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# A Geometric Uncertainty Principle with an Application to Pleijel's Estimate

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**Abstract.** Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain and $\Omega = \bigcup_{i=1}^{N} \Omega_i$ be a partition. Denote the Fraenkel asymmetry by $0 \leq \mathcal{A}(\Omega_i) \leq 2$ and write

$$D(\Omega_i) := \frac{|\Omega_i| - \min_{1 \leq j \leq N} |\Omega_j|}{|\Omega_i|}$$

with $0 \leq D(\Omega_i) \leq 1$. For $N$ sufficiently large depending only on $\Omega$, there is an uncertainty principle

$$\left( \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}(\Omega_i) \right) + \left( \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) \right) \geq \frac{1}{60000}.$$

The statement remains true in dimensions $n \geq 3$ for some constant $c_n > 0$. As an application, we give an (unspecified) improvement of Pleijel's estimate on the number of nodal domains of a Laplacian eigenfunction and an improved inequality for a spectral partition problem.

## 1. Introduction

### 1.1. Motivation

It is easy to partition $\mathbb{R}^2$ into sets of equal measure that are 'almost' disks (the hexagonal packing, for example) and it is also possible to decompose $\mathbb{R}^2$ into disks of different size (Apollonian packings)—but obviously not both at the same time. We are interested in a quantitative descriptions of this phenomenon.

This question turns out to have some relevance in the calculus of variations, in particular in the study of vibrations of a membrane $\Omega \subset \mathbb{R}^2$ as well as in spectral partition problems: given an eigenfunction $\phi$ of the Laplacian $-\Delta$ with Dirichlet boundary conditions on $\Omega$, what is the maximal number of connected components of $\Omega \backslash \{x \in \Omega : \phi(x) = 0\}$? Our quantitative study of

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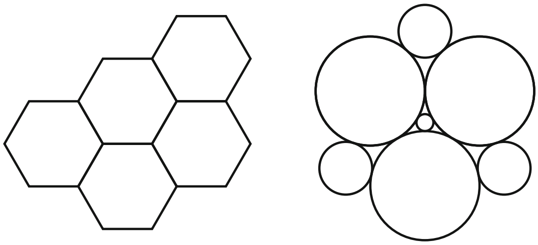

Figure 1. A partition into sets of equal measure and a partition into disks (only large disks visible)

this simple geometric principle in terms of Fraenkel asymmetry and size is very much motivated by the applicability to nodal domain estimates—it could be of interest to capture the same phenomenon in other geometrically natural quantities (Fig. 1).

### 1.2. Geometric Notions

Let $n \geq 2$. Consider an open, bounded domain $\Omega \subset \mathbb{R}^n$ with a given decomposition

$$\Omega = \bigcup_{i=1}^{N} \Omega_i. \tag{1}$$

We require two quantities to measure:

1. The deviation of $\Omega_i$ from a ball.
2. The deviation of $|\Omega_i|$ from.

$$\min_{1 \leq j \leq N} |\Omega_j|. \tag{2}$$

In measuring how much a set deviates from a ball, Fraenkel asymmetry has recently become an increasingly central notion (i.e., [10]); given a domain $\Omega \subset \mathbb{R}^n$, its Fraenkel asymmetry is defined via

$$\mathcal{A}(\Omega) := \inf_{B} \frac{|\Omega \triangle B|}{|\Omega|}, \tag{3}$$

where the infimum ranges over all disks $B \subset \mathbb{R}^n$ with $|B| = |\Omega|$ and $\triangle$ is the symmetric difference

$$\Omega \triangle B = (\Omega \backslash B) \cup (B \backslash \Omega). \tag{4}$$

Fraenkel asymmetry is scale invariant

$$0 \leq \mathcal{A}(\Omega) \leq 2. \tag{5}$$

As for deviation in size, we define the deviation from the smallest element in the partition via

$$D(\Omega_i) := \frac{|\Omega_i| - \min_{1 \leq j \leq N} |\Omega_j|}{|\Omega_i|}, \tag{6}$$

which is scale invariant as well and satisfies

$$0 \leq D(\Omega_i) \leq 1. \tag{7}$$

### 1.3. Main Result

Our main result states that for partitions of $\Omega$ into a large number of sets, an *average* element of the partition needs to have either its Fraenkel asymmetry $\mathcal{A}(\Omega_i)$ or its deviation from the smallest element $D(\Omega_i)$ bounded away from 0 by a *universal* constant. This statement obviously fails if we only pick one of the two terms: any set can be decomposed into $N$ sets of measure $|\Omega|/N$ each or each set can be decomposed into disks of different radii with an arbitrarily small measure of different shape (packings of Apollonian type).

**Theorem 1.** *Suppose $\Omega \subset \mathbb{R}^n$ is an open and bounded domain and*

$$\Omega = \bigcup_{i=1}^{N} \Omega_i \tag{8}$$

*with measurable sets $\Omega_i$ satisfying*

$$\Omega_i \cap \Omega_j = \emptyset \qquad \text{for} \quad i \neq j. \tag{9}$$

*There exists a universal constant $c_n > 0$ depending only on the dimension and a constant $N_0 \in \mathbb{N}$ depending only on $\Omega$ such that for $N \geq N_0$*

$$\left( \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}(\Omega_i) \right) + \left( \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) \right) \geq c_n. \tag{10}$$

*In particular,*

$$c_2 \geq \frac{1}{60000}. \tag{11}$$

*Remarks.*

- Taking $\Omega$ to be the union of a finite number of disjoint balls of equal radius shows that such a statement can only hold for $N$ sufficiently large depending on $\Omega$.
- There are no assumptions whatsoever on the shape of $\Omega_j$—they need not be connected.
- Fraenkel asymmetry turns the problem into a non-local one as the 'missing' measure $\Omega \triangle B$ can be arbitrarily spread over the plane: this is why we believe that any argument yielding a substantially improved constant will need to be based on significantly new ideas. Indeed, our proof will essentially only be a 'non-local perturbation' of a local argument, but not truly non-local itself (hence the small constant).
- What can be said about the optimal constant $c_n$? A natural candidate for an extremizer in $\mathbb{R}^2$ is the hexagonal tiling, which suggests that maybe

  $$c_2 \sim 0.074465754\ldots \tag{12}$$

  As packing density of spheres decreases in higher dimensions, we consider it extremely natural to conjecture that

  $$c_2 \leq c_3 \leq \cdots \tag{13}$$

- The following interesting question is due to Almut Burchard: suppose the hexagonal packing was indeed a minimizer; we can introduce a parameter $\alpha > 0$ and look for minimizers of
$$\alpha \left( \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}(\Omega_i) \right) + \left( \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) \right). \tag{14}$$
It seems reasonable to conjecture that the hexagonal packing will then be a minimizer for every $0 < \alpha \leq 1$. However, it is easy to see that there will be some $\alpha_0 \geq 1$ such that the hexagonal packing is no longer minimizing for any $\alpha > \alpha_0$. What happens at the transition? Which configurations minimize the expression then?

