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Quantum Ergodicity for a Point Scatterer on the Three-Dimensional Torus

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Abstract. Consider a point scatterer (the Laplacian perturbed by a delta-potential) on the standard three-dimensional flat torus. Together with the eigenfunctions of the Laplacian which vanish at the point, this operator has a set of new, perturbed eigenfunctions. In a recent paper, the author was able to show that all of the perturbed eigenfunctions are uniformly distributed in configuration space. In this paper we prove that almost all of these eigenfunctions are uniformly distributed in phase space, i.e. we prove quantum ergodicity for the subspace of the perturbed eigenfunctions. An analogue result for a point scatterer on the two-dimensional torus was recently proved by Kurlberg and Ueberschär.

1. Introduction

Consider a point scatterer on the standard three-dimensional flat torus $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$, which is formally given by

$$- \Delta + \alpha\delta_{x_0} \qquad (1.1)$$

where $-\Delta$ is the associated Laplacian on $\mathbb{T}^3$, δ_{x_0} is the Dirac potential at x_0 and α is a coupling parameter.

We want to study quantum ergodicity of this system, which is a key question in the field of quantum chaos. A classical result regarding quantum ergodicity is Schnirelman's theorem [4,14,19], which asserts that for classically ergodic systems, the quantum counterpart is quantum ergodic, i.e. almost all eigenstates are uniformly distributed in phase space. However our system is not classically ergodic but an intermediate system; its classical dynamics is nearly integrable.

Rigorously, a point scatterer is obtained as a self-adjoint extension of the Laplacian acting on functions vanishing near x_0. Such extensions are parametrized by $\phi \in (-\pi, \pi]$, where $\phi = \pi$ corresponds to the standard Laplacian [$\alpha = 0$ in (1.1)]. For $\phi \neq \pi$, the eigenfunctions of the corresponding operator consist of eigenfunctions of the Laplacian which vanish at x_0 (and are

not affected by the scatterer, so they are related to the unperturbed problem), and new, perturbed eigenfunctions. Since the latter are the only eigenfunctions which feel the scatterer, we have to consider only them to study the physics of the perturbed problem.

In a recent paper [18], the author was able to prove quantum (unique) ergodicity regarding the perturbed eigenfunctions of a point scatterer on the standard three-dimensional flat torus, but only for observables which do not depend on the momentum. Our goal now is to prove quantum ergodicity regarding the perturbed eigenfunctions in full phase space.

We remark that a two-dimensional analogue of the theorem in [18] (i.e. quantum ergodicity in configuration space) was proved for a general two-dimensional flat torus by Rudnick and Ueberschär [12]; in a recent work by Kurlberg and Ueberschär, they prove an analogue of the theorem in the current paper for the standard two-dimensional flat torus.

The unit (co)tangent bundle of $\mathbb{T}^3$ is the compact metric space $S^*\mathbb{T}^3 \simeq \mathbb{T}^3 \times S^2$, on which we have the Liouville probability measure μ which is the normalized product of the Lebesgue measure m on $\mathbb{T}^3$ and the Lebesgue measure σ on S^2. Observables are smooth, zero-th order ξ-homogeneous functions $a(x,\xi)$ on $S^*\mathbb{T}^3$. To quantize them, we use the notion of pseudo-differential operators on $\mathbb{T}^3$, which will be discussed in greater detail in Sect. 3 below. As we will see, quantizing $a \in C^\infty\left(S^*\mathbb{T}^3\right)$ leads to an operator $\mathrm{Op}(a)$, which is a bounded operator on $L^2\left(\mathbb{T}^3\right)$.

We now state the main theorem of this paper. For every $\phi \in (-\pi,\pi)$, let Λ_ϕ be the set of perturbed eigenvalues of the point scatterer, with the corresponding L^2-normalized eigenfunctions g_λ ($\lambda \in \Lambda_\phi$). We prove the following.

Theorem 1.1. *Fix $\phi \in (-\pi,\pi)$. There is a subset $\Lambda_{\phi,\infty} \subseteq \Lambda_\phi$ of density one, so that for all $a \in C^\infty\left(S^*\mathbb{T}^3\right)$,*

$$\langle \mathrm{Op}(a)\, g_\lambda, g_\lambda \rangle \to \int\limits_{S^*\mathbb{T}^3} a\mathrm{d}\mu$$

as $\lambda \to \infty$ along $\Lambda_{\phi,\infty}$.

We actually prove a more general statement: let $\mathcal{N}_3$ be the set of integers which are sums of three squares (these are the eigenvalues of the Laplacian), and let Λ be any increasing sequence whose elements interlace with the elements of $\mathcal{N}_3$. For any $\lambda \in \Lambda$, define g_λ to be the L^2-normalized Green's function: $g_\lambda = \frac{G_\lambda}{\|G_\lambda\|_2}$, where $G_\lambda = (\Delta + \lambda)^{-1} \delta_{x_0}$.

Theorem 1.2. *There is a subset $\Lambda_\infty \subseteq \Lambda$ of density one so that for all $a \in C^\infty\left(S^*\mathbb{T}^3\right)$,*

$$\langle \mathrm{Op}(a)\, g_\lambda, g_\lambda \rangle \to \int\limits_{S^*\mathbb{T}^3} a\mathrm{d}\mu$$

as $\lambda \to \infty$ along Λ_∞.

This is of interest since in the physics literature one considers ϕ which is not fixed, but varies as $\lambda \to \infty$ (see [15,16]); since the only condition for Λ in Theorem 1.2 is that its elements interlace with the elements of $\mathcal{N}_3$, the theorem will still hold in such cases.

