Ann. Henri Poincaré 15 (2014), 1903–1918

1424-0637/14/101903-16
published online December 10, 2013
DOI 10.1007/s00023-013-0296-y

Annales Henri Poincaré

Geometry of Normal Graphs in Euclidean Space and Applications to the Penrose Inequality in Minkowski

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Abstract. The Penrose inequality in Minkowski is a geometric inequality relating the total outer null expansion and the area of closed, connected and spacelike codimension-two surfaces $\boldsymbol{\mathcal{S}}$ in the Minkowski spacetime, subject to an additional convexity assumption. In a recent paper, Brendle and Wang A (Gibbons–Penrose inequality for surfaces in Schwarzschild Spacetime. arXiv:1303.1863, 2013) find a sufficient condition for the validity of this Penrose inequality in terms of the geometry of the orthogonal projection of $\boldsymbol{\mathcal{S}}$ onto a constant time hyperplane. In this work, we study the geometry of hypersurfaces in n-dimensional Euclidean space which are normal graphs over other surfaces and relate the intrinsic and extrinsic geometry of the graph with that of the base hypersurface. These results are used to rewrite Brendle and Wang's condition explicitly in terms of the time height function of $\boldsymbol{\mathcal{S}}$ over a hyperplane and the geometry of the projection of $\boldsymbol{\mathcal{S}}$ along its past null cone onto this hyperplane. We also include, in Appendix, a self-contained summary of known and new results on the geometry of projections along the Killing direction of codimension two-spacelike surfaces in a strictly static spacetime.

1. Introduction

The *Penrose inequality in Minkowski* refers to a geometric inequality for a class of codimension-two spacelike surfaces $\boldsymbol{\mathcal{S}}$ embedded in the $(n+2)$-dimensional Minkowski spacetime $(\mathcal{M}^{1,n+1}, \eta)$. The surfaces are restricted to be closed, connected, orientable and spacetime convex in the sense that their second fundamental form along one of its future directed null normals is non-positive[1]. It follows [2] that this property can only happen for one such future null

[1] Our sign conventions are such that the second fundamental form of sphere in $\mathbb{R}^3$ with respect to the outer normal is positive definite.

direction. Indeed, if both future null second fundamental forms were non-positive then the mean curvature vector H of the surface would be future causal and not-identically vanishing (because closed codimension-two surfaces in Minkowski cannot be totally geodesic). Thus, $\boldsymbol{\mathcal{S}}$ would be future trapped and not minimal, which cannot occur, e.g. by the results in [3]. The future directed null normal for which the second fundamental form is non-positive will be denoted by k (note that this field is defined up to an arbitrary positive scaling) and referred as the future directed *inner* null normal. The future directed *outer* null normal ℓ is defined by the conditions of being null, orthogonal to $\boldsymbol{\mathcal{S}}$, future directed and satisfying $\langle k, \ell\rangle_\eta = -2$, where $\langle \cdot, \cdot\rangle_\eta$ is the scalar product with the Minkowski metric η. Then the Penrose inequality in Minkowski can be written as

$$\int_{\boldsymbol{\mathcal{S}}} -\langle H, \ell\rangle_\eta \langle k, \xi\rangle_\eta \boldsymbol{\eta_{\mathcal{S}}} \geq n(\omega_n)^{\frac{1}{n}} |\boldsymbol{\mathcal{S}}|^{\frac{n-1}{n}}, \tag{1}$$

where ω_n is the total area of a the unit n-sphere, $\boldsymbol{\eta_{\mathcal{S}}}$ is the induced measure on $\boldsymbol{\mathcal{S}}$, $|\boldsymbol{\mathcal{S}}|$ its total area and ξ is any choice of future directed unit timelike Killing vector in Minkowski (referred from now on as a time translation). Note that there is not just one Penrose inequality in Minkowski, but one for each choice of time translation ξ. When a distinction is necessary, we will refer to "the Penrose inequality with respect to ξ".

The physical motivation for this inequality comes from a construction due to Penrose [4] where an incoming null shell of dust matter propagates in the Minkowski spacetime along the null geodesics with tangent vector k. The null hypersurface Ω they sweep is smooth all the way from $\boldsymbol{\mathcal{S}}$ to past null infinity (here is where the condition of spacetime convexity becomes important). Then, the standard Penrose inequality relating total (Bondi) energy and area of trapped surfaces can be rewritten in terms of Minkowskian quantities only. Details on the construction can be found, for instance, in [4–7].

Despite its apparent simplicity, inequality (1) is still a difficult open problem. It was proved by Gibbons [8] in the case of surfaces lying on the hyperplane orthogonal to ξ, where it becomes precisely the classic Minkowski inequality for convex surfaces in Euclidean space. For surfaces lying in the past null cone of a point (denoted as "spherical case", although obviously the surface $\boldsymbol{\mathcal{S}}$ is not in general spherically symmetric), the inequality was proved by Tod [6] in spacetime dimension four by using suitable Sobolev inequalities and extended to arbitrary spacetime dimension (bigger than three) in [2] as a consequence of the Beckner inequality for spheres [9]. In fact, the spherical case can also be viewed as a particular case of a Penrose inequality for spacetimes admitting shear-free null hypersurfaces extending from the trapped surface to past null infinity proved by Sauter [10].

In spacetime dimension four, the inequality (1) has been established for a large class of surfaces [2] using a geodesic flow of surfaces along Ω starting on $\boldsymbol{\mathcal{S}}$ and adapting and extending previous ideas of Ludvigsen and Vickers [11] and Bergqvist [12]. Wang [13] has proved the inequality for surfaces lying on a spacelike hyperboloid of Minkowski with the properties of being mean convex and star-shaped with respect to the point of tangency of the hyperboloid with the foliation by constant time hyperplanes orthogonal to ξ. Very recently Brendle and Wang [1] have proved the inequality for another large class of surfaces, namely those lying on a timelike cylindrical hypersurface with generator ξ and base a convex surface in a constant time hyperplane orthogonal to ξ. These cylinders are called *convex static timelike hypersurfaces* in [1]. In fact, the case analyzed by the authors refers to a generalization of inequality (1) conjectured for the Schwarzschild spacetime, but the argument applies to the Minkowski situation. The main idea behind their result consists in performing a projection of $\boldsymbol{S}$ along the time translation ξ onto a constant time hyperplane Σ_{t_0}. By relating the geometry of $\boldsymbol{S}$ to the geometry of the projected surface $\overline{S}$ on Σ_{t_0}, inequality (1) becomes a consequence of the standard Minkowski inequality in Euclidean space provided $\overline{S}$ is convex. More precisely, Brendle and Wang prove the following result (the result is established in [1] in spacetime dimension four, but the argument is in fact dimensional independent):

Theorem 1 (S. Brendle & M.T. Wang). *Let $(\mathcal{M}^{1,n+1}, \eta)$ be the $(n+2)$-dimensional Minkowski spacetime with t a Minkowskian time defining a unit Killing $\boldsymbol{\xi} = -\mathrm{d}t$. Let $\boldsymbol{S}$ be a closed, connected, orientable and spacetime convex surface in $(\mathcal{M}^{1,n+1}, \eta)$ with contravariant metric γ^{-1}. Let $\pi : \mathcal{M}^{1,n+1} \rightarrow \Sigma_{t_0}$ be the orthogonal projection onto the hyperplane $\Sigma_{t_0} = \{t = t_0\}$ and define $\overline{S} = \pi(\boldsymbol{S})$. Denote by $\boldsymbol{\eta}_{\overline{\boldsymbol{S}}}$ its volume form and by $\overline{K}$ its second fundamental form as a hypersurface of $(n+1)$-Euclidean space with respect to the outer unit normal. Then the Penrose inequality with respect to ξ for $\boldsymbol{S}$ is equivalent to*

$$\int_{\overline{S}} \mathrm{tr}_{\mathrm{d}\pi(\gamma^{-1})} \overline{K}\, \boldsymbol{\eta}_{\overline{\boldsymbol{S}}} \geq n(\omega_n)^{\frac{1}{n}} |\boldsymbol{S}|^{\frac{n-1}{n}} \tag{2}$$

and holds if $\overline{S}$ is convex.