### 1.4. Variants and Extensions

There are many possible variations and extensions. We can write Fraenkel asymmetry as
$$\mathcal{A}(\Omega) = \inf_{x \in \mathbb{R}^n} \frac{|\Omega \triangle (B + x)|}{|\Omega|}, \tag{15}$$
where $B$ is the ball centered at the origin scaled in such a way that $|B| = |\Omega|$. However, this definition can be easily generalized by considering other sets $K$ instead of the ball if one corrects for the arising lack of rotational symmetry, i.e.,
$$\mathcal{A}_K(\Omega) := \inf_{x \in \mathbb{R}^n} \inf_{R \in \mathcal{R}} \frac{|\Omega \triangle (RK + x)|}{|\Omega|}, \tag{16}$$
where $K$ is scaled in such a way that $|K| = |\Omega|$ and $\mathcal{R}$ is the set of all rotations. The proof of our main statement is quite robust: it immediately allows to prove the following variant.

**Theorem 2.** *Let $K \subset \mathbb{R}^n$ be a bounded, convex set with a smooth boundary containing no line segment. Then there exists a constant $c(K) > 0$ such that for any open, bounded $\Omega \subset \mathbb{R}^n$ and any decomposition*
$$\Omega = \bigcup_{i=1}^{N} \Omega_i \tag{17}$$
*with measurable sets $\Omega_i$ satisfying*
$$\Omega_i \cap \Omega_j = \emptyset \qquad \text{for} \quad i \neq j \tag{18}$$
*and $N$ sufficiently large, there is a geometric uncertainty principle*
$$\left( \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}_K(\Omega_i) \right) + \left( \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) \right) \geq c(K). \tag{19}$$

This is certainly not the most general form of the theorem. Let $\mathcal{S}$ be the set of bounded sets in $\mathbb{R}^n$ such that $\mathbb{R}^n$ can be partitioned into translations and rotations of $\mathcal{S}$. Suppose $K$ is a bounded set satisfying
$$\inf_{S \in \mathcal{S}} \mathcal{A}_S(K) > \varepsilon \tag{20}$$

for some $\varepsilon > 0$. Does this already imply a geometric uncertainty principle for $\mathcal{A}_K$ with a constant depending only on $\varepsilon$?

## 2. Application to Spectral Problems

### 2.1. Introduction

Consider an open, bounded domain $\Omega \subset \mathbb{R}^2$. The Laplacian operator with Dirichlet conditions gives rise to a sequence of eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ and associated eigenfunctions $(\phi_n)_{n\in\mathbb{N}}$, where

$$-\Delta\phi_n = \lambda_n\phi_n \quad \text{in } \Omega \tag{21}$$

$$\phi_n = 0 \qquad \text{on } \partial\Omega. \tag{22}$$

Laplacian eigenfunctions are of great intrinsic interest and have been extensively studied. One natural question is to find bounds on the number of connected components of

$$\Omega \backslash \{x \in \Omega : \phi_n(x) = 0\}. \tag{23}$$

Let us denote this quantity by $N(\phi_n)$. There are no non-trivial lower bounds on $N(\phi_n)$ in general. Denoting the smallest positive zero of the Bessel function by $j \sim 2.40\ldots$, the known upper bounds are as follows:

$$N(\phi_n) \le n \qquad \text{(Courant, 1924)} \tag{24}$$

$$\limsup_{n\to\infty} \frac{N(\phi_n)}{n} \le \left(\frac{2}{j}\right)^2 \qquad \text{(Pleijel, 1956)} \tag{25}$$

$$\limsup_{n\to\infty} \frac{N(\phi_n)}{n} \le \left(\frac{2}{j}\right)^2 - 3\cdot 10^{-9} \qquad \text{(Bourgain, 2013),} \tag{26}$$

where $(2/j)^2 \le 7/10$. Polterovich [15] suggests that the optimal constant might be $2/\pi \sim 0.63$ with equality for a rectangle (this example has also been noted Bérard [1] and probably others). It seems natural to assume that a domain $\Omega \subset \mathbb{R}^2$ giving rise to a large number of nodal domains needs to have a completely integrable geodesic flow. Some numerical experiments in this direction have been carried out by Blum et al. [4].

### 2.2. Pleijel's Argument

Pleijel's argument [14] is short and simple. Suppose the eigenfunction $\phi_n$ induces a partition

$$\Omega = \bigcup_{i=1}^{N} \Omega_i. \tag{27}$$

Then, by the Faber–Krahn inequality,

$$\lambda_n(\Omega) \ge \lambda_1(\Omega_i) \ge \lambda_1(B), \tag{28}$$

where $B$ is the disk satisfying $|B| = |\Omega_i|$. However, $\lambda_1(B)$ can be explicitly computed and the inequality then implies a lower bound on $|B|$. Combining this with Weyl's law $\lambda_n \sim 4\pi n/|\Omega|$, yields the result. Of course, this argument is only sharp if we have a decomposition of $\Omega$ into disks of equal radius.

### 2.3. Bourgain's Argument

Bourgain [5] employs a spectral stability estimate due to Hansen and Nadirashvili, which is formulated in terms of the inradius of a domain: for a non-empty, bounded domain $\Omega \subset \mathbb{R}^2$, we have

$$\lambda_1(\Omega) \geq \left[1 + \frac{1}{250}\left(1 - \frac{r_i(\Omega)}{r_o(\Omega)}\right)^3\right]\lambda_1(\Omega_0), \tag{29}$$

where $\Omega_0$ is the ball with $|\Omega_0| = |\Omega|$, $r_0(\Omega)$ is the radius of $\Omega_0$ and $r_i$ the inradius of $\Omega$. The second ingredient is a packing result due to Blind [3]: the packing density of a collection of disks in the plane with radii $a_1, a_2, \ldots$ satisfying $a_i \geq (3/4)a_j$ for all $i, j$ is bounded from above by $\pi/\sqrt{12}$. These two results imply the improvement.

