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Spectral Projections of the Complex Cubic Oscillator

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Abstract. We prove the spectral instability of the complex cubic oscillator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + ix^3 + i\alpha x$ for non-negative values of the parameter α, by getting the exponential growth rate of $\|\Pi_n(\alpha)\|$, where $\Pi_n(\alpha)$ is the spectral projection associated with the nth eigenvalue of the operator. More precisely, we show that for all non-negative α

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\Pi_n(\alpha)\| = \frac{\pi}{\sqrt{3}}.$$

1. Introduction

We consider the complex cubic oscillator

$$\mathcal{A}_\alpha = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + ix^3 + i\alpha x, \quad \alpha \in \mathbb{R} \tag{1.1}$$

on the real line. We define $\mathcal{A}_\alpha$ by extension of the operator

$$\mathcal{A}_\alpha^0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + ix^3 + i\alpha x, \quad \mathcal{D}(\mathcal{A}_\alpha^0) = \mathcal{C}_0^\infty(\mathbb{R}),$$

which is accretive, so we can define $\mathcal{A}_\alpha := \overline{\mathcal{A}_\alpha^0}$ as its closure. $\mathcal{A}_\alpha$ is then maximally accretive, with domain

$$\mathcal{D}(\mathcal{A}_\alpha) = H^2(\mathbb{R}) \cap L^2(\mathbb{R}; x^6 \mathrm{d}x).$$

The cubic oscillator presented here has been studied in [11] and [21]. It also belongs to the class of operators considered in [19]. Let us mention [14] as well, which deals with a quadratic perturbation of the cubic ix^3 potential.

The operator $\mathcal{A}_\alpha$ has compact resolvent, and its eigenvalues $(\lambda_n(\alpha))_{n\geq 1}$ are simple in the sense of the geometric multiplicity.

The properties of the complex cubic oscillator and its variants (the potential $x^2 + ix^3$, for instance) have been widely studied in the past few years (see [3–6, 9–11, 14, 16, 19, 21, 22]). As a non-selfadjoint operator, it has a surprising property: its spectrum is purely real for $\alpha \geq 0$ (see [4] for numerical observations and [19] for a rigorous proof). This property is suspected to be related with the so-called $\mathcal{PT}$-symmetry of the operator, namely

$$\mathcal{PT}\mathcal{A}_\alpha = \mathcal{A}_\alpha\mathcal{PT},$$

where $\mathcal{P}$ and $\mathcal{T}$, denoting, respectively, the spatial symmetry and time inversion operators, act as follows:

$$(\mathcal{P}u)(x) = u(-x) \quad \text{and} \quad (\mathcal{T}u)(x) = \overline{u(x)}.$$

The complex cubic oscillator is a toy model in the study of $\mathcal{PT}$-symmetric operators.

One of the main questions arising from this property of real spectrum is the following: does $\mathcal{A}_\alpha$ share some other similarities with selfadjoint operators? More precisely, does the family of eigenfunctions form a basis of $L^2(\mathbb{R})$ in some sense? Is the spectrum stable under perturbations of the operator? What can one say about the behavior of the eigenvalues for negative values of α? Some of these questions have already been answered, while other have been stated as conjectures. For instance, it has been established in [16] that the eigenfunctions of $\mathcal{A}_\alpha$ do not form a Riesz basis, as well as the existence of non-trivial pseudospectra.

The properties of the spectrum of $\mathcal{A}_\alpha$ for negative α have not been completely understood yet. Numerical simulations (see [9–11]), reproduced on Fig. 1, suggest that, for any $n \geq 1$, there exists a critical value $\alpha_n^{\text{crit}} < 0$ of the parameter such that $\lambda_n(\alpha)$ is real for $\alpha > \alpha_n^{\text{crit}}$. For $\alpha = \alpha_n^{\text{crit}}$, $\lambda_n(\alpha_n^{\text{crit}})$ seems to cross an adjacent eigenvalue, forming for $\alpha < \alpha_n^{\text{crit}}$ a complex conjugate pair lying away from the real axis. Regarding the analysis for large eigenvalues

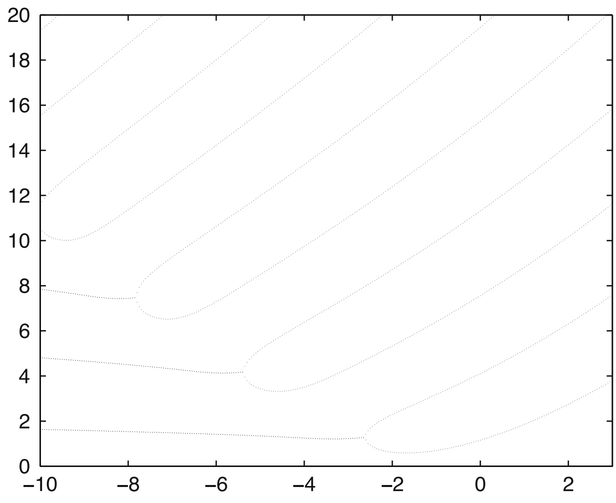

Figure 1. Real parts of the eigenvalues of $\mathcal{A}_\alpha$ as functions of α. Each pair of consecutive eigenvalues becomes non-real, complex conjugate on the left of the branch point

which we will perform in the following, the simulation suggests that, for any fixed $\alpha < 0$, the eigenvalues $\lambda_n(\alpha)$ are real for n large enough, but it does not seem to be proved yet. Therefore, we will only consider non-negative values of α in the following.

Our goal is to measure the spectral instability of the operator $\mathcal{A}_\alpha$. As mentioned above, the instability of the eigenvalues $\lambda_n(\alpha)$ has already been highlighted in [16] by proving the existence of non-trivial pseudospectra. We now want to understand more accurately this phenomenon, following the approach of [7,8] and [15].

To this purpose, we define the instability indices

$$\kappa_n(\alpha) = \|\Pi_n(\alpha)\|, \tag{1.2}$$

where $\Pi_n(\alpha)$ denotes the spectral projection of $\mathcal{A}_\alpha$ associated with the eigenvalue $\lambda_n(\alpha)$ (the eigenvalues being labeled in increasing order). We shall first consider the question of algebraic multiplicity for the eigenvalues $\lambda_n(\alpha)$, that is, whether there exist associated Jordan blocks or not. The algebraic simplicity of the eigenvalues has been proved for all $n \geq 1$ in [14] in the case of a potential of the form $ax^2 + i\sqrt{\beta}x^3$. Here, by an independent proof, we shall get the algebraic simplicity of $\lambda_n(\alpha)$, but only for n large enough, which will be enough to achieve the proof of our main statement. Hence, for n large enough, the expression

$$\kappa_n(\alpha) = \frac{\|u_n^\alpha\|^2}{|\langle u_n^\alpha, \bar{u}_n^\alpha\rangle|} \tag{1.3}$$

will hold, where u_n^α denotes an eigenfunction of $\mathcal{A}_\alpha$ associated with the eigenvalue $\lambda_n(\alpha)$ (see [2]). We will use this formula to prove the following theorem, which is the main statement of our work:

Theorem 1.1. *For all $\alpha \geq 0$, we have*

$$\lim_{n\to+\infty} \frac{1}{n} \log \kappa_n(\alpha) = \frac{\pi}{\sqrt{3}}. \tag{1.4}$$

Let us recall that the same question was considered in [7,8,15] in the case of anharmonic oscillators $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{e}^{i\theta}|x|^m$, $m > 0$, $|\theta| < \min\{(m+2)\pi/4, (m+2)\pi/2m\}$. More precisely, it has been proved that the spectral projections of these operators grow faster than any power of n as $n \to \infty$ [7], and the exponential growth rate was precisely obtained for $m = 2$ in [8] and for every even exponent m in [15].

