J.M., Terras, V.: Form factors of the $XXZ$ Heisenberg spin-1/2 finite chain. Nucl. Phys. B **554**, 647 (1999)

[19] Heisenberg, W.: Zur Theorie des Ferromagnetismus. Z. Phys. **49**, 619 (1928)

[20] Bethe, H.: Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette. Z. Phys. **71**, 205 (1931)

[21] Hulthen, L.: Uber das Austauschproblem eines Kristalls. Ark. Mat. Astron. Fys. **26**, 1 (1938)

[22] Orbach, R.: Linear antiferromagnetic chain with anisotropic coupling. Phys. Rev. **112**, 309 (1958)

[23] Walker, L.R.: Antiferromagnetic linear chain. Phys. Rev. **116**, 1089 (1959)

[24] Yang, C.N., Yang, C.P.: One-dimensional chain of anisotropic spin-spin interactions. I. Proof of Bethe's hypothesis for ground state in a finite system. Phys. Rev. **150**, 321 (1966)

[25] Yang, C.N., Yang, P.C.: One-dimensional chain of anisotropic spin-spin interactions. II. Properties of the ground-state energy per lattice site for an infinite system. Phys. Rev. **150**, 327 (1966)

[26] Gaudin, M.: La Fonction d'onde de Bethe. Masson, Paris (1983). ISBN: 9782225796074

[27] Lieb, E.H.,Mattis, D.C.: Mathematical Physics in One Dimension. Academic, New-York (1966). ISBN:978-0124487505

[28] Maillet, J.M., Terras, V.: On the quantum inverse scattering problem. Nucl. Phys. B **575**, 627 (2000)

[29] Izergin, A.G., Kitanine, N., Maillet, J.M., Terras, V.: Spontaneous magnetization of the $XXZ$ Heisenberg spin-1/2 chain. Nucl. Phys. B **554**, 679 (1999)

[30] Kitanine, N., Maillet, J.M., Terras, V.: Correlation functions of the $XXZ$ Heisenberg spin-1/2 chain in a magnetic field. Nucl. Phys. B **567**, 554 (2000)

[31] Kitanine, N., Maillet, J.M., Slavnov, N.A., Terras, V.: Spinspin correlation functions of the $XXZ$-1/2 Heisenberg chain in a magnetic field. Nucl. Phys. B **641**, 487 (2002)

[32] Kitanine, N., Maillet, J.M., Slavnov, N.A., Terras, V.: Correlation functions of the $XXZ$ spin-1/2 Heisenberg chain at the free fermion point from their multiple integral representations. Nucl. Phys. B **642**, 433 (2002)

[33] Kitanine, N., Maillet, J.M., Slavnov, N.A., Terras, V.: Emptiness formation probability of the $XXZ$ spin-1/2 Heisenberg chain at $\Delta = 1/2$. J. Phys. A **35**, L385 (2002)

[34] Kitanine, N. Maillet, J., M., Slavnov N., A., Terras, V.: Large distance asymptotic behaviour of the emptiness formation probability of the $XXZ$ spin-1/2 Heisenberg chain. J. Phys. A **35**, L753 (2002)

[35] Kitanine, N., Maillet, J.M., Slavnov, N.A., Terras, V.: Master equation for spinspin correlation functions of the $XXZ$ chain. Nucl. Phys. B **712**, 600 (2005)

[36] Kitanine, N., Maillet, J.M., Slavnov, N.A., Terras, V.: Dynamical correlation functions of the $XXZ$ spin-1/2 chain. Nucl. Phys. B **729**, 558 (2005)

[37] Kitanine, N., Maillet, J.M., Slavnov, N.A., Terras, V.: On the spin–spin correlation functions of the $XXZ$ spin-1/2 infinite chain. J. Phys. A **38**, 7441 (2005)

[38] Kitanine, N., Maillet, J.M., Slavnov, N.A., and Terras, V.: Exact results for the $\sigma^z$ two-point function of the $XXZ$ chain at $\Delta = 1/2$. J. Stat. Mech. L09002 (2005)