### 2.4. An Improved Pleijel Estimate

Exploiting stability estimates for the Faber–Krahn inequality in terms of Fraenkel asymmetry, we are able to prove the following result.

**Corollary 1.** *There exists a constant* $\varepsilon_0 > 0$ *such that*

$$\limsup_{n\to\infty} \frac{N(\phi_n)}{n} \leq \left(\frac{2}{j}\right)^2 - \varepsilon_0. \tag{30}$$

An explicit value for $\varepsilon_0$ would follow from an explicit constant in a Faber–Krahn stability result involving Fraenkel asymmetry (these constants are known to exist but have not yet been determined explicitly). Given the general interest in this question, we are confident that such a result will be eventually obtained. Much like Bourgain, however, we consider the underlying geometry more interesting than the actual numerical value—particularly in light of the following obstruction.

### 2.5. An Obstruction

Take $\Omega = [0,1]^2$ of unit measure and cover it using the hexagonal covering (with obvious modifications at the boundary). Numerical computations (e.g., [13]) give that the first Laplacian eigenvalue of a hexagon $H$ satisfies

$$\lambda_1(H) \sim \frac{18.5762}{|H|}. \tag{31}$$

The Weyl law gives

$$\lambda_n(\Omega) \sim 4\pi n. \tag{32}$$

We can place $N$ hexagons of size $|H|$ in $\Omega$, where

$$N|H| = 1. \tag{33}$$

Since we need to have $\lambda_n(\Omega) \geq \lambda_1(H)$, this implies

$$4\pi n \sim \frac{18.5762}{|H|} \tag{34}$$

and thus

$$N = \frac{1}{|H|} \sim \frac{4\pi}{18.5762} n \sim 0.676\ldots n. \tag{35}$$

As a consequence, any type of argument that leads to an improved Pleijel inequality with a constant smaller than $0.67\ldots$ will need to employ completely different arguments: the arguments given by Pleijel, Bourgain and this paper argue based on the *assumption* that a partition of $\Omega$ into nodal domains is given. However, such a partition could very well be the hexagonal partition. Arguments leading to a better constant than $0.676\ldots$ will need to *explain* why, say, an eigenfunction on a domain will not have eigenfunctions corresponding to a partition into hexagons.

### 2.6. Spectral Minimal Partitions

The problem of spectral minimal partitions is as follows: given a smooth, bounded domain $\Omega \subset \mathbb{R}^n$ and an integer $k \in \mathbb{N}$, find among all partitions of $\Omega$ into $k$ disjoint domains

$$\Omega = \bigcup_{i=1}^{k} \Omega_i \tag{36}$$

the one minimizing

$$\max_{1 \le i \le k} \lambda_1(\Omega_i). \tag{37}$$

It is conjectured that in two dimensions the minimal partitions should asymptotically behave like hexagonal tilings (with the exception of the boundary, which becomes negligible as $k \to \infty$). We refer to Caffarelli and Lin [7], Helffer et al. [12] and a survey of Helffer [11]. One basic inequality [12, Proposition 6.1] following immediately from Pleijel's estimate is that

$$\max_{1 \le i \le k} \lambda_1(\Omega_i) \ge k \frac{\pi j^2}{|\Omega|}. \tag{38}$$

Bourgain remarks that his argument also allows to slightly improve the constant in this inequality. As a second quantity that is sometimes minimized (see e.g., Caffarelli and Lin [7] or Bérard and Helffer [2]), one can consider the average and establish a strengthened Pleijel-type estimate

$$\max_{1 \le i \le k} \lambda_1(\Omega_i) \ge \frac{1}{k} \sum_{i=1}^{k} \lambda_1(\Omega_i) \ge k \frac{\pi j^2}{|\Omega|}. \tag{39}$$

This inequality, too, can be strengthened.

**Corollary 2.** *There exists a $\varepsilon_0 > 0$ such that for any smooth, bounded domain $\Omega \subset \mathbb{R}^2$ and all $k$ sufficiently large (depending on $\Omega$)*

$$\frac{1}{k} \sum_{i=1}^{k} \lambda_1(\Omega_i) \ge (\pi j^2 + \varepsilon_0) \frac{k}{|\Omega|}. \tag{40}$$

## 3. Proof of Theorem 1 in Two Dimensions

This section contains a complete proof of the main statement in dimension $n = 2$: the proof will track all arising constants. This takes up most of the text and contains all the ideas of this paper—the argument is robust, and the necessary (and rather easy) modifications to obtain more general results will then be given in subsequent sections.

### 3.1. Two Possible Strategies

There seems to be a very natural way to prove the statement; however, we did not manage to fully quantify all the steps and had to find another argument. We record our original idea nonetheless in the hope of building additional insight.

*Sketch of an idea.* The inequality can be regarded as a probabilistic statement. Pick a random domain weighted according to size (i.e., choosing a random point of the domain, the probability of picking $\Omega_i$ is $|\Omega_i|/|\Omega|$). Our statement can be read as a lower bound on the expectation of the random variable

$$\mathcal{A}(\Omega_i) + D(\Omega_i). \tag{41}$$

This motivates the following argument. Pick a random domain: either it already has large Fraenkel asymmetry (in which case we are done) or it does not and behaves quite disk-like. In the second case, we look at its neighbouring domains. If there are few adjacent domains, at least one of them touches along a long arc of the boundary meaning that the neighbouring domain has large Fraenkel asymmetry (two disks touch in at most one point). If there are many neighbours, either most are significantly smaller (making our randomly chosen domain big in comparison and giving the statement) or some will need to get squeezed together because there is not enough room (creating a large Fraenkel asymmetry). We believe that such a strategy, properly implemented, could give a relatively sharp constant—however, making all these steps quantitative seems complicated.

*Sketch of a different idea: our proof.* We chose a different approach of a more global nature: given a decomposition, we immediately switch to a collection of $N$ disks by taking disks realizing the Fraenkel asymmetry for each partition. Then, we show that:

- There are few very large elements: the size of neighbourhood of the union of all disks whose size is bounded away from the smallest element in the partition by a constant factor can be bounded from above.
- Ignoring the large sets (of which there are few), the Fraenkel balls of small sets usually do not overlap too much; the exceptional set is small.

Removing all large disks and all overlapping disks, we may shrink the remaining disks such that no two of them overlap: the resulting disk packing cannot have too high a density.

### 3.2. Defining Quantities

The limes inferior in the statement guarantees that boundary effects coming from $\partial\Omega$ become negligible and we will ignore the boundary throughout the proof (equivalently, we could have phrased the statement for periodic partitions of $\mathbb{R}^2$).

