Ann. Henri Poincaré 14 (2013), 1573–1597

1424-0637/13/061573-25
published online December 6, 2012
DOI 10.1007/s00023-012-0222-8

Annales Henri Poincaré

Effective Dynamics of an Electron Coupled to an External Potential in Non-relativistic QED

Volker Bach, Thomas Chen, Jérémy Faupin,
Jürg Fröhlich and Israel Michael Sigal

This paper is dedicated to the memory of Walter Hunziker—teacher and friend.

Abstract. In the framework of non-relativistic QED, we show that the renormalized mass of the electron (after having taken into account radiative corrections) appears as the kinematic mass in its response to an external potential force. Specifically, we study the dynamics of an electron in a slowly varying external potential and with slowly varying initial conditions and prove that, for a long time, it is accurately described by an associated effective dynamics of a Schrödinger electron in the same external potential and for the same initial data, with a kinetic energy operator determined by the renormalized dispersion law of the translation-invariant QED model.

1. Introduction

In this paper we show that the renormalized mass of the electron, taking into account radiative corrections due to its interaction with the quantized electromagnetic field, and the kinematic mass appearing in its response to a slowly varying external potential force are identical. Our analysis is carried out within the standard framework of non-relativistic quantum electrodynamics (QED). The renormalized electron mass, m_{ren}, is defined as the inverse curvature at zero momentum of the energy (dispersion law), $E(p)$, of a dressed electron as a function of its momentum p (no external potentials are present), i.e.,

The visit of J. Fröhlich at the Institute of Advanced Study in Princeton is supported by 'The Fund For Math' and 'The Monell Foundation'.

$m_{\mathrm{ren}} = E''(0)^{-1}$, while the kinematic mass of the electron enters the (effective) dynamical equations when it moves under the influence of an external potential force.

Our starting point is the dynamics generated by the Hamiltonian, H^V, describing a non-relativistic electron interacting with the quantized electromagnetic field and moving under the influence of a slowly varying potential, V_ϵ. We consider the time evolution of dressed one-electron states parameterized by wave functions $u_0^\epsilon \in H^1(\mathbb{R}^3)$, with $\|u_0^\epsilon\|_{L^2} = 1$ and $\|\nabla u_0^\epsilon\|_{L^2} \leq \epsilon^\kappa$, with $0 \leq \kappa < \frac{1}{3}$, and prove that their evolution is accurately approximated, during a long interval of time, by an effective Schrödinger dynamics generated by the one-particle Schrödinger operator

$$H_{\mathrm{eff}} := E(-i\nabla_x) + V_\epsilon(x)\,, \tag{1.1}$$

with kinetic energy given by the dispersion law $E(p)$. This result is in line with the general idea that any kind of physical dynamics is an effective dynamics that can ultimately be derived from a more fundamental theory. While results of a similar nature have been proven for quantum-mechanical particles interacting with *massive* bosons [26], ours is the first result covering electrons interacting with photons (or, more generally, *massless* bosons) and revealing effects of radiative corrections to the electron mass. Our derivation relies in an essential way on recent regularity results on the mass shell, i.e., the ground-state energy and the corresponding ground-state vector as a function of total momentum [9,10]. An interesting result on the effective dynamics of two heavy particles interacting via exchange of massless bosons has previously been obtained in [27]. We refer to [1,4,5,12,14–21,23,25] for further related works.

In the usual model of non-relativistic QED, the Hilbert space of states of a system consisting of a single electron and arbitrarily many photons (described in the Coulomb gauge) is given by

$$\mathcal{H} := L^2(\mathbb{R}^3) \otimes \mathfrak{F}\,, \tag{1.2}$$

where $L^2(\mathbb{R}^3)$ is the Hilbert space of square-integrable wave functions describing the electron degrees of freedom, (electron spin is neglected for notational convenience). The space $\mathfrak{F}$ is the Fock space of physical states of photons,

$$\mathfrak{F} := \bigoplus_{n \geq 0} \mathfrak{F}_n.$$

Here $\mathfrak{F}_n := \mathrm{Sym}(\, L^2(\mathbb{R}^3 \times \{+,-\}\,)\,)^{\otimes n}$ denotes the physical Hilbert space of states of n photons. The Hamiltonian acting on the space $\mathcal{H}$ is given by the expression

$$H^V := H + V_\epsilon \otimes \mathbf{1}_f\,, \tag{1.3}$$

where H is the generator of the dynamics of a single, freely moving non-relativistic electron minimally coupled to the quantized electromagnetic field, i.e.,

$$H := \frac{1}{2}(-i\nabla_x \otimes \mathbf{1}_f + \sqrt{\alpha}A(x))^2 + \mathbf{1}_{el} \otimes H_f\,, \tag{1.4}$$

and where $V_\epsilon(x) := V(\epsilon x)$ is a slowly varying potential, with $\epsilon > 0$ small; its precise properties are formulated in Theorem 1.1 below. Furthermore,

$$A(x) := \sum_\lambda \int_{|k|\le 1} \frac{\mathrm{d}k}{|k|^{1/2}} \{\epsilon_\lambda(k)\, e^{ikx} \otimes a_\lambda(k) + h.c.\} \tag{1.5}$$

denotes the quantized electromagnetic vector potential in the Coulomb gauge with an ultraviolet cutoff imposed, $|k| \le 1$, and

$$H_f \ := \sum_\lambda \int \mathrm{d}k\, |k|\, a_\lambda^*(k)\, a_\lambda(k) \tag{1.6}$$

is the photon Hamiltonian. In Eqs. (1.5) and (1.6), $a_\lambda^*(k), a_\lambda(k)$ are the usual photon creation- and annihilation operators, $\lambda = \pm$ indicates photon helicity, and $\epsilon_\lambda(k)$ is a polarization vector perpendicular to k corresponding to helicity λ. We note that all results in this paper hold for sufficiently small values of the fine structure constant, $0 < \alpha \ll 1$.

We observe that the Hamiltonian H is translation-invariant, in the sense that H commutes with translations, $T_y : \Psi(x) \to e^{iy\cdot P_f}\Psi(x+y)$, for $y \in \mathbb{R}^3$, where $P_f := \sum_\lambda \int \mathrm{d}k\, k\, a_\lambda^*(k) a_\lambda(k)$ is the momentum operator of the quantized radiation field. Hence H commutes with the total momentum operator

$$P_{tot} \ := \ -i\nabla_x \otimes \mathbf{1}_f \ + \ \mathbf{1}_{el} \otimes P_f, \tag{1.7}$$

of the electron and the photon field: $[H, P_{tot}] = 0$. It follows that H can be decomposed as a direct integral

$$UHU^{-1} = \int_{\mathbb{R}^3}^{\oplus} H(p)\mathrm{d}p, \tag{1.8}$$

of fiber operators, $H(p)$, over the spectrum of P_{tot}, where $H(p)$ is defined on the fiber space $\mathcal{H}_p \cong \mathfrak{F}$ in the direct integral decomposition, $\mathcal{H} \cong \int_{\mathbb{R}^3}^{\oplus} \mathrm{d}p\, \mathcal{H}_p$, of $\mathcal{H}$. The operator $U : \mathcal{H} \to \int^{\oplus} \mathrm{d}p\, \mathcal{H}_p$ is a generalized Fourier transform defined on smooth, rapidly decaying functions,

$$(U\Psi)(p) := (Fe^{iP_f\cdot x}\Psi)(p) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i(p-P_f)\cdot x}\Psi(x)\mathrm{d}x, \tag{1.9}$$

where F is the standard Fourier transform for Hilbert space-valued functions,

$$(F\Psi)(p) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ip\cdot x}\Psi(x)\mathrm{d}x.$$

For smooth, rapidly decaying vector-valued functions $\Phi(p) \in \mathcal{H}$, its inverse is given by

$$(U^{-1}\Phi)(x) := e^{-iP_f\cdot x}(F^{-1}\Phi)(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ix\cdot(p-P_f)}\Phi(p)\mathrm{d}p. \tag{1.10}$$

We note that

$$(UH\Psi)(p) = H(p)(U\Psi)(p), \quad (UP_{tot}\psi)(p) \ = \ p\,(U\psi)(p). \tag{1.11}$$

Since U is the composition of two unitary operators, $e^{iP_f \cdot x}$ and the standard Fourier transform F, it is unitary, too, and Eq. (1.10) defines its inverse.