The proof of Theorem 1.1 lies on WKB estimates of the eigenfunctions in the complex plane. This method has already been used in [15] in the even anharmonic case. However, here we will have to manage the sub-principal term $i\alpha x$ in the potential.

Some results from [16] can be recovered immediately from Theorem 1.1:

Corollary 1.2. *For all $\alpha \geq 0$, the eigenfunctions of $\mathcal{A}_\alpha$ do not form a Riesz basis.*

Proof. Let $(u_n^\alpha)_{n\geq 1}$ be a family of eigenfunctions for $\mathcal{A}_\alpha$ associated with the eigenvalues $(\lambda_n(\alpha))_{n\geq 1}$. Let us recall that $(u_n^\alpha)_{n\geq 1}$ is said to be a Riesz basis if it spans a dense subset of $L^2(\mathbb{R})$ and if there exists $C > 0$ such that, for all $\phi \in L^2(\mathbb{R})$,

$$C^{-1}\sum_{n=1}^{+\infty}|\langle\phi, u_n^\alpha\rangle|^2 \leq \|\phi\|^2 \leq C\sum_{n=1}^{+\infty}|\langle\phi, u_n^\alpha\rangle|^2. \tag{1.5}$$

According to Lemma 3.1 and Proposition 3.2 (which provides algebraic simplicity for large eigenvalues of $\mathcal{A}_\alpha$), we can choose the eigenfunctions u_n^α such that, for $n, m \geq 1$ and n large enough, $\langle u_n^\alpha, \overline{u_m^\alpha}\rangle = \delta_{n,m}$. Hence according to (3.1), we have $\kappa_n(\alpha) = \|u_n^\alpha\|^2$ for n large enough. Using that $\kappa_n(\alpha) \to +\infty$ as $n \to +\infty$, it is then straightforward to check that the sequence $\phi_n = \overline{u_n^\alpha}$ cannot satisfy (1.5). □

Furthermore, the pseudospectra in the neighborhood of an eigenvalue are known to grow proportionally to the corresponding instability index (see [2,20]). Hence the exponential growth obtained in Theorem 1.1 enables us to confirm the presence of nontrivial pseudospectra [16] and to somehow describe its shape near the eigenvalues.

Section 2 is devoted to the estimates on the eigenfunctions needed to prove Theorem 1.1. The proof itself is achieved in Sect. 3.

2. Asymptotic Behavior of the Eigenfunctions

2.1. Preliminary Scale Change

Let us first perform the following scale change. Let us recall that for all $\alpha \geq 0$, the spectrum of $\mathcal{A}_\alpha$ is real, and let us denote the eigenvalues, labeled in increasing order, by $\lambda_n(\alpha)$. We set

$$\begin{cases} h_n = \lambda_n(\alpha)^{-5/6} \\ \tilde{x} = h_n^{2/5}x. \end{cases} \tag{2.1}$$

The operator $(\mathcal{A}_\alpha - \lambda_n(\alpha))$ then writes

$$\begin{aligned} -h_n^{4/5}\frac{\mathrm{d}^2}{\mathrm{d}\tilde{x}^2} + ih_n^{-6/5}\tilde{x}^3 + i\alpha h_n^{-2/5}\tilde{x} - \lambda_n(\alpha) \\ = h_n^{-6/5}\left(-h_n^2\frac{\mathrm{d}^2}{\mathrm{d}\tilde{x}^2} + i\tilde{x}^3 + i\alpha h_n^{4/5}\tilde{x} - 1\right), \end{aligned}$$

and we are reduced to the study of the kernel of

$$\mathcal{A}_\alpha(h) = -h^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + ix^3 + i\alpha h^{4/5}x - 1.$$

An eigenfunction u_n^α of $\mathcal{A}_\alpha$ associated with $\lambda_n(\alpha)$ can be written as

$$u_n^\alpha(x) = \psi_\alpha(h_n^{2/5}x, h_n) = \psi_\alpha(\lambda_n(\alpha)^{-1/3}x, h_n), \tag{2.2}$$

where $\psi_\alpha(\cdot, h_n)$ is a solution of

$$\mathcal{A}_\alpha(h_n)\psi_\alpha(\cdot, h_n) = 0, \quad \psi_\alpha(\cdot, h_n) \in L^2(\mathbb{R}). \tag{2.3}$$

Notice that the condition $\psi_\alpha(\cdot, h_n) \in L^2(\mathbb{R})$, together with (2.3), ensures that $\psi_\alpha(\cdot, h_n)$ belongs to the domain $\mathcal{D}(\mathcal{A}_\alpha(h_n)) = \mathcal{D}(\mathcal{A}_\alpha)$ (see for instance Theorem 2.1 below). Thus, we will now work on these solutions ψ_α.

From now on, α is assumed to be fixed and non-negative.

2.2. Behavior of the Eigenfunctions Away from the Turning Points

In this subsection, we determine the global asymptotic behavior of the solutions $\psi_\alpha(x, h)$ of

$$\mathcal{A}_\alpha(h)\psi_\alpha(x, h) = 0, \quad \psi_\alpha(\cdot, h) \in L^2(\mathbb{R}) \tag{2.4}$$

as $h \to 0$.

More precisely, we want to understand the behavior of ψ_α in a domain of the complex plane avoiding the zeroes (called turning points of the equation) of the potential

$$V_\alpha(x, h) = ix^3 + i\alpha h^{4/5} x - 1.$$

Let $x_+^\alpha(h)$, $x_-^\alpha(h)$ and $x_{\mathbf{i}}^\alpha(h)$ denote the zeroes of $V_\alpha(\cdot, h)$, respectively, starting at $h = 0$ from the zeroes $x_+^0 = \mathrm{e}^{-i\pi/6}$, $x_-^0 = \mathrm{e}^{-5i\pi/6}$ and $x_{\mathbf{i}}^0 = i$ of the potential

$$V_0(x) = ix^3 - 1.$$

Note that for h small enough, $x_\pm^\alpha(h)$, $x_{\mathbf{i}}^\alpha(h)$ are simple zeroes of $V_\alpha(\cdot, h)$.

To understand the asymptotic properties of the solutions of (2.4), it will be useful to analyze the geometry of the level curves (Stokes lines) of the function

$$x \mapsto \mathrm{Re} \int_{x_+^\alpha(h)}^{x} \sqrt{V_\alpha(z, h)}\, \mathrm{d}z,$$

where $\sqrt{V_\alpha}$ is holomorphic in

$$D_h^\alpha = \mathbb{C} \setminus \bigcup_{\sigma \in \{+,-,\mathbf{i}\}} \{(1 + r)x_\sigma^\alpha(h) : r > 0\},$$

and $\sqrt{V_\alpha(0, h)} = i$.

The path of integration is included in D_h^α.

Let us notice that $x_+^\alpha(h)$ and $x_-^\alpha(h)$ belong to a common, bounded Stokes line, joining the two points

$$\mathrm{Re} \int_{x_-^\alpha(h)}^{x_+^\alpha(h)} \sqrt{V_\alpha(z, h)}\, \mathrm{d}z = 0.$$

Let us denote this line by $\ell_f^\alpha(h)$. It is the only bounded Stokes line for $\mathcal{A}_\alpha$ (see Fig. 2).