We assume w.l.o.g. that $|\Omega| = 1$. For a point $x \in \mathbb{R}^2$ and a set $A \subset \mathbb{R}^2$, we abbreviate

$$\|x - A\| := \inf_{y \in A} \|x - y\|. \tag{42}$$

We introduce two numbers $c_1, c_2 > 0$ that will serve as threshold values for 'being big' and 'strong overlap' and we will keep them as variables throughout the proof; however, a minimization problem towards the end of the proof will motivate us to set

$$c_1 = \frac{1}{250} \quad \text{and} \quad c_2 = \frac{7}{250} \tag{43}$$

and the reader can substitute these values throughout the proof if he wishes to. Their role is as follows: we call $\Omega_i$ 'big', if

$$|\Omega_i| \geq (1 + c_1) \min_{1 \leq j \leq N} |\Omega_j|. \tag{44}$$

The constant $c_2$ will serve as a measure of overlap: two disks with centers in $x, y \in \mathbb{R}^2$ and radii $r_1, r_2$ will be considered to have 'large' overlap if

$$|x - y| \leq (1 - c_2)(r_1 + r_2). \tag{45}$$

We define a natural length scale $\eta_0$. *Everything* in this problem and this proof is scale invariant and, correspondingly, the actual size of $\eta_0$ is completely irrelevant throughout the proof: the variable cancels in the end. However, we consider it helpful to imagine a fixed length scale $\eta_0$ at which everything plays out and will phrase all arising quantities in terms of $\eta_0$, which we define via

$$\pi \eta_0^2 = \min_{1 \leq i \leq N} |\Omega_i|. \tag{46}$$

The proof will be carried out via contradiction, we assume

$$\left( \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}(\Omega_i) \right) + \left( \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) \right) \leq c \tag{47}$$

for some small constant $c$ and show that this will lead to a contradiction if $c$ is small enough. It makes sense to be slightly more careful, and therefore we assume that for all $d_1, d_2 \geq 0$ with $c = d_1 + d_2$

$$\left( \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}(\Omega_i) \right) \leq d_1 \tag{48}$$

and

$$\left( \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{D}(\Omega_i) \right) \leq d_2. \tag{49}$$

We assign to each of the sets $\Omega_1, \dots, \Omega_n$ a disk $B_1, B_2, \dots, B_n$ such that $|B_i| = |\Omega_i|$ and

$$\mathcal{A}(\Omega_i) = \frac{|\Omega_i \triangle B_i|}{|\Omega_i|}. \tag{50}$$

Note that a disk $B_i$ need not be uniquely determined by $\Omega_i$ (if there is more than one possible choice, we pick an arbitrary one and fix it for the rest of the proof). Each of these disks $B_i$ has a center $x_i$ and a radius $r_i \geq \eta_0$.

### 3.3. The Union of Large Sets has Small Measure

Here, we prove a simple statement: the measure occupied by 'large' sets (in the sense of (44)) is small. Note that the statement is indeed for the measure and not the number of large sets, which could be small.

**Lemma 1.** *We have*

$$\left| \bigcup_{|\Omega_i| > (1+c_1)\pi\eta_0^2} \Omega_i \right| \leq \frac{d_2}{c_1} + d_2. \tag{51}$$

*Proof.* From (46), (49) and the definition of $D(\Omega_i)$, we get that

$$d_2 \geq \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) = \sum_{i=1}^{N} (|\Omega_i| - \pi\eta_0^2) = 1 - N\pi\eta_0^2 \tag{52}$$

and therefore

$$N \geq \frac{1 - d_2}{\pi\eta_0^2}. \tag{53}$$

Now, let us suppose that $0 \leq M \leq N$ elements of the partition are 'small' in the sense of satisfying $|\Omega_i| \leq (1 + c_1)\pi\eta_0^2$. We wish to show that $M$ itself has to be big. Trivially,

$$\left| \bigcup_{|\Omega_i| \leq (1+c_1)\pi\eta_0^2} \Omega_i \right| \geq M\pi\eta_0^2. \tag{54}$$

The remaining measure is divided among big sets, hence the number of 'big' elements is at most the remaining measure divided by the smallest possible area a 'big' set can have

$$N - M \leq \frac{1 - M\pi\eta_0^2}{(1 + c_1)\pi\eta_0^2} \tag{55}$$

and thus, in total,

$$\frac{1 - d_2}{\pi\eta_0^2} \leq N = M + (N - M) \leq M + \frac{1 - M\pi\eta_0^2}{(1 + c_1)\pi\eta_0^2}. \tag{56}$$

Rewriting gives

$$M \geq \frac{c_1 - d_2 - d_2c_1}{c_1\pi\eta_0^2}, \tag{57}$$

which implies

$$\left| \bigcup_{|\Omega_i| \leq (1+c_1)\pi\eta_0^2} \Omega_i \right| \geq \frac{c_1 - d_2 - d_2c_1}{c_1} \tag{58}$$

and therefore, since $|\Omega| = 1$,

$$\left| \bigcup_{|\Omega_i| > (1+c_1)\pi\eta_0^2} \Omega_i \right| \leq \frac{d_2}{c_1} + d_2. \tag{59}$$

□

### 3.4. A Neighbourhood of the Union of Large Sets has Small Measure

In the last section, we have seen that measure of the set of large disks is small. However, we actually require a slightly stronger statement showing that an entire neighbourhood of that set is still small. For future use, we define the index set $I$ of partition elements with 'big' measure

$$I = \left\{ i \in \{1, \ldots, N\} : |\Omega_i| \geq (1 + c_1)\pi\eta_0^2 \right\}. \tag{60}$$

**Lemma 2.** *A* $2\eta_0-$*neighbourhood of* $\bigcup_{i \in I} B_i$ *has small measure: we have*

$$\left| \left\{ x \in \Omega : \left\| x - \bigcup_{i \in I} B_i \right\| \leq 2\eta_0 \right\} \right| \leq \frac{9d_2}{c_1} + 9d_2. \tag{61}$$

*Proof.* This argument is very simple: the $2\eta_0-$neighbourhood of a disk with radius $r$ has measure $(r + 2\eta_0)^2\pi$. The worst case is precisely the case, where all $B_i$ are well separated such that their $2\eta_0-$neighbourhoods do not intersect (otherwise: move the disks apart to create a neighbourhood with bigger measure). In this case, the total measure gets amplified by factor

$$\frac{\left(\sqrt{1 + c_1} + 2\right)^2 \eta_0^2 \pi}{(1 + c_1)\eta_0^2 \pi} \leq 9 \tag{62}$$

and the result follows from (51). □

### 3.5. Most Small Sets have Well-Separated Balls

By now, we have a good control on the 'large' disks and their neighbourhood: we can (mentally and later in the proof literally) remove them from the stage and consider the remaining small disks. It remains to control their intersections.