Approximating an observable with a linear combination of the functions $Y_{l,m}(\xi)\, e^{ix \cdot \zeta}$, where $Y_{l,m}$ are spherical harmonics, the question at hand reduces to an arithmetic one. The main case is $\zeta = 0$ (the other cases will follow from the proof of proposition 3.9 in [18]), for which a new arithmetic ingredient is used—an estimate due to Duke, by which he proved in [5,6] the equidistribution of integer lattice points on a sphere of radius $\sqrt{n}$, $n \not\equiv 0, 4, 7\ (8)$, $n \to \infty$ (conjectured by Linnik, and proved independently by Golubeva and Fomenko [8,9]). It will be combined with Siegel's lower bound for $r_3(n)$, the number of representations of n as a sum of three squares $[n \not\equiv 0, 4, 7\ (8)]$ [17], which was also a key ingredient in proving the analogue result in configuration space in [18].

We comment that all this is very different from the two-dimensional case, where arithmetic questions about spheres are replaced by questions about circles. The two-dimensional analogue to Duke's estimate is the theorem of Erdös and Hall [7] about the distribution of lattice points on circles; an obvious complication constructing the density one sequence Λ_∞ of perturbed eigenvalues is due to the fact that the theorem of Erdös and Hall is not deterministic, i.e. lattice points on circles are equidistributed only for an unspecified density one set of compatible ns.

2. Point Scatterers on the Torus

Let $\mathbb{T}^3 = \mathbb{R}^3 / 2\pi \mathbb{Z}^3$ be the standard flat torus. A rigorous definition of the operator (1.1) can be found in [18], following [3,12]. For the convenience of the reader we give here a brief summary:

Let $D_0 = C_0^\infty\left(\mathbb{T}^3 \setminus \{x_0\}\right)$ be the domain of C^∞ functions vanishing in a neighbourhood of x_0, and define an operator on $L^2\left(\mathbb{T}^3\right)$ by $-\Delta_{x_0} = -\Delta_{|D_0}$. For the adjoint of $-\Delta_{x_0}$ we have

$$\mathrm{Dom}\left(-\Delta_{x_0}^*\right) = H^2\left(\mathbb{T}^3 \setminus \{x_0\}\right)$$
$$= \left\{ f \in L^2\left(\mathbb{T}^3\right) : \exists A \in \mathbb{C},\ -\Delta f + A\delta_{x_0} \in L^2\left(\mathbb{T}^3\right) \right\},$$

and the self-adjoint extensions of $-\Delta_{x_0}$ are indexed by a parameter $\phi \in (-\pi, \pi]$; the domain of the corresponding operators $-\Delta_{\phi, x_0}$ contains the functions $f \in \mathrm{Dom}\left(-\Delta_{x_0}^*\right)$ such that

$$\exists a \in \mathbb{C},\ f(x) = a\left(\cos\frac{\phi}{2} \cdot \frac{-1}{4\pi |x - x_0|} + \sin\frac{\phi}{2}\right) + o(1), \quad x \to x_0$$

and their action on $f \in \mathrm{Dom}\left(-\Delta_{\phi, x_0}\right)$ is given by

$$-\Delta_{\phi, x_0} f = -\Delta f + A\delta_{x_0} = -\Delta f + a\cos\frac{\phi}{2}\delta_{x_0}, \tag{2.1}$$

so we define a point scatterer to be one of these extended operators $-\Delta_{\phi, x_0}$.

We remark that for $\phi = \pi$ we have $\mathrm{Dom}\left(-\Delta_{\pi,x_0}\right) = H^2\left(\mathbb{T}^3\right)$ and $-\Delta_{\pi,x_0}f = -\Delta f$, so this extension recovers the standard Laplacian $-\Delta_\infty$ on the domain $H^2\left(\mathbb{T}^3\right)$ (which is the unique self-adjoint extension of $-\Delta_{|C^\infty(\mathbb{T}^3)}$).

The functions $(2\pi)^{-3/2}\,e^{i\xi\cdot x}$ $\left(\xi \in \mathbb{Z}^3\right)$ form an orthonormal basis of eigenfunctions of $-\Delta_\infty$ for $L^2\left(\mathbb{T}^3\right)$. The corresponding eigenvalues are the norms $|\xi|^2$ of vectors in $\mathbb{Z}^3$, i.e. the set $\mathcal{N}_3$ of integers which are sums of three squares, and each eigenvalue is of multiplicity $r_3(n)$ which is the number of representations of $n = a^2 + b^2 + c^2$ with $a, b, c \in \mathbb{Z}$ integers.

For the perturbed operator (2.1) with $\phi \neq \pi$, we still have the nonzero eigenvalues from the unperturbed problem ($0 \neq \lambda \in \sigma\left(-\Delta_\infty\right)$), with multiplicities decreased by one, as well as a new set $\Lambda = \Lambda_\phi$ of eigenvalues (referred to as the perturbed eigenvalues), each appearing with multiplicity one, with the corresponding eigenfunctions being multiples of the Green's function

$$G_\lambda\left(x, x_0\right) = (\Delta + \lambda)^{-1}\,\delta_{x_0}.$$

The main tool we use for studying the Green's functions G_λ is their L^2-expansion, which is given by

$$G_\lambda(x, x_0) = -\frac{1}{8\pi^3}\sum_{\xi \in \mathbb{Z}^3}\frac{\exp\left(i\xi\cdot\left(x - x_0\right)\right)}{|\xi|^2 - \lambda}.$$

We denote by

$$g_\lambda(x) := \frac{G_\lambda\left(x, x_0\right)}{\|G_\lambda\|_2}$$

the L^2-normalized Green's function, or the normalized perturbed eigenfunctions of the scatterer.