We define creation- and annihilation operators, $b^*_\lambda(k)$ and $b_\lambda(k)$, on the fiber spaces $\mathcal{H}_p$ by

$$b_\lambda(k) := U e^{ikx} a_\lambda(k) U^{-1}, \quad b^*_\lambda(k) := U e^{-ikx} a^*_\lambda(k) U^{-1}\,, \tag{1.12}$$

i.e.,

$$\begin{aligned} (U e^{ikx} a_\lambda(k)\Psi)(p) &= b_\lambda(k)(U\Psi)(p) \\ (U e^{-ikx} a^*_\lambda(k)\Psi)(p) &= b^*_\lambda(k)(U\Psi)(p), \end{aligned} \tag{1.13}$$

for $\Psi \in \mathcal{H}$. Obviously, the operator-valued distributions $b_\lambda(k)$ and $b^*_\lambda(k)$ commute with P_{tot}. Thus, the operators $b^{(*)}_\lambda(f) := \int b^{(*)}_\lambda(k)\, \widehat{f}(k) \mathrm{d}k$ map the fiber spaces $\mathcal{H}_p$ to themselves for any test function f. The fact that these operators satisfy the usual canonical commutation relations is obvious. The Fock space constructed from the operators $b^{(*)}_\lambda(f)$, $f \in L^2(\mathbb{R}^3 \times \{+,-\})$, and the vacuum vector Ω is denoted by $\mathfrak{F}^b$.

From abstract theory, the fiber operators $H(p), p \in \mathbb{R}^3$, are nonnegative self-adjoint operators acting on $\mathcal{H}_p \cong \mathfrak{F}^b$. Their explicit form is determined in the next section. We define $E(p) = \inf \mathrm{spec} H(p)$, for all $p \in \mathbb{R}^3$, and

$$\mathcal{S} := \left\{ p \in \mathbb{R}^3 \,\middle|\, |p| \le \frac{1}{3} \right\}. \tag{1.14}$$

Making use of approximate ground states, $\Phi^\rho(p), \rho > 0$, (dressed by a cloud of soft photons with frequencies below ρ) of the operators $H(p)$, which will be defined in (2.14), we introduce a family of maps $\mathcal{J}^\rho_0 : L^2(\mathbb{R}^3) \mapsto \mathcal{H}$, from the space $L^2(\mathbb{R}^3)$ of square-integrable one-particle wave functions, u, to a subspace of *dressed one-electron states*, $\widehat{u}\, \Phi^\rho$, as

$$\begin{aligned} \mathcal{J}^\rho_0(u)(x) &:= (U^{-1}\, \chi_{\mathcal{S}_\mu}\, \widehat{u}\, \Phi^\rho)(x) \\ &= (2\pi)^{-3/2} \int \mathrm{d}p\, \widehat{u}(p)\, e^{ix(p-P_f)}\, \chi_{\mathcal{S}_\mu}(p)\, \Phi^\rho(p), \end{aligned} \tag{1.15}$$

where $\chi_{\mathcal{S}_\mu}$ is a smooth approximate characteristic function of the set $S_\mu := (1-\mu)\mathcal{S} \subset \mathcal{S} \subset \mathbb{R}^3$, $(0 < \mu < 1)$.

In this paper we study the time evolution of one-electron states, $\mathcal{J}^\rho_0(u^\epsilon_0)$, where u^ϵ_0 is a slowly varying one-particle wave function, dressed by an *infrared cloud* of photons with frequencies below ρ. More precisely, we study solutions of the Schrödinger equation

$$i\partial_t \Psi(t) = H^V\, \Psi(t)\,, \quad \text{with } \Psi(0) = \mathcal{J}^\rho_0(u^\epsilon_0). \tag{1.16}$$

The key idea is to relate the solution $\Psi(t) = e^{-itH^V} \mathcal{J}^\rho_0(u^\epsilon_0)$ of this Schrödinger equation to the solution of the Schrödinger equation

$$i\partial_t u^\epsilon_t = H_{\mathrm{eff}}\, u^\epsilon_t, \quad \text{with } u^\epsilon_{t=0} = u^\epsilon_0, \tag{1.17}$$

corresponding to the one-particle Schrödinger operator (1.1), where we recall that $H_{\mathrm{eff}} = E(-i\nabla_x) + V_\epsilon(x)$, with $E(p)$ as defined above. We consider the comparison state

$$\mathcal{J}_0^\rho(u_t^\epsilon) \in \mathcal{H}, \tag{1.18}$$

where $u_t^\epsilon := e^{-itH_{\rm eff}} u_0^\epsilon$ is the solution of (1.17) and show that $\Psi(t)$ remains close to $\mathcal{J}_0^\rho(u_t^\epsilon)$, for a long time. The choice of initial data satisfying

$$\|u_0^\epsilon\|_{L^2(\mathbb{R}^3)} = 1 \quad \text{and} \quad \|\nabla u_0^\epsilon\|_{L^2(\mathbb{R}^3)} \leq \epsilon^\kappa, \;\; 0 \leq \kappa < \frac{1}{3}, \tag{1.19}$$

guarantees that $\widehat{u}_t^\epsilon$ remains concentrated in $\mathcal{S}$ during the time scales relevant for this problem, provided the support of $\widehat{u}_0^\epsilon$ is contained in $\mathcal{S}$.

Theorem 1.1. *Let $0 < \epsilon < 1/3, 0 \leq \kappa < 1/3$ and assume that $u_0^\epsilon \in L^2(\mathbb{R}^3)$ obeys (1.19). Assume, furthermore, that $V \in L^\infty(\mathbb{R}^3;\mathbb{R})$ is such that $\widehat{V} \in L^1(\mathbb{R}^3)$ and that $\widehat{V}$ is supported in the unit ball,*

$$\operatorname{supp}(\widehat{V}) \subset \big\{k \in \mathbb{R}^3 \,|\, |k| \leq 1\big\}. \tag{1.20}$$

Let $0 < \delta < 2(\frac{1}{3} - \kappa)$, and choose $\rho = \rho_\epsilon := \epsilon^{\frac{2}{3}-\delta}$.

Then there exists $0 < \alpha_\delta \ll 1$ such that, for all $0 \leq \alpha \leq \alpha_\delta$, the bound

$$\|e^{-itH^V} \mathcal{J}_0^{\rho_\epsilon}(u_0^\epsilon) - \mathcal{J}_0^{\rho_\epsilon}(e^{-itH_{\rm eff}} u_0^\epsilon)\|_{\mathcal{H}} \leq C_\delta \big(\epsilon^{\frac{1}{3}-\frac{\delta}{2}+\kappa}\, t + \epsilon^{\frac{4}{3}-\frac{\delta}{2}} t^2\big)\,, \tag{1.21}$$

holds for all times $t \geq 0$. In particular, for all $0 \leq t \leq \epsilon^{-2/3}$, we have that

$$\|e^{-itH^V} \mathcal{J}_0^{\rho_\epsilon}(u_0^\epsilon) - \mathcal{J}_0^{\rho_\epsilon}(e^{-itH_{\rm eff}} u_0^\epsilon)\|_{\mathcal{H}} \leq C_\delta \epsilon^{\frac{1}{3}-\frac{\delta}{2}+\kappa}\, t. \tag{1.22}$$

Remark 1.2. We note that for this result, the regularity properties of the dressed electron states are crucial, as described in (1.30), below.