On the other hand, there are seven unbounded Stokes lines starting from $x_\pm^\alpha(h)$, $x_{\mathbf{i}}^\alpha(h)$, with the five asymptotic directions as $|x| \to +\infty$,

$$D_k = \arg^{-1}\left\{\frac{\pi}{10} + \frac{2k\pi}{5}\right\}, \quad k = 0, \ldots, 4.$$

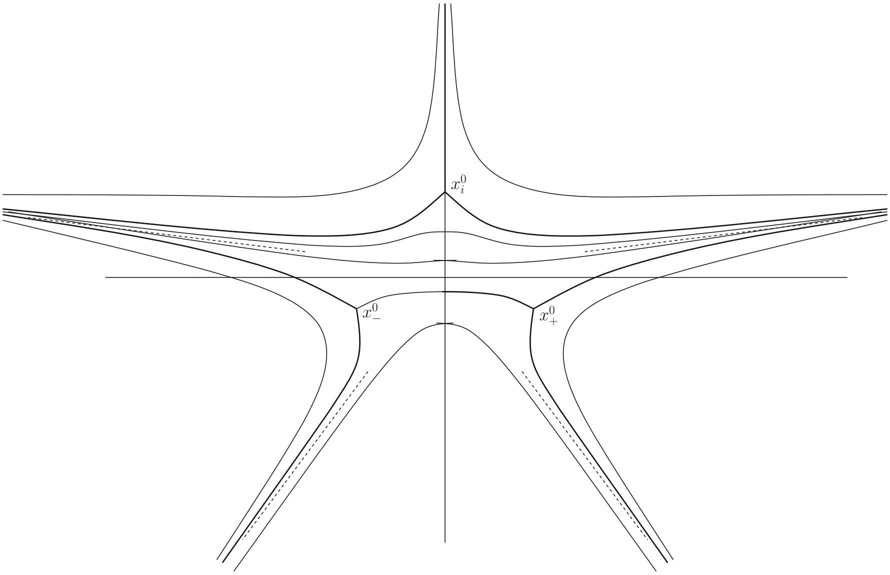

FIGURE 2. Stokes lines of the operator $\mathcal{A}_0 = -h^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + ix^3 - 1$. *Bold lines* are those starting from the turning points. *Dashed lines* are the asymptotic directions D_k, $k = 0, \ldots, 4$

Among those Stokes lines, one is starting from $x_{\mathbf{i}}^{\alpha}(h)$ and has asymptotic direction $D_1 = i\mathbb{R}^+$; let us denote it by $\ell_{\mathbf{i}}^{\alpha}(h)$. Notice that for $h = 0$, $\ell_{\mathbf{i}}^{\alpha}(0) = i[1, +\infty[$.

For $\varepsilon > 0$, let

$$\ell_{f,\varepsilon}^0 = \{x \in \mathbb{C} : d(x, \ell_f^0(0)) < \varepsilon\}, \tag{2.5}$$

and

$$\ell_{\mathbf{i},\varepsilon}^0 = \{x \in \mathbb{C} : d(x, i[1, +\infty[) < \varepsilon\}. \tag{2.6}$$

Hence, for all $\varepsilon > 0$ fixed, there exists $h_0 > 0$ such that, for all $h \in]0, h_0[$,

$$\ell_f^{\alpha}(h) \subset \ell_{f,\varepsilon}^0, \ \ell_{\mathbf{i}}^{\alpha}(h) \subset \ell_{\mathbf{i},\varepsilon}^0. \tag{2.7}$$

Finally, let

$$\Gamma_\varepsilon = \mathbb{C} \backslash (\ell_{f,\varepsilon}^0 \cup \ell_{\mathbf{i},\varepsilon}^0). \tag{2.8}$$

In the following theorem, $(h_n)_{n \geq 1}$ is the sequence defined in (2.1).

Theorem 2.1. *Let $\varepsilon > 0$ be fixed. There exists $N \geq 1$ such that, for all $n \geq N$, there exists a unique solution $\psi_1^{\alpha}(x, h_n) \in L^2(\mathbb{R})$ of*

$$\mathcal{A}_\alpha(h_n)\psi_1^{\alpha}(\cdot, h_n) = 0 \tag{2.9}$$

satisfying

$$\psi_1^\alpha(x,h_n)=\frac{\mathrm{e}^{-i\pi/8}}{x^{3/4}}(1+o(1))\exp\left(-\frac{1}{h_n}\int\limits_{x_+^\alpha(h_n)}^{x}\sqrt{V_\alpha(z,h_n)}\,\mathrm{d}z\right)\tag{2.10}$$

as $|x|\to+\infty$ *in* Γ_ε*, uniformly with respect to* $n\geq N$*.*

Moreover, there exists a sequence $(u_j^\alpha)_{j\geq1}$ *of functions, holomorphic on* Γ_ε*, such that, for every* $j_0\geq1$ *and* $x\in\Gamma_\varepsilon$*,*

$$\psi_1^\alpha(x,h_n)=\frac{1}{V_\alpha(x,h_n)^{1/4}}\exp\left(-\frac{1}{h_n}\int\limits_{x_+^\alpha(h_n)}^{x}\sqrt{V_\alpha(z,h_n)}\,\mathrm{d}z\right)$$
$$\times\left(1+\sum_{j=1}^{j_0}u_j^\alpha(x)h_n^j+R_{j_0+1}(x,h_n)\right),\tag{2.11}$$

where $|u_j^\alpha(x)|=\mathcal{O}(|x|^{-5j/2})$ *and* $|R_{j_0+1}(x,h)|\leq C|x|^{-5(j_0+1)/2}h^{j_0+1}$*.*

In particular the expansion (2.11) holds uniformly for $x\in\mathbb{R}$.

Proof. We apply Theorem 3.1, Chap. 10, p. 366 of [17].

Let

$$S(x)=\int\limits_{x_+^0}^{x}\sqrt{iz^3-1}\,\mathrm{d}z,$$

where $x_+^0=x_+^\alpha(0)$, and let $\Lambda_\pm$ be the set of points $x\in\mathbb{C}$ such that there exists a path γ_x joining $\pm\infty$ to x such that $\operatorname{Re}S\circ\gamma_x$ is increasing (canonical path). Let $\Lambda_\pm(\varepsilon)=\{x\in\Lambda_\pm : d(x,\partial\Lambda_\pm)\geq\varepsilon\}$ (see Fig. 3). We then notice that

$$\Gamma_\varepsilon=\Lambda_+(\varepsilon)\cup\Lambda_-(\varepsilon).$$

According to Theorem 3.1, Chap. 10, p. 366 of [17], there exists $h_0>0$ such that, for $h\in]0,h_0[$, any solution $\psi_\pm^\alpha(\cdot,h)\in L^2(\mathbb{R}^\pm)$ satisfies (2.10) and (2.11) in $\Lambda_\pm(\varepsilon)$, up to a multiplicative constant $c_\pm(h)\in\mathbb{C}$, and with $h\to0$ instead of the sequence $(h_n)_n$. Indeed, to check that the bound (3.04) in [17] on the remainder term of order k is of size $\mathcal{O}(h^k)$, we check that the conditions (i)–(iv) p. 370 are satisfied, which can be done by observing that the function

$$\sigma_\alpha(x,h):=\frac{1}{V_\alpha(x,h)^{3/4}}\left[\frac{1}{V_\alpha(x,h)^{1/4}}\right]''$$

satisfies, for some $k>0$,

$$|\sigma_\alpha(x,0)|\leq\frac{k}{1+|x|^5}\quad\text{and}\quad\sigma_\alpha(x,h)=\sigma_\alpha(x,0)(1+\mathcal{O}(h^{4/5}))$$

uniformly for $x\in\Lambda_\pm(\varepsilon)$.