The aim of this paper is to analyze this case in further detail by writing out the condition of $\overline{S}$ being convex explicitly in terms of the height function of $\boldsymbol{S}$ (defined below) and the geometry of the convex surface S obtained by intersecting the null hypersurface Ω ruled by the null geodesics starting at on $\boldsymbol{S}$ with tangent vector $-k$ and the hyperplane Σ_{t_0}. This requires analyzing the geometry of $\overline{S}$ as a graph over S. Although a purely Euclidean calculation we have not been able to find the result in the literature and most of our work consists in relating the induced metric and second fundamental forms of $\overline{S}$ to those of S. We devote Sect. 3 to present these calculations and the consequences they have on the Penrose inequality on convex static timelike hypersurfaces. Our main result is Theorem 3, where the explicit differential inequality that the height function of $\boldsymbol{S}$ needs to satisfy for a surface $\boldsymbol{S}$ to lie on a convex static timelike hypersurface is obtained.

Our second aim in this paper consists in presenting in a concise and unified manner the geometric relationship between the geometry of the codimension-two surface $\boldsymbol{S}$ and its projection $\overline{S}$, which lies at the core of the argument by Brendle and Wang. Some of these results have already appeared in several places in the literature, but in a somewhat scattered manner and not in a completely exhaustive form. Given the potential usefulness for such "vertical" projection in other areas of physics and geometry, we believe it to be convenient to present all the results in a unified and complete manner. We do this for an arbitrary strictly static spacetime in the Appendix.

2. Geometry of Normal Graphs on Hypersurfaces of $\mathbb{E}^{n+1}$

The following conventions and notation are used: if S is any embedded spacelike submanifold in a semi-Riemannian manifold, we denote by γ and D the induced metric and corresponding covariant derivative. The second fundamental form and mean curvature vectors are $\vec{K}(X,Y) := -(\nabla_X Y)^\perp$ and $H := \mathrm{tr}_\gamma K$, where X, Y are tangent vector fields to S and $\perp$ denotes the normal component to S. If ν is a vector field orthogonal to S, the extrinsic curvature along ν is $K^\nu(X,Y) := \langle \nu, \vec{K}(X,Y)\rangle$. All manifolds and tensors are assumed to be smooth.

In this section, the ambient manifold is the $(n+1)$-Euclidean space $(\mathbb{E}^{n+1}, g_E)$, $n \geq 2$. The flat connection is denoted by ∇ and the corresponding (global) parallel transport by $\mathcal{T}_{p_1\to p_2} : T_{p_1}\mathbb{E}^{n+1} \longrightarrow T_{p_2}\mathbb{E}^{n+1}$, for any $p_1, p_2 \in \mathbb{E}^{n+1}$. Obviously, in Cartesian coordinates $\{x^\alpha\}$, $(\alpha = 1, \ldots, n+1)$ this map simply preserves the coefficients of any vector $V \in T_{p_1}\mathbb{E}^{n+1}$ in the basis $\{\partial_{x^\alpha}\}$. $\{x^\alpha\}$ will always refer to a Cartesian coordinate system.

Consider two embedded submanifolds S and $\overline{S}$ in $\mathbb{E}^{n+1}$ and assume there is diffeomorphism $\psi : S \longrightarrow \overline{S}$. The following result relates the tangential covariant derivative of vector fields along S (not necessarily tangent to S) with the corresponding parallelly transported vector field on $\overline{S}$. This result will play an important role below.

Lemma 1. *Let S, $\overline{S}$ and ψ as above. Let Z be a vector field along S. Consider X a vector field tangent to S and define $\mathcal{T}Z|_{\psi(p)} := \mathcal{T}_{p\to\psi(p)} Z_p$ $\forall p \in S$. Then*

$$\mathcal{T}_{p\to\psi(p)}(\nabla_X Z|_p) = (\nabla_{\mathrm{d}\psi(X)} \mathcal{T}Z)|_{\psi(p)}, \tag{3}$$

Proof. The left-hand side of (3) is

$$\mathcal{T}_{p\to\psi(p)}(\nabla_X Z|_p) = \mathcal{T}_{p\to\psi(p)}\left(X(Z^\alpha)\partial_{x^\alpha}|_p\right) = X(Z^\alpha)|_p \partial_{x^\alpha}|_{\psi(p)}. \tag{4}$$

On the other hand, on $\overline{S}$ we have $\mathcal{T}Z = Z^{*\alpha}\partial_{x^\alpha}$, where $Z^{*\alpha} = Z^\alpha \circ \psi^{-1}$. Viewing Z^α as scalar functions we can also write $Z^{*\alpha} = (\psi^{-1})^\star(Z^\alpha)$. Its covariant derivative along $\mathrm{d}\psi|_p(X)$ is

$$\begin{aligned}(\nabla_{\mathrm{d}\psi(X)}\mathcal{T}Z)|_{\psi(p)} &= \nabla_{\mathrm{d}\psi(X)}\left(Z^{*\alpha}\partial_{x^\alpha}\right)|_{\psi(p)}\\ &= \mathrm{d}\psi(X)((\psi^{-1})^\star(Z^\alpha))\partial_{x^\alpha}|_{\psi(p)}\\ &= X(Z^\alpha)|_p\partial_{x^\alpha}|_{\psi(p)}\end{aligned} \tag{5}$$

which is the same as (4). □

Assume now that S is an orientable hypersurface and select a unit normal vector field ν. Choose a smooth function $\sigma : S \longrightarrow \mathbb{R}$ and consider the set of points at signed distance σ from each $p \in S \subset \mathbb{E}^{n+1}$ along the normal $\nu(p)$. The congruence of normal geodesics to S meets no focal points for distances σ satisfying the bounds

$$1 + \sigma\kappa_A > 0 \qquad A = 1, \dots, n \tag{6}$$

where $\{\kappa_A\}$ are the principal curvatures of S. Assuming this bound from now on, we have that the map $\psi' : S \to \mathbb{E}^{n+1}$ defined by

$$\psi'(p) = p + \sigma(p)\nu(p), \tag{7}$$

(where we are obviously using the affine structure of $\mathbb{E}^{n+1}$) is such that $\overline{S} := \psi'(S)$ is an embedded hypersurface of Euclidean space, and, in fact, a graph over S. Our aim is to relate the induced metrics and second fundamental forms of S and $\overline{S}$.

It is clear that the restriction of ψ' onto its image is a diffeomorphism between S and $\overline{S}$, which will be denoted by ψ. Let $X \in \mathfrak{X}(S)$ be a vector field tangent to S and define $\overline{X} := \mathrm{d}\psi(X)$, which is obviously tangent to $\overline{S}$.