Remark 1.3. Theorem 1.1 implies that, for all $\delta' > 0$ such that $\delta' < \frac{1}{3} - \frac{\delta}{2} + \kappa$

$$\|e^{-itH^V} \mathcal{J}_0^{\rho_\epsilon}(u_0^\epsilon) - \mathcal{J}_0^{\rho_\epsilon}(e^{-itH_{\rm eff}} u_0^\epsilon)\|_{\mathcal{H}} \leq C_\delta\, \epsilon^{\delta'} \tag{1.23}$$

holds for all times t with $0 \leq t \leq \epsilon^{-(\frac{1}{3}-\frac{\delta}{2}+\kappa)+\delta'}$.

Remark 1.4. The initial conditions in Theorem 1.1 are chosen such that the initial momentum is $O(\epsilon^\kappa)$. The conditions on the external potential imply that the expected force, and, thus, the acceleration, is of order $O(\epsilon)$. Hence, at time t, the momentum is of order $O(\epsilon^\kappa) + O(\epsilon t)$, and therefore the action, $E(p)t - E(0)t \approx \frac{1}{2m_{ren}} p^2 t$, is of order $O(\epsilon^{2\kappa} t) + O(\epsilon^2 t^3)$. Hence, if $\frac{1}{3} - \kappa > \frac{\delta}{2}$ and $t \leq \epsilon^{-1+\kappa}$, then this term is much larger than the error term in Eq. (1.21).

To make this remark more precise, we define the operator $\widetilde{H}_{\rm eff} := E(0) + V(\epsilon x)$, and consider the difference between $e^{-itH_{\rm eff}}$ and $e^{-it\widetilde{H}_{\rm eff}}$. We write $e^{-itH_{\rm eff}} - e^{-it\widetilde{H}_{\rm eff}}$ as the integral of a derivative,

$$e^{-itH_{\rm eff}} - e^{-it\widetilde{H}_{\rm eff}} = -i \int_0^t \mathrm{d}s\, e^{-i(t-s)H_{\rm eff}}\, (E(p) - E(0)) e^{-is\widetilde{H}_{\rm eff}}\,, \tag{1.24}$$

and use that $E'(0) = 0$ so that $cp^2 < E(p) - E(0) = \frac{1}{2m_{\rm ren}} p^2(1 + o(1)) < Cp^2$ (see Proposition 2.1, below). Then, using

$$e^{is\widetilde{H}_{\rm eff}} p^2 e^{-is\widetilde{H}_{\rm eff}} = p^2 + 2\epsilon\, p \cdot (\nabla V)(\epsilon x) s + \epsilon^2 \Delta V(\epsilon x) s^2\,, \tag{1.25}$$

we find that

$$e^{-itH_{\text{eff}}} - e^{-it\widetilde{H}_{\text{eff}}} = A_t + O(\epsilon t^2 p) + O(\epsilon^2 t^3), \tag{1.26}$$

where $A_t = O(tp^2)$. Adding the second and third terms on the r.h.s. of (1.26) to the error estimated by (1.22), we observe that

$$\begin{aligned} O(t\epsilon^{\frac{1}{3}-\frac{\delta}{2}+\kappa}) + O(\epsilon t^2 p) + O(\epsilon^2 t^3) &= O\big[t(\epsilon^{\frac{1}{3}-\frac{\delta}{2}+\kappa} + \epsilon^{\frac{1}{3}+\kappa} + \epsilon^{\frac{2}{3}})\big] \\ &= O(t\epsilon^{\frac{1}{3}-\frac{\delta}{2}+\kappa}), \end{aligned} \tag{1.27}$$

provided that $0 < t \leq \epsilon^{-2/3}$. Assuming that A_t is not only bounded above by $O(tp^2)$, but is actually of order

$$\|A_t\, u_0^\epsilon\| \geq C\, t\, \epsilon^{2\kappa}\,, \tag{1.28}$$

with $C \equiv C(u_0^\epsilon, m_{\text{ren}}) > 0$ depending on the initial data and on the renormalized mass m_{ren}, we can compare this contribution to (1.27) and observe that

$$\frac{\|e^{-itH^V} \mathcal{J}_0^{\rho_\epsilon}(u_0^\epsilon) - \mathcal{J}_0^{\rho_\epsilon}(e^{-it\widetilde{H}_{\text{eff}}}\, u_0^\epsilon) - \mathcal{J}_0^{\rho_\epsilon}(A_t u_0^\epsilon)\|_{\mathcal{H}}}{\|A_t u_0^\epsilon\|} \leq O(\epsilon^{\frac{1}{3}-\frac{\delta}{2}-\kappa}) \tag{1.29}$$

provided $\epsilon^{-2\kappa} \leq t \leq \epsilon^{-2/3}$. Thus our estimate allows us to separate the main contribution of the dynamics from the error terms on a suitable time scale.

1.1. Outline of Proof Strategy

To prove Theorem 1.1, we introduce an *infrared regularized* version of the model defined by (1.3), (1.4), obtained by restricting the integration domain in the quantized electromagnetic vector potential (1.5) to the region $\{\sigma \leq |k| \leq 1\}$, for an arbitrary infrared cutoff $\sigma > 0$. Thereby, we obtain infrared regularized Hamiltonians H_σ^V and H_σ, as well as an infrared regularized family of maps $\mathcal{J}_\sigma^\rho$ corresponding to $\mathcal{J}_0^\rho$.

We note that, unlike $H(p)$, the infrared cut-off fiber Hamiltonian $H_\sigma(p)$ has a ground-state $\Psi_\sigma(p) \in \mathcal{H}_p \cong \mathfrak{F}$, for every $p \in \mathcal{S}$ and for $\sigma > 0$, but $\Psi_\sigma(p)$ does not possess a limit in $\mathcal{H}_p \cong \mathfrak{F}$, as $\sigma \searrow 0$, when $p \neq 0$. In particular, we expect that the number of photons in the state $\Psi_\sigma(p)$ diverges, as $\sigma \searrow 0$, (thus the lack of convergence of $\Psi_\sigma(p)$ in $\mathfrak{F}$). This is a well-known aspect of the *infrared problem* in QED [8–11,22]. It is remedied by applying a dressing transformation, $W_{\nabla E_\sigma(p)}^{\sigma,\rho}$, defined in (2.14), below, to $\Psi_\sigma(p)$, where $E_\sigma(p) = \inf \operatorname{spec} H_\sigma(p)$. The resulting vector, $\Phi_\sigma^\rho(p) := W_{\nabla E_\sigma(p)}^{\sigma,\rho} \Psi_\sigma(p)$, describes an *infraparticle* (or *dressed electron) state* containing infrared photons with frequencies in $[\sigma, \rho]$. As $\sigma \searrow 0$, the limit

$$\Phi^\rho(p) = \lim_{\sigma \to 0} \Phi_\sigma^\rho(p) \tag{1.30}$$

exists in $\mathfrak{F}$, for all $p \in \mathcal{S}$; see Proposition 2.2. This allows us to construct the map $\mathcal{J}_0^\rho$ as the limit of the maps $\mathcal{J}_\sigma^\rho$, as $\sigma \searrow 0$. Note that, while $\Psi_\sigma(p)$ does not converge in $\mathfrak{F}$ as $\sigma \searrow 0$ when $p \neq 0$, we have that $\lim_{\sigma \searrow 0} E_\sigma(p) = E(p)$.

We note that $\Phi_\sigma^\rho(p)$ is the ground-state eigenvector of the fiber Hamiltonian

$$K_\sigma^\rho(p) := W_{\nabla E_\sigma(p)}^{\sigma,\rho}\, H_\sigma(p)\, (W_{\nabla E_\sigma(p)}^{\sigma,\rho})^* \tag{1.31}$$

which is obtained by applying to $H_\sigma(p)$ the *Bogoliubov transformation* corresponding to the dressing transformation $W^{\sigma,\rho}_{\nabla E_\sigma(p)}$.