[39] Kitanine, N., Maillet, J.M., Slavnov, N.A., Terras, V.: On the algebraic Bethe Ansatz approach to the correlation functions of the $XXZ$ spin-1/2 Heisenberg chain. In: Recent Progress in Solvable Lattice Models, RIMS Sciences Project Research 2004 on Method of Algebraic Analysis in Integrable Systems, RIMS, Kyoto, Kokyuroku, 1480, 14 (2006). hep-th/0505006

[40] Kitanine, N., Kozlowski, K., Maillet, J.M., Slavnov, N.A., Terras, V.: On correlation functions of integrable models associated with the six-vertex $R$-matrix. J. Stat. Mech. P01022 (2007)

[41] Kitanine, N.: Correlation functions of the higher spin $XXX$ chains. J. Phys. A Math. Gen. **34**, 8151 (2001)

[42] Castro-Alvaredo, O.A., Maillet, J.M.: Form factors of integrable Heisenberg (higher) spin chains. J. Phys. A **40**, 7451 (2007)

[43] Kitanine, N., Kozlowski, K.K., Maillet, J.M., Niccoli, G., Slavnov, N.A., Terras, V.: Correlation functions of the open $XXZ$ chain: I. J. Stat. Mech. P10009 (2007)

[44] Kozlowski, K.K.: On the emptiness formation probability of the open $XXZ$ spin-1/2 chain. J. Stat. Mech. P02006 (2008)

[45] Kitanine, N., Kozlowski, K.K., Maillet, J.M., Niccoli, G., Slavnov, N.A., Terras, V.: Correlation functions of the open $XXZ$ chain: II. J. Stat. Mech. P07010 (2008)

[46] Sklyanin, E.K.: Boundary conditions for integrable quantum systems. J. Phys. A Math. Gen. **21**, 2375 (1988)

[47] Cherednik, I.V.: Factorizing particles on a half-line and root systems. Theor. Math. Phys. **61**, 977 (1984)

[48] Kulish, P.P., Sklyanin, E.K.: The general Uq(sl(2)) invariant $XXZ$ integrable quantum spin chain. J. Phys. A Math. Gen. **24**, L435 (1991)

[49] Mezincescu, L., Nepomechie, R.: Integrability of open spin chains with quantum algebra symmetry. Int. J. Mod. Phys. A **6**, 5231 (1991)

[50] Kulish, P.P., Sklyanin, E.K.: Algebraic structures related to reflection equations. J. Phys. A Math. Gen. **25**, 5963 (1992)

[51] Ghoshal, S., Zamolodchikov, A.: Boundary $S$ matrix and boundary state in two-dimensional integrable quantum field theory. Int. J. Mod. Phys. A **9**, 3841 (1994)

[52] Ghoshal, S., Zamolodchikov, A.: Errata: boundary S matrix and boundary state in two-dimensional integrable quantum field theory. Int. J. Mod. Phys. A **9**, 4353 (1994)

[53] Tarasov, V.: Cyclic monodromy matrices for the $R$ matrix of the six vertex model and the chiral Potts model with fixed spin boundary conditions. Int. J. Mod. Phys. A **07**, 963 (1992)

[54] Niccoli, G., Teschner, J.: The sine–Gordon model revisited: I. J. Stat. Mech. P09014 (2010)

[55] Niccoli, G.: Reconstruction of Baxter Q-operator from Sklyanin SOV for cyclic representations of integrable quantum models. Nucl. Phys. B **835**, 263 (2010)

[56] Niccoli, G.: Completeness of Bethe ansatz by Sklyanin SOV for cyclic representations of integrable quantum models. JHEP **03**, 123 (2011)

[57] Bazhanov, V.V., Stroganov, Yu.G.: Chiral Potts model as a descendant of the six-vertex model. J. Stat. Phys. **59**, 799 (1990)

[58] Baxter, R.J., Bazhanov, V.V., Perk, J.H.H.: Functional relations for the transfer matrices of the chiral Potts model. Int. J. Mod. Phys. B **4**, 803 (1990)

[59] Baxter, R.J.: Transfer matrix functional relations for the generalized $\tau_2$ $(t_q)$ model. J. Stat. Phys. **117**, 1 (2004)