For the purposes of this section, it is convenient to transport parallelly $\overline{X}$ from $\psi(p)$ to p because this will allow us to perform all calculations in a single manifold. Thus, let us define the vector field $\widetilde{X} \in \mathfrak{X}(S)$ as $\widetilde{X}|_p := \mathcal{T}_{\psi(p)\to p}(\overline{X}|_{\psi(p)})$. The first aim is to relate $\widetilde{X}$ with X. Consider any curve $c(s)$ in S passing through $p \in S$ with tangent vector $X|_p$. From the definition of ψ, the curve $\overline{c} := \psi \circ c$ has tangent vector at $\psi(p)$ given by $\mathcal{T}_{p\to\psi(p)}(X + \sigma\nabla_X\nu|_p + X(\sigma)\nu)$. Recalling that the Weingarten map $\boldsymbol{K} : T_pS \longrightarrow T_pS$ is defined by $\boldsymbol{K}(X) := \nabla_X\nu$ we conclude

$$\widetilde{X}|_p = (X + \sigma\nabla_X\nu + X(\sigma)\nu)|_p = (\boldsymbol{Id} + \sigma\boldsymbol{K} + \mathrm{d}\sigma \otimes \nu)\,(X))|_p. \tag{8}$$

From the geometric construction of $\overline{S}$ it is intuitively clear that the normal vector $\overline{\nu}$ orthogonal to $\overline{S}$ must satisfy $g_E(\widetilde{\nu}|_p, \nu|_p) \neq 0$ for all $p \in S$, where $\widetilde{\nu}|_p := \mathcal{T}_{\psi(p)\to p}(\overline{\nu}|_{\psi(p)})$. For a rigorous proof we use (8) as follows. Given that $\mathcal{T}_{p\to\psi(p)}$ is an isometry, (8) implies the following identity, valid for any $X \in T_pS$:

$$\begin{aligned} 0 &= g_E(\overline{\nu}, \overline{X})|_{\psi(p)} = g_E(\mathcal{T}_{p\to\psi(p)}(\widetilde{\nu}), \mathcal{T}_{p\to\psi(p)}(\widetilde{X}))|_{\psi(p)} = g_E(\widetilde{\nu}, \widetilde{X})|_p \\ &= g_E(\widetilde{\nu}, (\boldsymbol{Id} + \sigma\boldsymbol{K})(X))|_p + \mathrm{d}\sigma(X) g_E(\widetilde{\nu}, \nu)|_p. \end{aligned} \tag{9}$$

Assume there is $p \in S$ such that $\widetilde{\nu}|_p \in T_pS$ (i.e. $g_E(\widetilde{\nu}, \nu) = 0$). Then $g_E(\widetilde{\nu}, (\boldsymbol{Id} + \sigma\boldsymbol{K})(X))|_p = 0$ for any $X \in T_pS$, which is a contradiction with the fact that the bound (6) implies that the endomorphism $\boldsymbol{Id} + \sigma\boldsymbol{K}$ is invertible.

Let us choose the orientation of $\overline{\nu}$ so that $W := g_E(\widetilde{\nu}, \nu) > 0$ on S. Thus, we can decompose $\widetilde{\nu} = W(\nu - T)$ on S, where $T \in \mathfrak{X}(S)$ is a tangent vector field. Equation (9) implies

$$T = (\boldsymbol{Id} + \sigma\boldsymbol{K})^{-1}(\mathrm{grad}_\gamma(\sigma)), \tag{10}$$

where $\mathrm{grad}_\gamma(\sigma)$ is the gradient of σ with respect to the induced metric γ. For notational simplicity, define the invertible endomorphism $\boldsymbol{C} := (\boldsymbol{Id} + \sigma\boldsymbol{K})$ so that $T = \boldsymbol{C}^{-1}(\mathrm{grad}_{\gamma}(\sigma))$. The condition of $\overline{\nu}$ being unit fixes W to satisfy $W^2(1+\gamma(T,T)) = 1$, which, given our choice of normal in $\overline{S}$, implies

$$W = \frac{1}{\sqrt{1+\gamma(T,T)}}. \tag{11}$$

We are ready to prove our main result of this section, which relates the geometry of the graph $\overline{S}$ with the geometry of its base S.

Theorem 2. *Consider the hypersurfaces S, $\overline{S}$ of Euclidean space $(\mathbb{E}^{n+1}, g_E)$ with signed distance function σ and diffeomorphism ψ, as above. The respective induced metrics γ and $\overline{\gamma}$ and second fundamental forms K and $\overline{K}$ with respect to the normals ν and $\overline{\nu}$ are related by*

$$\psi^{\star}(\overline{\gamma}) = \gamma + 2\sigma K + \sigma^2 K \circ K + \mathrm{d}\sigma \otimes \mathrm{d}\sigma, \tag{12}$$

$$\begin{aligned}\frac{1}{W}\psi^{\star}(\overline{K}) &= K + \sigma K \circ K + \sigma D\, K(\cdot, T, \cdot) + \mathrm{d}\sigma \otimes K(T,\cdot)\\ &\quad + K(T,\cdot)\otimes \mathrm{d}\sigma - \mathrm{Hess}_{\gamma}(\sigma),\end{aligned} \tag{13}$$

where T and W are defined in (10, 11), $K \circ K$ is the trace of $K \otimes K$ in the second and third indices, D is the Levi–Civita derivative of γ and $\mathrm{Hess}_{\gamma}(\sigma)$ is the Hessian of σ in this metric.

Remark. These expressions reduce to well-known results when either σ is constant or when the base surface is a hyperplane.

Remark. It is interesting that the symmetry of $\overline{K}$ for any σ is equivalent to the Codazzi identity $D\, K(X_1, \cdot, X_3) = D\, K(X_3, \cdot, X_1)$ for S. So, properties of normal graphs can be used to derive curvature identities on the base hypersurface, which usually would require different methods.

Proof. Let $X, Y \in \mathfrak{X}(S)$ be arbitrary tangent vector fields. We start with (12). With the notation above, and using that the parallel transport is an isometry:

$$\begin{aligned}\psi^{\star}(\overline{\gamma})(X,Y)|_{\psi(p)} &= \overline{\gamma}\left(\mathrm{d}\psi|_p(X), \mathrm{d}\psi|_p(Y)\right) = g_E(\overline{X}, \overline{Y})|_{\psi(p)} = g_E(\widetilde{X}, \widetilde{Y})|_p\\ &= g_E(\boldsymbol{C}(X), \boldsymbol{C}(Y))|_p + \mathrm{d}\sigma \otimes \mathrm{d}\sigma|_p(X,Y)\\ &= \gamma((\boldsymbol{Id} + \sigma\boldsymbol{K})(X), (\boldsymbol{Id} + \sigma \boldsymbol{K})(Y))|_p + \mathrm{d}\sigma\otimes\mathrm{d}\sigma|_p(X,Y),\end{aligned}$$

where in the fourth equality we used (8). This establishes (12). To prove (13) we first apply Lemma 1 to find the identity

$$\begin{aligned}g_E(\nabla_X \widetilde{\nu}, \widetilde{Y})|_p &= g_E\left(\mathcal{T}_{p\to\psi(p)}(\nabla_X\widetilde{\nu}), \mathcal{T}_{p\to\psi(p)}(\widetilde{Y})\right)|_{\psi(p)}\\ &= g_E\left(\nabla_{\overline{X}}\overline{\nu}, \overline{Y}\right)|_{\psi(p)} = \overline{K}(\overline{X}, \overline{Y})|_{\psi(p)} = \psi^{\star}(\overline{K})(X,Y)|_p.\end{aligned} \tag{14}$$

To evaluate the left-hand side we recall the fundamental identity, $\nabla_X Y = D_X Y - K(X,Y)\nu$, valid for any pair of tangential vector fields. Given that $g_E(\widetilde{\nu}, \widetilde{Y}) = 0$, the left-hand side of (14) becomes

$$\begin{aligned}g_E(\nabla_X\widetilde{\nu}, \widetilde{Y}) &= \frac{X(W)}{W} g_E(\widetilde{\nu}, \widetilde{Y}) + W g_E\left(\nabla_X(\nu - T), \mathrm{d}\sigma(Y)\nu + \boldsymbol{C}(Y)\right)\\ &= W\gamma\left(\boldsymbol{K}(X) - D_X T, \boldsymbol{C}(Y)\right) + W K(X,T)\mathrm{d}\sigma(Y).\end{aligned} \tag{15}$$