**Lemma 3.** *The union of 'small' disks* $B_i$*,* $i \neq I$*, for which there exists another 'small' disk such that they intersect strongly in the sense of* (45) *is bounded by*

$$\left| \bigcup_{i \notin I} \left\{ B_i : \exists_{j \notin I} i \neq j : \ |x_i - x_j| \leq (1 - c_2)(r_i + r_j) \right\} \right| \leq \frac{20\pi}{37} \frac{1 + c_1}{c_2^{3/2}} d_1 \tag{63}$$

*Proof.* For simplicity, we introduce the index set

$$J = \left\{ i \notin I : \exists B_i \ \exists_{j \notin I} \ i \neq j : \ |x_i - x_j| \leq (1 - c_2)(r_i + r_j) \right\}. \tag{64}$$

We will now derive an upper bound on the measure of the set, which we now can abbreviate as $\cup_{j \in J} B_j$, using nothing but the inequality (49)

$$\sum_{i=1}^{N} |\Omega_i| \mathcal{A}(\Omega_i) \leq d_1. \tag{65}$$

Suppose $i \in J$. Then there exists a $j \in J$ such that the balls $B_i, B_j$ have controlled radius (this follows automatically from the fact that both disks are 'small')

$$\eta_0 \leq r_i, r_j \leq \sqrt{1 + c_1}\eta_0 \tag{66}$$

and intersect in a quantitatively controlled way

$$|x_i - x_j| \le (1 - c_2)(r_i + r_j). \tag{67}$$

Then the intersection $B_i \cap B_j$ is of interest: if the Fraenkel asymmetry of $\Omega_i$ is to be small, then almost all of its measure should be contained in $B_i$, but the very same reasoning also holds for $\Omega_j$ and $B_j$. In particular, since every point in the intersection can only belong to one of the two sets, we have

$$|\Omega_i|\mathcal{A}(\Omega_i) + |\Omega_j|\mathcal{A}(\Omega_j) \ge |B_i \cap B_j|. \tag{68}$$

It remains to compute the quantity $|B_i \cap B_j|$. Using scaling invariance, we may assume $\eta_0 = 1$. We are then dealing with two disks in the Euclidean plane whose radii $r_1, r_2$ are bounded from below by 1 and whose centers $x_1, x_2$ satisfy

$$d := |x_1 - x_2| \le (1 - c_2)(r_1 + r_2). \tag{69}$$

Elementary Euclidean geometry yields

$$\begin{aligned} |B_i \cap B_j| = r_1^2 \arccos\left(\frac{d^2 + r_1^2 - r_2^2}{2dr_1}\right) + r_2^2 \arccos\left(\frac{d^2 + r_2^2 - r_1^2}{2dr_2}\right) \\ - \frac{1}{2}\sqrt{(-d + r_1 + r_2)(d + r_1 - r_2)(d - r_1 + r_2)(d + r_1 + r_2)}. \end{aligned} \tag{70}$$

Easy but tedious calculations give that the quantity is decreasing in both radii and as such minimized for $r_1 = r_2 = 1$. This is then a one-dimensional function in $c_2$ and it is easy to show that for $c_2 \le 0.05$ the function is

$$2 \arccos(1 - c_2) - 2\sqrt{(2 - c_2)(1 - c_2)^2 c_2} \ge \frac{37}{10} c_2^{\frac{3}{2}}. \tag{71}$$

Recalling the normalization $\eta_0 = 1$, we get the scale-invariant estimate

$$|B_i \cap B_j| \ge \frac{37}{10} c_2^{\frac{3}{2}} \eta_0. \tag{72}$$

A priori, the intersection patterns of $\{B_i : i \in J\}$ can be very complicated. However, there is a very simple monotonicity: we can remove areas, where three or more balls intersect and arrange the balls in (possibly more than one) chain. This increases the area and decreases the area of intersection. By the same argument, the area further increases if we assume that any disk in $\{B_i : i \in J\}$ touches precisely one other disk (i.e., the intersection pattern reduces to that of pairs of disks intersecting each other and no disk). Any such (i.e., intersecting) pair of disks $B_i, B_j$ satisfies (Fig. 2)

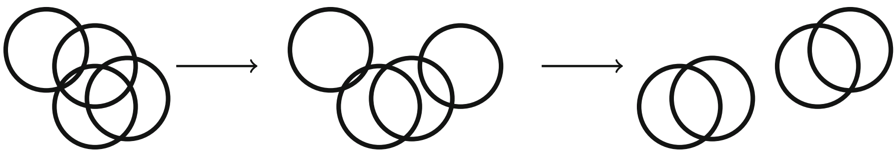

Figure 2. Increasing area while decreasing average Fraenkel asymmetry

$$|B_i \cup B_j| \le 2(1+c_1)\pi\eta_0^2 \tag{73}$$

as well as

$$|B_i \cap B_j| \ge \frac{37}{10} c_2^{\frac{3}{2}} \eta_0, \tag{74}$$

which is connected via (68) to the sum

$$\sum_{i=1}^{N} |\Omega_i| \mathcal{A}(\Omega_i) \le d_1. \tag{75}$$

Thus

$$\left| \bigcup_{j\in J} B_j \right| \le \left[2(1+c_1)\pi\eta_0^2\right] \frac{d_1}{\frac{37}{10} c_2^{3/2} \eta_0^2} = \frac{20\pi}{37} \frac{1+c_1}{c_2^{3/2}} d_1. \tag{76}$$

□

### 3.6. Bounds on the Size of the Neighbourhood of Strongly Intersecting Small Disks

By applying the very same reasoning as in Sect. 3.4, we could argue that by considering an entire $2\eta_0$-neighbourhood the measure gets amplified by a factor of at most 9. This is perfectly reasonable but can actually be improved as we are now dealing with disks intersecting other disks. We are thus studying the following problem: given two disks $B_1, B_2$ with radii $r_1, r_2 \ge \eta_0$ intersecting in precisely one point, what bounds can be proven on

$$\frac{|\left\{x \in \mathbb{R}^2 : \|x - (B_1 \cup B_2)\| \le 2\eta_0\right\}|}{|B_1| + |B_2|} \le ? \tag{77}$$

