Ann. Henri Poincaré 15 (2014), 1013–1036

1424-0637/14/051013-24
published online June 16, 2013
DOI 10.1007/s00023-013-0265-5

Annales Henri Poincaré

Fixed Points of Compact Quantum Groups Actions on Cuntz Algebras

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Abstract. Given an action of a Compact Quantum Group (CQG) on a finite dimensional Hilbert space, we can construct an action on the associated Cuntz algebra. We study the fixed point algebra of this action, using Kirchberg classification results. Under certain conditions, we prove that the fixed point algebra is purely infinite and simple. We further identify it as a C^*-algebra, compute its K-theory and prove a "stability property": the fixed points only depend on the CQG via its fusion rules. We apply the theory to $SU_q(N)$ and illustrate by explicit computations for $SU_q(2)$ and $SU_q(3)$. This construction provides examples of free actions of CQG (or "principal noncommutative bundles").

1. Introduction

Our original motivation for this article was to find an explicit construction of free actions of Compact Quantum Groups (CQGs—see Woronowicz' articles [33,34,36]) in the sense of Ellwood's work [14] (see also the recent article [9] by De Commer and Yamashita).

Another motivation for the study of such fixed point algebras is provided by the duality results of Doplicher and Roberts [11,12] for actions of (ordinary) compact groups on C^*-algebras and the considerable amount of work they generated (see for instance [3,26,27]). A basic step of Doplicher–Roberts' duality theory is to consider "canonical actions" of compact groups on Cuntz algebras (as defined in [7]).

These two motivations are combined by considering canonical actions of *quantum groups* on Cuntz algebras. This direction was first explored using Hopf algebras and multiplicative unitaries by Cuntz [8]. However, our approach is closer to the article by Konishi et al. [18]. Starting with a unitary representation α of dimension d of a CQG $\mathbb{G}$, we obtain an action of $\mathbb{G}$ on the Cuntz algebra $\mathcal{O}_d$ by a process extending that of Doplicher and Roberts.

The previous construction was studied by Marciniak [21], in the case of the irreducible representation α of $\mathbb{G} = SU_q(2)$ on $\mathbb{C}^2$. He gave an explicit

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description of the fixed point algebra by means of generators and relations. The general case of the natural irreducible representation of $\mathbb{G} = SU_q(d)$ on $\mathbb{C}^d$ was treated explicitly by Paolucci in her article [23], relying on the Tannaka–Krein duality for CQGs (as established by Woronowicz' seminal article [35]) and the analysis of the classical case expounded in [11]. She gave an explicit family of generators based on the infinite braid group B_∞. She further expanded her results in joint work [6] with Carey and Zhang. Working in the algebraic setting, they gave the same kind of description in the case of classical quantum groups $\mathbb{G} = SU_q(d), SO_q(d)$ or $SP_q(d)$. Moreover, they managed to recover the "q" in $(0, 1)$ of $\mathbb{G}$ from the fixed point algebra.

We take a rather different approach to this problem. Indeed, we stay strictly in the C^*-algebraic domain and try to obtain an abstract C^*-isomorphism. For this abstract identification, we depend on the remarkable classification results by Kirchberg and Phillips (see [16,19]) allowing to recover a C^*-isomorphism from K-theoretic properties. In this article, our reference regarding classification theory is the book by Rørdam and Størmer [30].

To prove that the fixed point algebras are simple and purely infinite and to perform the required K-theoretic computations, we use the notion of "crossed product by endomorphism" (denoted $A \rtimes_\sigma \mathbb{N}$) which first appeared in [24]. Rørdam elaborated on this notion in series of article (e.g. [13,29]) on which we rely here.

In the course of this article, we resort to previous work by Banica [1] to define R^+-isomorphic CQGs and by Wassermann [31] for the computation of K-theory.

The main results of this article come from two directions:

- Under some hypotheses, we prove that the fixed point algebra $\mathcal{O}^\alpha$ is simple and purely infinite (Theorem 4.6).
- The same hypotheses ensure that $\mathcal{O}^\alpha$ is a crossed product by endomorphism (Corollary 4.7) and thus we can compute its K-theory (Theorem 5.4).

Combining these two threads of results, an unexpected "stability theorem" (Theorem 6.5) appears, namely: as a C^*-algebra, the fixed point algebra $\mathcal{O}^\alpha$ only depends on $\mathbb{G}$ via its fusion rules. We show that these results apply to the natural representation of $SU_q(d)$ and discuss two explicit examples, the cases of $\mathbb{G} = SU_q(2)$ and $\mathbb{G} = SU_q(3)$. In this setting, this "stability result" for C^*-algebras contrasts sharply with the algebraic case treated in [6], as we discuss in Remark 8.6. Finally, we prove that this construction indeed yields free actions of CQGs.

After reviewing the construction of the fixed point algebra $\mathcal{O}^\alpha$ (Sect. 2), stating properly the hypotheses of our theorems and introducing the notion of crossed product by endomorphisms (Sect. 3), we prove the simplicity and purely infiniteness properties in Sect. 4, which leads us to identify the fixed points with such a crossed product. We proceed with the computation of K-theory (Sect. 5). The next Sect. 6 focuses on the combination of the previous parts to obtain the main results. We then discuss our hypotheses, show

how to apply our theory to $SU(N)$ and study two explicit examples (Sect. 7). Going back to our original motivation, we conclude in Sect. 8 with sufficient conditions to get a free action and a comparison with the algebraic case.

2. Review of Fixed Points Algebras

In this article, all tensor products of C^*-algebras are *minimal* tensor products. We consider a Compact Quantum Group (CQG) denoted by $\mathbb{G}$, i.e. a separable unital C^*-algebra $C(\mathbb{G})$ together with a unital $*$-algebra homomorphism $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ which satisfies coassociativity and cancellation properties—for more details on these objects and their representations, see [33,36].

We also fix a *unitary representation* α of $\mathbb{G}$ on a Hilbert space $\mathscr{H}$ of finite dimension d i.e. a $C(\mathbb{G})$-valued $d \times d$ matrix $(\alpha_{ij}) \in M_d(C(\mathbb{G}))$ satisfying a coassociativity property as well as equations $\sum_k \alpha_{ki}^* \alpha_{kj} = \delta_{ij}$ and $\sum_k \alpha_{ik} \alpha_{jk}^* = \delta_{ij}$. In particular, for any $\mathbb{G}$, we have a trivial representation denoted by ϵ, defined by the unit of $C(\mathbb{G})$. In the rest of this article, we only use unitary representations which we, therefore, simply call "representations".

If (e_i) is an orthonormal basis of $\mathscr{H}$, we define the associated *action* of $\mathbb{G}$ on $\mathscr{H}$ as the map $\delta_\alpha : \mathscr{H} \to \mathscr{H} \otimes C(\mathbb{G})$ given by $\delta_\alpha(e_i) = \sum_j e_j \otimes \alpha_{ji}$. Note that our conventions differ from those of [1,23].

Using this definition of representations for CQGs, Woronowicz introduced several notions which are very similar to the case of compact groups. In particular, given two representations $\alpha \in M_{d_1}(C(\mathbb{G}))$ and $\beta \in M_{d_2}(C(\mathbb{G}))$ acting on the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, a linear map T between $\mathscr{H}_1$ and $\mathscr{H}_2$ is an *intertwiner* (see Proposition 2.1 p. 629 of [33]) if

$$(T \otimes 1)\delta_\alpha = \delta_\beta T.$$

He also defined *irreducible representations* and proved a *Schur's lemma-type* theorem. Given two representations π_1 and π_2 of $\mathbb{G}$ on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, he constructed the unitary tensor product representation $\pi_1 \otimes \pi_2$ on $\mathscr{H}_1 \otimes \mathscr{H}_2$—see Section 2 of [33]. Furthermore, Woronowicz defined and proved the existence of the *Haar measures* for general CQGs—see [36] Theorem 1.3.

We denote by $R^+(\mathbb{G})$ the *fusion semiring* of (equivalence classes of) finite dimensional representations of $\mathbb{G}$ equipped with addition and tensor product (compare Definition 1.2, [1]). The following is an avatar of Definition 2.1 in [1]:

Definition 2.1. Given two CQGs $\mathbb{G}_1$ and $\mathbb{G}_2$, we say that they are *R^+-isomorphic* if an isomorphism of semirings $R^+(\mathbb{G}_1) \simeq R^+(\mathbb{G}_2)$ exists.

In other words, $\mathbb{G}_1$ and $\mathbb{G}_2$ have the same *fusion rules*, i.e. there is a bijection Φ between their irreducible representations and it is compatible with the tensor product. As an example, the q-deformations $SU_q(N)$ are R^+-isomorphic to the original $SU(N)$ (see [1], Theorem 2.1 and references therein).

The Cuntz algebra $\mathcal{O}_d$ is defined [7] as the universal unital C^*-algebra generated by d elements $(S_j)_{1\leqslant j\leqslant d}$ which satisfy $S_i^* S_j = \delta_{ij}$ and $\sum_j S_j S_j^* = 1$. The case $d = 1$ yields the algebra $C(S^1)$, which is a very special case, thus we assume from now on that $d \geqslant 2$. If (e_i) is an orthonormal basis of a Hilbert space $\mathscr{H}$ of dimension d, we define an injective linear map $\varphi : \mathscr{H} \to \mathcal{O}_d$ by $e_i \mapsto S_i$. This map preserves the scalar product in the sense that if $v, w \in \mathscr{H}$, then

$$\langle v, w\rangle = \varphi(v)^*\varphi(w) \in \mathbb{C}1 \subseteq \mathcal{O}_d.$$

Following [11], we extend this map to the iterated tensor products and set

$$\begin{aligned}&\mathscr{H}^{\otimes \ell}(\mathscr{H}^*)^{\otimes k}\\ &\quad := \mathrm{Span}\{S_{i_1}\dots S_{i_\ell} S_{j_1}^* \dots S_{j_k}^* | 1 \leqslant i_1,\dots,i_\ell, j_1,\dots,j_k \leqslant d\} \subseteq \mathcal{O}_d.\end{aligned}$$

Taking $(\ell, k) = (1, 0)$ (resp. $(\ell, k) = (0, 1)$) we recover $\mathscr{H}$ (resp. $\mathscr{H}^*$). These spaces satisfy $\mathscr{H}^*\mathscr{H} \subseteq \mathbb{C}1 \subseteq \mathcal{O}_d$. Thus, we can identify $\mathscr{H}^{\otimes \ell}(\mathscr{H}^*)^{\otimes k}$ with the space of linear operators from $\mathscr{H}^{\otimes k}$ to $\mathscr{H}^{\otimes \ell}$. Moreover, an inclusion $\mathscr{H}^{\otimes \ell}(\mathscr{H}^*)^{\otimes k} \hookrightarrow \mathscr{H}^{\otimes(\ell+1)}(\mathscr{H}^*)^{\otimes(k+1)}$ is defined by

$$S_{i_1}\dots S_{i_\ell} S_{j_1}^* \dots S_{j_k}^* \mapsto \sum_p S_{i_1}\dots S_{i_\ell} S_p S_p^* S_{j_1}^* \dots S_{j_k}^*.$$

We make extensive use of these identifications. We denote by $\mathcal{O}_d^{\mathrm{alg}}$ the algebraic unital $*$-algebra generated by S_j. Its elements $T \in \mathcal{O}_d^{\mathrm{alg}}$ are called *algebraic element*.