The first term is immediately $W\gamma(\boldsymbol{K}(X), \boldsymbol{C}(Y)) = W(K + \sigma K \circ K)(X, Y)$. To elaborate the second term, we use that the endomorphism $\boldsymbol{C}$ is symmetric with respect to γ, i.e. $\gamma(X_1, \boldsymbol{C}(X_2)) = \gamma(\boldsymbol{C}(X_1), X_2)$. Thus,

$$\begin{aligned}
-\gamma(D_X T, \boldsymbol{C}(Y)) &= -\gamma((\boldsymbol{C} \circ D_X \boldsymbol{C}^{-1})(\operatorname{grad}_\gamma(\sigma)), Y) - \gamma(D_X \operatorname{grad}_\gamma(\sigma), Y) \\
&= \gamma((D_X \boldsymbol{C})(T), Y) - \operatorname{Hess}_\gamma(\sigma)(X, Y) \\
&= \mathrm{d}\sigma(X) K(T, Y) + \sigma D\, K(X, T, Y) - \operatorname{Hess}_\gamma(\sigma)(X, Y),
\end{aligned}$$

where in the first equality we used (10) and in the second equality $-\boldsymbol{C} \circ (D_X \boldsymbol{C}^{-1}) = (D_X \boldsymbol{C}) \circ \boldsymbol{C}^{-1}$. Inserting this into (15) yields the result. $\square$

Remark. The Riemannian character of the ambient Euclidean space has only been used when evaluating $g_E(\nu, \nu)$ and $g_E(\overline{\nu}, \overline{\nu})$. With the same arguments as before, let S be an embedded submanifold of the Minkowski spacetime $(\mathcal{M}^{1,n+1}, \eta)$ with non-degenerate induced metric γ and unit normal ν satisfying $\langle \nu, \nu \rangle_\eta = \epsilon$ with $\epsilon = \pm 1$. $\overline{S}$ is constructed as before, where the orientation of the unit normal $\overline{\nu}$ is selected so that it satisfies $\langle \widetilde{\nu}, \nu \rangle_\eta = \epsilon W$, with $W > 0$. Under these conditions:

$$\psi^\star(\overline{\gamma}) = \gamma + 2\sigma K + \sigma^2 K \circ K + \epsilon \mathrm{d}\sigma \otimes \mathrm{d}\sigma, \tag{16}$$

$$\begin{aligned}
\frac{1}{W}\psi^\star(\overline{K}) &= K + \sigma K \circ K + \sigma D\, K(\cdot, T, \cdot) + \mathrm{d}\sigma \otimes K(T, \cdot) \\
&\quad + K(T, \cdot) \otimes \mathrm{d}\sigma - \epsilon \operatorname{Hess}_\gamma(\sigma),
\end{aligned} \tag{17}$$

where all definitions are as before and the decomposition $\widetilde{\nu} = W(\nu - T)$ still holds, but this time T reads $T = \epsilon(\boldsymbol{Id} + \sigma \boldsymbol{K})^{-1}(\operatorname{grad}_\gamma(\sigma))$ and W is

$$W = \frac{1}{\sqrt{1 + \epsilon\gamma(T, T)}}.$$

The condition $1 + \epsilon\gamma(T, T) > 0$ is necessary for $\overline{S}$ to be of the same causal character as S.

3. Matching Two Different Projections

Let $(\mathcal{M}^{1,n+1}, \eta)$ be the $(n+2)$-dimensional Minkowski spacetime $(n \geq 2)$. Choose a Minkowskian time t so that we can define a unit Killing $\boldsymbol{\xi} = -\mathrm{d}t$. The constant time hyperplanes $\{t = t_0\}$ will be denoted by Σ_{t_0}. Let the codimension-two surface S and its normal null frame $\{k, \ell\}$ be as in the Introduction. The convex surface $S \hookrightarrow \Sigma_{t_0}$ defines uniquely a null hypersurface Ω (defined as *spacetime convex null hypersurface* in [2]) and, then, any spacelike surface $\boldsymbol{\mathcal{S}}$ embedded in Ω is defined uniquely by the time height function over Σ_{t_0}, namely the function $\tau := t|_{\boldsymbol{\mathcal{S}}} - t_0$. This function is defined on $\boldsymbol{\mathcal{S}}$. However, there is a natural diffeomorphism that maps $\boldsymbol{\mathcal{S}}$ to S via null geodesics in Ω tangent to k, so that any geometric information can be transferred from $\boldsymbol{\mathcal{S}}$ onto S and vice versa. This applies to any scalar function f and in particular to τ. We will use indistinctly the same name for both functions, the precise meaning being clear from the context. For any closed *spacetime convex* surface

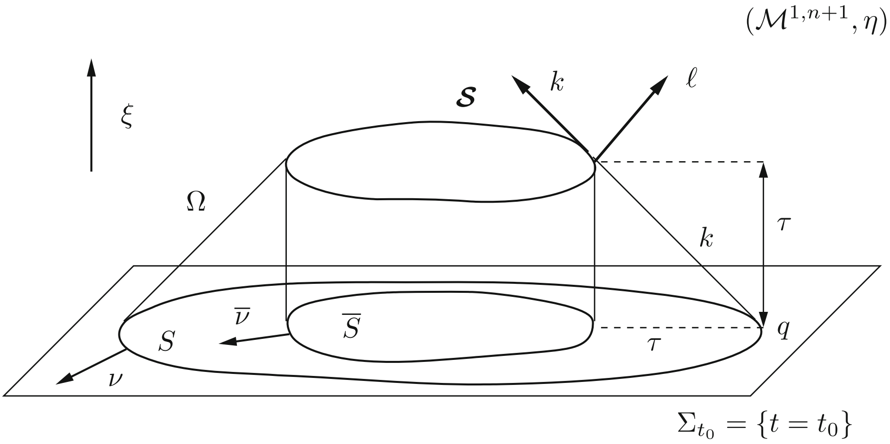

Figure 1. Schematic figure combining both projections: the spacetime convex surface $\boldsymbol{\mathcal{S}}$ is projected along Ω onto Σ_{t_0}, with $S=\Omega\cap\Sigma_{t_0}$. $\overline{S}$ is obtained by projecting $\boldsymbol{\mathcal{S}}$ along the Killing ξ. $\{k,\ell\}$ are normalized so that $\langle k,\ell\rangle_\eta=-2$

$\boldsymbol{\mathcal{S}}$, $\overline{S}$ must be embedded in Σ_{t_0} (otherwise two different points of $\boldsymbol{\mathcal{S}}$ with different time heights would project the same point onto Σ_{t_0} which is impossible given that they lie on a smooth null hypersurface). We can apply Theorem 2 to relate the geometry of S and $\overline{S}$ as follows:

Theorem 3 (Sufficient condition for the Penrose inequality in Minkowski in terms of spacetime convex geometry). *Let $(\mathcal{M}^{1,n+1},\eta)$ be the (n+2)-dimensional Minkowski spacetime with t a Minkowskian time defining a unit Killing $\boldsymbol{\xi}=-\mathrm{d}t$. Let $\boldsymbol{\mathcal{S}}$ be a closed, connected, orientable and spacetime convex surface in $(\mathcal{M}^{1,n+1},\eta)$ and Ω the convex null hypersurface containing $\boldsymbol{\mathcal{S}}$. Consider $S:=\Omega\cap\Sigma_{t_0}$ and let K be its second fundamental form as an Euclidean surface of Σ_{t_0} with respect to its outer unit normal ν (see Fig. 1), D the Levi-Civita connection of the metric γ of S, and grad$_\gamma(\tau)$ and Hess$_\gamma(\tau)$ the gradient and Hessian of τ in the metric γ respectively, where $\tau:=t|_{\boldsymbol{\mathcal{S}}}-t_0$. If the tensor*

$$\mathcal{T}=K-\tau K\circ K-\tau D\,K(\cdot,T,\cdot)-\mathrm{d}\tau\otimes K(T,\cdot)-K(T,\cdot)\otimes\mathrm{d}\tau+\mathrm{Hess}_\gamma(\tau) \tag{18}$$

is positive semidefinite, where $T=-(\boldsymbol{Id}-\tau\boldsymbol{K})^{-1}(grad_\gamma(\tau))$, then the Penrose inequality with respect to ξ holds for $\boldsymbol{\mathcal{S}}$.