[60] Albertini, G., McCoy, B.M., Perk, J.H.H.: Eigenvalue spectrum of the superintegrable chiral Potts model. In: Jimbo, M., Miwa, T., Tsuchiya, A. (eds.) Integrable Systems in Quantum Field Theory and Statistical Mechanics (Adv. Stud. Pure Math. vol. 19), pp. 1–55. Kinokuniya, Tokyo (1989). ISBN: 9780123853424

[61] Albertini, G., McCoy, B.M., Perk, J.H.H.: Commensurate-incommensurate transition in the ground state of the superintegrable chiral Potts model. Phys. Lett. A **135**, 159 (1989)

[62] Albertini, G., McCoy, B.M., Perk, J.H.H.: Level crossing transitions and the massless phases of the superintegrable chiral Potts chain. Phys. Lett. A **139**, 204 (1989)

[63] Au-Yang, H., McCoy, B.M., Perk, J.H.H., Tang, S., Yan, M.-L.: Commuting transfer matrices in the chiral Potts models: solutions of star-triangle equations with genus >1. Phys. Lett. A **123**, 219 (1987)

[64] Baxter, R.J., Perk, J.H.H., Au-Yang, H.: New solutions of the star-triangle relations for the chiral-Potts model. Phys. Lett. A. **128**, 138 (1988)

[65] Au-Yang, H., Perk, J.H.H.: Onsager's star triangle equation: master key to integrability. In: Jimbo, M., Miwa, T., Tsuchiya, A. (eds.) Integrable Systems in Quantum Field Theory and Statistical Mechanics (Adv. Stud. Pure Math. vol. 19), pp. 57-94. Kinokuniya, Tokyo (1989). ISBN: 9780123853424

[66] von Gehlen, G., Rittenberg, V.: $Z_n$-symmetric quantum chains with an infinite set of conserved charges and $Z_n$ zero modes. Nucl. Phys. B **257**, 351 (1985)

[67] Perk, J.H.H.: Star-triangle equations, quantum Lax pairs, and higher genus curves. In: Gunning, R.C., Ehrenpreis, L. (eds.) Proceedings of 1987 Summer Research Institute on Theta Functions (Proc. Symp. Pure Math. vol. 49), pp. 341–354. American Mathematical Society, Providence (1989). ISBN: 9780821814857

[68] Baxter, R.J.: The superintegrable chiral potts model. Phys. Lett. A **133**, 185 (1989)

[69] Baxter, R.J.: Superintegrable chiral Potts model: thermodynamic properties, an inverse model, and a simple associated Hamiltonian. J. Stat. Phys. **57**, 1 (1989)

[70] Baxter, R.J., Bazhanov, V.V., Perk, J.H.H.: Functional relations for transfer matrices of the chiral Potts model. Int. J. Mod. Phys. B **4**, 803870 (1990)

[71] Bazhanov, V.V., Bobenko, A., Reshetikhin, N.: Quantum discrete sine-Gordon model at roots of 1: integrable quantum system on the integrable classical background. Commun. Math. Phys. **175**, 377 (1996)

[72] Bazhanov, V.V.: Chiral Potts model and the discrete sine–Gordon model at roots of unity. Adv. Stud. Pure Math. **61**, 91–123 (2011)

[73] Bazhanov, V.V., Sergeev, S.: A master solution of the quantum Yang–Baxter equation and classical discrete integrable equations. Adv. Theor. Math. Phys. **16**, 65–95 (2012)

[74] McCoy, B.M., Perk, J.H.H., Tang, S., Sah, C.H.: Commuting transfer matrices for the four-state self-dual chiral Potts model with a genus-three uniformizing fermat curve. Phys. Lett. A **125**, 9 (1987)

[75] Au-Yang, H., McCoy, B.M., Perk, J.H.H., Tang, S.: Solvable models in statistical mechanics and Riemann surfaces of genus greater than one. In: Kashiwara, M., Kawai, T. (eds.) Papers Dedicated to Professor Mikio Sato on the Occasion of his Sixtieth Birthday, vol. I, pp. 29–40. Academic, San Diego (1988). ISBN: 9780124004658

[76] Tarasov, V.O.: Transfer matrix of the superintegrable chiral Potts model. Bethe ansatz spectrum. Phys. Lett. A **147**, 487 (1990)