One can see that perturbed eigenvalues are the solutions to the equation

$$\sum_{\xi \in \mathbb{Z}^3}\left\{\frac{1}{|\xi|^2 - \lambda} - \frac{|\xi|^2}{|\xi|^4 + 1}\right\} = c_0\tan\frac{\phi}{2} \tag{2.2}$$

with

$$c_0 = \sum_{\xi \in \mathbb{Z}^3}\frac{1}{|\xi|^4 + 1}$$

so the perturbed eigenvalues interlace with the elements of

$$\mathcal{N}_3 = \{0 = n_0 < n_1 < n_2 < \cdots\},$$

and denoting them by $\lambda_k = \lambda_k^\phi$ we can write

$$\lambda_0 < n_0 < \lambda_1 < n_1 < \lambda_2 < \cdots < n_k < \lambda_{k+1} < n_{k+1} < \ldots.$$

Recall that a subset $\Lambda' = \{\lambda_{j_k}\} \subseteq \Lambda$ is of density a ($0 \leq a \leq 1$) in Λ if

$$\lim_{J\to\infty}\frac{1}{J}\#\{k \in \mathbb{N} : j_k \leq J\} = a$$

or equivalently

$$\lim_{X \to \infty} \frac{\# \{\lambda \in \Lambda' : \lambda \le X\}}{\# \{\lambda \in \Lambda : \lambda \le X\}} = a.$$

3. Pseudo-Differential Operators on the Torus

To quantize observables $a \in C^\infty \left(S^* \mathbb{T}^3\right)$, one can use the special structure of $\mathbb{T}^3$ as a compact group, to get a global definition of pseudo-differential operators on $\mathbb{T}^3$, instead of using the theory of (Hörmander's) pseudo-differential operators on $\mathbb{R}^3$ locally, which could be rather inconvenient. This idea goes back to Agranovich [1], and it was proved by McLean [11] that both definitions of pseudo-differential operators on the torus are equivalent. A recent monograph by Ruzhansky and Turunen [13] gives a very comprehensive treatment of this subject—we present here the basic definitions of pseudo-differential operators on $\mathbb{T}^n$ (and in particular on $\mathbb{T}^3$) using their notations.

Let $\sigma : \mathbb{Z}^n \to \mathbb{C}$, and let e_j be the standard basis elements of $\mathbb{R}^n$. Let Δ_{ξ_j} be the partial difference operator defined by

$$\Delta_{\xi_j} \sigma \left(\xi\right) = \sigma \left(\xi + e_j\right) - \sigma \left(\xi\right),$$

and for a multi-index α define

$$\Delta_\xi^\alpha = \Delta_{\xi_1}^{\alpha_1} \cdots \Delta_{\xi_n}^{\alpha_n}.$$

Following the notation in [13], for $m \in \mathbb{R}$, $0 \le \delta, \rho \le 1$, define $S_{\rho,\delta}^m \left(\mathbb{T}^n \times \mathbb{Z}^n\right)$ to be the set of functions $a \left(x, \xi\right)$ which are smooth in x for all $\xi \in \mathbb{Z}^n$, and satisfy

$$\left|\Delta_\xi^\alpha \partial_x^\beta a \left(x, \xi\right)\right| \le C_{a,\alpha,\beta,m} \left\langle \xi \right\rangle^{m - \rho |\alpha| + \delta |\beta|}$$

for every $x \in \mathbb{T}^n$, α, β multi-indices, and $\xi \in \mathbb{Z}^n$. Here, $\left\langle \xi \right\rangle = \left(1 + |\xi|^2\right)^{1/2}$.

For every symbol $a \left(x, \xi\right) \in S_{\rho,\delta}^m \left(\mathbb{T}^n \times \mathbb{R}^n\right)$, define a toroidal symbol $\widetilde{a} : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ by the restriction $\widetilde{a} = a|_{\mathbb{T}^n \times \mathbb{Z}^n}$; it is not hard to show that $\widetilde{a} \left(x, \xi\right) \in S_{\rho,\delta}^m \left(\mathbb{T}^n \times \mathbb{Z}^n\right)$. Define

$$\begin{aligned} \mathrm{Op} \left(a\right) f \left(x\right) &= \mathrm{Op} \left(\widetilde{a}\right) f \left(x\right) \\ &= \sum_{\xi \in \mathbb{Z}^n} \mathrm{e}^{ix \cdot \xi} a \left(x, \xi\right) \widehat{f} \left(\xi\right). \end{aligned}$$

One verifies (see [13]) that the operator $\mathrm{Op} \left(a\right) : C^\infty \left(\mathbb{T}^n\right) \to C^\infty \left(\mathbb{T}^n\right)$ is well defined and continuous. Note that for a symbol $a \left(x, \xi\right) = \sum_{|\alpha| \le m} a_\alpha \left(x\right) \xi^\alpha$ we get that

$$\begin{aligned} \mathrm{Op} \left(a\right) f \left(x\right) &= \sum_{|\alpha| \le m} a_\alpha \left(x\right) \sum_{\xi \in \mathbb{Z}^3} \mathrm{e}^{ix \cdot \xi} \xi^\alpha \widehat{f} \left(\xi\right) \\ &= \sum_{|\alpha| \le m} a_\alpha \left(x\right) \sum_{\xi \in \mathbb{Z}^3} \mathrm{e}^{ix \cdot \xi} \widehat{\left(-i\partial\right)^\alpha} f \left(\xi\right) \\ &= \sum_{|\alpha| \le m} a_\alpha \left(x\right) \left(-i\partial\right)^\alpha f \left(x\right) \end{aligned}$$

as one would expect.