Proof. Observe that in the Euclidean hyperplane Σ_{t_0} we can obtain $\overline{S}$ as a graph over S moving inwards along the inner normal to S. Indeed, let ν and $\overline{\nu}$ be the outer unit normals of S and $\overline{S}$. Moving along geodesics tangent to k in the past null cone Ω a time height τ with respect to Σ_{t_0} is equivalent to the projected trajectory moving inwards the same signed distance τ (see Fig. 1). Thus, we can apply Theorem 2 with $\sigma=-\tau$ and conclude that $\mathcal{T}=\frac{1}{W}\psi^*(\overline{K})$

with $W > 0$. The validity of the Penrose inequality for $\boldsymbol{\mathcal{S}}$ is then a consequence of Theorem 1. $\square$

To get a flavour of the range of applicability of this result, let us consider a few examples. Consider a closed, axially symmetric convex surface S in a spacelike hyperplane Σ_{t_0} of four-dimensional Minkowski spacetime $\mathcal{M}^{1,3}$, and assume that this surface is a cylinder between two parallel planes $z = z_0$ and $z = z_1$ orthogonal to the axis of symmetry. Let ρ_0 be the radius of the cylinder. In cylindrical coordinates $\{\varphi, z\}$, (18) becomes, in the region $z_0 \leq z \leq z_1$,

$$\mathcal{T}_{AB} = (\rho_0 - \tau)\boldsymbol{\delta}^{\varphi}_{A}\boldsymbol{\delta}^{\varphi}_{B} + \tau_{,AB} + \frac{\tau_{,\varphi}}{\rho_0 - \tau}(\tau_{,A}\boldsymbol{\delta}^{\varphi}_{B} + \tau_{,B}\boldsymbol{\delta}^{\varphi}_{A}), \tag{19}$$

with $\boldsymbol{\delta}$ the Kronecker delta. Assuming τ also axially symmetric, then $\mathcal{T}$ is positive semidefinite if and only if $\tau_{,zz} \geq 0$. So, any smooth axially symmetric surface $\boldsymbol{\mathcal{S}}$ projecting to S along the past null cone and for which τ is a constant τ_1 on $z \geq z_1$, a constant τ_0 on $z \leq z_0$ and fulfills $\tau_{,zz} \geq 0$ on $z \in [z_0, z_1]$, satisfies the Penrose inequality (with respect to the time translation orthogonal to the hyperplane Σ_{t_0}).

Another simple example is obtained when S is a sphere of radius r_0 in Σ_{t_0}. In spherical coordinates $\{\theta, \varphi\}$ (we are again in four spacetime dimensions) non-negativity of the tensor $\mathcal{T}$ reads

$$\mathcal{T}_{AB} = \frac{r_0 - \tau}{r_0^2}\gamma_{AB} + D_A D_B \tau + \frac{2}{r_0 - \tau}\tau_{,A}\tau_{,B} \geq 0$$

which, in the case that $\boldsymbol{\mathcal{S}}$ is axially symmetric, becomes (after adapting the spherical coordinates so that $\tau(\theta)$)

$$(r_0 - \tau)^2 + (r_0 - \tau)\tau_{,\theta\theta} + 2(\tau_{,\theta})^2 \geq 0 \qquad (r_0 - \tau)\sin\theta + \cos\theta\tau_{,\theta} \geq 0. \tag{20}$$

Let us solve these inequalities in the strictly convex case (i.e. with strict inequalities in (20)). With the definition $z(\theta) := (r_0 - \tau(\theta))\cos\theta$, the second inequality becomes $z_{,\theta} < 0$, which can be inverted to define $\theta(z)$. With the definition $\rho(z) := (r_0 - \tau(\theta))\sin\theta|_{\theta(z)}$, the first inequality becomes, after a straightforward computation, $\rho_{,zz} < 0$. Note also that $\rho - z\rho_{,z} = -\frac{(r_0-\tau)^2}{z_{,\theta}} > 0$ as a consequence of their definitions. Conversely, let $\rho(z)$ satisfy $\rho_{,zz} < 0$ and $\rho - z\rho_{,z} > 0$. Define $z(\theta)$ by $\cos\theta = z\left(\sqrt{z^2 + \rho(z)^2}\right)^{-1}|_{z=z(\theta)}$ (the condition $\rho - z\rho_{,z} > 0$ is used here) and construct a function $\tau(\theta)$ be means of $\tau = r_0 - \sqrt{z^2 + \rho(z)^2}|_{z=z(\theta)}$. Then the surface $\boldsymbol{\mathcal{S}}$ defined by this time height over the sphere S satisfies the Penrose inequality.

We note that the Penrose inequality for surfaces $\boldsymbol{\mathcal{S}}$ lying in the past null cone of a point in the Minkowski spacetime has been established in full generality in [6] (for dimension 4) and [2] (in any dimension). So, the second example above does not extend in any way the class of surfaces for which the inequality holds. However, besides giving us an idea of the proportion of surfaces in the null cone case covered by Theorem 1, it also provides a method to construct a wide family of axially symmetric surfaces $\boldsymbol{\mathcal{S}}$ for which the Penrose inequality holds. Indeed, assume now that S is axially symmetric and consider axially symmetric functions τ on S so that $\overline{S}$ is strictly convex. Let e_z be the unit field tangent to the axis of symmetry and e_ρ the unit field radially outward from the axis of symmetry. Define the two functions on S

$$z(p) := g_E(x - \tau\nu, e_z)|_p, \qquad \rho(p) := g_E(x - \tau\nu, e_\rho)|_p, \tag{21}$$

where x is the position vector of a point p on S with respect to an origin on the axis of symmetry and ν the outward normal at p. The strict inequality $\mathcal{T} > 0$ is equivalent to (i) z being a coordinate on S away from points on the axis of symmetry and (ii) $\rho(z)$ satisfying $\rho_{,zz} < 0$. Conversely, given any function $\rho(z)$ satisfying $\rho_{,zz} < 0$, if there are two maps $z, \tau : S \to \mathbb{R}$ solving the algebraic Eq. (21) with $\rho(p) := \rho(z(p))$, then the spacetime surface $\boldsymbol{\mathcal{S}}$ defined by this time height function over S satisfies the Penrose inequality. The algebraic equations will be solvable provided the parametric surface $\{\rho(z), z, \varphi\}$ in cylindrical coordinates is a normal graph over S. It is obvious that this is not always the case, so restrictions are necessary. In the spherical case above, this restriction is precisely $\rho - z\rho_z > 0$.

As already mentioned, the first case where the Penrose inequality in Minkowski was proved is due to Gibbons [8], who considered convex surfaces $\boldsymbol{\mathcal{S}}$ lying on a spacelike hyperplane and established the Penrose inequality with respect to the Killing orthogonal to the hyperplane. This case is immediately covered by Theorem 1. In fact, this theorem also implies the validity of the Penrose inequality for $\boldsymbol{\mathcal{S}}$ with respect to any other time translation, as we show next.