In Theorem 2.3, below, we prove that an estimate similar to (1.21) is satisfied for the infrared regularized model, namely

$$\begin{aligned}&\|e^{-itH^V_\sigma}\mathcal{J}^\rho_\sigma(u^\epsilon_0)-\mathcal{J}^\rho_\sigma(e^{-itH_{\mathrm{eff},\sigma}}u^\epsilon_0)\|_{\mathcal{H}}\\&\quad\leq C_\delta(1+\ln(\rho^{-1}))\epsilon^{\frac{2}{3}-\delta}t+C\,\alpha^{\frac{1}{2}}\,\rho^{\frac{1}{2}}t(\epsilon^\kappa+\epsilon t)\,,\end{aligned}\tag{1.32}$$

holds uniformly in the infrared cutoff σ and the cut-off $\rho>\sigma$. This result crucially uses the regularity properties of the dressed electron states $\Phi^\rho_\sigma(p)$, which allow us to take advantage of the fact that V_ϵ is slowly varying. An additional key ingredient is the bound $\|(H_\sigma(p)-K^\rho_\sigma(p))\Phi^\rho_\sigma(p)\|_{\mathfrak{F}}\leq C\alpha^{\frac{1}{2}}\,\rho^{\frac{1}{2}}|p|$, for $p\in\mathcal{S}$, proven in Appendix A. In (1.32) we take $\rho=\rho_\epsilon:=\epsilon^{\frac{2}{3}-\delta}$ and absorb $\ln(\rho^{-1})$ into $\epsilon^{\frac{2}{3}-\delta}$.

In Sect. 3, we control the limit $\sigma\searrow 0$, thus concluding the proof of Theorem 1.1. This requires control of the radiation emitted by the electron due to its acceleration in the external potential V_ϵ, in the limit $\sigma\searrow 0$.

2. Infrared Cut-off and Construction of $\Phi^\rho(p)$

As noted in the introduction, we analyze the original dynamics by first imposing an infrared (IR) cut-off, and controlling the dynamics generated by the resulting Hamiltonian. Thus, we define the IR regularized Hamiltonian

$$H^V_\sigma=H_\sigma+V_\epsilon(x)\otimes\mathbf{1}_f\,,\tag{2.1}$$

where

$$H_\sigma:=\frac{1}{2}\big(-i\nabla_x\otimes\mathbf{1}_f+\sqrt{\alpha}A_\sigma(x)\,\big)^2+\mathbf{1}_{el}\otimes H_f\tag{2.2}$$

is the generator of the dynamics of a single, freely moving non-relativistic electron minimally coupled to the electromagnetic radiation field. In (2.2),

$$A_\sigma(x)=\sum_\lambda\int_{\sigma\leq|k|\leq 1}\frac{\mathrm{d}k}{|k|^{1/2}}\,\{\epsilon_\lambda(k)\,e^{ikx}\otimes a_\lambda(k)+h.c.\}\tag{2.3}$$

denotes the quantized electromagnetic vector potential with an infrared and ultraviolet cutoff corresponding to $\sigma\leq|k|\leq 1$. Since $V\in L^\infty(\mathbb{R}^3)$ is a bounded operator, $D(H^V_\sigma)=D(H_\sigma)=D(-\Delta_x\otimes\mathbf{1}_f+\mathbf{1}_{el}\otimes H_f)$. The results in this paper are proven for sufficiently small values of the finestructure constant, $0<\alpha\ll 1$.

The Hamiltonian H_σ is also translation invariant and, similarly to H, can be represented as the fiber integral

$$UH_\sigma U^{-1}=\int^\oplus_{\mathbb{R}^3}H_\sigma(p)\mathrm{d}p,\tag{2.4}$$

over the spectrum of P_{tot}, defined on the fiber integral $\int^{\oplus} \mathrm{d}p\, \mathcal{H}_p$, with fibers $\mathcal{H}_p \cong \mathfrak{F}^b$. The decomposition (2.4) is equivalent to

$$(UH_\sigma\Psi)(p) = H_\sigma(p)(U\Psi)(p). \tag{2.5}$$

Again, by abstract theory, the fiber Hamiltonians $H_\sigma(p), p \in \mathbb{R}^3$, are self-adjoint operators on $\mathcal{H}_p \cong \mathfrak{F}^b$. Written in terms of the creation- and annihilation operators on the fiber space, they are given by

$$H_\sigma(p) = \frac{1}{2}\big(p - P_f^b - \sqrt{\alpha}A_\sigma^b\big)^2 + H_f^b \tag{2.6}$$

where

$$H_f^b := \sum_\lambda \int \mathrm{d}k\, |k|\, b_\lambda^*(k)\, b_\lambda(k), \quad P_f^b := \sum_\lambda \int \mathrm{d}k\, k\, b_\lambda^*(k)\, b_\lambda(k) \tag{2.7}$$

and

$$A_\sigma^b := \sum_\lambda \int_{\sigma \le |k| \le 1} \frac{\mathrm{d}k}{|k|^{1/2}}\, \{\epsilon_\lambda(k)\, b_\lambda(k) + h.c.\}. \tag{2.8}$$

Henceforth, we will drop the superscripts "b" from the notation.

While $H(p)$ has a ground state only for $p = 0$, it is proven in [2,6] that, for $p \in \mathcal{S} := \{p \in \mathbb{R}^3 | |p| \le 1/3\}$ and $\sigma > 0, H_\sigma(p)$ has a non-degenerate (fiber) ground state. This motivates the introduction of the cut-off. Properties of the fiber ground-state energy, $E_\sigma(p) = \inf \mathrm{spec} H_\sigma(p)$, are given in the following proposition proven in [2,6,9,10]:

Proposition 2.1. *There exists a constant $0 < \alpha_0 \ll 1$ such that for all $0 < \alpha \le \alpha_0$, the infimum of the spectrum of the fiber Hamiltonian,*

$$E_\sigma(p) = \inf \mathrm{spec} H_\sigma(p), \tag{2.9}$$

satisfies:

1. *For any $\sigma > 0, E_\sigma \in C^2(\mathcal{S})$, and for all $p \in \mathcal{S} = \{p \in \mathbb{R}^3 \,|\, |p| \le \frac{1}{3}\}, E_\sigma(p)$ is a simple eigenvalue.*
2. *There exists a constant $c < \infty$ such that, for any $p \in \mathcal{S}$ and $\sigma \ge 0$, we have that*

$$|\nabla_p E_\sigma(p) - p| \le c\,\alpha\,|p|\,, \quad \text{and} \quad 1 - c\,\alpha \le \partial_{|p|}^2 E_\sigma(p) \le 1\,. \tag{2.10}$$

3. *The following limit exists in $C^2(\mathcal{S})$*

$$\lim_{\sigma \searrow 0} E_\sigma(\,\cdot\,) \;=\; E(\,\cdot\,). \tag{2.11}$$

We let $\Psi_\sigma(p) \in \mathfrak{F}$, with $\|\Psi_\sigma(p)\|_{\mathfrak{F}} = 1$, denote the normalized fiber ground state corresponding to $E_\sigma(p)$,

$$H_\sigma(p)\Psi_\sigma(p) = E_\sigma(p)\,\Psi_\sigma(p)\,, \tag{2.12}$$

for $p \in \mathcal{S}$. For $0 < \sigma < \rho \le 1$ and $p \in \mathcal{S}$, we introduce the Weyl operators

$$W_{\nabla E_\sigma(p)}^{\sigma,\rho} := \exp\left[\alpha^{\frac{1}{2}} \sum_\lambda \int_{\sigma \le |k| \le \rho} \mathrm{d}k\, \frac{\nabla E_\sigma(p) \cdot \epsilon_\lambda(k) b_\lambda(k) - h.c.}{|k|^{1/2}(|k| - \nabla E_\sigma(p) \cdot k)}\right], \tag{2.13}$$

with $\nabla E_\sigma(p) \equiv \nabla_p E_\sigma(p)$, which are unitary on $\mathfrak{F}$, for $\sigma > 0$. Moreover, we define dressed electron states

$$\Phi_\sigma^\rho(p) := W^{\sigma,\rho}_{\nabla E_\sigma(p)}\, \Psi_\sigma(p). \tag{2.14}$$

For $p \in \mathcal{S}$, we define the Bogoliubov-transformed fiber Hamiltonians

$$K_\sigma^\rho(p) := W^{\sigma,\rho}_{\nabla E_\sigma(p)}\, H_\sigma(p)\, (W^{\sigma,\rho}_{\nabla E_\sigma(p)})^*. \tag{2.15}$$

It is convenient to define $K_\sigma^\rho(p) := H_\sigma(p)$, for $p \in \mathbb{R}^3 \setminus \mathcal{S}$.