A pointwise continuous S^1-action γ on a C^*-algebra B is called a *gauge action*. It yields the *spectral subspaces* $B^{(k)}$ defined by:

$$B^{(k)} := \{T \in B | \forall z \in S^1 \subseteq \mathbb{C}, \gamma_z(T) = z^k T\}.$$

If $T \in B^{(k)}$, we says that T is a *gauge-homogeneous* element of *total gauge* k.

In the case of $\mathcal{O}_d$, such a gauge action is defined on generators by $\gamma_z(S_j) = zS_j$, yielding spaces $\mathcal{O}_d^{(k)}$. We use the short hand notation $\mathcal{F} := \mathcal{O}_d^{(0)}$ and call the elements of this set *gauge-invariant*. This algebra contains $\mathcal{F}^\ell := \mathscr{H}^{\otimes \ell}(\mathscr{H}^*)^{\otimes \ell}$, which can be identified with the algebra of matrices $M_{d^\ell}(\mathbb{C})$ by the previous discussion.

Actions of CQGs on C^*-algebras were first defined in [5, 25].

Definition 2.2. An action $\underline{\delta}$ of a CQG on a C^*-algebra B is a faithful unital $*$-homomorphism

$$\underline{\delta} : B \to B \otimes C(\mathbb{G})$$

satisfying the coaction condition $(\mathrm{Id} \otimes \Delta) \circ \underline{\delta} = (\underline{\delta} \otimes \mathrm{Id}) \circ \underline{\delta}$ as well as the density condition

$$\overline{(1 \otimes C(\mathbb{G}))\underline{\delta}(B)} = B \otimes C(\mathbb{G}).$$

We quote Theorem 1 of [18]:

Theorem 2.3. *For a given representation α of $\mathbb{G}$ on a Hilbert space $\mathscr{H}$ of dimension d, there is an action $\underline{\delta} : \mathcal{O}_d \to \mathcal{O}_d \otimes C(\mathbb{G})$ of $\mathbb{G}$ on $\mathcal{O}_d$ defined by*

$$\underline{\delta}(S_i) = \sum_{j=1}^{d} S_j \otimes \alpha_{ji}.$$

By construction, $\underline{\delta}$ preserves the spaces $\mathscr{H}^{\otimes \ell}(\mathscr{H}^*)^{\otimes k}$. It also induces actions δ_k of $\mathbb{G}$ on $\mathscr{H}^{\otimes k}$. The associated representations are called $\alpha^{\otimes k}$—indeed, they are the iterated tensor products of α.

We now introduce our main object of interest:

Definition 2.4. The fixed point algebra $\mathcal{O}^\alpha$ for the action $\underline{\delta}$ is defined as:

$$\mathcal{O}^\alpha := \{T \in \mathcal{O}_d : \underline{\delta}(T) = T \otimes 1\} \subseteq \mathcal{O}_d.$$

In particular, if α is the trivial representation on $\mathbb{C}^d$, given by $\alpha_{ij} = \delta_{ij}$, then we recover $\mathcal{O}^\alpha = \mathcal{O}_d$.

It is readily checked [23] that just like in the "regular" group case [11], the gauge action γ on $\mathcal{O}_d$ "commutes" with the coaction of $\mathbb{G}$ on $\mathcal{O}_d$, therefore, $\mathcal{O}^\alpha$ also carries a gauge action and we set $\mathcal{F}^\alpha := \mathcal{O}^\alpha \cap \mathcal{F}$. We use the notation $\mathcal{F}^{\alpha,\ell} := \mathcal{O}^\alpha \cap \mathscr{H}^{\otimes \ell}(\mathscr{H}^*)^{\otimes \ell}$. The following already appeared as Proposition 3.4 of [21], extending results of [11]:

Lemma 2.5. *The elements of $\mathcal{O}^\alpha \cap \mathscr{H}^{\otimes \ell}(\mathscr{H}^*)^{\otimes k}$ are precisely the intertwiners of the representations $\alpha^{\otimes \ell}$ and $\alpha^{\otimes k}$.*

Denote by $(\alpha^{\otimes \ell}, \alpha^{\otimes k})$ the intertwiners between the representations $\alpha^{\otimes \ell}$ and $\alpha^{\otimes k}$. The previous Lemma states that $\mathcal{O}^\alpha \cap \mathscr{H}^{\otimes \ell}(\mathscr{H}^*)^{\otimes k} = (\alpha^{\otimes \ell}, \alpha^{\otimes k})$.

We adapt Lemma 5 of [18]:

Definition 2.6. If $\mathbb{G}$ is a compact quantum group, the *conditional expectation* $\mathbb{E}_{\mathbb{G}}$ associated with the action α is:

$$\mathbb{E}_{\mathbb{G}}(T) = (\mathrm{Id}_{\mathcal{O}_d} \otimes h)\alpha(T),$$

where h is the Haar measure on $\mathbb{G}$.

$\mathbb{E}_{\mathbb{G}}$ defines a projection of norm 1 from $\mathcal{O}_d$ onto $\mathcal{O}^\alpha$ (see [21] p.611). It sends algebraic elements to algebraic elements and $\mathscr{H}^{\otimes \ell}(\mathscr{H}^*)^{\otimes \ell}$ to $\mathscr{H}^{\otimes \ell}(\mathscr{H}^*)^{\otimes \ell} \cap \mathcal{O}^\alpha$.

We refer to the book [30] (Definition 1.2.1) for a definition of Approximately Finite dimensional algebras or AF-algebras. The conditional expectation $\mathbb{E}_{\mathbb{G}}$ is used in the proof of the following Lemma 6 of [18]:

Lemma 2.7. *If α is a representation of the CQG $\mathbb{G}$, then the algebraic elements of $\mathcal{O}^\alpha$ are dense in $\mathcal{O}^\alpha$. Moreover,*

- *any positive $T \in \mathcal{O}^\alpha$ can be approximated by a positive and algebraic $T_\alpha \in \mathcal{O}^\alpha$;*
- *$\mathcal{F}^\alpha$ is an AF-algebra and $\mathcal{F}^\alpha = \lim_{\rightarrow} \mathcal{F}^{\alpha,\ell}$. Any positive $T \in \mathcal{F}^\alpha$ can be approximated by a positive algebraic $T_\alpha \in \mathcal{F}^{\alpha,\ell}$, for ℓ large enough.*

Remark 2.8. Using the notations introduced below Lemma 2.5, we can express the second point as $\mathcal{F}^\alpha = \lim_\to(\alpha^\ell, \alpha^\ell)$. It is more generally true that for $k \in \mathbb{Z}, (\mathcal{O}^\alpha)^{(k)} = \lim_\to(\alpha^\ell, \alpha^{k+\ell})$.

Proof. Take an element $T \in \mathcal{O}^\alpha \subseteq \mathcal{O}_d$. By definition of $\mathcal{O}_d$, we can find an algebraic element $T' \in \mathcal{O}_d$ s.t. $\|T' - T\| \leqslant \varepsilon$. Using the conditional expectation $\mathbb{E}_\mathbb{G}$ associated with α and setting $T_\alpha := \mathbb{E}_\mathbb{G}(T')$ the following holds:

$$\|T_\alpha - T\| = \|\mathbb{E}_\mathbb{G}(T') - T\| = \|\mathbb{E}_\mathbb{G}(T' - T)\| \leqslant \|T' - T\| \leqslant \varepsilon. \tag{2.1}$$

Hence $T_\alpha \in \mathcal{O}^\alpha$ is an algebraic approximation of T in $\mathcal{O}^\alpha$.

If $T \in \mathcal{O}^\alpha$ is positive, we can consider its square root $B \in \mathcal{O}^\alpha$ with $T = B^*B$ and $B = B^*$. Applying the above argument to B, we get an algebraic approximation B_α of B. Finally, $T = B^*B$ is approximated by the positive algebraic element $T_\alpha := B_\alpha^* B_\alpha \in \mathcal{O}^\alpha$.

Since $\mathcal{F}$ is an AF-algebra and $\mathcal{F} = \lim_\to \mathcal{F}^\ell$, for any $T \in \mathcal{F}^\alpha \subseteq \mathcal{F}$ and any $\varepsilon > 0$, there is a $T' \in \mathcal{F}^\ell$ for ℓ large enough such that $\|T - T'\| \leqslant \varepsilon$. Considering then $T_\alpha := \mathbb{E}_\mathbb{G}(T')$ and using the estimate (2.1), we see that T_α is an algebraic approximation of T. Moreover, since $T' \in \mathcal{F}^\ell, T_\alpha \in \mathcal{F}^{\alpha,\ell}$. For positive elements, considering the square root as above proves the required property. □

Remark 2.9. Any selfadjoint element $T \in \mathcal{F}^\alpha$ can be written $T = T_+ - T_-$ where both T_+ and T_- are positive. It is, therefore, clear that any selfadjoint $T \in \mathcal{F}^\alpha$ can be approximated by a selfadjoint $T_0 \in \mathcal{F}^{\alpha,\ell}$ for ℓ large enough. Since the latter algebra is finite dimensional, T_0 has finite spectrum and $\mathcal{F}^\alpha$ has real rank zero, according to Theorem V.3.2.9 p. 453 of [4].

The conditional expectation $\mathbb{E}_\mathbb{G}$ leads to the following, which we adapt from the proof of Proposition 2.1 in [10]:

Lemma 2.10. *The algebra $\mathcal{O}^\alpha$ is nuclear and separable.*

Proof. Separability is an immediate consequence of Lemma 2.7. Regarding nuclearity, the argument is that $\mathcal{O}^\alpha$ is a subalgebra of the nuclear algebra $\mathcal{O}_d$ and that there is a conditional expectation $\mathbb{E}_\mathbb{G} : \mathcal{O}_d \to \mathcal{O}^\alpha$. □

3. Conditions and Endomorphism Crossed Products

Notation 3.1. Given a representation α of $\mathbb{G}$, we call $\mathcal{T}_\alpha$ the set of (classes of) irreducible representations appearing in the iterated tensor products $\alpha^{\otimes \ell}$ for $\ell \in \mathbb{N}$.

Our results rely on the following conditions:

(C1) For any $\beta \in \mathcal{T}_\alpha$, we can find $\beta' \in \mathcal{T}_\alpha$ s.t. the representation $\beta \otimes \beta'$ possesses a nonzero invariant vector.

(C2) There are integers $N, k_0 \in \mathbb{N}$ such that $\alpha^{\otimes N}$ is contained in $\alpha^{\otimes(N+k_0)}$ and for all integers k, ℓ with $0 < k < k_0, (\alpha^{\otimes \ell}, \alpha^{\otimes(\ell+k)}) = \{0\}$.