To conclude, we have seen in Sect. 2.1 that if $\lambda_n(\alpha)$ denotes the nth eigenvalue of $\mathcal{A}_\alpha$, and if

$$h_n=\lambda_n(\alpha)^{-5/6},\tag{2.12}$$

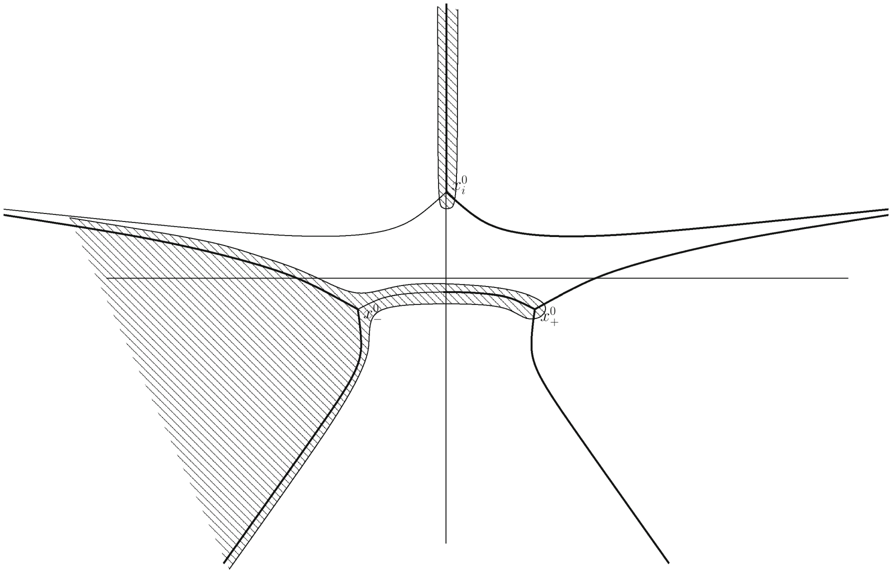

Figure 3. The domain $\Lambda_+(\varepsilon)$ (*unshaded domain*). $\Lambda_-(\varepsilon)$ is obtained from $\Lambda_+(\varepsilon)$ by applying the symmetry of axis $i\mathbb{R}$

then there exists, for all $n \geq 1$, a solution $\psi_1^\alpha(\cdot, h_n) \in L^2(\mathbb{R})$ of (2.9). Then, according to the previous arguments, $\psi_1^\alpha(\cdot, h_n)$ satisfies (2.10) and (2.11) in $\Lambda_+(\varepsilon)$ and $\Lambda_-(\varepsilon)$ up to respective constants $c_+(h)$ and $c_-(h)$. Comparing these expressions for $x \in \Lambda_+(\varepsilon) \cap \Lambda_-(\varepsilon)$, we see that $c_+(h) = c_-(h)$, and the statement follows by choosing $c_+(h) = c_-(h) = 1$. □

The asymptotic expansion (2.11) does not hold in the neighborhood of the bounded Stokes line $\ell_f^\alpha(h)$. In order to determine the behavior of a solution on $\ell_f^\alpha(h)$, we have to take into account the presence of terms of the form

$$V_\alpha(x,h)^{-1/4} \exp\left(+\frac{1}{h}\int_{x_+^\alpha(h)}^{x} \sqrt{V_\alpha(z,h)}\,\mathrm{d}z\right)$$

in its expression. Those terms, exponentially small as $h^{-1}\mathrm{Re}\int_{x_+^\alpha(h)}^{x} \sqrt{V_\alpha(z,h)}\,\mathrm{d}z \to -\infty$, are significant on $\ell_f^\alpha(h)$. In the following subsection, we consider solutions which oscillate along $\ell_f^\alpha(h)$. We will obtain an asymptotic expression which also holds in a neighborhood of the turning points $x_\pm^\alpha(h)$.

2.3. Behavior of the Eigenfunctions in the Neighborhood of the Turning Points

In the neighborhood of a turning point, the previous asymptotic expansions are no longer available. We will now use an approximation of the solutions involving the Airy function Ai.

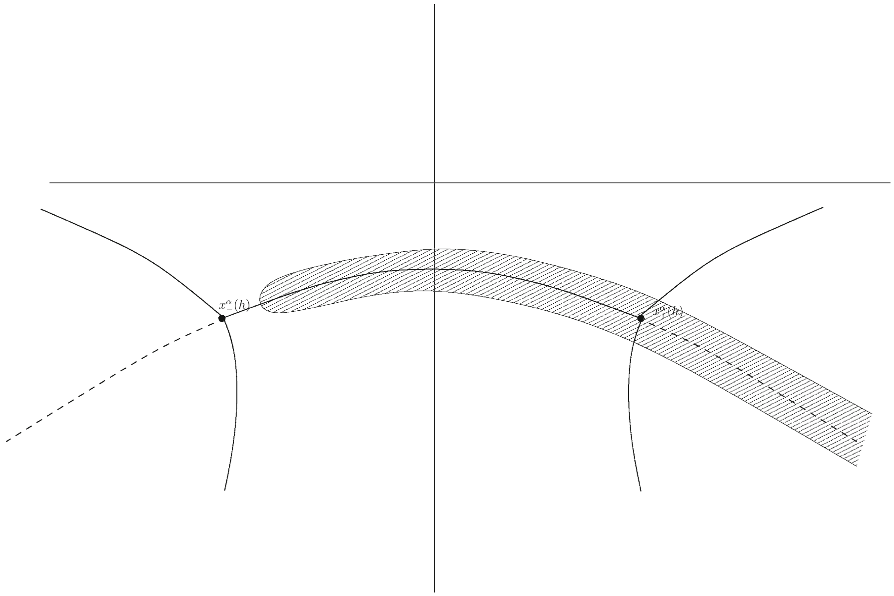

FIGURE 4. The domain $\mathcal{D}_+(\delta, \eta)$ (*shaded domain*). The line joining $x_-^\alpha(h)$ to $x_+^\alpha(h)$ is the finite Stokes line $\ell_f^\alpha(h)$. The *two dashed lines* represent the anti-Stokes lines $\tilde{\ell}_-^\alpha(h)$ (*on the left*) and $\tilde{\ell}_+^\alpha(h)$ (*on the right*)

We introduce the anti-Stokes lines starting from $x_\pm^\alpha(h)$, defined as the level curves of the function

$$x \mapsto \operatorname{Im} \int_{x_+^\alpha(h)}^{x} \sqrt{V_\alpha(z,h)}\, \mathrm{d}z$$

containing $x_\pm^\alpha(h)$. A local analysis near the turning points shows that there exist three anti-Stokes lines starting from $x_\pm^\alpha(h)$, and we will denote by $\tilde{\ell}_\pm^\alpha(h)$ (see Fig. 4) the one that satisfies

$$\forall x \in \tilde{\ell}_\pm^\alpha(h), \quad \int_{x_\pm^\alpha(h)}^{x} \sqrt{V_\alpha(z,h)}\, \mathrm{d}z > 0.$$

As in the previous subsection, we define a neighborhood of the line $\tilde{\ell}_\pm^0(0)$ by

$$\tilde{\ell}_{\pm,\delta}^0 = \{x \in \mathbb{C} : d(x, \tilde{\ell}_\pm^0(0)) < \delta\}, \tag{2.13}$$

and we have $\tilde{\ell}_\pm^\alpha(h) \subset \tilde{\ell}_{\pm,\delta}^0$ for h small enough.