In particular, we use this quantization for zero-th order positively homogeneous symbols $a(x,\xi) \in S_{1,0}^0 (\mathbb{T}^3 \times \mathbb{R}^3)$ (in the sense that $a(x,\lambda\xi) = a(x,\xi)$ for $|\xi| \geq 1$ and $\lambda \geq 1$). Since $a(x,\xi)$ is a smooth ξ-homogeneous function of order zero, we can identify it with a smooth function on the unit cotangent bundle of $\mathbb{T}^3$ and write $a \in C^\infty (S^*\mathbb{T}^3)$.

One proves that in this case, $\mathrm{Op}(a)$ extends to a bounded operator on $L^2(\mathbb{T}^3)$. Moreover, we have the following result on the L^2-norm of $\mathrm{Op}(a)$ (Theorem 4.8.1 in [13]): Let $k \in \mathbb{N}$ and $k > n/2$. Let $a : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ be such that $\left| \partial_x^\beta a(x,\xi) \right| \leq C$ for all $(x,\xi) \in \mathbb{T}^n \times \mathbb{Z}^n$ and all $|\beta| \leq k$. Then, the operator $\mathrm{Op}(a)$ extends to a bounded linear operator on $L^2(\mathbb{T}^n)$, and there exists a constant C_k (which depends only on k) such that

$$\|\mathrm{Op}(a)\|_{L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)}^2 \leq C_k \sum_{|\alpha| \leq k} \sup_{y \in \mathbb{T}^n} \sup_{\xi \in \mathbb{Z}^n} \left| \partial_y^\alpha a(y,\xi) \right|^2.$$

Thus, for $n = 3$, $k = 2$, $a \in S_{1,0}^0 (\mathbb{T}^3 \times \mathbb{R}^3)$ we have

$$\|\mathrm{Op}(a)\|_{L^2(\mathbb{T}^3) \to L^2(\mathbb{T}^3)}^2 \leq C \sum_{|\alpha| \leq 2} \sup_{y \in \mathbb{T}^3} \sup_{\xi \in \mathbb{Z}^3} \left| \partial_y^\alpha a(y,\xi) \right|^2. \tag{3.1}$$

4. Bounds For the Green's Functions and Truncation

We collect some auxiliary results proved in [18]:

We have the following lower bound for the L^2-norm of the Green's function:

$$\|G_\lambda\|_2^2 \gg \lambda^{1/2 - \varepsilon}.$$

For $L > 0$, define the truncated Green's function by

$$G_{\lambda,L}(x) = -\frac{1}{8\pi^3} \sum_{\left| |\xi|^2 - \lambda \right| < L} \frac{\mathrm{e}^{i\xi \cdot (x - x_0)}}{|\xi|^2 - \lambda}$$

and the L^2-normalized truncated Green's function by $g_{\lambda,L} = \frac{G_{\lambda,L}}{\|G_{\lambda,L}\|_2}$.

For $L = \lambda^\delta$, $\delta > 0$, we have

$$\|g_\lambda - g_{\lambda,L}\|_2 \to 0$$

as $\lambda \to \infty$ and

$$\|G_{\lambda,L}\|_2 = \|G_\lambda\|_2 (1 + o(1)).$$

It is also proved that for all $f \in C^\infty (\mathbb{T}^3)$

$$|\langle f g_\lambda, g_\lambda \rangle - \langle f g_{\lambda,L}, g_{\lambda,L} \rangle| \leq 2 \|f\|_\infty \|g_\lambda - g_{\lambda,L}\|_2,$$

and a similar proof shows that for all $a \in C^\infty (S^*\mathbb{T}^3)$

$$|\langle \mathrm{Op}(a) g_\lambda, g_\lambda \rangle - \langle \mathrm{Op}(a) g_{\lambda,L}, g_{\lambda,L} \rangle|$$
$$\leq 2 \|\mathrm{Op}(a)\|_{L^2(\mathbb{T}^3) \to L^2(\mathbb{T}^3)} \|g_\lambda - g_{\lambda,L}\|_2. \tag{4.1}$$

5. Basis Elements

The following lemma shows that we can approximate a smooth function on $\mathbb{T}^3 \times S^2$ by a finite linear combination of functions of the form $e_{\zeta,l,m}(x,\xi) = Y_{l,m}(\xi) e^{ix\cdot\zeta}$, where $Y_{l,m}(\xi)$ is the spherical harmonic of degree l and order m (normalized so that $\int_{S^2} |Y_{l,m}|^2 d\sigma = 1$).

Lemma 5.1. *For all $a \in C^\infty (\mathbb{T}^3 \times S^2)$, $\varepsilon > 0$, there exist N_1, N_2 and*

$$P(x,\xi) = \sum_{|\zeta| \leq N_1} \sum_{l \leq N_2} \sum_{|m| \leq l} c_{\zeta,l,m} e_{\zeta,l,m}(x,\xi) \in C^\infty (\mathbb{T}^3 \times S^2),$$

such that for all $x \in \mathbb{T}^3$, $\xi \in S^2$ and for all multi-index α where $|\alpha| \leq 2$, we have $|\partial_x^\alpha (a - P)(x,\xi)| < \varepsilon$.