This problem can be explicitly solved using elementary calculus and reduces to a case-distinction and two integrations; we leave the details to the interested reader. Carrying out the calculations gives

$$\frac{|\left\{x \in \mathbb{R}^2 : \|x - (B_1 \cup B_2)\| \le 2\eta_0\right\}|}{|B_1| + |B_2|} \le \frac{9}{2} + \frac{2\sqrt{2}}{\pi} + \frac{9}{\pi}\arcsin\left(\frac{1}{3}\right) \tag{78}$$

$$\sim 6.37\ldots \tag{79}$$

$$\le \frac{32}{5} \tag{80}$$

with equality for $r_1 = r_2 = 1$. Arguing as in Sect. 3.4 and using (63), we get

$$\left| \left\{ x \in \Omega : \left\| x - \bigcup_{j\in J} B_j \right\| \le 2\eta_0 \right\} \right| \le \frac{128\pi}{37} \frac{(1+c_1)}{c_2^{3/2}} d_1 \tag{81}$$

### 3.7. Finding a Dense Disk Packing

We conclude our argument by deriving the existence of a disk packing in the plane with impossible properties. Here, we employ an aforementioned result of Blind [3] that also played a role in Bourgain's argument and was mentioned before: the packing density of a collection of disks in the plane with radii $a_1, a_2, \ldots$ satisfying $a_i \ge (3/4)a_j$ for all $i, j$ is bounded from above by $\pi/\sqrt{12}$.

A rough outline of the remainder of the argument is as follows:

1. Consider the set of Fraenkel disks $\{B_i : 1 \leq i \leq N\}$
2. Remove all 'big' disks
3. Remove all remaining 'small' disks strongly intersecting another small disk
4. Shrink all remaining disks by a factor $(1 - c_2)$.

This leaves us with a set of *disjoint* disks in the Euclidean plane with roughly the same radius and we can apply Blind's result—the argument has one big flaw, of course, removing elements from a set does *not* increase its packing density (just think of a hexagonal packing of disks: if we remove the little triangle-shaped gaps between the disks, packing density goes up to 1).

We counter the problem by not only removing 'big' disks or 'small' disks strongly intersecting other small disks, but an entire $2\eta_0-$neighbourhood of these sets as well. Doing this is equivalent to assuming that while we created holes in the middle of the set, these holes are of such a shape that within a neighbourhood we can actually achieve packing density 1.

From (61) and (81), we get that the set

$$\Omega^* := \Omega \setminus \left( \left\{ x \in \Omega : \left\| x - \bigcup_{i \in I} B_i \right\| \leq 2\eta_0 \right\} \cup \left\{ x \in \Omega : \left\| x - \bigcup_{j \in J} B_j \right\| \leq 2\eta_0 \right\} \right) \tag{82}$$

satisfies

$$|\Omega^*| \geq 1 - \left( \frac{9d_2}{c_1} + 9d_2 + \frac{128\pi}{37} \frac{(1+c_1)}{c_2^{3/2}} d_1 \right). \tag{83}$$

$\Omega^*$ consists of disks with radii satisfying

$$\eta_0 \leq r_i \leq \sqrt{1+c_1}\eta_0 \tag{84}$$

and with the additional property that the centers of any two disks are well separated

$$|x_i - x_j| \geq (1 - c_2)(r_i + r_j). \tag{85}$$

By shrinking all these disks by a factor of $1 - c_2$ while keeping their center in the same place, they become disjoint. Thus, from Blind's result

$$|(1-c_2)\Omega^*| \leq (1-c_2)^2 \left[ 1 - \left( \frac{9d_2}{c_1} + 9d_2 + \frac{128\pi}{37} \frac{(1+c_1)}{c_2^{3/2}} d_1 \right) \right] \leq \frac{\pi}{\sqrt{12}}. \tag{86}$$

We need to find a set of parameters, for which the inequality fails. Indeed, setting

$$c_1 = \frac{1}{250} \quad \text{and} \quad c_2 = \frac{7}{250}, \tag{87}$$

we get for any $d_1, d_2 \geq 0$ with

$$d_1 + d_2 = \frac{1}{60000}, \tag{88}$$

that

$$(1-c_2)^2\left[1-\left(\frac{9d_2}{c_1}+9d_2+\frac{128\pi}{37}\frac{(1+c_1)}{c_2^{3/2}}d_1\right)\right]\geq\frac{\pi}{\sqrt{12}}+\frac{1}{1000}. \tag{89}$$

This contradiction proves the statement. □

*Remark.* The weakest point in the argument is certainly the last step, where we remove an entire $2\eta_0-$neighbourhood. Intuition suggests that we should be able that maybe even removing merely a $\eta_0$-neighbourhood should be more than sufficient; however, we have not been able to make a progress on that question, which would certainly be the most natural starting point if one wanted to improve the constant using arguments along these lines.

## 4. Proof of the General Case

Here, we give a proof of Theorem 2 in general dimensions (which contains Theorem 1 for $n \geq 3$ as a special case). This section essentially recapitulates the previous argument without caring about the actual numerical values at all. The new ingredient is the following insight: in the proof of Theorem 1, after a careful geometric analysis, we did end up with the inequality

$$(1-c_2)^2\left[1-\left(\frac{9d_2}{c_1}+9d_2+\frac{128\pi}{37}\frac{(1+c_1)}{c_2^{3/2}}d_1\right)\right]\geq\frac{\pi}{\sqrt{12}}+\frac{1}{1000}. \tag{90}$$

The crucial point is the following: no matter what actual numerical values are placed in front, by choosing $d_2 \ll c_1$ and $d_1 \ll c_2$, the inequality will always be false for $c_1, c_2$ sufficiently close to 1 by simple continuity. In the previous proof, it was our goal to keep $d_1, d_2$ as large as possible, but once we discard this concern we can be much more wasteful in the actual geometric estimates.