The dressed electron states $\Phi_\sigma^\rho(p)$, for $p \in \mathcal{S}$, are the ground states of the Bogoliubov-transformed fiber Hamiltonians $K_\sigma^\rho(p)$, defined in (2.15), i.e.,

$$K_\sigma^\rho(p)\, \Phi_\sigma^\rho(p) = E_\sigma(p)\, \Phi_\sigma^\rho(p). \tag{2.16}$$

The properties of these states are described in the following proposition:

Proposition 2.2. *For any $p \in \mathcal{S}, 0 < \rho \le 1$, and for sufficiently small values of the finestructure constant $0 < \alpha \ll 1$, the ground-state eigenvector $\Phi_\sigma^\rho(p)$ satisfies*

1. *The strong limit*

$$\Phi^\rho(p) := \lim_{\sigma \to 0} \Phi_\sigma^\rho(p) \tag{2.17}$$

   exists in $\mathfrak{F}$.

2. *For $\theta < \frac{2}{3}$, the vectors $\Phi_\sigma^\rho(p)$ are θ-Hölder continuous in p,*

$$\sup_{p,q \in \mathcal{S}} \frac{\| \Phi_\sigma^\rho(p) - \Phi_\sigma^\rho(q) \|}{|p-q|^\theta} \le C(\theta)\ \ln \frac{1}{\rho} < \infty\,, \tag{2.18}$$

   uniformly in σ, with $0 \le \sigma < \rho \le 1$.

The proof of θ-Hölder continuity for $\theta < \frac{2}{3}$ is given in Sect. 5; (see also [9,10,22] for earlier results covering the range $\theta < \frac{1}{4}$, in the case where $\rho = 1$).

For arbitrary $u \in L^2(\mathbb{R}^3)$ (with Fourier transform denoted by $\widehat{u}$), we define the linear map

$$\mathcal{J}_\sigma^\rho : u \mapsto (2\pi)^{-3/2} \int_{\mathcal{S}} \mathrm{d}p\, \widehat{u}(p)\, e^{ix(p-P_f)}\, \chi_{\mathcal{S}_\mu}(p)\, \Phi_\sigma^\rho(p)\,, \tag{2.19}$$

where x is the electron position, $\chi_{\mathcal{S}_\mu}$ is a smooth approximate characteristic function of the set

$$\mathcal{S}_\mu := (1-\mu)\, \mathcal{S} \subset \mathcal{S} \subset \mathbb{R}^3, \tag{2.20}$$

and $0 < \mu < 1$. Note that $\mathcal{J}_\sigma^\rho : L^2(\mathbb{R}^3) \to \mathcal{M} \subset \mathcal{H}$, where

$$\mathcal{M} := \left\{ (2\pi)^{-3/2} \int_{\mathbb{R}^3} \mathrm{d}p\, \widehat{u}(p)\, e^{ix(p-P_f)}\, \chi_{\mathcal{S}_\mu}(p)\, \Phi_\sigma^\rho(p) \,\Big|\, u \in L^2(\mathbb{R}^3) \right\}, \tag{2.21}$$

the subspace of vectors in $\mathcal{H}$ supported on the one-particle shell of the operator $\int_{\mathcal{S}}^{\oplus} \mathrm{d}p\, K_\sigma^\rho(p)$. We also note that in (2.21) we do not require that $\mathrm{supp}(\widehat{u}) \subset \mathcal{S}_\mu$; instead, we cutoff $\widehat{u}$ outside the region $\mathcal{S}_\mu$ by multiplying it by $\chi_{\mathcal{S}_\mu}$.

Furthermore, we introduce the one-particle Schrödinger operator

$$H_{\mathrm{eff},\sigma} := E_{\mathrm{eff},\sigma}(-i\nabla_x) + V_\epsilon(x). \tag{2.22}$$

Here, the kinetic energy operator is defined by

$$E_{\mathrm{eff},\sigma}(p) := E_\sigma(p)\ ,\ \ p \in \mathcal{S}\,, \tag{2.23}$$

and suitably extended to $p \in \mathbb{R}^3 \setminus \mathcal{S}$. Note that the restriction of $E_{\mathrm{eff},\sigma}$ to $\mathcal{S}$ is twice continuously differentiable, $E_{\mathrm{eff},\sigma}|_{\mathcal{S}} \in C^2(\mathcal{S})$; see Proposition 2.1.

As a first step towards proving Theorem 1.1, we prove the following result:

Theorem 2.3. *Under the conditions of Theorem 1.1, there exists $\alpha_\delta > 0$ such that, for all $0 \le \alpha \le \alpha_\delta$, the bound (1.32) holds uniformly in the infrared cutoff $\sigma > 0$ and the cutoff $\rho > \sigma$.*

Proof. Our proof makes crucial use of the properties of the fiber ground-state energy $E_\sigma(p)$ and of the corresponding dressed electron states $\Phi^\rho_\sigma(p)$, for $p \in \mathcal{S}$, given in Propositions 2.1 and 2.2 above. We define the operator K^ρ_σ acting on $\mathcal{H}$,

$$K^\rho_\sigma := \int^{\oplus} K^\rho_\sigma(p)\,\mathrm{d}p, \tag{2.24}$$

and the perturbed operator $K^V_\sigma := K^\rho_\sigma + V_\epsilon$. Note that the operator K^ρ_σ has the property that

$$K^\rho_\sigma \mathcal{J}^\rho_\sigma = \mathcal{J}^\rho_\sigma E_{\mathrm{eff},\sigma}(-i\nabla). \tag{2.25}$$

We write the difference on the LHS of (1.32) as the integral of a derivative, substitute $H^V_\sigma \to H^V_\sigma - K^V_\sigma + K^V_\sigma$ inside the integral, and group terms suitably to obtain

$$\begin{aligned} &e^{-itH^V_\sigma}\,\mathcal{J}^\rho_\sigma(u^\epsilon_0) - \mathcal{J}^\rho_\sigma(e^{-itH_{\mathrm{eff},\sigma}}\,u^\epsilon_0) \\ &\quad= -i\,e^{-itH^V_\sigma}\int_0^t \mathrm{d}s\,e^{isH^V_\sigma}\,(H^V_\sigma\,\mathcal{J}^\rho_\sigma(u^\epsilon_s) - \mathcal{J}^\rho_\sigma(H_{\mathrm{eff}}u^\epsilon_s)) \\ &\quad=: \phi^1(t) + \phi^2(t), \end{aligned} \tag{2.26}$$

where $u^\epsilon_s := e^{-isH_{\mathrm{eff},\sigma}}\,u^\epsilon_0$ and

$$\phi^1(t) := -i\,e^{-itH^V_\sigma}\int_0^t \mathrm{d}s\,e^{isH^V_\sigma}\,\big(\,H_\sigma - K^\rho_\sigma\,\big)\,\mathcal{J}^\rho_\sigma(\,u^\epsilon_0\,)\,, \tag{2.27}$$

where we have used the cancelation of V in $H^V_\sigma - K^V_\sigma = H_\sigma - K^\rho_\sigma$, and

$$\phi^2(t) := -i\,e^{-itH^V_\sigma}\int_0^t \mathrm{d}s\,e^{isH^V_\sigma}\,\big(K^V_\sigma\,\mathcal{J}^\rho_\sigma(u^\epsilon_s) - \mathcal{J}^\rho_\sigma(H_{\mathrm{eff}}u^\epsilon_s)\big).$$

The first term on the r.h.s. of (2.26) accounts for the radiation of infrared photons, while the second term accounts for the influence of the external potential

V_ϵ on the full QED dynamics $\Psi(t) = e^{-itH_\sigma^V}\mathcal{J}_\sigma^\rho(u_0^\epsilon)$, as compared with the effective Schrödinger evolution $e^{-itH_{\mathrm{eff},\sigma}}u_0^\epsilon$.