Theorem 4. *Let $\boldsymbol{\mathcal{S}}$ be a closed, connected and convex surface embedded in a spacelike hyperplane $\Sigma'_{t'_0} \hookrightarrow \mathcal{M}^{1,3}$. Let ξ be any unit time translation (not necessarily orthogonal to $\Sigma'_{t'_0}$). Then the Penrose inequality with respect to ξ holds for $\boldsymbol{\mathcal{S}}$.*

Proof. Let ν' be the outward normal to $\boldsymbol{\mathcal{S}}$ in $\Sigma'_{t'_0}$. Since a hyperplane is totally geodesic, the second fundamental form vector of $\boldsymbol{\mathcal{S}}$ is $K = K^{\nu'}\nu'$, where $K^{\nu'}$ is positive semidefinite. Choose any hyperplane Σ_{t_0} orthogonal to ξ and define $\overline{S}$ as the orthogonal projection of $\boldsymbol{\mathcal{S}}$ onto Σ_{t_0}. To prove the theorem it suffices to show that $\overline{S}$ is convex, i.e. that its second fundamental form $\overline{K}$ with respect to the unit outer normal $\overline{\nu}$ in Σ_{t_0} is non-negative. From Proposition 1 in Sect. 4 (with $V = 1$, as we are in Minkowski) we have

$$K^{\overline{\nu}} = \pi^\star(\overline{K}),$$

where $\pi : \boldsymbol{\mathcal{S}} \to \overline{S}$ is the projection along ξ, $\overline{\nu}$ is the parallel extension along ξ of the normal vector $\overline{\nu}$ of $\overline{S}$ evaluated on $\boldsymbol{\mathcal{S}}$ and $K^{\overline{\nu}} := \langle K, \overline{\nu}\rangle_\eta$. Thus, $\overline{K}$ is non-negative if and only if $\langle \nu', \overline{\nu}\rangle_\eta$ is non-negative. Now, both ν' and $\overline{\nu}$ are normal to $\boldsymbol{\mathcal{S}}$, spacelike and unit. Since they belong to a two-dimensional Lorentzian space, $\langle \nu', \overline{\nu}\rangle_\eta$ vanishes nowhere, and, hence, has constant sign. For the choice $\xi = \xi'$, i.e. the time translation normal to $\Sigma'_{t'_0}$, we obviously have $\nu' = \overline{\nu}$ and the sign is positive. Since ξ can be obtained from ξ' by a smooth deformation, and $\overline{\nu}$ also changes smoothly, it is impossible that the sign of $\langle \nu', \overline{\nu}\rangle_\eta$ changes from $+1$ to -1, and the theorem is proved. □

This theorem implies a Minkowski type inequality for $S' := \boldsymbol{\mathcal{S}}$ as a convex surface of Euclidean space. Indeed, the Killing vector ξ can be decomposed as $\xi = \sqrt{1+|v|^2}\,\xi' + v$ where v is a translation of Euclidean space $(\mathbb{E}^3, g_E)$ (identified with the hyperplane $\Sigma'_{t'_0}$). With the definition of null vectors $k = \xi' - \nu'$ and $\ell = \xi' + \nu'$ on S' and, given that the mean curvature vector of S' is $H'\nu'$, where H' is the mean curvature of $S' \hookrightarrow \mathbb{E}^3$, the Penrose inequality (1) with respect to ξ becomes

$$\int_{S'} H' f \boldsymbol{\eta}_{\boldsymbol{S'}} \geq \sqrt{16\pi |S'|}, \tag{22}$$

where

$$f = \sqrt{1+|v|^2} + g_E(\nu', v).$$

Obviously, when $v = 0$ we recover the standard Minkowski inequality. The validity of this inequality suggests that it might be worth studying for which functions f Minkowski type inequalities of the form (22) hold for arbitrary convex surfaces of Euclidean space. We note in this respect different, but somewhat related results in [14].

Acknowledgements

Financial support under the projects FIS2012-30926 (MICINN) and P09-FQM-4496 (Junta de Andalucía and FEDER funds) are acknowledged. A.S. acknowledges the Ph.D. grant AP2009-0063 (MEC).

4. Appendix

One of the ingredients in Theorem 1 of Brendle and Wang is a computation relating the extrinsic curvatures of a codimension-two spacelike surface S embedded in a strictly static spacetime and its projection $\overline{S}$ onto a hypersurface of constant static time. A similar and more exhaustive analysis in the case of Minkowski spacetime was carried out in connection with a new definition of quasi-local mass in [15,16]. Results of this type in the general static case have also appeared in [17]. However, to the best of our knowledge, no systematic account of the relation between all the intrinsic and extrinsic geometric properties of S and its projection $\overline{S}$ has appeared in the literature, neither in the Minkowski nor in the general strictly static case. We devote this Appendix to doing so.

Let $(\mathcal{M}, g)$ be an $(n+2)$-dimensional spacetime with a Killing vector field ξ which is everywhere timelike and hypersurface orthogonal. We choose ξ to be future directed. The covariant derivative of $(\mathcal{M}, g)$ is denoted as $\nabla^{\mathcal{M}}$ and $\langle \cdot, \cdot \rangle$ is used for scalar products with g. The norm $V > 0$ of the Killing vector is defined by $\langle \xi, \xi \rangle = -V^2$.

Consider a codimension-two spacelike surface S in $\mathcal{M}$. Since all calculations are local we can assume without loss of generality that S is embedded, and that there exists a time function $t : \mathcal{M} \to \mathbb{R}$ such that $\boldsymbol{\xi} = -V^2 dt$ (this follows from the integrability of $\boldsymbol{\xi}$ and locality). Choose any $t_0 \in \mathbb{R}$ and let $\Sigma_{t_0} = \{t = t_0\}$. The projection $\overline{S}$ of S onto Σ_{t_0} along the orbits of ξ defines a codimension-two surface which again can be taken to be embedded (after restricting S if necessary). Thus, we have a diffeomorphism $\pi : S \to \overline{S}$ defined by projection along ξ. The induced metrics and covariant derivative on S (resp. $\overline{S}$) are denoted as γ and D (resp. $\overline{\gamma}$ and $\overline{D}$). The function $\tau := t|_S - t_0$ and $V_S := V|_S$ will play an important role in relating the geometry of the two surfaces. As before, scalar functions on S will be transferred to $\overline{S}$ by means of π while keeping their names. The precise meaning will follow from the context.

For any vector field $X \in \mathfrak{X}(S)$ we denote its projection $d\pi(X) \in \mathfrak{X}(\overline{S})$ as $\overline{X}$. Given any such vector $\overline{X}$ we extend it along the orbits of the Killing vector by Lie transport along ξ, i.e. solving $[\xi, \overline{X}] = 0$. Again we keep the same name for the extension. Note that $\overline{X}$ is everywhere orthogonal to ξ. With these definitions it is straightforward that, at any $p \in S$,

$$X|_p = \overline{X}(\tau)\xi|_p + \overline{X}|_p. \tag{23}$$

As a consequence, the metrics γ and $\overline{\gamma}$ are related by

$$\gamma(X,Y)|_p = \langle \overline{X}(\tau)\xi + \overline{X}, \overline{Y}(\tau)\xi + \overline{Y}\rangle|_p = \left(\pi^*(\overline{\gamma}) - V_S^2 \mathrm{d}\tau \otimes \mathrm{d}\tau|_p\right)(X,Y)|_p$$

where we have used $\mathrm{d}\pi|_p(X) = \overline{X}|_{\pi(p)}$ and $\overline{X}(\tau) = \mathrm{d}\tau(X)$. So, we conclude

$$\gamma = \pi^\star(\overline{\gamma}) - V_S^2 \mathrm{d}\tau \otimes \mathrm{d}\tau. \tag{24}$$

The inverse metrics are then related by

$$\overline{\gamma}^{-1} = \mathrm{d}\pi(\gamma^{-1}) - \frac{V_S^2}{W^2}\mathrm{grad}_{\overline{\gamma}}(\tau) \otimes \mathrm{grad}_{\overline{\gamma}}(\tau), \qquad W := \sqrt{1 - V_S^2|\mathrm{d}\tau|^2_{\overline{\gamma}}},$$

which has, as immediate consequences,

$$\mathrm{d}\pi(\mathrm{grad}_{\gamma}(\tau)) = \frac{1}{W^2}\mathrm{grad}_{\overline{\gamma}}(\tau), \quad |\mathrm{d}\tau|^2_\gamma = \frac{|\mathrm{d}\tau|^2_{\overline{\gamma}}}{W^2}, \quad W = \frac{1}{\sqrt{1 + V_S^2|\mathrm{d}\tau|^2_\gamma}}. \tag{25}$$