Proof. Let $\varepsilon > 0$. Expanding a to its Fourier series in the variable x, for every $\xi \in S^2$ we have

$$a(x,\xi) = \sum_{\zeta \in \mathbb{Z}^3} a_\zeta(\xi) e^{ix\cdot\zeta} \tag{5.1}$$

where $a_\zeta(\xi) = \frac{1}{8\pi^3} \int_{\mathbb{T}^3} a(x,\xi) e^{-ix\cdot\zeta} dx$, with convergence in the sense of $L^2(\mathbb{T}^3)$. Moreover, since $a \in C^\infty (\mathbb{T}^3 \times S^2)$, integration by parts yields that $a_\zeta(\xi) \ll |\zeta|^{-k}$ for all k (where the implied constant is independent of ξ), so that the series in (5.1) is uniformly convergent in $\mathbb{T}^3 \times S^2$. Since uniform convergence implies L^2 convergence to the same (equivalence class of) function, and since for every ξ both sides of (5.1) are continuous functions on $\mathbb{T}^3$, we conclude that the series uniformly converges in $\mathbb{T}^3 \times S^2$ to $a(x,\xi)$. We also get that for every multi-index α, we have

$$\partial_x^\alpha a(x,\xi) = \sum_{\zeta \in \mathbb{Z}^3} a_\zeta(\xi) \partial_x^\alpha e^{ix\cdot\zeta}$$

$$= \sum_{\zeta \in \mathbb{Z}^3} a_\zeta(\xi) (i\zeta)^\alpha e^{ix\cdot\zeta} \tag{5.2}$$

and the series in (5.2) is uniformly convergent in $\mathbb{T}^3 \times S^2$ to $\partial_x^\alpha a(x,\xi)$, so we can find N_1 such that for all x,ξ and for all multi-index α where $|\alpha| \leq 2$, we have

$$\left| \partial_x^\alpha \left(a(x,\xi) - \sum_{|\zeta| \leq N_1} a_\zeta(\xi) e^{ix\cdot\zeta} \right) \right| < \frac{\varepsilon}{2}.$$

For every ζ, we have the spherical harmonics expansion:

$$a_\zeta(\xi) = \sum_{l=0}^\infty \sum_{|m| \leq l} c_{\zeta,l,m} Y_{l,m}(\xi) \tag{5.3}$$

where $c_{\zeta,l,m} = \int_{S^2} a_\zeta(\xi) \overline{Y_{l,m}(\xi)} d\sigma$, with convergence in the sense of $L^2(S^2)$. Since $a(x,\xi) \in C^\infty (\mathbb{T}^3 \times S^2)$, we easily see that for all ζ we have $a_\zeta(\xi) \in C^\infty (S^2)$, and hence $\sum_{|m| \leq l} c_{\zeta,l,m} Y_{l,m}(\xi) \ll l^{-k}$ for all k (see [2], for example),

so the sum in (5.3) is uniformly convergent in S^2 for all ζ and, again, it must converge to $a_\zeta(\xi)$. We conclude that there exists N_2 such that for all $|\zeta| \leq N_1$ and for all ξ we have

$$\left| a_\zeta(\xi) - \sum_{l \leq N_2} \sum_{|m| \leq l} c_{\zeta,l,m} Y_{l,m}(\xi) \right| < \frac{\varepsilon}{2N_1^5}$$

and if we denote

$$P(x,\xi) = \sum_{|\zeta| \leq N_1} \sum_{l \leq N_2} \sum_{|m| \leq l} c_{\zeta,l,m} e_{\zeta,l,m}(x,\xi),$$

we get that for all x, ξ and for all multi-index α such that $|\alpha| \leq 2$, we have

$$
\left| \partial_x^\alpha (a - P)(x,\xi) \right| \leq \left| \partial_x^\alpha \left(P(x,\xi) - \sum_{|\zeta| \leq N_1} a_\zeta(\xi) \, \mathrm{e}^{ix \cdot \zeta} \right) \right| + \frac{\varepsilon}{2}
$$

$$
= \left| \partial_x^\alpha \left(\sum_{|\zeta| \leq N_1} \left(a_\zeta(\xi) - \sum_{l \leq N_2} \sum_{|m| \leq l} c_{\zeta,l,m} Y_{l,m}(\xi) \right) \mathrm{e}^{ix \cdot \zeta} \right) \right| + \frac{\varepsilon}{2}
$$

$$
\leq N_1^2 \sum_{|\zeta| \leq N_1} \left| a_\zeta(\xi) - \sum_{l \leq N_2} \sum_{|m| \leq l} c_{\zeta,l,m} Y_{l,m}(\xi) \right| + \frac{\varepsilon}{2} < \varepsilon.
$$

$\square$

We can think of $e_{\zeta,l,m}(x,\xi) = Y_{l,m}(\xi) \, \mathrm{e}^{ix \cdot \zeta}$ as a zero-th order positively homogeneous symbol in $S_{1,0}^0 \left(\mathbb{T}^3 \times \mathbb{R}^3 \right)$, by extending $Y_{l,m}(\xi)$, $(l,m) \neq (0,0)$ homogeneously to the domain $|\xi| \geq 1$, and arbitrarily to the domain $|\xi| < 1$; the function $Y_{0,0}(\xi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$ extends to $\xi \in \mathbb{R}^3$ in an obvious way. To prove our main theorem, we will now see that it suffices to show a simpler version of it on the functions $e_{\zeta,l,m}(x,\xi)$.

Proposition 5.2. *Suppose that there is a subset $\Lambda_\infty \subseteq \Lambda$ of density one so that for all ζ, l, m with at least one of them nonzero we have*

$$\langle Op(e_{\zeta,l,m}) g_{\lambda,L}, g_{\lambda,L} \rangle \to 0$$

as $\lambda \to \infty$ along Λ_∞, then Theorem 1.2 follows. Here $0 < \delta < 1$, $L = \lambda^\delta$.