Condition (C2′) can alternatively be stated as (see Lemma 3.2 below):

(C2′) For any $k \in \mathbb{N} \setminus \{0\}$, if $(\mathcal{O}^\alpha)^{(k)}$ is nontrivial, there is an isometry in it. Moreover, not all $(\mathcal{O}^\alpha)^{(k)}$ are trivial.

A few comments on these conditions:

- In Condition (C1), saying that "$\beta \otimes \beta'$ possesses a nonzero invariant vector" amounts to stating that the dual (or contragredient) β' of β is in $\mathcal{T}_\alpha$. In other words, $\mathcal{T}_\alpha$ is closed under duality hence it is the representation category of some quantum group quotient of $\mathbb{G}$.
- Condition (C1) is non-trivial. Indeed some CQGs do not satisfy Condition (C1): consider $\mathbb{G} = U(1)$ and its representation $\mu : z \mapsto z$. Clearly, in T_μ we get all the representations $z \mapsto z^n$ for $n > 1$ but we do not recover the representation $z \mapsto z^{-1}$ which would satisfy (C1).
- Condition (C1) is satisfied for semisimple Lie groups and finite groups, as discussed in Sect. 7—see Propositions 7.2 and 7.4.
- Given an inclusion of representations $\alpha \leqslant \beta$, it is clear that for any representation $\gamma, \alpha \otimes \gamma \leqslant \beta \otimes \gamma$. Hence, if Condition (C2) is satisfied for N_0, then it is satisfied for any larger $N \geqslant N_0$.

Lemma 3.2. *Assuming (C2′), if k_0 is the smallest strictly positive integer such that $(\mathcal{O}^\alpha)^{(k)}$ is nontrivial, then the spectral subspace $(\mathcal{O}^\alpha)^{(k)}$ is nontrivial if and only if k_0 divides k.*

Conditions (C2) and (C2′) are equivalent.

To prove the equivalence, we will need the following:

Definition 3.3. We define the *Fourier coefficients maps* $m_k : \mathcal{O}^\alpha \to (\mathcal{O}^\alpha)^{(k)}$ for $k \in \mathbb{Z}$ by setting

$$m_k(T) := \int_{S^1} z^{-k} \gamma_z(T) dz.$$

We have $\|m_k(T)\| \leqslant \|T\|$ and the restriction of m_k to $(\mathcal{O}^\alpha)^{(k)}$ is the identity.

Some remarks on this definition:

- The conditional expectation associated with the gauge action is simply m_0.
- The maps m_k send algebraic expressions to algebraic expressions.

Proof. Under Condition (C2′), the integer k_0 exists thus there is an isometry $\nu \in \mathscr{H}^{\otimes(N+k_0)}(\mathscr{H}^*)^{\otimes N} \cap \mathcal{O}^\alpha$. Considering ν^p and $(\nu^*)^p$ for $p \in \mathbb{N}$, this shows that for all multiple k of k_0, $(\mathcal{O}^\alpha)^{(k)}$ is nontrivial.

Conversely, if $k \leqslant \ell$ and both $(\mathcal{O}^\alpha)^{(k)}$ and $(\mathcal{O}^\alpha)^{(\ell)}$ are nontrivial, denoting u and v isometries in each of these spaces, vu^* and uv are nontrivial elements of $(\mathcal{O}^\alpha)^{(\ell-k)}$ and $(\mathcal{O}^\alpha)^{(\ell+k)}$, respectively. Indeed $v^*(vu^*)u = 1$ and $(uv)^*(uv) = 1$ since both u and v are isometries. This shows that the set of all k such that $(\mathcal{O}^\alpha)^{(k)}$ is nontrivial is a subgroup of $\mathbb{Z}$ and thus has the form $k_0\mathbb{Z}$.

The next step is to prove that (C2′) implies (C2). The first point of the lemma provides us with an integer k_0 and an isometry $\nu \in (\mathcal{O}^\alpha)^{(k_0)}$. We need to turn it into an algebraic element $\mu \in \mathscr{H}^{\otimes(N+k_0)}(\mathscr{H}^*)^{\otimes N} \cap \mathcal{O}^\alpha$

to recover (C2). Lemma 2.7 proves that there is an algebraic approximation $\nu_0 \in \mathcal{O}^\alpha$ of ν. We thus get an integer N large enough such that $\nu' := m_{k_0}(\nu_0) \in \mathscr{H}^{\otimes(N+k_0)}(\mathscr{H}^*)^{\otimes N}$, we can estimate:

$$\|\nu' - \nu\| = \|m_{k_0}(\nu_0 - \nu)\| \leqslant \|\nu_0 - \nu\|.$$

Consequently, $(\nu')^*\nu'$ is close to $\nu^*\nu = 1$ and for a suitable choice of ν_0, it is, therefore, invertible in $\mathscr{H}^{\otimes(N+k_0)}(\mathscr{H}^*)^{\otimes(N+k_0)} \cap \mathcal{O}^\alpha$ in which both $\nu^*\nu = 1$ and $(\nu')^*\nu'$ lie. Finally, setting $\mu = \nu'\big((\nu')^*\nu'\big)^{-1/2}$ yields the required isometry in $\mathscr{H}^{\otimes(N+k_0)}(\mathscr{H}^*)^{\otimes(N+k_0)} \cap \mathcal{O}^\alpha$. It remains to show that for all integers k, ℓ with $0 < k < k_0$, there are no nonzero intertwiners between $\alpha^{\otimes \ell}$ and $\alpha^{\otimes(\ell+k)}$, but if such intertwiners exist, Lemma 2.5 implies that $(\mathcal{O}^\alpha)^{(k)} \neq \{0\}$. This completes the proof of (C2$'$) $\implies$ (C2).

Assuming (C2), it appears that $(\mathcal{O}^\alpha)^{(k_0)}$ contains an algebraic isometry $\nu \in \mathscr{H}^{\otimes(N+k_0)}(\mathscr{H}^*)^{\otimes N} \cap \mathcal{O}^\alpha$. This proves that not all $(\mathcal{O}^\alpha)^{(k)}$ are trivial and that if k_0 divides k, then $(\mathcal{O}^\alpha)^{(k)}$ is nontrivial. It now suffices to prove that all other $(\mathcal{O}^\alpha)^{(k)}$ are trivial. By the same process as above, if $(\mathcal{O}^\alpha)^{(k)} \neq \{0\}$ then using the maps m_k, we can find ℓ large enough such that there is a nonzero algebraic intertwiner $T \in \mathscr{H}^{\otimes(\ell+k)}(\mathscr{H}^*)^{\otimes \ell} \cap \mathcal{O}^\alpha$. Since ν is an isometry,

$$\nu T \neq 0 \iff T \neq 0 \iff T\nu^* \neq 0.$$

Iterating this argument, we reduce the problem to the case of a nonzero algebraic intertwiner $T \in (\mathcal{O}^\alpha)^{(k)}$ for $0 < k < k_0$. But such map cannot exist according to Condition (C2) and, therefore, the equivalence is established. □

The previous conditions are related to those appearing in [29, Proposition 2.1] by the following:

Lemma 3.4. *If α satisfies Condition (C2), then*

- *There is an isometry $\nu \in \mathscr{H}^{\otimes(N+k_0)}(\mathscr{H}^*)^{\otimes N} \cap \mathcal{O}^\alpha$,*
- *the expression $\sigma(T) = \nu T \nu^*$ defines an injective corner endomorphism $\sigma : \mathcal{F} \to \mathcal{F}$. It restricts to an injective corner endomorphism of $\mathcal{F}^\alpha$, also denoted by σ.*

Proof. Condition (C2) imposes the existence of a norm preserving map $\psi : \mathscr{H}^N \to \mathscr{H}^{N+k_0}$ intertwining the $\mathbb{G}$-action. In particular, $\psi^*\psi = \mathrm{Id}_{\mathscr{H}^N}$. From Lemma 2.5, we see that we can interpret ψ as an element $\nu \in \mathscr{H}^{\otimes(N+k_0)}(\mathscr{H}^*)^{\otimes N} \cap \mathcal{O}^\alpha$. Moreover, $\nu^*\nu$ corresponds to $\psi^*\psi$ and thus $\nu^*\nu = 1$.

Regarding σ, notice first that ν has gauge k_0 thus $\sigma(T)$ as gauge 0, i.e. it is an element of $\mathcal{F}$. Since $\nu^*\nu = 1, \sigma$ is an injective $*$-endomorphism. Moreover, $\sigma(1) = \nu\nu^*$ thus for any $T \in \mathcal{F}^\alpha$,

$$\sigma(T) = \nu T \nu^* = \nu\nu^*\nu T \nu^*\nu\nu^* \in \sigma(1)\mathcal{F}^\alpha\sigma(1)$$

and the image of σ is precisely $\sigma(1)\mathcal{F}^\alpha\sigma(1)$.

Finally, $\nu \in \mathcal{O}^\alpha$ and thus if $T \in \mathcal{F}^\alpha$ then $\sigma(T) \in \mathcal{F}^\alpha$. □

Using this injective endomorphism, we can form the endomorphism crossed product $\mathcal{F}^\alpha \rtimes_\sigma \mathbb{N}$ as defined in [13, Section 2, p. 327]:

Definition 3.5. Given a C^*-algebra A and an injective endomorphism $\sigma : A \to A$, the *endomorphism crossed product* $A \rtimes_\sigma \mathbb{N}$ is the universal C^*-algebra generated by a copy of A and an isometry s such that $sas^* = \sigma(a)$ for all $a \in A$.

To define such a universal C^*-algebra, all generators must have finite bounds on their norms [4, II.8.3]. This property is satisfied for $a \in A \subseteq A \rtimes \mathbb{N}$ and s is required to be an isometry thus $\|s\| = 1$.

4. Simple and Purely Infinite Algebras

In this section, we prove that under Conditions (C1) and (C2), the algebras $\mathcal{O}^\alpha$ are simple and purely infinite.

Definition 4.1. A C^*-algebra C is called *simple and Purely Infinite* (PI) if for any nonzero $T \in C$ and any $\varepsilon > 0$, there are $A, B \in C$ s.t. $\|ATB - 1\| \leqslant \varepsilon$.

The above definition implies that C is simple. Indeed, when ATB is close enough to 1, it is invertible and thus we can find A', B' such that $A'TB' = 1$.