Using the direct integral decomposition, we obtain

$$\phi^1(t) = -i\,(2\pi)^{-\frac{3}{2}}e^{-itH_\sigma^V} \times \int_0^t \mathrm{d}s\, e^{isH_\sigma^V}\int_{\mathcal{S}} \mathrm{d}p\, \widehat{u}_s^\epsilon(p)\, e^{i(p-P_f)x}(H_\sigma - K_\sigma^\rho)(p)\,\chi_{\mathcal{S}_\mu}(p)\Phi_\sigma^\rho(p), \tag{2.28}$$

so that

$$\|\phi^1(t)\|_{\mathcal{H}} \le \sup_{p\in\mathcal{S}}\left\{\frac{1}{|p|}\,\|(H_\sigma - K_\sigma^\rho)(p)\,\Phi_\sigma^\rho(p)\|_{\mathfrak{F}}\right\}\int_0^t \|\,\nabla u_s^\epsilon\,\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}s. \tag{2.29}$$

We note that thanks to $\chi_{\mathcal{S}_\mu}$ in (2.28), which cuts off the tail of u_s^ϵ outside of $\mathcal{S}_\mu$, the supremum in (2.29) can be taken only for $p\in\mathcal{S}_\mu$, respectively, $\mathcal{S}$.

In Appendix A we prove the following key result:

$$\sup_{p\in\mathcal{S}}\left\{\frac{1}{|p|}\,\|(H_\sigma - K_\sigma^\rho)(p)\,\Phi_\sigma^\rho(p)\|_{\mathfrak{F}}\right\} \le C\alpha^{\frac{1}{2}}\,\rho^{\frac{1}{2}}\,, \tag{2.30}$$

uniformly in $\sigma \ge 0$. Furthermore, we have the estimate

$$\int_0^t \|\,\nabla u_s^\epsilon\,\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}s \le C\; t\,(\epsilon^\kappa + \epsilon t)\,, \tag{2.31}$$

as shown below in (2.37)–(2.39), using the condition $\|\nabla u_0^\epsilon\|_{L^2(\mathbb{R}^3)} \le \epsilon^\kappa$ on u_0^ϵ, and the fact that the potential V satisfies (1.20). We obtain

$$\|\phi^1(t)\|_{\mathcal{H}} \le C\,t\,(\epsilon^\kappa + \epsilon t)\,\alpha^{\frac{1}{2}}\,\rho^{\frac{1}{2}}, \tag{2.32}$$

which yields the second contribution to the r.h.s. of (1.32).

For the second term on the r.h.s. of (2.26), using the fiber decomposition and the equation $K_\sigma^\rho(p)\,\Phi_\sigma^\rho(p) = E_\sigma(p)\,\Phi_\sigma^\rho(p)$, we have that

$$\phi^2(t) = -i\,e^{-itH_\sigma^V}\int_0^t \mathrm{d}s\,e^{isH_\sigma^V}\,\big(V_\epsilon\mathcal{J}_\sigma^\rho(u_s^\epsilon) - \mathcal{J}_\sigma^\rho(V_\epsilon u_s^\epsilon)\big). \tag{2.33}$$

Let $\|\Phi\|_{C^\theta(\mathcal{S})} := \sup_{p,q\in\mathcal{S}}\frac{\|\,\Phi(p)-\Phi(q)\,\|}{|p-q|^\theta}$.

In (2.40)–(2.47) below, we prove an estimate of the form

$$\|\phi^2(t)\|_{\mathcal{H}} \le t\,C\,\|\widehat{|\nabla|^\theta V_\epsilon}\|_{L^1(\mathbb{R}^3)}\,(1 + \|\Phi_\sigma^\rho\|_{C^\theta(\mathcal{S})})\,, \tag{2.34}$$

for $\theta < \frac{2}{3}$. The key point here is that the θ-Hölder continuity of the fiber ground-state $\Phi_\sigma^\rho(p)$ enables us to gain a θ derivative of the potential, yielding $\|\widehat{|\nabla|^\theta V_\epsilon}\|_{L^1(\mathbb{R}^3)} \le C\epsilon^\theta$. Using the θ-Hölder continuity of $\Phi_\sigma^\rho(\,\cdot\,)$, which holds uniformly in σ, with $0<\sigma<\rho$, and the fact that

$$\|\widehat{|\nabla|^\theta V}\|_{L^1(\mathbb{R}^3)} \le \gamma\,, \quad \text{where} \quad \gamma := \|\widehat{V}(k)\|_{L^1} < \infty\,, \tag{2.35}$$

(see (1.20)) and using $\|\Phi^\rho_\sigma\|_{C^\theta(\mathcal{S})} \le C_\delta\,(1+\ln(\rho^{-1}))$, which we prove in Proposition 5.4, we arrive at

$$\|\phi^2(t)\|_{\mathcal{H}} \le C_\delta\, t\, \epsilon^\theta\,(1+\ln(\rho^{-1})), \tag{2.36}$$

which yields the first term on the RHS of (1.32). □

Proof of (2.31). To verify (2.31), a simple calculation shows that

$$\nabla u^\epsilon_s = e^{-isH_{\mathrm{eff},\sigma}}\,\nabla u^\epsilon_0 - i\int_0^s \mathrm{d}v\, e^{-ivH_{\mathrm{eff},\sigma}}\,\nabla V_\epsilon(x)\, e^{-i(s-v)H_{\mathrm{eff},\sigma}}\, u^\epsilon_s. \tag{2.37}$$

Using that $\|\nabla u^\epsilon_0\|_{L^2} \le \epsilon^\kappa$ and that

$$\|\nabla V_\epsilon\|_{L^\infty} = \|\widehat{\nabla V_\epsilon}\|_{L^1} \le \gamma\,\epsilon\,, \tag{2.38}$$

we conclude that

$$\|\,\nabla u^\epsilon_s\,\|_{L^2} \le C\,(\epsilon^\kappa + \epsilon s), \tag{2.39}$$

and thus (2.31). □

Proof of (2.34). In what follows we use the notation

$$(U\Psi)(p) = \widehat{\Psi}(p) \quad \text{and} \quad (U^{-1}\Phi)(x) = \Phi^\vee(x).$$

We define

$$\psi_s := V_\epsilon \mathcal{J}^\rho_\sigma(u^\epsilon_s) - \mathcal{J}^\rho_\sigma(V_\epsilon u^\epsilon_s). \tag{2.40}$$

Using the definition of $\mathcal{J}^\rho_\sigma$ and computing the Fourier transform, we find that

$$\widehat{\psi}_s(p) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} \mathrm{d}q\, \widehat{V}_\epsilon(p-q)\,\widehat{u}(s,q)\,\big(\chi_{\mathcal{S}_\mu}(q)\Phi^\rho_\sigma(q) - \chi_{\mathcal{S}_\mu}(p)\Phi^\rho_\sigma(p)\big). \tag{2.41}$$

By relations (2.33) and (2.40) and the unitarity of the generalized Fourier transform we have that

$$\|\phi^2(t)\|_{\mathcal{H}} \le \int_0^t \mathrm{d}s\,\|\psi_s\|_{L^2_x\otimes\mathfrak{F}} = \int_0^t \mathrm{d}s\,\|\widehat{\psi_s}\|_{L^2_p\otimes\mathfrak{F}}. \tag{2.42}$$