The bound $1 - V_S^2|\mathrm{d}\tau|^2_{\overline{\gamma}} > 0$ (necessary for W to be real) is a consequence of S being spacelike everywhere. It is also immediate to show that the respective volume forms $\boldsymbol{\eta_S}$ and $\boldsymbol{\eta_{\overline{S}}}$ are related by

$$\boldsymbol{\eta_S} = W\boldsymbol{\eta_{\overline{S}}}. \tag{26}$$

In order to study the relation between the extrinsic geometries of S and $\overline{S}$ it is useful to choose a basis of the normal bundle of each surface. Concerning $\overline{S}$, the natural choice is $\{\overline{\nu}, V_S^{-1}\xi|_S\}$, where $\overline{\nu}$ is a unit normal of $\overline{S}$ as a hypersurface in Σ_{t_0}. We denote by $\overline{K}$ the second fundamental form of $\overline{S}$ along $\overline{\nu}$. Concerning S, the Lie constant extension $\overline{\nu}$ along the Killing ξ defines a spacelike and unit normal to S, still denoted by $\overline{\nu}$. For the second vector, note that $\xi|_S$ is nowhere tangent to S and hence its normal component $\xi^\perp$ in the orthogonal decomposition $T_p\mathcal{M} = T_pS \oplus N_pS$ is nowhere zero and, in fact, timelike. From $\boldsymbol{\xi} = -V^2\mathrm{d}t$ we have, for any $X \in T_pS$, $\langle \xi|_S, X\rangle = -V_S^2\mathrm{d}\tau(X)$ which means that the tangential component of $\xi|_S$ is $-V_S^2\mathrm{grad}_\gamma(\tau)$, or equivalently

$\xi^\perp = \xi|_S + V_S^2 \mathrm{grad}_\gamma(\tau)$. Following [16] we denote by u the future directed unit vector tangent to $\xi^\perp$. Its explicit form is

$$u = \frac{W}{V_S}\left(\xi|_S + V_S^2 \mathrm{grad}_\gamma(\tau)\right) \tag{27}$$

as a consequence of u being unit and orthogonal to $\mathrm{grad}_\gamma(\tau)$ and the property $\langle \xi, \xi \rangle = -V^2$. We note that $\{\overline{\nu}, u\}$ defines an orthonormal basis of the normal bundle of S.

The extrinsic geometry of S is encoded into its second fundamental form vector K and the connection of the normal bundle $\boldsymbol{\alpha}$. For the basis above, this geometric information is in turn given by the two symmetric tensors $K^u := \langle K, u\rangle$, $K^{\overline{\nu}} := \langle K, \overline{\nu}\rangle$ and the one-form $\boldsymbol{\alpha_{\overline{\nu}}}(X) := \langle \nabla^{\mathcal{M}}_X \overline{\nu}, u\rangle$, $X \in \mathfrak{X}(S)$. The following proposition relates these objects with the geometry of the projected surface:

Proposition 1. *With the notation above,*

$$K^{\overline{\nu}} = \pi^*(\overline{K}) - V_S \overline{\nu}(V)|_S \mathrm{d}\tau \otimes \mathrm{d}\tau, \tag{28}$$

$$\begin{aligned} K^u =& \frac{1}{W}\left(\mathrm{d}V_S \otimes \mathrm{d}\tau + \mathrm{d}\tau \otimes \mathrm{d}V_S + V_S \pi^*(\mathrm{Hess}_{\overline{\gamma}}(\tau))\right) \\ & - \frac{V_S^2}{W} \mathrm{d}V_S(\mathrm{grad}_{\overline{\gamma}}(\tau)) \mathrm{d}\tau \otimes \mathrm{d}\tau, \qquad W = \sqrt{1 - V_S^2 |\mathrm{d}\tau|^2_{\overline{\gamma}}} \end{aligned} \tag{29}$$

$$\boldsymbol{\alpha_{\overline{\nu}}} = \frac{1}{W}\left(V_S \pi^\star(\overline{K}(\mathrm{grad}_{\overline{\gamma}}(\tau), \cdot)) - \overline{\nu}(V)|_S \mathrm{d}\tau\right). \tag{30}$$

Proof. Inserting (23) in the defining expression $K^{\overline{\nu}}(X,Y) = \langle \nabla^{\mathcal{M}}_X \overline{\nu}, Y\rangle$ gives, after using $\overline{X}(\tau) = \mathrm{d}\tau(X)$,

$$K^{\overline{\nu}}(X,Y) = \mathrm{d}\tau(Y)\langle \nabla^{\mathcal{M}}_X \overline{\nu}, \xi\rangle + \mathrm{d}\tau(X)\langle \nabla^{\mathcal{M}}_\xi \overline{\nu}, \overline{Y}\rangle + \langle \nabla^{\mathcal{M}}_{\overline{X}} \overline{\nu}, \overline{Y}\rangle. \tag{31}$$

Now, $\langle \nabla^{\mathcal{M}}_X \overline{\nu}, \xi\rangle = \overline{X}(\tau)\langle \nabla^{\mathcal{M}}_\xi \overline{\nu}, \xi\rangle + \langle \nabla^{\mathcal{M}}_{\overline{X}} \overline{\nu}, \xi\rangle = \overline{X}(\tau)\langle \nabla^{\mathcal{M}}_\xi \overline{\nu}, \xi\rangle$, the second equality following from Σ_{t_0} being totally geodesic. To elaborate this further, we note that $d\boldsymbol{\xi} = 2V^{-1}\mathrm{d}V \wedge \boldsymbol{\xi}$ as a consequence of $\boldsymbol{\xi} = -V^2 \mathrm{d}t$. Hence

$$\nabla^{\mathcal{M}}_{\overline{\nu}} \boldsymbol{\xi} = \frac{1}{2}\mathrm{d}\boldsymbol{\xi}(\overline{\nu}, \cdot) = \frac{\overline{\nu}(V)|_S}{V_S}\boldsymbol{\xi}, \tag{32}$$

where in the first equality we used the Killing equations and in the second the orthogonality of $\overline{\nu}$ and ξ. Raising indices and recalling that $[\xi, \overline{\nu}] = 0$ we conclude

$$\nabla^{\mathcal{M}}_\xi \overline{\nu} = \nabla^{\mathcal{M}}_{\overline{\nu}} \xi = \frac{\overline{\nu}(V)|_S}{V_S}\xi, \tag{33}$$

and therefore

$$\langle \nabla^{\mathcal{M}}_X \overline{\nu}, \xi\rangle = -V_S \overline{\nu}(V)|_S \mathrm{d}\tau(X). \tag{34}$$

Using these expressions, the first term in (31) becomes $-V_S\overline{\nu}(V)|_S(\mathrm{d}\tau \otimes \mathrm{d}\tau)(X,Y)$, while the second term vanishes. Finally, the last term gives the second fundamental form of $\overline{S}$ and (28) follows (to our knowledge, this identity appeared for the first time in [1]).