Proposition 4.2. *If α is a representation of $\mathbb{G}$ which satisfies (C1), then for any nonzero gauge-invariant projection $P \in \mathcal{O}^\alpha \cap \mathscr{H}^{\otimes \ell}(\mathscr{H}^*)^{\otimes \ell}$, we can find L large enough and $y \in \mathcal{O}^\alpha \cap \mathscr{H}^{\otimes L}$ s.t. $Py = y, y^*Py = 1$ and $y^*y = 1$.*

Proof. A projection $P \in \mathcal{O}^\alpha$ in $\mathscr{H}^{\otimes \ell}(\mathscr{H}^*)^{\otimes \ell}$ always defines a α-invariant Hilbert space in $\mathscr{H}^{\otimes \ell}$—and, therefore, a representation of $\mathbb{G}$. Indeed, the finite dimensional Hilbert space $\mathscr{K} := P\mathscr{H}^{\otimes \ell}$ satisfies:

$$\underline{\delta}(Px) = \underline{\delta}(P)\underline{\delta}(x) = (P \otimes 1)\underline{\delta}(x) \in \mathscr{K} \otimes C(\mathbb{G}), \tag{4.1}$$

thereby inducing a representation of $\mathbb{G}$ by restriction. By construction, the restriction of $\underline{\delta}$ to $\mathscr{K}$ has a decomposition into irreducible representations which only involves (subrepresentations of) iterated tensor products of α, i.e. elements of $\mathcal{T}_\alpha$. Pick one such irreducible representation $\beta \in \mathcal{T}_\alpha$. From Condition (C1), we get $q \in \mathbb{N}$ and β' in the decomposition of $\alpha^{\otimes q}$ such that $\beta \otimes \beta'$ possesses an invariant vector.

Using an argument similar to (4.1), it is easy to show that $\mathscr{K} \otimes \mathscr{H}^{\otimes q} \simeq P\mathscr{H}^{\otimes(\ell+q)}$ carries a representation of $\mathbb{G}$ induced from α. If we decompose this representation, using the tensor product $\mathscr{K} \otimes \mathscr{H}^{\otimes q}$ we find:

$$(\beta \oplus t) \otimes (\beta' \oplus u) \simeq (\beta \otimes \beta') \oplus (\beta \otimes u) \oplus (t \otimes \beta') \oplus (t \otimes u)$$

where t and u are sums of irreducible representations. Property (C1) thus implies that $\mathscr{K} \otimes \mathscr{H}^{\otimes q}$ contains a nonzero invariant vector.

Let us pick such an invariant vector $y \neq 0$ in $\mathscr{K} \otimes \mathscr{H}^{\otimes q} \simeq P\mathscr{H}^{\otimes(\ell+q)} \subseteq \mathscr{H}^{\otimes(\ell+q)}$. Without loss of generality, we can assume that $\|y\| = 1$, i.e. $y^*y = 1$. By construction, $\underline{\delta}(y) = y \otimes 1$ so $y \in \mathcal{O}^\alpha$. Since y is in $P\mathscr{H}^{\otimes(\ell+q)}$, we have $Py = y$. Hence

$$y^*y = 1 \qquad y^*Py = (Py)^*Py = y^*y = 1.$$

□

Proposition 4.3. *If α satisfies Condition (C1), then for any positive nonzero $T \in \mathcal{F}^\alpha$, we can find an integer L large enough and $z \in (\mathcal{O}^\alpha)^{(L)}$ s.t.*

$$z^*Tz = 1.$$

Proof. Given any nozero positive $T \in \mathcal{F}^\alpha$, we can find an ε-close positive $T_0 \in \mathcal{F}^{\alpha,\ell}$ with $T_0 \neq 0$ using Lemma 2.7. Since T_0 is a normal element of the finite dimensional algebra $\mathcal{F}^{\alpha,\ell}$, we can write it as a finite sum:

$$T_0 = \sum_j \lambda_j P_j,$$

where the P_j are its spectral projections and the λ_i are its eigenvalues. Pick λ_j, the largest (nonzero) eigenvalue and apply Proposition 4.2 to P_j to get $y \in \mathscr{H}^{\otimes L} \cap \mathcal{O}^\alpha$ s.t. $y^*P_jy = 1$. Since P_j is orthogonal to the other spectral projections, for $i \neq j$, we have:

$$y^*P_i = (P_jy)^*P_i = y^*P_jP_i = 0$$

and thus:

$$y^*T_0y = y^*\left(\sum_i \lambda_i P_i\right)y = \lambda_j y^*P_jy = \lambda_i 1.$$

We just have to renormalise y to get 1. For simplicity, we also denote by y the renormalised element of $(\mathcal{O}^\alpha)^{(L)}$, whose norm is $\|T_0\|^{-1/2}$. Since $\|T - T_0\| \leqslant \varepsilon$ we get

$$\|y^*Ty - 1\| = \|y^*Ty - y^*T_0y\| \leqslant (\|T\| - \varepsilon)^{-1/2}\varepsilon.$$

For ε small enough, y^*Ty is, therefore, an invertible positive element of $\mathcal{F}^\alpha$ and setting $z := y(y^*Ty)^{-1/2}$, we get $z \in (\mathcal{O}^\alpha)^{(L)}$ and $z^*Tz = 1$. $\square$

Corollary 4.4. *If α satisfies Conditions (C1) and (C2), then for each non-zero hereditary subalgebra B of $\mathcal{F}^\alpha$, there is a projection in B which is equivalent (in $\mathcal{F}^\alpha$) to $\sigma^L(1)$ for some $L \in \mathbb{N}$.*

In other words, for any nonzero positive $T \in \mathcal{F}^\alpha$, we can find $z' \in \mathcal{F}^\alpha$ and L large enough such that $z'T(z')^* = \sigma^L(1)$.

Proof. Using the same notations as in Proposition 4.3, the existence of the nonzero element $z \in (\mathcal{O}^\alpha)^{(L)}$ together with Lemma 3.2 ensures that k_0 divides L. Writing $L = pk_0$, we set $z' := \nu^p z^*$. This is gauge-invariant, i.e. an element of $\mathcal{F}^\alpha$ which satisfies $z'T(z')^* = \sigma^L(1)$. $\square$

To apply Theorem 2.1 of [13], it remains to prove that σ^m is outer for all $m \in \mathbb{N} \setminus \{0\}$. To fit our case within the setting of this article, we introduce the inductive system

$$\mathcal{F}^\alpha \xrightarrow{\sigma} \mathcal{F}^\alpha \xrightarrow{\sigma} \mathcal{F}^\alpha \xrightarrow{\sigma} \cdots$$

as well as its inductive limit $\overline{\mathcal{F}^\alpha} := \lim_\rightarrow \mathcal{F}^\alpha$. We denote by $\mu_n^\alpha : \mathcal{F}^\alpha \to \overline{\mathcal{F}^\alpha}$ the associated system of morphisms. An automorphism $\overline{\sigma}$ of $\overline{\mathcal{F}^\alpha}$ is defined by $\overline{\sigma}(\mu_n^\alpha(a)) = \mu_n^\alpha(\sigma(a)) = \mu_{n-1}^\alpha(a)$.

Lemma 4.5. *Under Condition (C2), for all* $m \in \mathbb{N} \setminus \{0\}$, *the automorphism* $\overline{\sigma}^m$ *is outer.*

Proof. Our proof of this fact will rely on the inductive system

$$\mathcal{F} \xrightarrow{\sigma} \mathcal{F} \xrightarrow{\sigma} \mathcal{F} \xrightarrow{\sigma} \cdots$$

whose inductive limit we denote by $\overline{\mathcal{F}}$. It comes with a system of morphisms denoted by $\mu_n : \mathcal{F} \to \overline{\mathcal{F}}$. There is a natural map $\overline{\varphi} : \overline{\mathcal{F}^\alpha} \to \overline{\mathcal{F}}$ defined via the natural inclusions $\varphi_n : \mathcal{F}^\alpha \to \mathcal{F}$ applied to the defining inductive systems:

$$\begin{tikzcd}
\mathcal{F}^\alpha \arrow[r, "\sigma"] \arrow[d, "\varphi_1"] & \mathcal{F}^\alpha \arrow[r, "\sigma"] \arrow[d, "\varphi_2"] & \mathcal{F}^\alpha \arrow[r, "\sigma"] \arrow[d, "\varphi_3"] & \cdots \arrow[r] & \overline{\mathcal{F}^\alpha} \arrow[d, "\overline{\varphi}"] \\
\mathcal{F} \arrow[r, "\sigma"] & \mathcal{F} \arrow[r, "\sigma"] & \mathcal{F} \arrow[r, "\sigma"] & \cdots \arrow[r] & \overline{\mathcal{F}}.
\end{tikzcd}$$

The squares appearing in the above diagram are all commutative. The map $\overline{\varphi}$ is characterised by $\overline{\varphi} \circ \mu_n^\alpha = \mu_n \circ \varphi_n$. To compute the K-theory of $\overline{\mathcal{F}}$, we rely on the continuity of K-theory (Theorem 6.3.2 p. 98 of [28]), together with an estimation of the action of σ_* on the K-theory of $\mathcal{F}$. It is well-known that for $\mathcal{F}$, i.e. the UHF algebra of type d^∞, $K_1(\mathcal{F}) = 0$ and $K_0(\mathcal{F})$ is isomorphic to $\mathbb{Z}\left[\frac{1}{d}\right]$ (see e.g. [4, V.1.1.16, p. 400]), with the isomorphism implemented by the unique normalised trace τ on $\mathcal{F}$: to a projection e we associate $\tau(e)$.

Since $T \mapsto \tau(\sigma(T))$ is a trace on $\mathcal{F}$, there must be a real constant ρ such that $\tau(\sigma(T)) = \rho\tau(T)$ for all T. Considering $T = 1$ leads to $\rho = \tau(\nu\nu^*)$. Since by definition ν is an isometry between spaces of dimensions d^N and d^{N+k_0}, we obtain $\rho = d^{-k_0} < 1$. At the level of K-theory, it means that $\sigma_*(K) = \rho K$.

As σ_* is an automorphism of $K_0(\mathcal{F})$, it appears that the inductive limit defining the K-theory of $\overline{\mathcal{F}}$ is simply $K_0(\overline{\mathcal{F}}) = \mathbb{Z}[1/d]$ equipped with the system of morphisms $(\mu_n)_*(K) = (\sigma_*)^{-(n-1)}(K)$ (compare e.g. [28, Definition 6.2.2, p. 92]). In particular, all maps $(\mu_n)_*$ are injective. Moreover, the action of $\overline{\sigma}_*$ on $K_0(\overline{\mathcal{F}})$ is $\overline{\sigma}_*(K) = \rho K$ and for $m \neq 0$ the morphism $\overline{\sigma}_*^m$ leaves no point fixed.

Now, if $\overline{\sigma}^m$ is an inner morphism of $\overline{\mathcal{F}^\alpha}$ for $m \neq 0$, then its action on the K-theory of $\overline{\mathcal{F}^\alpha}$ must be trivial. In particular, we must have $\overline{\varphi}_*(K_0(\overline{\mathcal{F}^\alpha})) \subseteq \ker(\mathrm{Id} - (\overline{\sigma}^m)_*)$. However, we know that for $m \neq 0, \ker(\mathrm{Id} - (\overline{\sigma}^m)_*) = \{0\}$ and, therefore, $\overline{\varphi}$ must vanish, which is false.