Let $\eta > 0$ be such that $\eta < |x_+^0(0) - x_-^0(0)|$. Note that, for h small enough, it implies $\eta < |x_+^\alpha(h) - x_-^\alpha(h)|$. Then, for $\delta > 0$, we denote

$$\mathcal{D}_\pm(\delta,\eta) = \left(\ell^0_{f,\delta} \cap \{x \in \mathbb{C} : |x - x^0_\pm(0)| < \eta\}\right) \cup \tilde{\ell}^0_{\pm,\delta}. \tag{2.14}$$

This domain is represented on Fig. 4.

In the following statement and its proof, we use the notation

$$\zeta^\alpha_\pm(x,h) = \left(\frac{3}{2}\int_{x^\alpha_\pm(h)}^{x} \sqrt{V_\alpha(z,h)}\mathrm{d}z\right)^{2/3} \tag{2.15}$$

and

$$\tilde{\sigma}_\pm = \frac{1}{|V_\alpha|^{1/4}}\partial_x^2\left(\frac{1}{|V_\alpha|^{1/4}}\right) - \frac{5|V_\alpha|^{1/2}}{16|\zeta^\alpha_\pm|^3}, \tag{2.16}$$

which is defined for $x \neq x^\alpha_\pm(h)$.

Theorem 2.2. *Let $\alpha \in \mathbb{R}$. There exist positive constants $\delta > 0$ and $h_1 > 0$, and two solutions $\psi^\alpha_\pm(x,h)$ of equation*

$$\mathcal{A}_\alpha(h)\psi^\alpha_\pm(x,h) = \left(-h^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\alpha(x,h)\right)\psi^\alpha_\pm(x,h) = 0$$

such that, for all $h \in (0,h_1]$ and $x \in \mathcal{D}_\pm(\delta,\eta)$,

$$\psi^\alpha_\pm(x,h) = \left(\frac{\zeta^\alpha_\pm(x,h)}{V_\alpha(x,h)}\right)^{1/4} Ai\left(\frac{\zeta^\alpha_\pm(x,h)}{h^{2/3}}\right) + hr^\alpha_\pm(x,h), \tag{2.17}$$

where the function $r^\alpha_\pm$ satisfies, for all $h \in [0,h_1]$,

$$\begin{cases} \forall x \in \mathcal{D}_\pm(\delta,\eta)\backslash\ell^\alpha_f(h), & |r^\alpha_\pm(x,h)| \leq C^\alpha_\pm(x)\left|Ai\left(\frac{\zeta^\alpha_\pm(x,h)}{h^{2/3}}\right)\right|, \\ \forall x \in \mathcal{D}_\pm(\delta,\eta) \cap \ell^\alpha_f(h), & |r^\alpha_\pm(x,h)| \leq K^\alpha_\pm, \end{cases} \tag{2.18}$$

for some constant $K^\alpha_\pm > 0$, some function $C^\alpha_\pm(x)$ bounded in $\mathcal{D}_\pm(\delta_\pm,\eta)$ outside any open neighborhood of $\ell^0_f(0)$.

Proof. We work in the domain $\mathcal{D}_+(\delta,\eta)$, and we will possibly drop the index $+$ in the expressions. We shall apply Theorem 9.1, p. 417 in [17], with a h-dependent potential here. We introduce the following change of variable in $\mathcal{D}_+(\delta,\eta)$ (δ small enough will be determined in the following):

$$x \mapsto \zeta = \zeta(x,h) \tag{2.19}$$

for a fixed $h \in [0,h_0]$. We denote its inverse by

$$\zeta \mapsto x = x(\zeta,h). \tag{2.20}$$

The three Stokes lines starting from $x^\alpha_+(h)$ are mapped by (2.19) onto the half-lines

$$L_j = \arg^{-1}\left\{\frac{\pi}{3} + \frac{2j\pi}{3}\right\},$$

and the anti-Stokes line $\tilde{\ell}^\alpha_+(h)$ is mapped onto the half-line $[0,+\infty[$.

Let $a = +\infty$, and let $Z(a)$ be the set of points $\zeta \in \mathbb{C}$ such that there exists a complex path γ_ζ joining ζ to a, which coincides at infinity with $[0,+\infty[$, and such that $v \mapsto \operatorname{Re}\gamma_\zeta(v)^{3/2}$ is non-decreasing.

Then there exists $\delta > 0$ such that, for $h = 0$, $\zeta(\mathcal{D}_+(2\delta, \eta), 0) \subset Z(a)$. Since V_α has the form

$$V_\alpha(x, h) = V_0(x) + h^{4/5} v_\alpha(x, h), \tag{2.21}$$

where $|v_\alpha(x, h)| = o(|V_0(x)|)$ uniformly with respect to h as $|x| \to +\infty$, there exists $h_1 > 0$ such that for all $h \in]0, h_1[$,

$$\zeta(\mathcal{D}_+(\delta, \eta), h) \subset Z(a).$$

Thus, Theorem 9.1, p. 417 in [17], which applies for all $\zeta \in Z(a)$, ensures that there exists a solution

$$\psi^\alpha(x, h) = \left(\frac{\zeta(x, h)}{V_\alpha(x, h)}\right)^{1/4} W(\zeta(x, h), h),$$

where W has the form

$$\forall h \in (0, h_1],\ \forall \zeta \in \zeta(\mathcal{D}(\delta, \eta), h),\ W(\zeta, h) = Ai\left(\frac{\zeta}{h^{2/3}}\right) + h\varepsilon(\zeta, h). \tag{2.22}$$

In view of inequality (9.03), p. 418 in [17] (here applied with $n = 0$, $u = h^{-1}$ and ε_{2n+1} replaced by $h\varepsilon(\zeta, h)$), to prove that the function

$$r_+^\alpha(x, h) := \left(\frac{\zeta(x, h)}{V(x, h)}\right)^{1/4} \varepsilon(\zeta(x, h), h) \tag{2.23}$$

satisfies the bounds (2.18), it remains to check that there exists $M > 0$ such that, for all $h \in]0, h_1[$ and $\zeta \in \zeta(\mathcal{D}_+(\delta, \eta), h)$,

$$\int\limits_{x(\gamma_\zeta, h)} |\tilde{\sigma}(z, h)||\mathrm{d}z| \le M, \tag{2.24}$$

where $\tilde{\sigma}$ is the function defined in (2.16), and $x(\gamma_\zeta, h)$ denotes the image by (2.20) of the path γ_ζ defined above. Here we used the notation $|\mathrm{d}z| = |x(\gamma_\zeta, h)'(t)|\mathrm{d}t$.

Notice that the function $\tilde{\sigma}(x, h)$ is integrable at $x = x_\pm^\alpha(h)$, see for instance Lemma 3.1, p. 399 in [17]. Moreover, one can easily check that there exists $k > 0$ such that

$$|\tilde{\sigma}(x, 0)| \le \frac{k}{1 + |x|^{7/2}} \tag{2.25}$$

for $|x|$ large enough, $x \in \mathcal{D}_+(\delta, \eta)$. Thus, (2.24) follows from (2.21) and (2.25), and (2.18) is then proved. □

We now want to integrate the solution $\psi_\pm^\alpha$ over a path on which $\zeta(x, h)$ is real. In this purpose, we choose a $\mathcal{C}^1$ path $\gamma_h = \gamma_{h,\pm}^\alpha : [-d, +\infty[\to \mathbb{C}$ such that $\gamma_h(0) = x_\pm^\alpha(h)$,

$$\gamma_h([-d, +\infty[) = \bar{\mathcal{D}}_\pm(\delta, \eta) \cap (\ell_f^\alpha(h) \cup \tilde{\ell}_\pm^\alpha(h)), \tag{2.26}$$

and satisfying

$$\forall t \in [-d, +\infty[, \quad |\gamma_h'(t)| = 1. \tag{2.27}$$

Such a smooth path exists because both lines $\ell_f^\alpha(h)$ and $\tilde{\ell}_\pm^\alpha(h)$ reach the point $x_\pm^\alpha(h)$ with the same angle $-\frac{2}{3}\arg\sqrt{\partial_x V_\alpha(x_\pm^\alpha(h),h)}$ (modulo π).