Concerning K^u, its symmetry properties allows us to write $K^u(X,Y) = \frac{1}{2}(\langle \nabla^{\mathcal{M}}_X u, Y\rangle + \langle \nabla^{\mathcal{M}}_Y u, X\rangle)$, which after inserting (27) yields

$$\begin{aligned} K^u(X,Y) = \frac{W}{2V_S}(&\langle \nabla^{\mathcal{M}}_X \xi, Y\rangle + \langle \nabla^{\mathcal{M}}_Y \xi, X\rangle \\ &+\langle \nabla^{\mathcal{M}}_X (V_S^2 \mathrm{grad}_{\gamma}(\tau)), Y\rangle + \langle \nabla^{\mathcal{M}}_Y (V_S^2 \mathrm{grad}_{\gamma}(\tau)), X\rangle). \end{aligned}$$

The Killing equations imply that the first two terms cancel each other. Expanding the remaining terms it follows immediately

$$K^u = W\left(\mathrm{d}V_S \otimes \mathrm{d}\tau + \mathrm{d}\tau \otimes \mathrm{d}V_S + V_S \mathrm{Hess}_{\gamma}(\tau)\right). \tag{35}$$

In order to rewrite this in terms of the projected geometry, we need to find the relation between the Hessians of τ on each one of the surfaces. To that aim, recall that the difference between connections D and $\overline{D}$ on a given manifold defines a type $(1,2)$ tensor $\mathcal{Z}$ such that the following identity holds for any one-form $\boldsymbol{\omega}$ (see e.g. [18]):

$$(D\boldsymbol{\omega})(X,Y) - (\overline{D}\boldsymbol{\omega})(X,Y) = -\mathcal{Z}(\boldsymbol{\omega}, X, Y). \tag{36}$$

In our context, we can use $\pi^\star(\overline{\gamma})$ on S and the corresponding connection $\overline{D}$ it defines. Given the relation (24), a straightforward computation gives

$$\begin{aligned} \mathcal{Z}(\mathrm{d}\tau, \cdot, \cdot) = -V_S |\mathrm{d}\tau|^2_{\gamma}(&\mathrm{d}V_S \otimes \mathrm{d}\tau + \mathrm{d}\tau \otimes \mathrm{d}V_S \\ &+ V_S \pi^*(\mathrm{Hess}_{\overline{\gamma}}(\tau))) + V_S \mathrm{d}V_S(\mathrm{grad}_{\gamma}(\tau))\mathrm{d}\tau \otimes \mathrm{d}\tau \end{aligned}$$

Inserting this into (36) with $\boldsymbol{\omega} \to \mathrm{d}\tau$ and using (25) it follows

$$\begin{aligned} W^2 \mathrm{Hess}_{\gamma}(\tau) = \pi^*(\mathrm{Hess}_{\overline{\gamma}}(\tau)) &+ V_S |\mathrm{d}\tau|^2_{\overline{\gamma}}(\mathrm{d}\tau \otimes \mathrm{d}V_S + \mathrm{d}V_S \otimes \mathrm{d}\tau) \\ &- V_S \mathrm{d}V_S(\mathrm{grad}_{\overline{\gamma}}(\tau))\mathrm{d}\tau \otimes \mathrm{d}\tau. \end{aligned} \tag{37}$$

Combining this and (35) gives (29) at once.

It only remains to compute the connection 1-form $\alpha_{\overline{\nu}}(X) = \langle \nabla^{\mathcal{M}}_X \overline{\nu}, u\rangle$. Substituting (27) one finds

$$\begin{aligned} \alpha_{\overline{\nu}}(X) &= \frac{W}{V_S}\langle \nabla^{\mathcal{M}}_X \overline{\nu}, \xi + V_S^2 \mathrm{grad}_{\gamma}(\tau)\rangle \\ &= \frac{W}{V_S}\langle \nabla^{\mathcal{M}}_X \overline{\nu}, \xi\rangle + W V_S K^{\overline{\nu}}(\mathrm{grad}_{\gamma}(\tau), X) \\ &= -W\overline{\nu}(V)|_S \mathrm{d}\tau(X) + W V_S K^{\overline{\nu}}(\mathrm{grad}_{\gamma}(\tau), X), \end{aligned}$$

where in the last equality we used (34). Replacing (28) and using the first relation in (25) and the definition of W, then

$$\begin{aligned} \alpha_{\overline{\nu}}(X) = &-W\overline{\nu}(V)|_S \mathrm{d}\tau(X) \\ &+ W V_S\left(\pi^*(\overline{K})(\mathrm{grad}_{\gamma}(\tau), X) - V_S \overline{\nu}(V)|_S |\mathrm{d}\tau|^2_{\gamma} \mathrm{d}\tau(X)\right) \\ = &\frac{1}{W}\left(V_S \overline{K}(\mathrm{grad}_{\overline{\gamma}}(\tau), \mathrm{d}\pi(X)) - \overline{\nu}(V)|_S \mathrm{d}\tau(X)\right) \end{aligned}$$

as claimed. □

Remark. Although we have assumed ξ to be timelike, all the calculations in this section are similar when ξ is spacelike and nowhere zero. In particular the geometric relations between S and its projection $\overline{S}$ in a purely Riemannian context where $\langle \xi, \xi \rangle = V^2$ and $\boldsymbol{\xi} = V^2 \mathrm{d}t$ are

$$\begin{aligned}
\gamma &= \pi^\star(\overline{\gamma}) + V_S^2 \mathrm{d}\tau \otimes \mathrm{d}\tau, \\
\boldsymbol{\eta_S} &= W \boldsymbol{\eta_{\overline{S}}} \qquad W = \sqrt{1 + V_S^2 |\mathrm{d}\tau|^2_{\overline{\gamma}}}, \\
K^{\overline{\nu}} &= \pi^*(\overline{K}) + V_S \overline{\nu}(V)|_S \mathrm{d}\tau \otimes \mathrm{d}\tau, \\
K^u &= -\frac{1}{W} \left(\mathrm{d}V_S \otimes \mathrm{d}\tau + \mathrm{d}\tau \otimes \mathrm{d}V_S + V_S \pi^* (\mathrm{Hess}_{\overline{\gamma}}(\tau)) \right) \\
&\quad - \frac{V_S^2}{W} \mathrm{d}V_S (\mathrm{grad}_{\overline{\gamma}}(\tau)) \mathrm{d}\tau \otimes \mathrm{d}\tau, \\
\boldsymbol{\alpha_{\overline{\nu}}} &= \frac{1}{W} \left(-V_S \pi^\star (\overline{K}(\mathrm{grad}_{\overline{\gamma}}(\tau), \cdot)) + \overline{\nu}(V)|_S \mathrm{d}\tau \right),
\end{aligned}$$

where this time the unit vector u reads

$$u = \frac{W}{V_S} \left(\xi|_S - V_S^2 \mathrm{grad}_{\gamma}(\tau) \right).$$

Remark. Note that the expressions above contain all the information needed to relate any geometric quantity on S with geometric information on its projection $\overline{S}$. For instance, the mean curvature vector of S can be related to the projected geometry simply taking the trace in $K = K^{\overline{\nu}} \overline{\nu} - K^u u$ with the metric γ^{-1} and using (24) together with the results in Proposition 1. Similarly, the null second fundamental forms K^k, K^ℓ of S along a basis of null normals $\{k, \ell\}$ can be obtained directly from Proposition 1 after decomposing $\{k, \ell\}$ in the basis $\{\overline{\nu}, u\}$. The same applies to the corresponding null expansions.

Concerning the connection one-form, its behaviour under change of basis is not tensorial (being a connection), so it may be worth giving its explicit expression in a null basis $\{k, \ell\}$ of the form $k = f(-\overline{\nu} + u)$ and $\ell = f^{-1}(\overline{\nu} + u)$ where $f : S \to \mathbb{R} \setminus \{0\}$ is smooth. With the usual definition of connection one-form in this basis given by $\boldsymbol{s}(X) := \frac{1}{2} \langle \nabla^{\mathcal{M}}_X k, \ell \rangle$ we have

$$\boldsymbol{s}(X) = \frac{1}{2} \langle \nabla^{\mathcal{M}}_X k, \ell \rangle = \frac{1}{2} \langle \nabla^{\mathcal{M}}_X (-f\overline{\nu} + fu), f^{-1}\overline{\nu} + f^{-1} u \rangle = -\frac{X(f)}{f} - \alpha_{\overline{\nu}}(X),$$

and hence

$$\boldsymbol{s} = -\frac{\mathrm{d}f}{f} + \frac{1}{W} \left(\overline{\nu}(V)|_S \mathrm{d}\tau - V_S \pi^\star (\overline{K}(\mathrm{grad}_{\overline{\gamma}}(\tau), \cdot)) \right).$$

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Communicated by James A. Isenberg.

Received: July 19, 2012.

Accepted: October 8, 2013.