Indeed, taking the K-theory class $[1] \in K_0(\mathcal{F}^\alpha)$ the map φ_1 sends it to $[1] \in K_0(\mathcal{F})$. This K-theory class is nonzero in $K_0(\mathcal{F})$ and the map $(\mu_1)_* : K_0(\mathcal{F}) \to K_0(\overline{\mathcal{F}})$ is injective. The properties of inductive limits prove that

$$(\mu_1)_* \circ (\varphi_1)_*([1]) = \overline{\varphi}_* \circ (\mu_1^\alpha)_*([1]) \neq 0$$

and thus $\overline{\varphi}_*$ does not vanish. □

Theorem 4.6. *Let* α *be a representation of* $\mathbb{G}$ *which satisfies Conditions (C1) and (C2), the crossed product* $\mathcal{F}^\alpha \rtimes_\sigma \mathbb{N}$ *is simple and PI.*

Proof. Corollary 4.4 and Lemma 4.5 ensure that we can apply Theorem 2.1 of [13]. □

Corollary 4.7. *If α satisfies Conditions (C1) and (C2), the fixed point algebra $\mathcal{O}^\alpha$ is isomorphic to $\mathcal{F}^\alpha \rtimes_\sigma \mathbb{N}$ and, therefore, simple and purely infinite.*

Proof. Lemma 3.4 together with the universal property of $\mathcal{F}^\alpha \rtimes_\sigma \mathbb{N}$ appearing in Definition 3.5 ensure that there is a C^*-morphism $\Phi: \mathcal{F}^\alpha \rtimes_\sigma \mathbb{N} \to \mathcal{O}^\alpha$. By Theorem 4.6, the crossed product $\mathcal{F}^\alpha \rtimes_\sigma \mathbb{N}$ is simple and, therefore, Φ is injective.

To prove surjectivity, it suffices to prove that ν and $\mathcal{F}^\alpha$ generate $\mathcal{O}^\alpha$. This follows from the density of algebraic elements in $\mathcal{O}^\alpha$ (Lemma 2.7) and the observation that any gauge-homogeneous element can be written either $(\nu^*)^k a$ or $\nu^k a$ for some $k \in \mathbb{N}$ and $a \in \mathcal{F}^\alpha$. □

5. Computation of K-Groups

In this section, we compute the K-theory of $\mathcal{O}^\alpha$—which only depends on $\mathbb{G}$ up to R^+-isomorphism. Very analogous results for $\mathcal{F}^\alpha$ were first stated in the case of (ordinary) compact groups by Antony Wassermann in his PhD thesis [31] (see in particular III.3.(iii) p.103 and III.7. p.157). He also sketched a proof of the results for $\mathcal{O}^\alpha$. Here we give complete proofs of the results in the CQG setting.

To compute the K-theory of $\mathcal{O}^\alpha$, we start by describing the algebra $\mathcal{F}^\alpha$. By Lemma 2.7, this is an AF-algebra determined by $\lim_\to \mathcal{F}^{\alpha,\ell}$ and the associated Bratteli's diagram (see for instance [30, p. 13]). In particular, $K_1(\mathcal{F}^\alpha) = 0$. To evaluate $K_0(\mathcal{F}^\alpha)$, fix ℓ and consider the decomposition into irreducible representations of $\alpha^{\otimes \ell}$:

$$\alpha^{\otimes \ell} \simeq \bigoplus d_t(t), \tag{5.1}$$

where all irreducible representations (t) are in $\mathcal{T}_\alpha$ (by definition) and only a finite number of d_t are nonzero. From Lemma 2.5 together with the Schur's lemma for representations of $\mathbb{G}$, we get that $\mathcal{F}^{\alpha,\ell}$ is isomorphic to

$$\mathcal{F}^{\alpha,\ell} \simeq \bigoplus M_{d_t}(\mathbb{C}). \tag{5.2}$$

Hence, the different matrix components of $\mathcal{F}^\alpha$ are determined by the irreducible representations appearing in α^ℓ. The connecting maps $\varphi_\ell: \mathcal{F}^{\alpha,\ell} \to \mathcal{F}^{\alpha,\ell+1}$ are fully determined by the fusion rules $(t) \otimes \alpha$ for (t) appearing in (5.1). Indeed, if

$$(t) \otimes \alpha = m_1(\tau_1) \oplus \cdots \oplus m_p(\tau_p)$$

then the multiplicity of φ_ℓ between the (t) component of $\mathcal{F}^{\alpha,\ell}$ and the (τ_p) component of $\mathcal{F}^{\alpha,\ell+1}$ is m_p. For a more concrete illustration, refer to Sect. 7.1 of the present paper. Since K_0 is continuous, we recover $K_0(\mathcal{F}^\alpha) = \lim_\to K_0(\mathcal{F}^{\alpha,\ell})$ and thus

Proposition 5.1. *The K-theory of $\mathcal{F}^\alpha$ is fully determined by the fusion rules on $\mathbb{G}$. In other words, it only depends on $\mathbb{G}$ up to R^+-isomorphism (Definition 2.1).*

Remark 5.2. Of course, the K-theory of $\mathcal{F}^\alpha$ also depends on the choice of α. But if $\mathbb{G}$ and $\mathbb{G}'$ are R^+-isomorphic, then α is sent by the isomorphism to some representation α' of $\mathbb{G}'$, giving a meaning to the above proposition.

We have $K_0(\mathcal{F}^{\alpha,\ell}) \simeq \bigoplus_{t\in\mathcal{T}_\alpha^\ell} \mathbb{Z}$ where $\mathcal{T}_\alpha^\ell$ is the set of irreducible representations appearing in (5.1) with nonzero multiplicity. Hence, the elements of $K_0(\mathcal{F}^{\alpha,\ell})$ can be represented as formal sums $\sum_{t\in\mathcal{T}_\alpha^\ell} n_t(t)$. It is then natural to consider $\mathbb{Z}[\mathcal{T}_\alpha]$, the formal sums on $\mathcal{T}_\alpha$ with integer coefficients. It is obvious that if β and γ are irreducible representations in $\mathcal{T}_\alpha$, then the irreducible representations appearing in the decomposition of $\beta\otimes\gamma$ are also in $\mathcal{T}_\alpha$. Thus, there is a product in $\mathcal{T}_\alpha$ induced by the tensor product and $\mathbb{Z}[\mathcal{T}_\alpha]$ inherits a ring structure from it.

If the ring $\mathbb{Z}[\mathcal{T}_\alpha]$ is commutative—it is the case for compact groups and their R^+-deformations—further simplifications arise, involving the localised ring $\mathbb{Z}[\mathcal{T}_\alpha]\left[\frac{1}{\alpha}\right]$. The definition of this set as inductive limit, together with the expression of the connecting maps $K_0(\varphi_\ell): K_0(\mathcal{F}^{\alpha,\ell}) \to K_0(\mathcal{F}^{\alpha,\ell+1})$:

$$K_0(\varphi_\ell)\left(\sum n_t(t)\right) = \sum n_t\big((t)\otimes\alpha\big)$$

shows that $K_0(\mathcal{F}^\alpha)$ can be realised as a submodule of $\mathbb{Z}[\mathcal{T}_\alpha]\left[\frac{1}{\alpha}\right]$. More precisely (compare [31], p.103):

Remark 5.3. If the ring $\mathbb{Z}[\mathcal{T}_\alpha]$ is commutative, $K_0(\mathcal{F}^\alpha)$ is the set of all fractions $\left(\sum n_t(t)\right)/\alpha^{\otimes\ell}$ where all (t) in the sum appear with nonzero multiplicity in (5.1).

Going back to the general case, we can determine the K-theory of $\mathcal{O}^\alpha$:

Theorem 5.4. *If α satisfies Properties (C1) and (C2), the K-theory of $\mathcal{O}^\alpha$ is*

$$K_0(\mathcal{O}^\alpha) = \mathrm{Coker}(\mathrm{Id}-\sigma_*) \quad K_1(\mathcal{O}^\alpha) = \ker(\mathrm{Id}-\sigma_*), \tag{5.3}$$

where σ_* is the endomorphism of Lemma 3.4. If moreover $\mathbb{Z}[\mathcal{T}_\alpha]$ is an integral domain, then $K_1(\mathcal{O}^\alpha)=0$.

In any case, $K_*(\mathcal{O}^\alpha)$ only depends on $\mathbb{G}$ up to R^+-isomorphism, in the sense of Remark 5.2.

Proof. The algebra $\mathcal{F}^\alpha$ is unital and σ is a corner endomorphism of $\mathcal{F}^\alpha$ (Lemma 3.4), hence the K-theory of $\mathcal{F}^\alpha \rtimes_\sigma \mathbb{N}$ is determined by the six-term exact sequence (see Corollary 2.2 in [29]):

$$\begin{tikzcd} K_0(\mathcal{F}^\alpha) \arrow[r, "\mathrm{Id}-\sigma_*"] & K_0(\mathcal{F}^\alpha) \arrow[r, "\iota_*"] & K_0(\mathcal{O}^\alpha) \arrow[d, "\partial"] \\ K_1(\mathcal{O}^\alpha) \arrow[u, "\partial"] & K_1(\mathcal{F}^\alpha) \arrow[l, "\iota_*"'] & K_1(\mathcal{F}^\alpha), \arrow[l, "\mathrm{Id}-\sigma_*"'] \end{tikzcd} \tag{5.4}$$

where σ_* is the map in K-theory induced by functoriality from σ. Since $K_1(\mathcal{F}^\alpha)=0$, this sequence can actually be written:

$$0 \to K_1(\mathcal{O}^\alpha) \to K_0(\mathcal{F}^\alpha) \xrightarrow{1-\sigma_*} K_0(\mathcal{F}^\alpha) \to K_0(\mathcal{O}^\alpha) \to 0,$$

yielding the equalities (5.3).

To complete the computation, we describe the action of σ_* on $K_0(\mathcal{F}^{\alpha,\ell})$ for $\ell \geqslant N$—where N is the integer of Lemma 3.4. Given a projection $e = (e_{i,j}) \in M_n(\mathcal{F}^{\alpha,\ell})$ its image under σ_* is:

$$f = \sigma_*(e) = (\nu e_{i,j}\nu^*)_{i,j} \in M_n(\mathcal{F}^{\alpha,\ell+k_0}).$$

This element acts naturally on $\mathbb{C}^n \otimes \mathscr{H}^{\otimes(\ell+k_0)}$. We know from Condition (C2) that there is an injection $\psi : \mathscr{H}^{\otimes\ell} \to \mathscr{H}^{\otimes(\ell+k_0)}$ which preserves the $\mathbb{G}$-action. f is just the image of e by this injection. In particular, f decomposes into precisely the same number of components as e, with the same multiplicities. In other words, σ_* sends $\sum n_t(t) \in K_0(\mathcal{F}^{\alpha,\ell})$ to $\sum n_t(t) \in K_0(\mathcal{F}^{\alpha,\ell+k_0})$. Such a map exists because of Condition (C2): indeed, if the irreducible representation (t) appears in $\alpha^{\otimes\ell}$, then it also appears in $\alpha^{\otimes(\ell+k_0)}$.