Proof. Let $\varepsilon > 0$, and $a(x,\xi) \in C^\infty \left(S^* \mathbb{T}^3 \right)$. From Lemma 5.1, there is

$$P(x,\xi) = \sum_{|\zeta| \leq N_1} \sum_{l \leq N_2} \sum_{|m| \leq l} c_{\zeta,l,m} e_{\zeta,l,m}(x,\xi)$$

such that for all $x \in \mathbb{T}^3$, $\xi \in \mathbb{R}^3$, $|\xi| \geq 1$ and for all multi-index α where $|\alpha| \leq 2$, we have $|\partial_x^\alpha (a - P)(x,\xi)| < \varepsilon$. Without loss of generality we can assume that we have

$$\partial_x^\alpha P(x,0) = \partial_x^\alpha a(x,0) \tag{5.4}$$

for all $x \in \mathbb{T}^3$ and for all multi-index α, because for λ large enough

$$\mathrm{Op}\,(P)\,g_{\lambda,L}\,(x) = -\frac{1}{8\pi^3 \,\|G_{\lambda,L}\|_2} \sum_{\left||\xi|^2 - \lambda\right| < L} e^{i(x-x_0)\cdot\xi} P\,(x,\xi)\,\frac{1}{|\xi|^2 - \lambda}$$

does not depend on the value of $P\,(x,0)$ [and it is easy to change $P\,(x,\xi)$ in the domain $|\xi| < 1$ to get a new symbol $\tilde{P} \in C^\infty\left(S^*\mathbb{T}^3\right)$ satisfying (5.4)], so under this assumption the inequality $|\partial_x^\alpha\,(a - P)\,(x,\xi)| < \varepsilon$ holds for every $\xi \in \mathbb{Z}^n$. Since

$$\mathrm{Op}\,(e_{0,0,0}) = \frac{1}{2}\sqrt{\frac{1}{\pi}}\,\mathrm{id},$$

and for $(\zeta, l, m) \neq (0, 0, 0)$

$$\langle \mathrm{Op}\,(e_{\zeta,l,m})\,g_{\lambda,L}, g_{\lambda,L}\rangle \to 0$$

as $\lambda \to \infty$ along Λ_∞, we have

$$\langle \mathrm{Op}\,(P)\,g_{\lambda,L}, g_{\lambda,L}\rangle = \sum_{|\zeta|\le N_1}\sum_{l\le N_2}\sum_{|m|\le l} c_{\zeta,l,m}\,\langle \mathrm{Op}\,(e_{\zeta,l,m})\,g_{\lambda,L}, g_{\lambda,L}\rangle$$

$$\to \frac{1}{2}\sqrt{\frac{1}{\pi}}\,c_{0,0,0}$$

as $\lambda \to \infty$ along Λ_∞, and

$$\int_{S^*\mathbb{T}^3} P\mathrm{d}\mu = \frac{1}{\mathrm{area}\,(\mathbb{T}^3)\,\mathrm{area}\,(S^2)}$$

$$\sum_{|\zeta|\le N_1}\sum_{l\le N_2}\sum_{|m|\le l} c_{\zeta,l,m}\left(\int_{\mathbb{T}^3} e^{ix\cdot\zeta}\mathrm{d}m\right)\left(\int_{S^2} Y_{l,m}\,(\xi)\,\mathrm{d}\sigma\right)$$

$$= \frac{1}{\mathrm{area}\,(\mathbb{T}^3)\,\mathrm{area}\,(S^2)}\,c_{0,0,0}\left(\int_{\mathbb{T}^3} 1\mathrm{d}m\right)\left(\int_{S^2} Y_{0,0}\,(\xi)\,\mathrm{d}\sigma\right)$$

$$= \frac{1}{2}\sqrt{\frac{1}{\pi}}\,c_{0,0,0},$$

so

$$\langle \mathrm{Op}\,(P)\,g_{\lambda,L}, g_{\lambda,L}\rangle \to \int_{S^*\mathbb{T}^3} P\mathrm{d}\mu$$

as $\lambda \to \infty$ along Λ_∞. Thus for $\lambda \in \Lambda_\infty$ large enough, we have

$$\left|\langle \mathrm{Op}\,(P)\,g_\lambda, g_\lambda\rangle - \int_{S^*\mathbb{T}^3} P\mathrm{d}\mu\right| < \varepsilon + |\langle \mathrm{Op}\,(P)\,g_\lambda, g_\lambda\rangle - \langle \mathrm{Op}\,(P)\,g_{\lambda,L}, g_{\lambda,L}\rangle|$$

$$\le \varepsilon + 2\,\|\mathrm{Op}\,(P)\|_{L^2(\mathbb{T}^3)\to L^2(\mathbb{T}^3)}\,\|g_\lambda - g_{\lambda,L}\|_2$$

$$< C\varepsilon.$$

Using the bound in (3.1), we get that

$$
\begin{aligned}
|\langle \operatorname{Op}(a-P) g_\lambda, g_\lambda \rangle|^2 &\leq \|\operatorname{Op}(a-P) g_\lambda\|_2^2 \\
&\leq \|\operatorname{Op}(a-P)\|_{L^2(\mathbb{T}^3) \to L^2(\mathbb{T}^3)}^2 \\
&\leq C \sum_{|\alpha| \leq 2} \sup_{y \in \mathbb{T}^3} \sup_{\xi \in \mathbb{Z}^3} |\partial_y^\alpha (a-P)(y,\xi)|^2 \\
&< C\varepsilon^2
\end{aligned}
$$

for $\lambda \in \Lambda_\infty$ (we call all our constants C).