It is important to note that, for any function $f \in L^2(\mathbb{R}^3)$ with $\mathrm{supp}(f) \subset \mathcal{S}_\mu$,

$$\mathrm{supp}(\widehat{V}_\epsilon * f) \subset \mathcal{S}, \tag{2.43}$$

for $\epsilon \le \mu/3$, since we are assuming $\mathrm{supp}(\widehat{V}) \subset \{k||k| \le 1\}$, so that $\mathrm{supp}(\widehat{V}_\epsilon) \subset \{k||k| \le \epsilon\}$. Since the term in the integrand given by $(\widehat{u}^\epsilon_s\chi_{\mathcal{S}_\mu}\Phi^\rho_\sigma)(q)$ is supported in $q \in \mathcal{S}_\mu$, so that, by (2.43), its convolution with $\widehat{V}_\epsilon$ has support in $\mathcal{S}$, we find

$$\begin{aligned}\widehat{\psi}_s(p) &= (2\pi)^{-3/2}\\ &\quad\times \mathbf{1}_{\mathcal{S}}(p)\int_{\mathbb{R}^3}\mathrm{d}q\,\widehat{V}_\epsilon(p-q)\,\widehat{u}(s,q)\,(\chi_{\mathcal{S}_\mu}(q)\,\Phi^\rho_\sigma(q) - \chi_{\mathcal{S}_\mu}(p)\Phi^\rho_\sigma(p)),\end{aligned} \tag{2.44}$$

for $\epsilon \leq \mu/3$, where $\mathbf{1}_{\mathcal{S}}$ is the characteristic function of the set $\mathcal{S}$. Inserting $|p-q|^{\theta}|p-q|^{-\theta} = \mathbf{1}$ into (2.44), using the definition of $|\nabla|^{\theta}$ by its Fourier transform and using that, since $\chi_{\mathcal{S}_\mu}$ is a smooth function,

$$\sup_{p,q\in\mathcal{S}} |p-q|^{-\theta}\|\big(\chi_{\mathcal{S}_\mu}(q)\Phi^{\rho}_{\sigma}(q) - \chi_{\mathcal{S}_\mu}(p)\Phi^{\rho}_{\sigma}(p)\big)\|_{\mathfrak{F}} \leq C(1+\|\Phi^{\rho}_{\sigma}\|_{C^{\theta}(\mathcal{S})}), \tag{2.45}$$

we obtain the bound $\|\hat{\psi}_s\|_{L^2_x\otimes\mathfrak{F}} \leq C(1+\|\Phi^{\rho}_{\sigma}\|_{C^{\theta}(\mathcal{S})})\|\,|\mathbf{1}_{\mathcal{S}}\widehat{u}^{\epsilon}_s| * |\widehat{|\nabla|^{\theta}V_\epsilon}|\,\|_{L^2(\mathcal{S})}$. Next, using Young's inequality, $\|f*g\|_{L^r} \leq \|f\|_{L^1}\|g\|_{L^r}$, we find that

$$\|\widehat{\psi}_s\|_{L^2_x\otimes\mathfrak{F}} \leq C(1+\|\Phi^{\rho}_{\sigma}\|_{C^{\theta}(\mathcal{S})})\,\|\widehat{|\nabla|^{\theta}V_\epsilon}\|_{L^1(\mathbb{R}^3)} \sup_{s\in[0,t]}\|\mathbf{1}_{\mathcal{S}}\widehat{u}^{\epsilon}_s\|_{L^2(\mathbb{R}^3)}. \tag{2.46}$$

Finally, observing that

$$\|\mathbf{1}_{\mathcal{S}}\widehat{u}^{\epsilon}_s\|_{L^2(\mathbb{R}^3)} \leq \|\widehat{u}^{\epsilon}_s\|_{L^2(\mathbb{R}^3)} = \|u^{\epsilon}_s\|_{L^2(\mathbb{R}^3)} = \|u^{\epsilon}_0\|_{L^2(\mathbb{R}^3)} = 1\,, \tag{2.47}$$

by unitarity of $e^{-itH_{\mathrm{eff},\sigma}}$, and using (2.42), we arrive at (2.34). □

3. The Limit $\sigma \searrow 0$

In this section we remove the infrared cut-off from the evolution.

Proposition 3.1. *Under the conditions of Theorem 2.3, the strong limits*

$$s - \lim_{\sigma\searrow 0} e^{-itH^V_\sigma}\,\mathcal{J}^{\rho}_{\sigma}(u^{\epsilon}_0) = e^{-itH^V}\,\mathcal{J}^{\rho}_{0}(u^{\epsilon}_0) \tag{3.1}$$

and

$$s - \lim_{\sigma\searrow 0} \mathcal{J}^{\rho}_{\sigma}(e^{-itH_{\mathrm{eff},\sigma}}\,u^{\epsilon}_0) = \mathcal{J}^{\rho}_{0}(e^{-itH_{\mathrm{eff}}}\,u^{\epsilon}_0) \tag{3.2}$$

exist, for arbitrary $|t| < \infty$.

Proof. We write

$$e^{-itH^V_\sigma}\,\mathcal{J}^{\rho}_{\sigma}(u^{\epsilon}_0) - e^{-itH^V}\,\mathcal{J}^{\rho}_{0}(u^{\epsilon}_0) \tag{3.3}$$

$$= (e^{-itH^V_\sigma} - e^{-itH^V})\mathcal{J}^{\rho}_{0}(u^{\epsilon}_0) + e^{-itH^V_\sigma}(\mathcal{J}^{\rho}_{\sigma} - \mathcal{J}^{\rho}_{0})(u^{\epsilon}_0). \tag{3.4}$$

Clearly,

$$\begin{aligned}\left\|e^{-itH^V_\sigma}(\mathcal{J}^{\rho}_{\sigma} - \mathcal{J}^{\rho}_{0})(u^{\epsilon}_0)\right\| &= \|(\mathcal{J}^{\rho}_{\sigma} - \mathcal{J}^{\rho}_{0})(u^{\epsilon}_0)\| \\ &\leq \|u^{\epsilon}_0\|_{L^2}\sup_{p\in\mathcal{S}_\mu}\|\Phi^{\rho}_{\sigma}(p) - \Phi^{\rho}(p)\|_{\mathcal{F}}\,.\end{aligned}$$

Thus,

$$\lim_{\sigma\searrow 0}\left\|e^{-itH^V_\sigma}(\mathcal{J}^{\rho}_{\sigma} - \mathcal{J}^{\rho}_{0})(u^{\epsilon}_0)\right\| = 0\,,$$

follows from Proposition 5.1.

Next, we discuss the first term on the right side of (3.4). In order to prove that it converges to 0, as $\sigma \searrow 0$, it suffices to show that H^V_σ converges to H^V in the norm resolvent sense; (see [24, Theorem VIII.21]), i.e.,

$$\lim_{\sigma\searrow 0}\left\|(H^V_\sigma + i)^{-1} - (H^V + i)^{-1}\right\| = 0.$$

From the second resolvent equation and the fact that $\|(H_\sigma^V + i)^{-1}\| \leq 1$, it follows that

$$\left\|(H_\sigma^V + i)^{-1} - (H^V + i)^{-1}\right\| = \left\|(H^V + i)^{-1} \, Q_\sigma \, (H_\sigma^V + i)^{-1}\right\| , \tag{3.5}$$

where

$$Q_\sigma := H^V - H_\sigma^V \;=\; \alpha^{\frac{1}{2}} A_{<\sigma}(x) \cdot v_\sigma + \frac{\alpha}{2} (A_{<\sigma}(x))^2 \,,$$

and

$$v_\sigma := -i\nabla_x + \alpha^{\frac{1}{2}} A_\sigma(x)$$

is the velocity operator. Here $A_\sigma(x)$ is defined in (2.3), and

$$A_{<\sigma}(x) := \sum_\lambda \int\limits_{|k|\leq\sigma} \frac{\mathrm{d}k}{|k|^{1/2}} \, \{\epsilon_\lambda(k) \, e^{-ikx} \otimes a_\lambda(k) \; + \; h.c.\}. \tag{3.6}$$

In order to estimate the norm of $Q_\sigma(H^V + i)^{-1}$, we use the following well-known lemma:

Lemma 3.2. *Let $f, g \in \mathrm{L}^2(\mathbb{R}^3 \times \{+,-\}; \mathcal{B}(\mathcal{H}_{\mathrm{el}}))$ be operator-valued functions such that $\|(1 + |k|^{-1})^{1/2} f\|, \|(1 + |k|^{-1})^{1/2} g\| < \infty$. Then*

$$\|a^\#(f)(H_f + 1)^{-\frac{1}{2}}\| \leq \|(1 + |k|^{-1})^{\frac{1}{2}} f\|_{L^2}, \tag{3.7}$$

$$\|a^\#(f) a^\#(g)(H_f + 1)^{-1}\| \leq \|(1 + |k|^{-1})^{\frac{1}{2}} f\|_{L^2} \; \|(1 + |k|^{-1})^{\frac{1}{2}} g\|_{L^2}, \tag{3.8}$$

where $a^\#$ stands for a or a^*.