*Proof.* The argument is again by contradiction. $\eta_0$ plays a similar same role as before, we define it via

$$\eta_0 = \left(\min_{1\leq j\leq N}|\Omega_j|\right)^{1/n}. \tag{91}$$

The constant $c_1$ again determines whether a domain is 'big', which we define to be the case if

$$|\Omega_i| \geq (1+c_1)\min_{1\leq j\leq N}|\Omega_j|. \tag{92}$$

The precise meaning of $c_2$ is introduced further below. Arguing by contradiction we assume that

$$\left(\sum_{i=1}^{N}\frac{|\Omega_i|}{|\Omega|}\mathcal{A}_K(\Omega_i)\right)+\left(\sum_{i=1}^{N}\frac{|\Omega_i|}{|\Omega|}D(\Omega_i)\right)\leq c \tag{93}$$

and want to derive a contradiction for $c$ sufficiently small. Following the same argument as before, we again get a bound on the number of large sets

$$\left| \bigcup_{|\Omega_i| > (1+c_1)\eta_0^n} \Omega_i \right| \leq \frac{c}{c_1} + c. \tag{94}$$

Switching again to the Fraenkel bodies $K_1, \dots, K_N$, we wish to remove a $c_3\eta_0$ neighbourhood of any 'large' Fraenkel body $K_i$, where $c_3 < \infty$ is chosen such that $c_3\eta_0$ is many multiples of the diameter of a 'small' $K_i$ having measure at most $(1+c_1)\eta_0^n$. This allows us to bound the size of a $c_3\eta_0$ neighbourhood of

$$\bigcup_{|\Omega_i| > (1+c_1)\eta_0^n} K_i \tag{95}$$

by $c_4(c/c_1 + c)$ for some finite constant $c_4$. The constant $c_2$ now measures whether two 'small' Fraenkel bodies have large intersection, writing again

$$I = \{i \in \{1, \dots, N\} : |\Omega_i| \geq (1+c_1)\eta_0^n\}, \tag{96}$$

we consider

$$\bigcup_{i \notin I} \{K_i : \exists_j i \neq j \notin I : \ |(K_i \cap K_j)| \geq c_2\eta_0^n\}. \tag{97}$$

The same argument as before implies that for any two elements in the set, we get

$$\mathcal{A}_K(K_i)|K_i| + \mathcal{A}_K(K_j)|K_j| \geq c_2\eta_0^n. \tag{98}$$

Since

$$\left( \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}_K(\Omega_i) \right) \leq c, \tag{99}$$

this implies a bound on the measure of the set

$$\left| \bigcup_{i \notin I} \{K_i : \exists_j i \neq j \notin I : \ |(K_i \cap K_j)| \geq c_2\eta_0^n\} \right| \leq c_5 c \tag{100}$$

for some constant $c_5 < \infty$ and a bound of the form $c_6 c$ on the measure of its $c_3\eta_0$ neighbourhood. Finally, since the boundary of the convex body $K$ contains no line segment, we get that for every $\varepsilon_1 > 0$ there is a $\varepsilon_2 > 0$ such that any collection $K_1, K_2, \dots$ of non-overlapping rotated and scaled translates of $K$ in the plane with volumes $v_1, v_2, \dots$ satisfying

$$\inf_{i,j} \frac{v_i}{v_j} \geq 1 - \varepsilon_1 \tag{101}$$

has packing density at most $1 - \varepsilon_2$. Finally, there exists a constant $c_7$ such that for any two scaled, translated copies $K_1, K_2$ of $K$ with

$$|(K_i \cap K_j)| \leq c_2\eta_0^n, \tag{102}$$

the rescaled bodies $c_7K_1, c_7K_2$ (rescaling being done in a way to fix, say, their center of mass) satisfy

$$(c_7K_1) \cap (c_7K_2) = \emptyset. \tag{103}$$

Note that the optimal $c_7$ depends continuously on $c_2$ and tends to 1 as $c_2$ tends to 0. Now, following the same argument as before, we can derive the inequality

$$1 - \varepsilon_2 \geq c_7^n \left(1 - \frac{c_4 c}{c_1} - c_4 c - c_6 c\right). \tag{104}$$

The dependence is easy: pick some $0 < \varepsilon_1 \ll 1$. This yields $\varepsilon_2 > 0$. Given $\varepsilon_1$, pick $c_1 \ll \varepsilon_1$. We pick $c_2$ so small that $c_7^n > 1 - \varepsilon_2$. $c_4$ and $c_6$ are again externally given, but the inequality can now be seen to be false if $c = 0$. By continuity $c > 0$. □

### 4.1. Proof of the Improved Pleijel Estimate

The Corollary has a very simple proof: as in the proof of Pleijel's estimate, we get a lower bound on

$$\min_{1 \leq i \leq N} |\Omega_i| \tag{105}$$

from the Faber–Krahn inequality. Theorem 1 now implies that either not all elements in the partition are of that size (in which case some need to be bigger and their requirement for more spaces allows for a smaller number of nodal domains) or that some deviate from the disk in a controlled way (in which case stability estimates require them to have a larger measure).

*Proof.* Let

$$\Omega = \bigcup_{i=1}^{N} \Omega_i \tag{106}$$

be the decomposition introduced by a Laplacian eigenfunction with eigenvalue $\lambda \gg 1$, and let $\eta_0 = \eta_0(\lambda)$ be chosen in such a way that $\pi\eta_0^2 = |B|$, where $B$ is the disk such that $\lambda_1(B) = \lambda$. Theorem 1 yields that

$$\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \left(\mathcal{A}(\Omega_i) + D(\Omega_i)\right) \geq c \tag{107}$$

for some $c \geq 1/60000$; therefore, either

$$\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) \geq \frac{c}{2} \quad \text{or} \quad \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}(\Omega_i) \geq \frac{c}{2}. \tag{108}$$

Suppose the first inequality holds. Then

$$\frac{c}{2} \leq \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) = \frac{1}{|\Omega|}\left(|\Omega| - N\pi\eta_0^2\right) \tag{109}$$

in which case

$$N \leq \left(1 - \frac{c}{2}\right) \frac{|\Omega|}{\pi\eta_0^2}. \tag{110}$$

The fact that Pleijel's argument is sharp for a partition into equally sized disks (or, equivalently, Weyl's law) implies

$$\lim_{\lambda \to \infty} \frac{|\Omega|}{\pi\eta_0^2} = \left(\frac{2}{j}\right)^2 n \tag{111}$$

and this yields the result. Suppose the second inequality holds. We start with a simple Lemma.