The kernel of $1 - \sigma_*$ is characterised as all elements $x \in \mathbb{Z}[\mathcal{T}_\alpha]$ such that $x = \sigma_* x$ or equivalently $x\alpha^{\otimes k_0} = x$—where we chose a realisation of x inside some $\mathcal{F}^{\alpha,\ell}$ for ℓ large enough. When $\mathbb{Z}[\mathcal{T}_\alpha]$ is an integral domain—e.g. for compact connected Lie groups (see Corollary 2.8 p. 167 of [2]) and their R^+-deformations—the only solution to this equation is $x = 0$. Indeed $\alpha^{\otimes k_0} \neq \epsilon$ (trivial rep.) because of the condition $d > 1$ and thus in this case $K_1(\mathcal{O}^\alpha) = 0$. $\square$

6. Main Results and Bootstrap Class

We refer to [30] Definition 4.3.1 for the following:

Definition 6.1. A *Kirchberg algebra* is a PI, simple, nuclear and separable C^*-algebra.

Under some hypotheses, these algebras are fully classified by their K-theory, according to the Kirchberg–Phillips classification theorem. To make this assumption precise, we need the following definition, borrowed from Blackader [4, V.1.5.4]:

Definition 6.2. The (large) *bootstrap class* or Universal Coefficient Theorem (*UCT*) *class* is the smallest class $\mathscr{N}$ of separable nuclear C^*-algebras s.t.

(i) $\mathbb{C} \in \mathscr{N}$;
(ii) $\mathscr{N}$ is closed under inductive limit;
(iii) if $0 \to J \to A \to A/J \to 0$ is an exact sequence, and two C^*-algebras J, A or A/J are in $\mathscr{N}$, then so is the third;
(iv) $\mathscr{N}$ is closed under KK-equivalence.

We can now state Kirchberg–Phillips classification theorem, which first appeared in [16,19] (see also Theorem 8.4.1 (iv) p. 128 of [30]):

Theorem 6.3. *Let A and B be unital Kirchberg algebras in the UCT class $\mathscr{N}$. A and B are isomorphic if and only if there are isomorphisms*

$$\alpha_0 : K_0(A) \to K_0(B) \qquad \alpha_1 : K_1(A) \to K_1(B)$$

with $\alpha_0([1_A]_0) = [1_B]_0$. For each such pair of isomorphisms, there is an isomorphism $\varphi : A \to B$ with $K_0(\varphi) = \alpha_0$ and $K_1(\varphi) = \alpha_1$.

To apply the above Theorem to our situation, we need:

Lemma 6.4. *If α satisfies Conditions (C1) and (C2), the fixed point algebra $\mathcal{O}^\alpha$ is in the bootstrap class $\mathcal{N}$.*

Proof. Our proof relies on the techniques used in Sect. 4. First of all, as mentioned in [30], Section 3 p. 76, the crossed product $\mathcal{F}^\alpha \rtimes_\sigma \mathbb{N}$ is strongly Morita equivalent to the algebra $\overline{\mathcal{F}^\alpha} \rtimes_{\overline{\sigma}} \mathbb{Z}$. In this crossed product, $\overline{\mathcal{F}^\alpha}$ is an AF algebra because it is a countable limit of AF algebras (see [4] II.8.3.24 p. 167). Consequently, $\overline{\mathcal{F}^\alpha}$ is in the bootstrap class $\mathcal{N}$. As taking crossed products by $\mathbb{Z}$ leaves $\mathcal{N}$ invariant (see [4] V.1.5.4 p. 414), it follows that $\overline{\mathcal{F}^\alpha} \rtimes_{\overline{\sigma}} \mathbb{Z} \in \mathcal{N}$. Finally, strong Morita equivalences are KK-equivalences and, therefore, $\mathcal{O}^\alpha \simeq \mathcal{F}^\alpha \rtimes_\sigma \mathbb{N}$ (Corollary 4.7) is an element of $\mathcal{N}$. □

Theorem 6.5. *Let α be a representation of $\mathbb{G}$. If α satisfies Properties (C1) and (C2), then up to C^*-isomorphism, the fixed point algebra $\mathcal{O}^\alpha$ only depends on the R^+-isomorphism class of $\mathbb{G}$, in the sense of Remark 5.2.*

Proof. The result is immediate by combining Theorem 5.4, Lemma 2.10, Corollary 4.7 and Lemma 6.4 with Theorem 6.3. □

7. Discussion and Examples

In view of [11,12], the most interesting cases are the natural representations of $SU(N)$. In the following section, we show that Condition (C2) is satisfied for natural representations of $\mathbb{G} = SU(N)$ and more generally that our theory apply to these.

A crucial tool in applying our theory is the notion of *chain group* that we adapt from Baumgärtel and Lledó [3] (see Theorem 5.5 therein), where it was studied for (ordinary) compact groups:

Definition 7.1. The *chain group* $\mathfrak{C}(\mathbb{G})$ is defined as the set of equivalence classes $[t]$ of irreducible representations of $\mathbb{G}$ under the relation $\sim$, where $(t) \sim (t')$ if and only if we can find irreducible representations $(\tau_1), \dots, (\tau_n)$ s.t. (t) and (t') appear in the decomposition of the tensor product $(\tau_1) \otimes \cdots \otimes (\tau_n)$ into irreducible components. The tensor product induces a group structure on this set.

The proofs of Baumgärtel and Lledó [3] adapt seamlessly to the case of CQGs. The only difference is that in general $\mathfrak{C}(\mathbb{G})$ is no longer Abelian. The identity element of $\mathfrak{C}(\mathbb{G})$ is $e := [\epsilon]$, the class of the trivial representation. By construction, $\mathfrak{C}(\mathbb{G})$ only depends on the fusion rules of $\mathbb{G}$, i.e. on its R^+-isomorphism class.

If α is a single irreducible representation and (C1) is satisfied, then the *semigroup* generated in $\mathfrak{C}(\mathbb{G})$ by $[\alpha]$ is actually a group. This subgroup of $\mathfrak{C}(\mathbb{G})$ is then *finite* and *Abelian*. In the case of $\mathbb{G} = U(1)$, this property fails.

However, we can find two large classes on which (C1) is satisfied. First, we have semisimple Lie groups:

Proposition 7.2. *If $\mathbb{G}$ is a compact semisimple Lie group (or a R^+-deformation thereof), then it satisfies (C1) for any irreducible representation α.*

Remark 7.3. Under the hypothesis of the proposition, the chain group can be identified with the character group of the center of $\mathbb{G}$. The identification is defined by the restriction of the irreducible representation to the center of $\mathbb{G}$. This fact was first proved in [22, Theorem 3.1] (see also [3, (5.2) and Theorem 5.5]).

Proof. Without loss of generality, we assume that $\mathbb{G}$ is a compact semisimple Lie group. We are actually going to prove that for any representation β of $\mathbb{G}$, the tensor product $\beta^{\otimes N}$ contains the trivial representation ϵ, where N is the dimension of the representation β. Since $\mathbb{G}$ is a compact semisimple Lie group, it has a unique dimension 1 representation. If β acts on $\mathscr{H}$, $\beta^{\otimes N}$ induces a representation on $\bigwedge^N \mathscr{H} \simeq \mathbb{C}$. This concludes the proof. □

The second large class which satisfies Condition (C1) is finite groups:

Proposition 7.4. *If $\mathbb{G}$ is a finite group (or a R^+-deformation thereof), then it satisfies (C1) for any irreducible representation α.*

Remark 7.5. Property (C1) actually holds more generally for semisimple Hopf algebras over $\mathbb{C}$ as proved in the Theorem of paragraph 4.2 in [20].

Proof. For self-containment, we include a proof—which is similar to that of Proposition 7.2. Without loss of generality, we assume that $\mathbb{G}$ is a finite group. We start from any representation β and prove that some iterated tensor product $\beta^{\otimes N}$ contains the trivial representation.

If β acts on a n-dimensional Hilbert space $\mathscr{H}$, then it induces a dimension 1 representation on $\bigwedge^n \mathscr{H}$—which is included in $\beta^{\otimes n}$. Since this representation $\bigwedge^n \beta$ has dimension 1, for any $g \in \mathbb{G}$, we have:

$$(\wedge^n \beta(g))^{\otimes k} = \wedge^n \beta(g^k).$$

If we take $k = N$, the order of the group $\mathbb{G}$, then $(\bigwedge^n \beta)^{\otimes N}$ is the trivial representation. The result follows. □

For $d \geqslant 2$, we can describe the algebra associated with the natural representation of $\mathbb{G} = SU(d)$:

Proposition 7.6. *If $\mathbb{G} = SU(d)$ (or a R^+-deformation thereof) and $\alpha = \nu$ is the natural representation of $\mathbb{G}$ on $\mathbb{C}^d$, then $\mathcal{O}^\alpha$ is both a unital Kirchberg algebra in the UCT class $\mathscr{N}$ and an endomorphism crossed product whose K-theory is described by Theorem 5.4.*

Remark 7.7. This proposition complements the explicit description by generators of $\mathcal{O}^\alpha$ provided for this case by Paolucci (see [23, Lemma 7]).

Proof. Proposition 7.2 ensures that Condition (C1) is satisfied. If ν is the natural representation of $\mathbb{G} = SU(d)$, it is readily checked that $[\nu]$ is a generator of $\mathfrak{C}(\mathbb{G}) \simeq \mathbb{Z}/d\mathbb{Z}$. Consequently, $[\nu]$ has order d and which shows that for $0 < k < d$ and all $\ell \in \mathbb{N}$, $(\alpha^{\otimes \ell}, \alpha^{\otimes(\ell+k)}) = \{0\}$ since $[\nu]^k \neq e$ in $\mathfrak{C}(\mathbb{G})$.

Moreover, by definition of $\mathbb{G} = SU(d)$, the d-fold tensor product ν^d contains the trivial representation. This way, we obtain an inclusion $\alpha^{\otimes 0} \leqslant \alpha^{\otimes d}$ thus proving Condition (C2). The result follows from Theorem 5.4, Lemma 2.10, Corollary 4.7 and Lemma 6.4. $\square$

Our treatment of examples will rely on the following comparison of $\mathcal{O}^\alpha$ and $\mathcal{O}^{\alpha^{\otimes M}}$ for any integer M, which is also interesting per se:

Proposition 7.8. *Let α be a representation of a CQG $\mathbb{G}$.*

(i) *For any $M \geqslant 1$, there is an injection $\mathcal{O}^{\alpha^{\otimes M}} \to \mathcal{O}^\alpha$.*
(ii) *If all irreducible representations in α have the same nontrivial class in $\mathfrak{C}(\mathbb{G})$, which we denote $[\alpha] \neq e$, and M is the order of $[\alpha]$ inside $\mathfrak{C}(\mathbb{G})$, then $\mathcal{O}^{\alpha^{\otimes M}} \to \mathcal{O}^\alpha$ is surjective.*

Proof. Let us prove the injectivity: denote $d \in \mathbb{N}$ the dimension of the Hilbert space $\mathscr{H}$ underlying the representation α. By definition, $\mathcal{O}^\alpha \subseteq \mathcal{O}_d$ and $\mathcal{O}^{\alpha^{\otimes M}} \subseteq \mathcal{O}_{d^M}$. There is a C^*-algebra morphism $\mathcal{O}_{d^M} \to \mathcal{O}_d$ defined on generators by:

$$S_K \to S_{k_0} S_{k_1} S_{k_2} \ldots S_{k_{M-1}},$$

where $K \in \{1, 2, \ldots, d^M\}$ and $k_0, k_1, \ldots, k_{M-1} \in \{1, 2, \ldots, d\}$ are uniquely determined by $K = k_0 + k_1 d + k_2 d^2 + \cdots + k_{M-1} d^{M-1}$. Indeed, using the properties of the generators of $\mathcal{O}_d$:

$$\begin{aligned}(S_{k_0} S_{k_1} S_{k_2} \ldots S_{k_{M-1}})^* S_{k'_0} S_{k'_1} S_{k'_2} \ldots S_{k'_{M-1}} \\ = \delta_{k_0,k'_0} \delta_{k_1,k'_1} \delta_{k_2,k'_2} \ldots \delta_{k_{M-1},k'_{M-1}} 1\end{aligned}$$

and

$$\sum_{k_0,k_1,k_2,\ldots,k_{M-1}} S_{k_0} S_{k_1} S_{k_2} \ldots S_{k_{M-1}} (S_{k_0} S_{k_1} S_{k_2} \ldots S_{k_{M-1}})^* = 1.$$

The universal property of $\mathcal{O}_{d^M}$ then yields an injective morphism $\mathcal{O}_{d^M} \to \mathcal{O}_d$. Next step, we restrict and corestrict to a morphism $\mathcal{O}^{\alpha^{\otimes M}} \to \mathcal{O}^\alpha$.