We conclude that for $\lambda \in \Lambda_\infty$ large enough, we have

$$
\left| \langle \operatorname{Op}(a) g_\lambda, g_\lambda \rangle - \int_{S^*\mathbb{T}^3} a \mathrm{d}\mu \right| \leq |\langle \operatorname{Op}(a-P) g_\lambda, g_\lambda \rangle|
$$

$$
+ \left| \langle \operatorname{Op}(P) g_\lambda, g_\lambda \rangle - \int_{S^*\mathbb{T}^3} P \mathrm{d}\mu \right|
$$

$$
+ \left| \int_{S^*\mathbb{T}^3} P \mathrm{d}\mu - \int_{S^*\mathbb{T}^3} a \mathrm{d}\mu \right|
$$

$$
< C\varepsilon
$$

and the proposition follows. □

6. A Density One Set

By the theorem of Legendre and Gauss (see [10]), the Diophantine equation

$$
x_1^2 + x_2^2 + x_3^2 = n
$$

has solutions in integers x_i $(i = 1, 2, 3)$ if and only if n is not of the form $4^a (8k + 7)$ with $a \in \mathbb{Z}$, $a \geq 0$ and $k \in \mathbb{Z}$, and for all n, $r_3(4^a n) = r_3(n)$.

Equivalently, if we write $n = 4^a n_1$, with $4 \nmid n_1$, then n is a sum of three squares if and only if $n_1 \not\equiv 7$ (8), that is to say

$$
\mathcal{N}_3 = \{n \in \mathbb{N} : n = 4^a n_1, \, 4 \nmid n_1 \Rightarrow n_1 \not\equiv 7 \, (8)\},
$$

and $r_3(n) = r_3(n_1)$.

For $n \in \mathcal{N}_3$, write $n = 4^a n_1$ with $4 \nmid n_1$, and define

$$
\mathcal{N}_{\text{good}} = \left\{ n \in \mathcal{N}_3 : n_1 > n^{1/2} \right\}
$$

the set of "good" elements in $\mathcal{N}_3$, and $\mathcal{N}_{\text{bad}} = \mathcal{N}_3 \backslash \mathcal{N}_{\text{good}}$ the set of "bad elements". We show that there are very few "bad" elements in $\mathcal{N}_3$:

Lemma 6.1. $\# \{n \in \mathcal{N}_{\text{bad}} : n \leq X\} \leq X^{1/2} \log X.$

Proof. For $n \in \mathcal{N}_{\text{bad}}$, $n \leq X$, we have $n = 4^a n_1$ with $n_1 \leq n^{1/2} \leq X^{1/2}$ and $a = \log_4(n/n_1) \leq \log n / \log 4 \leq \log X$, so there are at most $X^{1/2} \log X$ possibilities for such n. □

For $\lambda \in \Lambda$, denote n_λ to be the closest element in $\mathcal{N}_3$ to λ (and if there are two elements with the same distance, take the smallest of them). Note that $|n_\lambda - \lambda| \leq 1.5$, and for $n \neq n_\lambda$ we have $|n - \lambda| \geq 0.5$.

Define $\Lambda_\infty = \{\lambda \in \Lambda : n_\lambda \in \mathcal{N}_{\text{good}}\}$. We show that:

Lemma 6.2. Λ_∞ *is a density one set in* Λ.

Proof. For every $\lambda \in \Lambda \backslash \Lambda_\infty$, we have $n_k < \lambda < n_{k+1}$, where either $n_\lambda = n_k$ or $n_\lambda = n_{k+1}$, and $n_\lambda \in \mathcal{N}_{\text{bad}}$. Thus, for every $n \in \mathcal{N}_{\text{bad}}$ where $n \leq X + 1.5$ there are at most two $\lambda \in \Lambda \backslash \Lambda_\infty$, $\lambda \leq X$ such that $n_\lambda = n$, so by Lemma 6.1 we have

$$\# \{\lambda \in \Lambda \backslash \Lambda_\infty : \lambda \leq X\} \leq 2 \# \{n \in \mathcal{N}_{\text{bad}} : n \leq X + 1.5\} \ll X^{1/2} \log X,$$

but

$$\# \{\lambda \in \Lambda : \lambda \leq X\} \geq \# \{n \leq X : n \not\equiv 0, 4, 7 \ (8)\} \asymp X,$$

so

$$\frac{\# \{\lambda \in \Lambda \backslash \Lambda_\infty : \lambda \leq X\}}{\# \{\lambda \in \Lambda : \lambda \leq X\}} \ll X^{-1/2} \log X$$

which tends to zero as $X \to \infty$. So $\Lambda \backslash \Lambda_\infty$ is a density zero set in Λ, and therefore Λ_∞ is a density one set in Λ. $\qquad\square$

7. Proving Theorem 1.2

We are only left to prove the following proposition:

Proposition 7.1. *Let* ζ, l, m *with at least one of them nonzero, and let* $0 < \delta < 1/28$, $L = \lambda^\delta$. *Then*

$$\langle Op(e_{\zeta,l,m}) \, g_{\lambda,L}, g_{\lambda,L} \rangle \to 0$$

as $\lambda \to \infty$ *along* Λ_∞.