Let us fix $\delta' \in]0,\delta[$, $\eta' \in]0,\eta[$, and $\chi_\pm \in \mathcal{C}^\infty(\mathbb{C},[0,1])$ with $\chi_\pm(x) = 1$ for $x \in \mathcal{D}_\pm(\delta',\eta')$ and $\operatorname{Supp}\chi_\pm \subset \mathcal{D}_\pm(\delta,\eta)$.

Lemma 2.3. *There exists $c_\pm^\alpha \neq 0$ such that, as $h \to 0$,*

$$\int\limits_{\gamma_{h,\pm}^\alpha} \psi_\pm^\alpha(x,h)^2 \chi_\pm^\alpha(x)\mathrm{d}x = c_\pm^\alpha h^{1/3}(1+o(1)). \tag{2.28}$$

Proof. Let us consider the case of ψ_+^α. We set $\zeta = \zeta_+^\alpha$, $\gamma_h = \gamma_{h,+}^\alpha$, $\chi = \chi_+^\alpha$ to simplify the notation.

We first apply the following change of variable, for a fixed $h \in [0,h_1]$:

$$[-d,+\infty[\ni t \mapsto \zeta := \zeta(\gamma_h(t),h) \in [-b_h,+\infty[,$$

where $[-b_h,+\infty[$ is the range of this function. Note that we have $\gamma_h(t) = x(\zeta,h)$, where $x(\cdot,h)$ is the inverse mapping (2.20).

Let b such that $b > b_h$ for all $h \in [0,h_1]$, and $\chi_h(\zeta) = \chi\circ x(\zeta,h)$, supported in $]-b,+\infty[$. Then,

$$\int\limits_{\gamma_h} \psi_+^\alpha(x,h)^2\chi(x)\mathrm{d}x = I_0(h) + hI_1(h) + h^2 I_2(h), \tag{2.29}$$

where

$$I_0(h) = \int\limits_{-b}^{+\infty} \frac{\zeta}{V_\alpha(x(\zeta,h),h)} Ai\left(\frac{\zeta}{h^{2/3}}\right)^2 \chi_h(\zeta)\mathrm{d}\zeta, \tag{2.30}$$

$$I_1(h) = 2\int\limits_{-b}^{+\infty} \frac{\zeta}{V_\alpha(x(\zeta,h),h)} Ai\left(\frac{\zeta}{h^{2/3}}\right) \varepsilon(\zeta,h)\chi_h(\zeta)\mathrm{d}\zeta \tag{2.31}$$

and

$$I_2(h) = \int\limits_{-b}^{+\infty} \frac{\zeta}{V_\alpha(x(\zeta,h),h)} \varepsilon(\zeta,h)^2\chi_h(\zeta)\mathrm{d}\zeta. \tag{2.32}$$

We recall that the Airy function is defined by

$$Ai(x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \mathrm{e}^{i(x\xi+\xi^3/3)}\mathrm{d}\xi;$$

hence

$$Ai\left(\frac{\zeta}{h^{2/3}}\right) = \frac{1}{2\pi h^{1/3}}\int\limits_{\mathbb{R}} \mathrm{e}^{\frac{i}{h}(\zeta\xi+\xi^3/3)}\mathrm{d}\xi.$$

Thus,

$$I_0(h) = \frac{1}{4\pi^2 h^{2/3}} \iiint\limits_{[-d,+\infty[\times\mathbb{R}^2} \frac{\zeta}{V_\alpha(x(\zeta,h),h)} e^{\frac{i}{h}\Phi(\zeta,\eta,\xi)}\chi_h(\zeta)\mathrm{d}\zeta\mathrm{d}\eta\mathrm{d}\xi, \tag{2.33}$$

where

$$\Phi(\zeta, \eta, \xi) = \zeta(\xi - \eta) + \frac{1}{3}(\xi^3 - \eta^3).$$

It is then straightforward to check that for all $\xi \in \mathbb{R}$, the function $\Phi(\cdot, \cdot, \xi)$ has a unique critical point $(-\xi^2, \xi)$, which is non-degenerate. Moreover, $\Phi(-\xi^2, \xi, \xi) = 0$. Thus, the stationary phase method with ξ fixed in (2.33) yields

$$I_0(h) = c_+^\alpha h^{1/3}(1 + o(1)), \ h \to 0, \tag{2.34}$$

where

$$c_+^\alpha = -(2\pi)^{-3/2} \int\limits_{\xi \in \mathbb{R}} \frac{\xi^2}{V_\alpha(x(-\xi^2, 0), 0)} \chi_0(-\xi^2) \mathrm{d}\xi. \tag{2.35}$$

Finally, using (2.18) and the asymptotic behavior of the Airy function as $z \to \pm\infty$ (see [1]), one can easily check that

$$hI_1(h) + h^2 I_2(h) = \mathcal{O}(h^{7/6}),$$

and the statement follows. □

2.4. Connection

In Sects. 2.2 and 2.3, we have determined the asymptotic behavior as $h \to 0$ of several solutions of (2.4). More precisely, we have built a solution $\psi_1^\alpha(\cdot, h_n) \in L^2(\mathbb{R})$ whose behavior is known in a domain Γ_ε avoiding a neighborhood of the bounded Stokes line $\ell_f^\alpha(h)$, and two solutions $\psi_\pm^\alpha(\cdot, h)$ whose asymptotic behavior is known in a neighborhood of $\ell_f^\alpha(h)$ avoiding the opposite turning point (see Theorem 2.2). We now want to connect these solutions, comparing their asymptotic expressions in the intersection of their domain of validity.

We first state the Bohr–Sommerfeld quantization rule, which gives a relation between the value of h_n and the index n. We will then use it to determine the coefficient relating the solutions ψ_1^α and $\psi_\pm^\alpha$. This lemma can be proved as Formula (25) in [13].

Lemma 2.4 (Bohr–Sommerfeld quantization rule).

$$\mathrm{Im} \int\limits_{x_-^\alpha(h_n)}^{x_+^\alpha(h_n)} \sqrt{V_\alpha(z, h_n)}\, \mathrm{d}z = \pi \left(n + \frac{1}{2}\right) h_n + \mathcal{O}(h_n^2). \tag{2.36}$$

We are now going to compare the asymptotic expressions of ψ_1^α and $\psi_\pm^\alpha$, for fixed h as $|x| \to +\infty$ along the lines $\tilde{\ell}_\pm^\alpha(h)$. Let $n \geq 1$ be large enough so that $\tilde{\ell}_-^\alpha(h_n) \subset \tilde{\ell}_{-,\delta}^0$, and let $x \in \tilde{\ell}_-^\alpha(h_n)$. We are then able to use the asymptotic expansion of the Airy function as $|z| \to +\infty$ [1], $|\arg z| < \pi$, with $z = \zeta_-^\alpha(x, h_n)$. If we denote $S_\pm^\alpha(x, h) = \int_{x_\pm^\alpha(h)}^x \sqrt{V_\alpha(z, h)}\, \mathrm{d}z$, expression (2.17) then writes

$$\begin{aligned}\psi_-^\alpha(x,h_n) &= \frac{h_n^{1/6}}{2\sqrt{\pi}V_\alpha(x,h_n)^{1/4}}\exp\left(-\frac{1}{h_n}S_-^\alpha(x,h_n)\right)(1+\mathcal{O}(S_-^\alpha(x,h_n)^{-3/2}))\\ &= \frac{h_n^{1/6}}{2\sqrt{\pi}}\exp\left(-\frac{1}{h_n}\int_{x_-^\alpha(h_n)}^{x_+^\alpha(h_n)}\sqrt{V_\alpha(z,h_n)}\,\mathrm{d}z\right)\psi_1^\alpha(x,h_n)\\ &\quad\times(1+\mathcal{O}(|x|^{-5/2})),\end{aligned} \tag{2.37}$$

where we used (2.11).