To prove that the range of $\mathcal{O}^{\alpha^{\otimes M}} \to \mathcal{O}_d$ is included in $\mathcal{O}^\alpha$, we rely on Lemma 2.7. It thus suffices to prove that the image of any algebraic element $T \in \mathcal{O}^{\alpha^{\otimes M}}$ is in $\mathcal{O}^\alpha$. Without loss of generality, we can assume that T is gauge-homogeneous of total gauge k. We can then find integers k_0, k_1 s.t.

$$T \in \mathscr{H}_M^{\otimes k_0} (\mathscr{H}_M^*)^{\otimes k_1}$$

where the difference $k_0 - k_1 = k$ is the total gauge of T in $\mathcal{O}^{\alpha^{\otimes M}}$ and $\mathscr{H}_M$ is the M-fold tensor product (over $\mathbb{C}$) of Hilbert spaces $\mathscr{H}_M := \mathscr{H}^{\otimes M}$. This is the Hilbert space on which $\alpha^{\otimes M}$ is represented. In this setting, T is in $\mathcal{O}^{\alpha^{\otimes M}}$ if and only if it intertwines $\mathscr{H}_M^{\otimes k_1}$ endowed with $(\alpha^{\otimes M})^{\otimes k_1}$ and $\mathscr{H}_M^{\otimes k_0}$ with $(\alpha^{\otimes M})^{\otimes k_0}$ (Lemma 2.5). But identifying $(\mathscr{H}_M)^{\otimes k_i}$ with $\mathscr{H}^{\otimes M k_i}$ and $(\alpha^{\otimes M})^{\otimes k_i}$ with $\alpha^{\otimes M k_i}$, it is clear that T can be seen as element of $\mathcal{O}^\alpha$—thus the induced morphism $\mathcal{O}^{\alpha^{\otimes M}} \to \mathcal{O}^\alpha$ is well defined. Moreover, it is injective since $\mathcal{O}_{d^M} \to \mathcal{O}_d$ is.

Note that the choice of the morphism $\mathcal{O}_{d^M} \to \mathcal{O}_d$ dictates the explicit form of the identification of $\mathscr{H}_M$ with $\mathscr{H}^{\otimes M}$. Alternatively, (i) follows from Remark 2.8.

Further assuming that all irreducible representations in α have the same class $[\alpha] \neq e$ in $\mathfrak{C}(\mathbb{G})$ and that the order of $[\alpha]$ is M, we prove surjectivity using the chain group. It suffices to show that any algebraic element T of $\mathcal{O}^\alpha$ which is homogeneous with total gauge k is reached. For such a T, we can always find k_0 and k_1 large enough to have

$$T \in \mathscr{H}^{\otimes k_0}(\mathscr{H}^*)^{\otimes k_1}$$

with $k = k_0 - k_1$, total gauge of T. If such a nonzero intertwiner exists, then in particular $[\alpha]^{k_0} = [\alpha]^{k_1}$ i.e. $[\alpha]^{k_0 - k_1} = e$ in $\mathfrak{C}(\mathbb{G})$. By definition of M, this is equivalent to $k = k_0 - k_1$ being a multiple of M. Upon iterating the inclusion

$$\mathscr{H}^{\otimes k_0}(\mathscr{H}^*)^{\otimes k_1} \hookrightarrow \mathscr{H}^{\otimes (k_0+1)}(\mathscr{H}^*)^{\otimes (k_1+1)},$$

we can further assume that k_0 and thus k_1 are multiples of M. In this case, it is clear that $T \in \mathcal{O}^\alpha$ can be lifted to $T' \in \mathcal{O}^{\alpha^{\otimes M}}$. $\square$

7.1. Example: $SU_q(2)$

In this subsection, we perform detailed computations regarding the case of $\mathbb{G} = SU_q(2)$, when α is the natural representation of $\mathbb{G}$ on $\mathbb{C}^2$. This algebra of fixed points was described explicitly in [18] and [21]. Our results complements the previous ones by providing an identification up to C^*-isomorphism.

In the rest of this subsection, we denote the $n+1$-dimensional irreducible representations of $\mathbb{G}$ by (n) for $n \in \mathbb{N}$. Hence, (0) is the trivial representation, (1) is the natural representation... We distinguish between odd and even representations, depending on the parity of (n). This distinction corresponds to the chain group $\mathfrak{C}(SU(2)) = \mathbb{Z}/2\mathbb{Z}$ (see [3, 5.1.1, p. 794]). The tensor product of irreducible representations in $SU_q(2)$ is determined by the Clebsch–Gordan formula ([34, Theorem 5.11]):

$$(k) \otimes (\ell) = (|k - \ell|) \oplus (|k - \ell| + 2) \oplus \cdots \oplus (k + \ell). \tag{7.1}$$

Parity is "compatible" with (7.1) and (1) is odd, thus for all $\ell, (\alpha^{\otimes \ell}, \alpha^{\otimes(\ell+1)}) = \{0\}$ and since $(1) \otimes (1) = (0) \oplus (2)$ we get $\alpha^{\otimes 0} \leqslant \alpha^{\otimes 2}$. This proves that Condition (C2) is satisfied. Note that precisely the same argument would apply to any (finite) sum of odd representations of $SU(2)$.

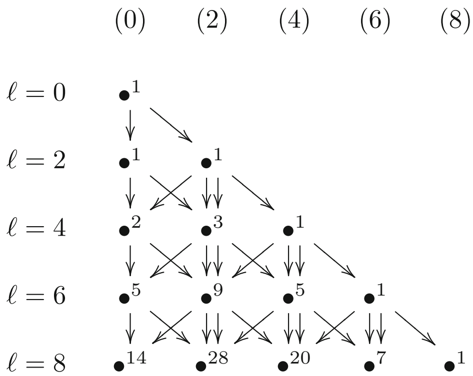
However, to compute the K-theory of $\mathcal{O}^\alpha$ in our case, it is more convenient to rely on Proposition 7.8 and reduce the problem to $\alpha = (1)^{\otimes 2} = (0) \oplus (2)$. As an explicit illustration of the method of Sect. 5, we compute the Bratteli diagram corresponding to $\mathcal{F}^\alpha = \lim_\to \mathcal{F}^{\alpha,\ell}$. Since α contains only even representations, no odd representation appears in $\mathcal{T}_\alpha$ (see Notation 3.1). Thus, there is no odd representation in the Bratteli diagram. Using (7.1), we can compute the iterated tensor powers $\alpha^{\otimes \ell}$:

$$\begin{aligned} \alpha^{\otimes 0} &= (0) & \alpha^{\otimes 2} &= 2.(0) \oplus 3.(2) \oplus (4) \\ \alpha^{\otimes 1} &= (0) \oplus (2) & \alpha^{\otimes 3} &= 5.(0) \oplus 9.(2) \oplus 5.(4) \oplus (6) \end{aligned}$$

From the above computations, we deduce as in (5.2) that

$$\mathcal{F}^{\alpha,0} = \mathbb{C} \qquad \mathcal{F}^{\alpha,2} = M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_4(\mathbb{C})$$
$$\mathcal{F}^{\alpha,1} = M_2(\mathbb{C}) \oplus \mathbb{C} \qquad \mathcal{F}^{\alpha,3} = M_5(\mathbb{C}) \oplus M_9(\mathbb{C}) \oplus M_5(\mathbb{C}) \oplus \mathbb{C}$$

Hence, iterated tensor powers and Bratteli diagram correspond to the same diagram:



It is well known that the representation ring of $SU(2)$ can be identified with $\mathbb{Z}[t]$ using $(n) \leftrightsquigarrow U_n(t/2)$ where U_n are the Chebyshev polynomials of the second kind. Explicitly

$$(0) \leftrightsquigarrow 1 \qquad (1) \leftrightsquigarrow t \qquad (2) \leftrightsquigarrow t^2 - 1 \qquad (3) \leftrightsquigarrow t^3 - 2t \qquad \ldots$$

One can easily check that the set $\mathcal{T}_\alpha$ actually contains all even representation. Moreover, the polynomials corresponding to even representations are even (as functions). Thus, we can change the variable and use $T := t^2$. In particular, $(0) \oplus (2) \leftrightsquigarrow T$.

From Remark 5.3, we see that $K_0(\mathcal{F}^\alpha)$ is a submodule of the fractions $\mathbb{Z}(T)$. It is easy to prove that $((0) \oplus (2))^{\otimes n}$ contains all irreducible representations $(2k)$ for $0 \leqslant k \leqslant n$. Since the polynomial corresponding to $(2k)$ has degree k in T and leading coefficient 1, we get:

Lemma 7.9. *$K_0(\mathcal{F}^\alpha)$ is the set of $\frac{P(T)}{T^n}$ where the polynomial $P(T) \in \mathbb{Z}[T]$ has degree at most n.*

We can now conclude:

Proposition 7.10. *When $\mathbb{G} = SU_q(2)$ and $\alpha = (1)$, the fixed point algebra $\mathcal{O}^\alpha$ is a Kirchberg algebra in the UCT class $\mathcal{N}$ whose K-theory is*

$$K_0(\mathcal{O}^\alpha) = \mathbb{Z} \qquad K_1(\mathcal{O}^\alpha) = 0.$$

Moreover, $[1_{\mathcal{O}^\alpha}]_0 = 1$ and, therefore, $\mathcal{O}^\alpha$ is C^*-isomorphic to the infinite Cuntz algebra $\mathcal{O}_\infty$.