[77] Kulish, P.P., Reshetikhin, N.Y., Sklyanin, E.K.: Yang–Baxter equation and representation theory: I. Lett. Math. Phys. **5**, 393 (1981)

[78] Kirillov, A.N., Reshetikhin, N.Y.: Exact solution of the integrable $XXZ$ Heisenberg model with arbitrary spin. I. The ground state and the excitation spectrum. J. Phys. A Math. Gen. **20**, 1565 (1987)

[79] Au-Yang, H., Perk, J.H.H.: Eigenvectors in the superintegrable model I: $\mathfrak{sl}_2$ generators. J. Phys. A Math. Theor. **41**, 275201 (2008)

[80] Au-Yang, H., Perk, J.H.H.: Eigenvectors in the superintegrable model II: ground-state sector. J. Phys. A Math. Theor. **42**, 375208 (2009)

[81] Nishino, A., Deguchi, T.: An algebraic derivation of the eigenspaces associated with an Ising-like spectrum of the superintegrable chiral Potts model. J. Stat. Phys. **133**, 587 (2008)

[82] Roan, S.S.: Eigenvectors of an arbitrary Onsager sector in super-integrable $\tau_2$ model and chiral Potts model (2010). arXiv:1003.3621

[83] Onsager, L.: Crystal statistics. I. A two-dimensional model with an order-disorder transition. Phys. Rev. **65**, 117 (1944)

[84] Fabricius, K., McCoy, B.M.: Evaluation parameters and Bethe roots for the six-vertex model at roots of unity. In: Kashiwara, M., Miwa, T. (eds.) MathPhys Odyssey (Progress in Math. Phys. vol. 23), p. 119. Birkhäuser, Basel (2001)

[85] Davies, B.: Onsager's algebra and superintegrability. J. Phys. A Math. Gen. **23**, 2245 (1990)

[86] Date, E., Roan, S.S.: The algebraic structure of the Onsager algebra. Czechoslov. J. Phys. **50**, 37 (2000)

[87] Roan, S.S.: The Onsager algebra symmetry of $T^{(j)}$-matrices in the superintegrable chiral Potts model. J. Stat. Mech. P09007 (2005)

[88] Nishino, A., Deguchi, T.: The L(sl2) symmetry of the BazhanovStroganov model associated with the superintegrable chiral Potts model. Phys. Lett. A **356**, 366 (2006)

[89] Roan, S.S.: Fusion operators in the generalized $T^{(2)}$-model and root-of-unity symmetry of the $XXZ$ spin chain of higher spin. J. Phys. A Math. Theor. **40**, 1481 (2007)

[90] Roan, S.S.: Duality and symmetry in chiral Potts model. J. Stat. Mech. P08012 (2009)

[91] Albertini, G., McCoy, B.M., Perk, J.H.H., Tang, S.: Excitation spectrum and order parameter for the integrable N-state chiral Potts model. Nucl. Phys. B **314**, 741 (1989)

[92] Baxter, R.J.: Derivation of the order parameter of the chiral Potts model. Phys. Rev. Lett. **94**, 130602 (2005)

[93] Baxter, R.J.: The order parameter of the chiral Potts model. J. Stat. Phys. **120**, 1 (2005)

[94] Jimbo, M., Miwa, T., Nakayashiki, A.: Difference equations for the correlation functions of the eight-vertex model. J. Phys. A Math. Gen. **26**, 2199 (1993)

[95] Baxter, R.J.: Exactly Solved Models in Statistical Mechanics. Academic, London (1982). ISBN: 9780120831821

[96] Baxter, R.J.: Corner transfer matrices in statistical mechanics. J. Phys. A Math. Theor. **40**, 12577 (2007)

[97] Baxter, R.J.: Algebraic reduction of the Ising model. J. Stat. Phys. **132**, 959 (2008)

[98] Baxter, R.J.: Some remarks on a generalization of the superintegrable chiral Potts model. J. Stat. Phys. **137**, 798 (2009)

[99] Au-Yang, H., Perk, J.H.H.: Identities in the superintegrable chiral Potts model. J. Phys. A Math. Theor. **43**, 025203 (2010)