The two solutions ψ_-^α and ψ_1^α, being both exponentially decreasing as $|x|\to+\infty$ along $\tilde{\ell}_-^\alpha(h_n)$, are necessarily colinear. Hence, (2.36) and (2.37) yield

$$\psi_-^\alpha(x,h_n) = \frac{(-1)^{n-1}i}{2\sqrt{\pi}}h_n^{1/6}\psi_1^\alpha(x,h_n)(1+\mathcal{O}(h_n)),\quad n\to+\infty. \tag{2.38}$$

Similarly, comparing the asymptotic representations of ψ_1^α and ψ_+^α as $|x|\to+\infty$ along $\tilde{\ell}_+^\alpha(h_n)$, we get

$$\psi_+^\alpha(x,h_n) = \frac{1}{2\sqrt{\pi}}h_n^{1/6}\psi_1^\alpha(x,h_n). \tag{2.39}$$

Due to these relations, we can integrate the square of the solution $\psi_1^\alpha(x,h_n)$ over the curve consisting in the union of the three lines $\tilde{\ell}_-^\alpha(h_n)$, $\ell_f^\alpha(h_n)$ and $\tilde{\ell}_+^\alpha(h_n)$,

$$\mathcal{L}_\alpha(h_n) = \tilde{\ell}_-^\alpha(h_n)\cup\ell_f^\alpha(h_n)\cup\tilde{\ell}_+^\alpha(h_n). \tag{2.40}$$

We choose $\eta>0$ such that $\eta<|x_+^0(0)-x_-^0(0)|$ and such that $\ell_{f,\delta}^0\subset\mathcal{D}_+(\delta,\eta)\cup\mathcal{D}_-(\delta,\eta)$. Let also $\eta'<|x_+^0(0)-x_-^0(0)|/2$ and $\delta'\in]0,\delta[$. We choose a partition of unity (χ_-,χ_+) such that, for all $h\in]0,h_1]$ and all $x\in\mathcal{L}_\alpha(h)$, $\chi_-(x)+\chi_+(x)=1$, and such that $\chi_\pm(x)=1$ for $x\in\mathcal{D}_\pm(\delta',\eta')$, and Supp $\chi_\pm\subset\mathcal{D}_\pm(\delta,\eta)$.

Then, according to (2.39) and (2.38), for all $x\in\mathcal{L}_\alpha(h_n)$,

$$\psi_1^\alpha(x,h_n)^2 = 4\pi h_n^{-1/3}(\psi_+^\alpha(x,h_n)^2\chi_+(x)-\psi_-^\alpha(x,h_n)^2\chi_-(x))(1+\mathcal{O}(h_n)) \tag{2.41}$$

as $n\to+\infty$.

Thus, we deduce the following lemma from (2.28), where $c_\alpha=c_+^\alpha+c_-^\alpha\neq0$ (see (2.35)):

Lemma 2.5. *For all $\alpha\in\mathbb{R}$, there exists $c_\alpha\neq0$ such that*

$$\int_{\mathcal{L}_\alpha(h_n)}\psi_1^\alpha(x,h_n)^2\mathrm{d}x = c_\alpha(1+o(1)) \tag{2.42}$$

as $n\to+\infty$.

In the last section, we gather the previous results to prove Theorem 1.1.

3. Estimate on the Instability Indices

Let us first recall the following general result, which will provide an explicit formula for the instability indices $\kappa_n(\mathcal{A}_\alpha)$, for n large enough (see [2]).

Lemma 3.1. *Let $\mathcal{A}$ be a closed operator on the Hilbert space $\mathcal{H}$, and $\lambda \in \sigma(\mathcal{A})$ a simple isolated eigenvalue. Let Π_λ be the spectral projection associated with λ, u_λ an eigenvector associated with λ, and u_λ^* an eigenvector of $\mathcal{A}^*$ associated with the eigenvalue $\bar{\lambda}$. Then*

(i) *Π_λ has rank 1 if and only if $\langle u_\lambda, u_\lambda^* \rangle \neq 0$.*
(ii) *In this case, we have*

$$\kappa(\lambda) := \|\Pi_\lambda\| = \frac{\|u_\lambda\| \|u_\lambda^*\|}{|\langle u_\lambda, u_\lambda^* \rangle|}. \tag{3.1}$$

We recall (see Sect. 2.1) that the eigenfunctions u_n^α associated with the nth eigenvalue $\lambda_n(\alpha) \in \mathbb{R}$ of $\mathcal{A}_\alpha$ have the form

$$u_n^\alpha(x) = \psi_\alpha(h_n^{2/5} x, h_n), \tag{3.2}$$

where

$$h_n = \lambda_n(\alpha)^{-5/6}, \tag{3.3}$$

and where $\psi_\alpha(\cdot, h_n) \in L^2(\mathbb{R})$ is a solution of $\mathcal{A}_\alpha(h_n)\psi_\alpha(\cdot, h_n) = 0$.

We normalize u_n^α so that

$$u_n^\alpha(x) = \psi_1^\alpha(h_n^{2/5} x, h_n), \tag{3.4}$$

where ψ_1^α is the solution introduced in Theorem 2.1.

We have

Proposition 3.2. *Let $\alpha \geq 0$. There exists $N \geq 1$ such that, for all $n \geq N$, the spectral projection $\Pi_n(\alpha)$ of $\mathcal{A}_\alpha$ associated with $\lambda_n(\alpha)$ has rank 1. Moreover, there exists $k_\alpha > 0$ such that the nth instability index satisfies*

$$\kappa_n(\alpha) = k_\alpha \|\psi_1^\alpha(\cdot, h_n)\|_{L^2(\mathbb{R})}(1 + o(1)), \quad n \to +\infty. \tag{3.5}$$

Proof. By deformation of the integration path, and using the exponential decay of $\psi_1^\alpha(x, h_n)$ as $|x| \to +\infty$ in the sectors $\arg^{-1}(]-3\pi/10, \pi/10[)$ and $\arg^{-1}(]9\pi/10, 13\pi/10[)$ (see Theorem 2.1), we get

$$\int_{\mathbb{R}} \psi_1^\alpha(x, h_n)^2 \mathrm{d}x = \int_{\mathcal{L}_\alpha(h_n)} \psi_1^\alpha(x, h_n)^2 \mathrm{d}x. \tag{3.6}$$

We then notice that $\mathcal{A}_\alpha^* \Gamma = \Gamma \mathcal{A}_\alpha$, where $\Gamma : u(x) \mapsto \overline{u(x)}$. Hence, we have $(u_n^\alpha)^*(x) = \overline{u_n^\alpha(x)}$, with the notation of Proposition 3.1. Thus, according to (3.4),

$$\langle u_n^\alpha, (u_n^\alpha)^* \rangle = h_n^{-2/5} \int_{\mathcal{L}_\alpha(h_n)} \psi_1^\alpha(x, h_n)^2 \mathrm{d}x.$$