Proof. From Proposition 7.6, we see that all conditions of Theorem 5.3 are satisfied. Thus, we get $K_1(\mathcal{O}^\alpha) = 0$ and $K_0(\mathcal{O}^\alpha) = \mathrm{Coker}(1 - \sigma_*)$. In our situation, it is readily checked that $\sigma_*\left(\frac{P(T)}{T^n}\right) = \frac{P(T)}{T^{n+1}}$. We are then left to evaluate the cokernel of

$$\frac{P(T)}{T^n} \mapsto \frac{P(T)(T-1)}{T^{n+1}}.$$

It is obvious from this expression that $\frac{Q(T)}{T^n}$ is in the image of $1-\sigma_*$ if and only if $Q(1)=0$. Thus, the cokernel is the image of the evaluation map $Q \mapsto Q(1)$, i.e. $\mathrm{Coker}(1-\sigma_*) = \mathbb{Z}$. It is also clear that with this identification, $[1_{\mathcal{O}^\alpha}]_0 = 1$. Applying Theorem 6.3 and known properties of $\mathcal{O}_\infty$, we get the result. □

7.2. Example: $SU_q(3)$

In this subsection, we use the notations and results presented in Wesslén [32]. The matrix Lie group $\mathbb{G} = SU(3)$ is simply connected and thus its representations correspond precisely to those of $\mathfrak{su}(3)$. We denote by (p,q) the representation with highest weight $p\lambda_1 + q\lambda_2$ where λ_i are the fundamental weights. In particular, the trivial representation ($\mathbf{1}$ in Physics notations) is $(0,0)$, the natural representation ($\mathbf{3}$) is $(1,0)$, its contragredient representation ($\overline{\mathbf{3}}$) is $(0,1)$. It follows from Wesslén [32] that:

Remark 7.11. For any irreducible representation β of $\mathbb{G} = SU(3)$, the tensor product $\beta^{\otimes 3}$ contains the trivial representation ϵ.

Therefore, if all irreducible representations in α have the same class $[\alpha] \neq e$ in $\mathfrak{C}(\mathbb{G})$, then $[\alpha]$ generates $\mathfrak{C}(\mathbb{G}) \simeq \mathbb{Z}/3\mathbb{Z}$ and all hypotheses of Proposition 7.8 are satisfied for $M = 3$. Moreover, from properties of the chain group $\mathfrak{C}(\mathbb{G})$ we see that for all $0 < k < 3$ and $\ell \in \mathbb{N}$, $(\alpha^{\otimes \ell}, \alpha^{\otimes(\ell+k)}) = \{0\}$. Therefore, Condition (C2) is satisfied by α.

However, just like in the previous example, the computations are more easily performed by relying on Proposition 7.8 and using $\alpha = (1,0)^{\otimes 3} = (0,0) \oplus 2.(1,1) \oplus (3,0)$. The representation ring of $\mathbb{G}$ is $\mathbb{Z}[\Lambda^1, \Lambda^2]$ where Λ^1 corresponds to $(1,0)$ and Λ^2 corresponds to $(0,1)$ (see for instance [2], Section 5 p.265). By construction, $\alpha^{\otimes 3}$ corresponds to $(\Lambda^1)^3$. Since $(0,1)$ is an irreducible component of $(1,0) \otimes (1,0)$, we have:

Lemma 7.12. *The group $K_0(\mathcal{F}^\alpha)$ is the set of $\frac{P(\Lambda^1,\Lambda^2)}{(\Lambda^1)^{3n}}$ where the polynomial $P(\Lambda^1,\Lambda^2) \in \mathbb{Z}[\Lambda^1,\Lambda^2]$ includes monomials of degree at most $3n$, Λ^1 counting as degree 1 and Λ^2 as degree 2, endowed with its usual addition.*

For instance, for $n = 1$, the possible polynomials are:

$$\frac{c_0 + c_1\Lambda^1 + c_2(\Lambda^1)^2 + c_3\Lambda^2 + c_4(\Lambda^1)^3 + c_5\Lambda^1\Lambda^2}{(\Lambda^1)^3}$$

where $c_0, \ldots, c_5 \in \mathbb{Z}$. Let us now prove:

Proposition 7.13. *When $\mathbb{G} = SU_q(3)$ and $\alpha = (1,0)$, the fixed point algebra $\mathcal{O}^\alpha$ is a Kirchberg algebra in the UCT class $\mathcal{N}$ whose K-theory is*

$$K_0(\mathcal{O}^\alpha) = \mathbb{Z}^3 \otimes \mathbb{Z}[\Lambda^2] \qquad\qquad K_1(\mathcal{O}^\alpha) = 0.$$

Proof. Proposition 7.6 and Theorem 6.5 apply, thus it suffices to treat the case of $\mathbb{G} = SU(3)$.

From Theorem 5.4, we get that $K_1(\mathcal{O}^\alpha) = 0$ and $K_0(\mathcal{O}^\alpha) = \mathrm{Coker}(1-\sigma_*)$. In our situation, it is readily checked that $\sigma_*\left(\frac{P(\Lambda^1,\Lambda^2)}{(\Lambda^1)^{3n}}\right) = \frac{P(\Lambda^1,\Lambda^2)}{(\Lambda^1)^{3(n+1)}}$. We are then left to evaluate the cokernel of

$$\frac{P(\Lambda^1,\Lambda^2)}{(\Lambda^1)^{3n}} \mapsto \frac{P(\Lambda^1,\Lambda^2)(1-(\Lambda^1)^3)}{(\Lambda^1)^{3(n+1)}}.$$

It is obvious from this expression that $\frac{Q(\Lambda^1,\Lambda^2)}{(\Lambda^1)^{3(n+1)}}$ is in the image of $1-\sigma_*$ if and only if

$$Q(1,\Lambda^2) = 0 \qquad Q(j,\Lambda^2) = 0 \qquad Q(j^2,\Lambda^2) = 0.$$

Thus, the cokernel is given by the evaluation map $\Phi : \mathbb{Z}[\Lambda^1,\Lambda^2] \to \mathbb{C}^3 \otimes \mathbb{Z}[\Lambda^2]$:

$$\Phi(Q) = \Big(Q(1,\Lambda^2), Q(j,\Lambda^2), Q(j^2,\Lambda^2)\Big).$$

Considering $Q(\Lambda^1,\Lambda^2) = (\Lambda^1)^p(\Lambda^2)^q$ for $p = 0,1,2$ and $q \in \mathbb{N}$, we see that the image of Φ is $\langle v_1, v_2, v_3\rangle \otimes \mathbb{Z}[\Lambda^2]$ where

$$v_1 = (1,1,1) \qquad v_2 = (1,j,j^2) \qquad v_3 = (1,j^2,j).$$

These vectors are linearly independent over $\mathbb{R}$ and thus over $\mathbb{Z}$. □

8. Final Remarks

We first comment on the case of finite groups:

Remark 8.1. For finite groups, there is another possibility to prove that the fixed point algebra is purely infinite. Indeed, if the finite group G acts by outer automorphisms, Lemma 10 of [17] applies, thereby proving that the crossed product $\mathcal{O}_d \rtimes G$ is simple and purely infinite. Now the unital fixed point algebra $\mathcal{O}_d^\alpha$ is isomorphic to a corner of $\mathcal{O}_d \rtimes G$ (see [4] II.10.4.18 and references therein) and thus $\mathcal{O}_d^\alpha$ is simple and purely infinite. It would be interesting to draw a more detailed comparison between the hypotheses of this argument and ours.

As a conclusion for this article, we discuss the consequences of our results for semisimple compact Lie groups and their R^+-deformations. We first recall the following Proposition 9.3 from [15, p.125]:

Proposition 8.2. *Let β be an irreducible representation of a semisimple compact Lie group G. Each irreducible representation of G occurs in some tensor power $\beta^{\otimes k}$ if and only if β is faithful.*

In particular, if G is simple, β is faithful if and only if the centre $Z(G)$ is faithfully represented on β.

The following first appeared as Definition 2.4 in [14] (see also [9]):

Definition 8.3. Given a CQG $\mathbb{G}$, an action $\underline{\delta} : A \to A \otimes C(\mathbb{G})$ on a C^*-algebra A is called *free* if $(A \otimes 1)\underline{\delta}(A)$ is dense in $A \otimes C(\mathbb{G})$.

Going back to our initial motivations concerning free actions, we prove:

Proposition 8.4. *If $\mathbb{G}$ is a semisimple compact Lie group (or a R^+-deformation thereof) and α is (a R^+-deformation of) a faithful representation of $\mathbb{G}$ on a d-dimensional Hilbert space $\mathscr{H}$, then the induced action $\underline{\delta}$ on $\mathcal{O}_d$ is free.*

Remark 8.5. This applies, in particular, to $\mathbb{G} = SU(d)$ and its natural representation on $\mathbb{C}^d$.

Proof. First assume that $\mathbb{G}$ is a semisimple compact Lie group. We can apply Proposition 8.2 to $\mathbb{G}$ and α. In this way, all irreducible representations of $\mathbb{G}$ appear in some tensor power $\alpha^{\otimes k}$ for k large enough. This is also true for any R^+-deformation of $\mathbb{G}$, since they share the same fusion rules. Thus for any irreducible representation defined by a matrix (β_{ij}), we can find an orthonormal family of vectors $v_j \in \mathscr{H}^{\otimes k}$ s.t.

$$\underline{\delta}(v_j) = \sum_i v_i \otimes \beta_{ij}.$$

Realising these vectors in $\mathcal{O}_d$, and picking any $T \in \mathcal{O}_d$ we get:

$$(Tv_i^* \otimes 1)\underline{\delta}(v_j) = (Tv_i^* \otimes 1)\left(\sum_k v_k \otimes \beta_{kj}\right) = T \otimes \beta_{ij}.$$

Relying on Theorem 1.2 of [36], it follows that $(\mathcal{O}_d \otimes 1)\underline{\delta}(\mathcal{O}_d \otimes 1)$ is dense in the tensor product $A \otimes C(\mathbb{G})$. □

We conclude with the following:

Remark 8.6. The stability result of Theorem 6.5 shows that the fixed point C^*-algebra is not a very fine invariant: indeed, it only "sees" the fusion rules of $\mathbb{G}$. Concretely, in the setting of the natural representation ν of $SU_q(d)$, we cannot retrieve the "q".

This situation is the exact opposite of Theorem 2 of [6]. In an algebraic version of the same setting, this result proves that the (algebraic) fixed point algebra characterises the (algebraic) quantum group. In other words, we can retrieve the "q" from the algebraic relations.

Acknowledgements

The author is deeply grateful to D. Bahns for her support and encouragement in general, and for our many discussions about this topic and her careful reading of this article in particular. He is indebted to J.-F. Planchat and J. Bichon for their remarks leading to Proposition 7.4 and Remark 7.5. He thanks J. Renault, R. Meyer, S. Albandik and the anonymous referees for providing references. The careful reviews and useful suggestions of the latters lead to significant improvements of the article, for which the author would like to express his gratitude.

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Communicated by Jean Bellissard.

Received: October 22, 2012.

Accepted: April 25, 2013.