[100] Au-Yang, H., Perk, J.H.H.: Quantum loop subalgebra and eigenvectors of the superintegrable chiral Potts transfer matrices. J. Phys. A Math. Theor. **44**, 025205 (2011)

[101] Au-Yang, H., Perk, J.H.H.: Super-integrable chiral Potts model: proof of the conjecture for the coefficients of the generating function G(t,u). arXiv:1108.4713v1

[102] Baxter, R.J.: A conjecture for the superintegrable chiral Potts model. J. Stat. Phys. **132**, 983 (2008)

[103] Iorgov, N., Pakuliak, S., Shadura, V., Tykhyy, Yu., von Gehlen, G.: Spin operator matrix elements in the superintegrable chiral Potts quantum chain. J. Stat. Phys. **139**, 743 (2009)

[104] Bugrij, A., Lisovyy, O.: Correlation function of the two-dimensional Ising model on a finite lattice: II. Theor. Math. Phys. **140**, 987 (2004)

[105] Iorgov, N.: Form factors of the finite quantum $XY$-chain. J. Phys. A Math. Theor. **44**, 335005 (2011)

[106] Baxter, R.J.: Spontaneous magnetization of the superintegrable chiral Potts model: calculation of the determinant $D_{PQ}$. J. Phys. A Math. Theor. **43**, 145002 (2010)

[107] Baxter, R.J.: Proof of the determinantal form of the spontaneous magnetization of the superintegrable chiral Potts model. ANZIAM J. **51**, 309 (2010)

[108] Dasmahapatra, S., Kedem, R., McCoy, B.: Spectrum and completeness of the three-state superintegrable chiral Potts model. Nucl. Phys. B **396**, 506 (1993)

[109] Albertini, G., Dasmahapatra, S., McCoy, B.: Spectrum and completeness of the intergable 3-state Potts model: a finite size study. Int. J. Mod. Phys. A **7**(supp01a), 1 (1992)

[110] Fateev, V.A., Zamolodchikov, A.B.: Self-dual solutions of the star-triangle relations in $Z_N$-models. Phys. Lett. A **92**, 37 (1982)

[111] Fabricius, K., McCoy, B.: Bethe's equation is incomplete for the $XXZ$ model at roots of unity. J. Stat. Phys. **103**, 647 (2001)

[112] Nepomechie, R.I., Ravanini, F.: Completeness of the Bethe Ansatz solution of the open $XXZ$ chain with nondiagonal boundary terms. J. Phys. A **36**, 11391 (2003)

[113] von Gehlen, G., Iorgov, N., Pakuliak, S., Shadura, V.: The Baxter–Bazhanov–Stroganov model: separation of variables and the Baxter equation. J. Phys. A Math. Gen. **39**, 7257 (2006)

[114] Iorgov, N.: Eigenvectors of open Bazhanov–Stroganov quantum chain. SIGMA **2**, 019 (2006)

[115] Gehlen, G.von , Iorgov, N., Pakuliak, S., Shadura, V., Tykhyy, Yu: Form-factors in the Baxter–Bazhanov–Stroganov model I: norms and matrix elements. J. Phys. A Math. Theor. **40**, 14117 (2007)

[116] von Gehlen, G., Iorgov, N., Pakuliak, S., Shadura, V., Tykhyy, Yu: Form factors in the Baxter–Bazhanov–Stroganov model II: Ising model on the finite lattice. J. Phys. A Math. Theor. **41**, 095003 (2008)

[117] von Gehlen, G., Iorgov, N., Pakuliak, S., Shadura, V.: Factorized finite-size Ising model spin matrix elements from separation of variables. J. Phys. A Math. Theor. **42**, 304026 (2009)

[118] Grosjean, N., Niccoli, G.: The $\tau_2$-model and the chiral Potts model revisited: completeness of Bethe equations from Sklyanin's SOV method. J. Stat. Mech. P11005 (2012)

[119] Alcaraz, F.C., Barber, M.N., Batchelor, M.T., Baxter, R.J., Quispel, G.R.W.: Surface exponents of the quantum $XXZ$, Ashkin–Teller and Potts models. J. Phys. A **20**, 6397 (1987)