Using (2.42) we then get, for n large enough, $|\langle u_n^\alpha, (u_n^\alpha)^* \rangle| > 0$, and the desired statement on the rank of $\Pi_n(\alpha)$ follows from Proposition 3.1, (i). Expression

(3.5) follows from (2.42) and Proposition 3.1, (ii), after the change of variable $x \mapsto h_n^{2/5} x$. □

Now it remains to determine an equivalent for the norm $\|\psi_1^\alpha(\cdot, h_n)\|_{L^2(\mathbb{R})}$ appearing in (3.5). We will do so using the expansion (2.11). Let us recall that this expansion is uniform with respect to $x \in \mathbb{R}$; hence we integrate

$$\|\psi_1^\alpha(\cdot, h_n)\|_{L^2(\mathbb{R})}^2 = (1 + o(1)) \int_{\mathbb{R}} a(x) \mathrm{e}^{-\varphi_\alpha(x, h_n)} \mathrm{d}x \tag{3.7}$$

as $n \to +\infty$, where

$$a(x) = \frac{1}{V_0(x)^{1/4}}$$

and

$$\varphi_\alpha(x, h) = \frac{2}{h} \mathrm{Re} \int_{x_+^\alpha(h)}^{x} \sqrt{V_\alpha(z, h)}\, \mathrm{d}z.$$

Lemma 3.3. *If $\alpha \geq 0$ then, as $n \to +\infty$,*

$$\|\psi_1^\alpha(\cdot, h_n)\|_{L^2(\mathbb{R})}^2 = \frac{\sqrt{2}}{2} \Gamma(1/4) h_n^{1/4} (1 + o(1)) \exp\left(\frac{C}{h_n} + \frac{\alpha r}{h_n^{1/5}}\right), \tag{3.8}$$

where

$$C = \int_0^1 \sqrt{1 - t^3}\, \mathrm{d}t \ > 0 \quad \textit{and} \quad r = \frac{1}{2} \int_0^1 \frac{t}{\sqrt{1 - t^3}}\, \mathrm{d}t. \tag{3.9}$$

Proof. Let us first assume that $\alpha > 0$. We shall apply the Laplace method with two parameters in [18] to determine the behavior as $h \to 0$ of the integral

$$I_\alpha(h) = \int_{\mathbb{R}} a(x) \mathrm{e}^{-\varphi_\alpha(x, h)} \mathrm{d}x$$

appearing in (3.7). We write $\varphi_\alpha(x, h) = \frac{1}{h} g_\alpha(x, \varepsilon(h))$ with $\varepsilon(h) = h^{4/5}$ and

$$g_\alpha(x, \varepsilon) = 2\mathrm{Re} \int_{\tilde{x}_+^\alpha(\varepsilon)}^{x} \sqrt{\tilde{V}_\alpha(z, \varepsilon)}\, \mathrm{d}z,$$

where we have denoted $\tilde{x}_+^\alpha(\varepsilon) = x_+^\alpha(\varepsilon^{5/4})$ and $\tilde{V}_\alpha(x, \varepsilon) = V_\alpha(x, \varepsilon^{5/4}) = ix^3 + i\alpha\varepsilon x - 1$.

The function g_α is $\mathcal{C}^\infty$ for $x \in \mathbb{R}$ and ε small enough. Moreover, $g_\alpha(\cdot, 0)$ has a unique critical point $x = 0$. Indeed,

$$\partial_x g_\alpha(x, 0) = 2\mathrm{Re} \sqrt{ix^3 - 1} = 0$$

if and only if $\arg(ix^3 - 1) = \pi$, that is $x = 0$.

We write

$$\varphi_\alpha(x, h) = \frac{1}{h} g_\alpha(x, 0) + \frac{\varepsilon(h)}{h} \partial_\varepsilon g_\alpha(x, 0) + \mathcal{O}\left(\frac{\varepsilon(h)^2}{h}\right), \tag{3.10}$$

and we easily check that the remainder term is uniform with respect to $x \in \mathbb{R}$. We also check that

$$\partial_x^2 g_\alpha(0,0) = \partial_x^3 g_\alpha(0,0) = 0 \ \text{ and } \ \partial_x^4 g_\alpha(0,0) = 6,$$

and that

$$\partial_x \partial_\varepsilon g_\alpha(0,0) = 0 \ \text{ and } \ \partial_x^2 \partial_\varepsilon g_\alpha(0,0) = \alpha.$$

Thus,

$$g_\alpha(x,0) - g_\alpha(0,0) = \frac{x^4}{4} + \mathcal{O}(|x|^5),\ \partial_\varepsilon g_\alpha(x,0) - \partial_\varepsilon g_\alpha(0,0) = \frac{\alpha x^2}{2} + \mathcal{O}(|x|^3).$$

We can then apply Theorem 2 in [18], with $\phi(x) = g_\alpha(x,0) - g_\alpha(0,0)$, $\psi(x) = -\partial_\varepsilon g_\alpha(x,0)$, $\nu = 4$, $\mu = 2$, $\lambda = 0$, and replacing h by h^{-1} and k by $\varepsilon(h)h^{-1} = h^{-1/5}$. This yields

$$\begin{aligned}&\|\psi_1^\alpha(\cdot, h_n)\|^2_{L^2(\mathbb{R})}\\ &\quad= \frac{\sqrt{2}}{2}\Gamma(1/4)h_n^{1/4}(1+o(1))\exp\left(-\frac{1}{h_n}(g_\alpha(0,0) + h_n^{4/5}\partial_\varepsilon g_\alpha(0,0)\right)\end{aligned}$$

In order to get the desired statement, it only remains to notice that $g_\alpha(0,0) = -C$ and $\partial_\varepsilon g_\alpha(0,0) = -\alpha r$, where C and r are the constants in (3.9).

In the case $\alpha = 0$, we check similarly that the Laplace method applies (see for instance [12]) and leads to the same statement. □

To conclude the proof of Theorem 1.1, we use the Bohr–Sommerfeld rule (2.36), which gives an asymptotic expansion for h_n. Let us compute the first few terms. By expanding the left-hand-side of (2.36), we get

$$\mathrm{Im} \int_{x_-^\alpha(h_n)}^{x_+^\alpha(h_n)} \sqrt{V_\alpha(z,h_n)}\,\mathrm{d}z = \sqrt{3}C - \sqrt{3}\alpha r h_n^{4/5} + \mathcal{O}(h_n^{8/5}),$$

where C and r are the constants in (3.9). Expression (2.36) then writes

$$h_n = \frac{\sqrt{3}C}{\pi\left(n+\frac{1}{2}\right)} - \frac{3^{9/10}\alpha r C^{4/5}}{\pi^{9/5}\left(n+\frac{1}{2}\right)^{9/5}} + \mathcal{O}((n+1/2)^{-13/5}). \tag{3.11}$$

Gathering (3.5), (3.8) and (3.11), and replacing C and r by their values

$$C = \frac{2\sqrt{3}\pi^{3/2}}{15\Gamma(2/3)\Gamma(5/6)} \ \text{ and } \ r = \frac{\Gamma(2/3)\Gamma(5/6)}{2\sqrt{\pi}},$$

we get the following statement, and Theorem 1.1 follows:

Theorem 3.4. *For all $\alpha \geq 0$, there exists a positive constant K_α such that*

$$\kappa_n(\alpha) = \frac{K_\alpha}{n^{1/4}}(1+o(1))\exp\left(\frac{\pi}{\sqrt{3}}n + \alpha c n^{1/5}\right) \tag{3.12}$$

as $n \to +\infty$, where

$$c = (5/2)^{1/5}\pi^{-3/5}\Gamma(2/3)^{6/5}\Gamma(5/6)^{6/5}.$$

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