**Lemma 4.** *We have*

$$\left| \bigcup_{\mathcal{A}(\Omega_i) \geq \frac{c}{6}} \Omega_i \right| \geq \frac{c}{6} |\Omega|. \tag{112}$$

*Proof of the Lemma.* Suppose the statement was false. Then, using $\mathcal{A}(\Omega_i) \leq 2$,

$$\begin{aligned} \frac{c}{2} &\leq \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}(\Omega_i) & (113) \\ &< \frac{2}{|\Omega|} \left| \bigcup_{\mathcal{A}(\Omega_i) \geq \frac{c}{6}} \Omega_i \right| + \frac{c}{6} \frac{1}{|\Omega|} \left| \bigcup_{\mathcal{A}(\Omega_i) \leq \frac{c}{6}} \Omega_i \right| & (114) \\ &\leq \frac{c}{3} + \frac{c}{6} = \frac{c}{2}. \end{aligned}$$

□

Now, we recall some stability estimates for the Faber–Krahn inequality in terms of Fraenkel asymmetry. Brasco et al. [6] (improving an earlier result of Fusco et al. [9]) have shown that

$$\frac{\lambda_1(\Omega) - \lambda_1(\Omega_0)}{\lambda_1(\Omega_0)} \gtrsim \mathcal{A}(\Omega)^2, \tag{115}$$

where $\Omega_0$ is again the disk with $|\Omega_0| = |\Omega|$.

Pick any domain $\Omega_i$ with $\mathcal{A}(\Omega_i) \geq c/6$ and use $B$ to denote the disk such that $|B| = |\Omega_i|$. The stability estimate

$$\frac{\lambda_1(\Omega_i) - \lambda_1(B)}{\lambda_1(B)} \geq C \cdot \mathcal{A}(\Omega_i)^2 \tag{116}$$

can be rewritten as

$$\lambda_1(\Omega_i) \geq \left(1 + C\frac{c^2}{36}\right) \lambda_1(B) \tag{117}$$

Recall that $\eta_0$ is chosen such that the disk $D$ of radius $\eta_0$ satisfies $\lambda_1(D) = \lambda_n(\Omega)$. However, by (117), we know that $\lambda_1(\Omega_i)$ is a multiplicative factor larger than the first eigenvalue of the disk of equal measure. In order for $\lambda_1(\Omega_i) \leq \lambda_n(\Omega)$ to still be satisfied, we require that

$$\frac{|\Omega_i|}{\pi \eta_0^2} \geq 1 + C\frac{c^2}{36}. \tag{118}$$

We use this as follows:

$$\Omega = \bigcup_{i=1}^{N} \Omega_i = \left( \bigcup_{\mathcal{A}(\Omega_i) \geq \frac{c}{6}} \Omega_i \right) \cup \left( \bigcup_{\mathcal{A}(\Omega_i) \leq \frac{c}{6}} \Omega_i \right) \tag{119}$$

By Pleijel's argument, the number of nodal domains in the second set is bounded from above by

$$\frac{1}{\pi\eta_0^2}\left|\bigcup_{\mathcal{A}(\Omega_i)\leq\frac{c}{6}}\Omega_i\right| \tag{120}$$

while (118) implies the number of nodal domains in the first set is bounded by

$$\frac{1}{\pi\eta_0^2}\frac{1}{1+C\frac{c^2}{36}}\left|\bigcup_{\mathcal{A}(\Omega_i)\geq\frac{c}{6}}\Omega_i\right|. \tag{121}$$

Using (112) and $|\Omega| = 1$, we get

$$\begin{aligned}\frac{1}{\pi\eta_0^2}&\frac{1}{1+C\frac{c^2}{36}}\left|\bigcup_{\mathcal{A}(\Omega_i)\geq\frac{c}{6}}\Omega_i\right| + \frac{1}{\pi\eta_0^2}\left|\bigcup_{\mathcal{A}(\Omega_i)\leq\frac{c}{6}}\Omega_i\right|\\ &\leq \frac{1}{\pi\eta_0^2}\frac{1}{1+C\frac{c^2}{36}}\frac{c}{6}+\frac{1}{\pi\eta_0^2}\left(1-\frac{c}{6}\right) &(122)\\ &\leq \left(1-\frac{c^3C}{216+6c^2C}\right)\frac{1}{\pi\eta_0^2}. &(123)\end{aligned}$$

By definition of $\eta_0$, the expression $(\pi\eta_0^2)^{-1}$ is precisely the upper bound of Pleijel on the number of nodal domains, and thus

$$N \leq \left(1-\frac{c^3C}{216+6c^2C}\right)\left(\frac{2}{j}\right)^2 n \tag{124}$$

for $n$ sufficiently big. □

### 4.2. Proof of the Spectral Partition Inequality

*Proof.* Suppose the statement was false. Then there exists some smooth, bounded $\Omega$ such that for any $\varepsilon > 0$ there are arbitrarily large $k$ such that there are partitions

$$\Omega = \bigcup_{i=1}^{k}\Omega_i \tag{125}$$

with

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_1(\Omega_i) \leq (\pi j^2+\varepsilon)\frac{k}{|\Omega|}. \tag{126}$$

Let us start by re-iterating the proof of the original estimate. The Faber–Krahn inequality implies

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_1(\Omega_i) \geq \frac{1}{k}\sum_{i=1}^{k}\frac{\pi j^2}{|\Omega_i|}. \tag{127}$$

The convexity of $x \to 1/x$ and the fact that

$$\sum_{i=1}^{k}|\Omega_i| = |\Omega| \tag{128}$$

immediately imply that

$$\frac{1}{k}\sum_{i=1}^{k}\frac{\pi j^2}{|\Omega_i|} \geq k\frac{\pi j^2}{|\Omega|} \tag{129}$$

with equality if and only if $|\Omega_i| = |\Omega|/k$ for all $1 \leq i \leq k$. We can quantify the notion of convexity a little bit. Indeed, assuming our desired spectral partition inequality to be false and given any $\delta > 0$, we can find a subsequence of partitions with

$$\frac{1}{k_j}\#\left\{1 \leq i \leq k_j : (1-\delta)|\Omega| \leq k_j|\Omega_i| \leq (1+\delta)|\Omega|\right\} \rightarrow 1. \tag{130}$$

This, however, means that we can find a subsequence of partitions with the property that

$$\left(\sum_{i=1}^{k_j}\frac{|\Omega_i|}{|\Omega|}D(\Omega_i)\right) \leq \delta', \tag{131}$$

where $\delta' > 0$ can be as small as we wish. Then, however, the geometric uncertainty principle implies that

$$\left(\sum_{i=1}^{k_j}\frac{|\Omega_i|}{|\Omega|}\mathcal{A}(\Omega_i)\right) \geq c_2 - \delta', \tag{132}$$

where $c_2 > 1/60000$ is the optimal constant in two dimensions. Then, however, arguing as before, we can improve on Pleijel's estimate. □

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