[120] Reshetikhin, N.Y.: A method of functional equations in the theory of exactly solvable quantum systems. Lett. Math. Phys. **7**, 205 (1983)

[121] Reshetikhin, N.Y.: The functional equation method in the theory of exactly soluble quantum systems. JETP **57**, 691 (1983)

[122] Mukhin, E., Tarasov, V., Varchenko, A.: Bethe algebra of homogeneous $XXX$ Heisenberg model has simple spectrum. Commun. Math. Phys. **288**, 1 (2009)

[123] Orlando, D., Reffert, S., Reshetikhin, N.: On domain wall boundary conditions for the $XXZ$ spin Hamiltonian. arXiv:0912.0348

[124] Korff, C.: Cylindric versions of specialised Macdonald functions and a deformed Verlinde algebra. Commun. Math. Phys. **318**, 173 (2013)

[125] Izergin, A.G., Korepin, V.E.: A lattice model related to the nonlinear Schroedinger equation. Dokl. Akad. Nauk **259**, 76 (1981). arXiv:0910.0295

[126] Slavnov, N.A.: Calculation of scalar products of wave functions and form factors in the framework of the alcebraic Bethe ansatz. Theor. Math. Phys. **79**, 502 (1989)

[127] Gutzwiller, M.: The quantum mechanical Toda lattice, II. Ann. Phys. **133**, 304 (1981)

[128] Pasquier, V., Gaudin, M.: The periodic Toda chain and a matrix generalization of the Bessel function recursion relations. J. Phys. A **25**, 5243 (1992)

[129] Kharchev, S., Lebedev, D.: Integral representation for the eigenfunctions of a quantum periodic Toda chain. Lett. Math. Phys. **50**, 53 (1999)

[130] Smirnov, F.: Structure of matrix elements in the quantum Toda chain. J. Phys. A Math. Gen. **31**, 8953 (1998)

[131] Bytsko, A., Teschner, J.: Quantization of models with non-compact quantum group symmetry: modular $XXZ$ magnet and lattice sine–Gordon model. J. Phys. A **39**, 12927 (2006)

[132] Faddeev, L.D., Kashaev, R.M.: Quantum dilogarithm. Mod. Phys. Lett. A **9**, 427 (1994)

[133] Faddeev, L.D.: Discrete Heisenberg–Weyl Group and modular group. Lett. Math. Phys. **34**, 249 (1995)

[134] Ruijsenaars, S.N.M.: First order analytic difference equations and integrable quantum systems. J. Math. Phys. **38**, 1069 (1997)

[135] Woronowicz, S.L.: Quantum exponential function. Rev. Math. Phys. **12**, 873 (2000)

[136] Ponsot, B., Teschner, J.: ClebschGordan and RacahWigner coefficients for a continuous series of representations of $U_q$ (sl(2,R)). Commun. Math. Phys. **224**, 613 (2001)

[137] Kashaev, R.M.: The non-compact quantum dilogarithm and the Baxter equations. J. Stat. Phys. **102**, 923 (2001)

[138] Kashaev, R.M.: The quantum dilogarithm and Dehn twists in quantum Teichmüller theory. In: Pakuliak, S., von Gehlen, G. (eds.) Integrable Structures of Exactly Solvable Two-Dimensional Models of Quantum Field Theory (Nato Science Series II: (Closed), vol. 35, pp. 211–221. Kluwer, Dordrecht (2001). ISBN: 978-0-7923-7183-0

[139] Bytsko, A., Teschner, J.: R-operator, co-product and Haar-measure for the modular double of $U_q$(sl(2,R)). Commun. Math. Phys. **240**, 171 (2003)

[140] Teschner, J.: Liouville theory revisited. Class. Quantum Gravity **18**, R153 (2001)

[141] Teschner, J.: A lecture on the Liouville vertex operators. Int. J. Mod. Phys. A **19**(supp02), 436 (2004)

[142] Volkov, A.Yu.: Noncommutative hypergeometry. Commun. Math. Phys. **258**, 257 (2005)

[143] Tarasov, V.O., Takhtadzhyan, I.A., Faddeev, L.D.: Local Hamiltonians for integrable quantum models on a lattice. Theor. Math. Phys. **57**(2), 1059 (1983)