Proof. We have

$$|\langle \mathrm{Op}(e_{\zeta,l,m}) G_{\lambda,L}, G_{\lambda,L} \rangle|$$

$$\asymp \left| \left\langle \sum_{||\xi|^2 - \lambda| < L} \frac{e^{i(x-x_0)\cdot\xi}}{|\xi|^2 - \lambda} e^{ix\cdot\zeta} Y_{l,m}\left(\frac{\xi}{|\xi|}\right), \sum_{||\eta|^2 - \lambda| < L} \frac{e^{i(x-x_0)\cdot\eta}}{|\eta|^2 - \lambda} \right\rangle \right|$$

$$\asymp \left| \sum_{\substack{||\xi|^2 - \lambda| < L \\ ||\xi+\zeta|^2 - \lambda| < L}} \frac{1}{\left(|\xi|^2 - \lambda\right)\left(|\xi+\zeta|^2 - \lambda\right)} Y_{l,m}\left(\frac{\xi}{|\xi|}\right) \right|.$$

First, assume that $\zeta \neq 0$:

The functions $Y_{l,m}$ are bounded on S^2, so

$$|\langle \mathrm{Op}\,(e_{\zeta,l,m})\,G_{\lambda,L}, G_{\lambda,L}\rangle| \ll \sum_{\substack{\left||\xi|^2-\lambda\right|<L \\ \left||\xi+\zeta|^2-\lambda\right|<L}} \frac{1}{\left||\xi|^2-\lambda\right|\left||\xi+\zeta|^2-\lambda\right|}$$

and therefore

$$|\langle \mathrm{Op}\,(e_{\zeta,l,m})\,g_{\lambda,L}, g_{\lambda,L}\rangle| \ll \frac{\sum_{\substack{\left||\xi|^2-\lambda\right|<L \\ \left||\xi+\zeta|^2-\lambda\right|<L}} \frac{1}{\left||\xi|^2-\lambda\right|\left||\xi+\zeta|^2-\lambda\right|}}{\|G_{\lambda,L}\|_2^2} \to 0$$

as $\lambda \to \infty$ by the proof of Proposition 3.9 in [18].

Assume now that $\zeta = 0$.

We have

$$|\langle \mathrm{Op}\,(e_{0,l,m})\,G_{\lambda,L}, G_{\lambda,L}\rangle| \asymp \left| \sum_{\left||\xi|^2-\lambda\right|<L} \frac{Y_{l,m}\left(\frac{\xi}{|\xi|}\right)}{\left(|\xi|^2-\lambda\right)^2} \right|$$

$$= \left| \sum_{|n-\lambda|<L} \frac{W_{l,m}\,(n)}{(n-\lambda)^2} \right|$$

where

$$W_{l,m}\,(n) = \sum_{|\xi|^2=n} Y_{l,m}\left(\frac{\xi}{|\xi|}\right).$$

If we write $n = 4^a n_1$ with $4 \nmid n_1$, then by Duke's estimate (see [5,6])

$$|W_{l,m}\,(n)| = |W_{l,m}\,(n_1)| \ll n_1^{13/28+\varepsilon} \leq n_1^{13/28+\varepsilon}$$

and by Siegel's theorem [17] $r_3\,(n) = r_3\,(n_1) \gg n_1^{1/2-\varepsilon}$, so

$$\frac{|W_{l,m}\,(n)|}{r_3\,(n)} \ll n_1^{-1/28+\varepsilon}.$$

For $\lambda \in \Lambda_\infty$, recall that n_λ is the closest element in $\mathcal{N}_3$ to λ (and if there are two elements with the same distance, take the smallest of them). From the definition of Λ_∞ we know that $n_\lambda \in \mathcal{N}_{\mathrm{good}}$.

Write

$$\sum_{|n-\lambda|<L} \frac{W_{l,m}\,(n)}{(n-\lambda)^2} = \sum_{\substack{|n-\lambda|<L \\ n\neq n_\lambda}} \frac{W_{l,m}(n)}{(n-\lambda)^2} + \frac{W_{l,m}(n_\lambda)}{(n_\lambda-\lambda)^2}.$$

Since for $n \neq n_\lambda$ we have $|n-\lambda| \geq 0.5$:

$$\left| \sum_{\substack{|n-\lambda|<L \\ n\neq n_\lambda}} \frac{W_{l,m}\,(n)}{(n-\lambda)^2} \right| \ll \sum_{|n-\lambda|<L} n^{13/28+\varepsilon} \ll \lambda^{13/28+\delta+\varepsilon},$$

then

$$\frac{\left|\sum_{\substack{|n-\lambda|<L \\ n\neq n_\lambda}} \frac{W_{l,m}(n)}{(n-\lambda)^2}\right|}{\|G_{\lambda,L}\|_2^2} \ll \frac{\lambda^{13/28+\delta+\varepsilon}}{\|G_\lambda\|_2^2} \ll \lambda^{-1/28+\delta+2\varepsilon}$$

which tends to zero for $\varepsilon > 0$ small enough, because $\delta < \frac{1}{28}$. For the other term, writing $n_\lambda = 4^a n_1$ with $4 \nmid n_1$, we have

$$\frac{\left|\frac{W_{l,m}(n_\lambda)}{(n_\lambda-\lambda)^2}\right|}{\|G_{\lambda,L}\|_2^2} \asymp \frac{\frac{|W_{l,m}(n_\lambda)|}{(n_\lambda-\lambda)^2}}{\|G_\lambda\|_2^2}$$

$$\asymp \frac{\frac{|W_{l,m}(n_\lambda)|}{(n_\lambda-\lambda)^2}}{\sum_{n=0}^\infty \frac{r_3(n)}{(n-\lambda)^2}}$$

$$\ll \frac{|W_{l,m}(n_\lambda)|}{r_3(n_\lambda)} \ll n_1^{-1/28+\varepsilon}$$

and, since $n_\lambda \in \mathcal{N}_{\text{good}}$, we have

$$n_1^{-1/28+\varepsilon} \le n_\lambda^{-1/56+\frac{\varepsilon}{2}} \ll \lambda^{-1/56+\frac{\varepsilon}{2}}$$

so this term also tends to zero as $\lambda \to \infty$ along Λ_∞. $\square$

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