[144] Oota, T.: Quantum projectors and local operators in lattice integrable models. J. Phys. A Math. Gen. **37**, 441 (2004)

[145] Kuznetsov, V.B.: Inverse Problem for sl(2) Lattices, Symmetry and Perturbation Theory, pp. 136–152. World Scientific (2002). arXiv:nlin/0207025

[146] Caux, J.-S., Maillet, J.-M.: Computation of dynamical correlation functions of Heisenberg chains in a magnetic field. Phys. Rev. Lett. **95**, 077201 (2005)

[147] Caux, J.-S., Hagemans, R., Maillet, J.-M.: Computation of dynamical correlation functions of Heisenberg chains: the gapless anisotropic regime. J. Stat. Mech. P09003 (2005)

[148] Pereira, R.G., Sirker, J., Caux, J.-S., Hagemans, R., Maillet, J.M., White, S.R., Affleck, I.: Dynamical spin structure factor for the anisotropic spin-1/2 Heisenberg chain. Phys. Rev. Lett. **96**, 257202 (2006)

[149] Hagemans, R., Caux, J.-S., Maillet, J. M.: How to calculate correlation functions of Heisenberg chains. In: Proceedings of the "Tenth Training Course in the Physics of Correlated Electron Systems and High-Tc Superconductors", Salerno, 2005, vol. 846, p. 245. AIP Conference Proceedings (2006)

[150] Pereira, R. G., Sirker, J., Caux, J.-S., Hagemans, R., Maillet, J.M., White, S.R., Affleck, I.: Dynamical structure factor at small $q$ for the $XXZ$ spin-1/2 chain. J. Stat. Mech. P08022 (2007)

[151] Sirker, J., Pereira, R.G., Caux, J.-S., Hagemans, R., Maillet, J.M., White, S.R., Affleck, I.: Boson decay and the dynamical structure factor for the $XXZ$ chain at finite magnetic field. Proc. SCES'07 Houst. Phys. B **403**, 1520 (2008)

[152] Caux, J.S., Calabrese, P., Slavnov, N.A.: One-particle dynamical correlations in the one-dimensional Bose gas. J. Stat. Mech. P01008 (2007)

[153] Bloch, F.: On the magnetic scattering of neutrons. Phys. Rev. **50**, 259 (1936)

[154] Schwinger, J.S.: On the magnetic scattering of neutrons. Phys. Rev. **51**, 544 (1937)

[155] Halpern, O., Johnson, M.H.: On the magnetic scattering of neutrons. Phys. Rev. **55**, 898 (1938)

[156] Van Hove, L.: Correlations in space and time and born approximation scattering in systems of interacting particles. Phys. Rev. **95**, 249 (1954)

[157] Van Hove, L.: Time-dependent correlations between spins and neutron scattering in ferromagnetic crystals. Phys. Rev. **95**, 1374 (1954)

[158] Marshall, W., Lovesey, S.W.: Theory of Thermal Neutron Scattering. Clarenton Press, Oxford (1971). ISBN: 9780198512547

[159] Balescu, R.: Equilibrium and Nonequilibrium Statistical Mechanics. Wiley, New York (1975). ISBN: 978-0471046004

[160] Kitanine, N., Kozlowski, K.K., Maillet, J.M., Slavnov, N.A., Terras, V.: On the thermodynamic limit of form factors in the massless $XXZ$ Heisenberg chain. J. Math. Phys. **50**, 095209 (2009)

[161] Kozlowski, K.K.: Fine structure of the asymptotic expansion of cyclic integrals. J. Math. Phys. **50**, 095205 (2009)

[162] Kitanine, N., Kozlowski, K.K., Maillet, J.M., Slavnov, N.A., Terras, V.: The thermodynamic limit of particle hole form factors in the massless $XXZ$ Heisenberg chain. J. Stat. Mech. P05028 (2011)

[163] Kozlowski, K.K., Maillet, J.M., Slavnov, N.A.: Long-distance behavior of temperature correlation functions in the one-dimensional Bose gas. J. Stat. Mech. P03018 (2011)

[164] Kozlowski, K.K., Maillet, J.M., Slavnov, N.A.: Correlation functions for one-dimensional bosons at low temperature. J. Stat. Mech. P03019 (2011)

[165] Kozlowski, K.K.: Low-T Asymptotic Expansion of the Solution to the Yang–Yang Equation. Lett. Math. Phys. (2013). doi:10.1007/s11005-013-0654-1

[166] Kozlowski, K.K.: On form factors of the conjugated field in the nonlinear Schrödinger model. J. Math. Phys. **52**, 083302 (2011)

[167] Kozlowski, K.K.: Large-distance and long-time asymptotic behavior of the reduced density matrix in the non-linear Schrödinger model. arXiv:1101.1626

[168] Kozlowski, K.K., Terras, V.: Long-time and large-distance asymptotic behavior of the currentcurrent correlators in the non-linear Schrdinger model. J. Stat. Mech. P09013 (2011)

[169] Kitanine, N., Kozlowski, K.K., Maillet, J.M., Slavnov, N.A., Terras, V.: A form factor approach to the asymptotic behavior of correlation functions in critical models. J. Stat. Mech. P12010 (2011)

[170] Kozlowski, K.K., Pozsgay, B.: Surface free energy of the open $XXZ$ spin-1/2 chain. J. Stat. Mech. P05021 (2012)

[171] Kitanine, N., Kozlowski, K.K., Maillet, J.M., Slavnov, N.A., Terras, V.: Riemann–Hilbert approach to a generalised sine kernel and applications. Commun. Math. Phys. **291**, 691 (2009)

[172] Kitanine, N., Kozlowski, K.K., Maillet, J.M., Slavnov, N.A., Terras, V.: Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions. J. Stat. Mech. P04003 (2009)

[173] Kozlowski, K.K.: Riemann–Hilbert approach to the time-dependent generalized sine kernel. Adv. Theor. Math. Phys. **15**, 1655 (2011)

[174] Babelon, O., Bernard, D., Smirnov, F.: Quantization of solitons and the restricted sine–Gordon model. Commun. Math. Phys. **182**, 319 (1996)

[175] Babelon, O., Bernard, D., Smirnov, F.: Null-vectors in integrable field theory. Commun. Math. Phys. **186**, 601 (1997)

[176] Babelon, O.: On the quantum inverse problem for the closed Toda chain. J. Phys. A **37**, 303 (2004)

[177] Sklyanin, E.: Bispectrality for the quantum open Toda chain. J. Phys. A Math. Theor. **46**, 382001 (2013)

[178] Kozlowski, K.K.: Aspects of the inverse problem for the Toda chain. arXiv:1307.4052

[179] Smirnov, F.: Quasi-classical study of form factors in finite volume. arXiv:hep-th/9802132

[180] Niccoli, G.: Form factors and complete spectrum of $XXX$ antiperiodic higher spin chains by quantum separation of variables. Nucl. Phys. B **870**, 397 (2013)

[181] Niccoli, G.: Form factors and complete spectrum of $XXX$ antiperiodic higher spin chains by quantum separation of variables. J. Math. Phys. **54**, 053516 (2013)

[182] Niccoli, G.: Non-diagonal open spin-1/2 $XXZ$ quantum chains by separation of variables: complete spectrum and matrix elements of some quasi-local operators. J. Stat. Mech. P10025 (2012)

[183] Faldella, S., Kitanine, N., Niccoli, G.: The complete spectrum and scalar products for the open spin-1/2 $XXZ$ quantum chains with non-diagonal boundary terms. J. Stat. Mech. P01011 (2014)

[184] Faldella, S., Niccoli, G.: SOV approach for integrable quantum models associated with general representations on spin-1/2 chains of the 8-vertex reflection algebra. J. Phys. A: Math. Theor. **47**, 115202 (2014)

[185] Niccoli, G.: An antiperiodic dynamical six-vertex model: I. Complete spectrum by SOV, matrix elements of the identity on separate states and connections to the periodic eight-vertex model. J. Phys. A Math. Theor. **46**, 075003 (2013)

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Communicated by Krzysztof Gawedzki.
Received: November 11, 2013.
Accepted: April 5, 2014.