In particular, using the Kato-Rellich theorem, one easily shows that, for α small enough, $D(H^V) = D(-\Delta_x \otimes I + I \otimes H_f) \subset D(H_f)$. Thus, we have that

$$\left\|(H_f + 1)(H^V + i)^{-1}\right\| \, \leq \, C,$$

which when combined with Lemma 3.2 yields

$$\left\| \frac{\alpha}{2} (A_{<\sigma}(x))^2 (H^V + i)^{-1} \right\| \leq C \, \alpha \, \sigma. \tag{3.9}$$

Likewise, one verifies that

$$\left\| \alpha^{\frac{1}{2}} A_{<\sigma}(x) \cdot v_\sigma (H^V + i)^{-1} \right\| \leq C \, \alpha^{\frac{1}{2}} \, \sigma^{\frac{1}{2}}, \tag{3.10}$$

since $0 \leq v_\sigma^2 \leq H^V + \|V\|_{L^\infty}$ is bounded relative to H^V. Estimates (3.9) and (3.10) yield

$$\left\|Q_\sigma(H^V + i)^{-1}\right\| \leq C \, \alpha^{\frac{1}{2}} \, \sigma^{\frac{1}{2}}.$$

By (3.5), we have shown that H_σ^V converges to H^V, as $\sigma \searrow 0$, in the norm resolvent sense. □

4. Proof of Theorem 1.1

In this section, we prove the bound in Theorem 1.1, which compares the full dynamics to the effective dynamics for the system without infrared cutoff. We have that

$$\begin{aligned} &\| e^{-itH^V} \mathcal{J}_0^\rho(u_0^\epsilon) - \mathcal{J}_0^\rho(e^{-itH_{\mathrm{eff}}} u_0^\epsilon) \|_{\mathcal{H}} \\ &\quad\le \| e^{-itH_\sigma^V} \mathcal{J}_\sigma^\rho(u_0^\epsilon) - \mathcal{J}_\sigma^\rho(e^{-itH_{\mathrm{eff},\sigma}} u_0^\epsilon) \|_{\mathcal{H}} \\ &\qquad + \| e^{-itH_\sigma^V} \mathcal{J}_\sigma^\rho(u_0^\epsilon) - e^{-itH^V} \mathcal{J}_0^\rho(u_0^\epsilon) \|_{\mathcal{H}} \\ &\qquad + \| \mathcal{J}_\sigma^\rho(e^{-itH_{\mathrm{eff},\sigma}} u_0^\epsilon) - \mathcal{J}_0^\rho(e^{-itH_{\mathrm{eff}}} u_0^\epsilon) \|_{\mathcal{H}}\,, \end{aligned} \tag{4.1}$$

for any t and $0 < \sigma < \rho \le 1$. It follows from Theorem 2.3 that the first term on the r.s. of the inequality sign is bounded by $C_\delta\,(1+\ln(\rho^{-1}))\,\epsilon^{\frac{2}{3}-\delta}\,t + C\,\alpha^{\frac{1}{2}}\,\rho^{\frac{1}{2}}\,t\,(\epsilon^\kappa + \epsilon t)$, uniformly in $\sigma > 0$.

From Proposition 3.1, it follows that the second and third terms on the r.s. converge to zero, as $\sigma \searrow 0$. By taking σ to zero, we thus conclude that

$$\begin{aligned} &\|e^{-itH^V} \mathcal{J}_0^\rho(u_0^\epsilon) - \mathcal{J}_0^\rho(e^{-itH_{\mathrm{eff}}} u_0^\epsilon) \|_{\mathcal{H}} \\ &\quad\le C_\delta\,(1+\ln(\rho^{-1}))\,\epsilon^{\frac{2}{3}-\delta}\,t + C\,\alpha^{\frac{1}{2}}\,\rho^{\frac{1}{2}}\,t\,(\epsilon^\kappa + \epsilon t)\,. \end{aligned} \tag{4.2}$$

Due to our choice $\rho = \epsilon^{\frac{2}{3}-\delta}$, this concludes the proof of Theorem 1.1. We note that in the inequality (1.21), the logarithmic term $\ln(\rho_\epsilon^{-1})$ has been absorbed by an arbitrary small shift of δ, which we do not keep track of notationally. □

5. Hölder Continuity of the Ground State

We recall that $\Phi_\sigma^\rho(p)$ denotes a normalized ground state of the Bogoliubov transformed fiber Hamiltonian $K_\sigma^\rho(p) = W_{\nabla E_\sigma(p)}^\rho\, H_\sigma(p)\,(W_{\nabla E_\sigma(p)}^\rho)^*$, with infrared cutoff $\sigma > 0$ (see (2.15)). Our aim in this appendix is to prove that, for a suitable choice of the vectors $\Phi_\sigma^\rho(p)$, the map $p \mapsto \Phi_\sigma^\rho(p)$ is θ-Hölder continuous, for $\theta < 2/3$.

For $\rho = 1$, we set

$$\Phi_\sigma(p) := \Phi_\sigma^1(p), \qquad K_\sigma(p) := K_\sigma^1(p). \tag{5.1}$$

We remark that

$$K_\sigma^\rho(p) = \big(W_{\nabla E_\sigma(p)}^{\rho,1}\big)^*\, K_\sigma(p)\, W_{\nabla E_\sigma(p)}^{\rho,1}\,, \quad \Phi_\sigma^\rho(p) = \big(W_{\nabla E_\sigma(p)}^{\rho,1}\big)^*\, \Phi_\sigma(p)\,, \tag{5.2}$$

where we recall that $W_{\nabla E_\sigma(p)}^{\rho,1}$ is defined in (2.13).

Letting

$$\mathfrak{F}_\sigma := \bigoplus_{n\ge 0} \mathrm{Sym}(L^2(\{k\in\mathbb{R}^3, |k|\ge\sigma\}\times\{+,-\}))^{\otimes n} \tag{5.3}$$

denote the Fock space of photons of energies $\ge \sigma$, and identifying $\mathfrak{F}_\sigma$ with a subspace of $\mathfrak{F}$, we observe that $K_\sigma(p)$ leaves $\mathfrak{F}_\sigma$ invariant. Let $\widetilde{K}_\sigma(p)$ denote the restriction of $K_\sigma(p)$ to $\mathfrak{F}_\sigma$. An important property, proven in [3,10,13], is that there is an energy gap of size $\eta\sigma$, where $\eta > 0$ is uniform in $\sigma \searrow 0$, in the spectrum of $\widetilde{K}_\sigma(p)$ above the ground-state energy $E_\sigma(p)$. Moreover, one can choose

$$\Phi_\sigma(p) = \widetilde{\Phi}_\sigma(p) \otimes \Omega_{<\sigma}\,, \tag{5.4}$$

in the representation $\mathfrak{F} \simeq \mathfrak{F}_\sigma \otimes \mathfrak{F}_{<\sigma}$, where

$$\mathfrak{F}_{<\sigma} := \bigoplus_{n\geq 0} \mathrm{Sym}(\, L^2(\{k \in \mathbb{R}^3, |k| \leq \sigma\} \,\times\, \{+,-\}\,)\,)^{\otimes n}. \tag{5.5}$$

Now, let Ω_σ denote the vacuum sector in $\mathfrak{F}_\sigma$ and $\widetilde{\Pi}_\sigma(p)$ be the rank-one projection onto the eigenspace associated with $E_\sigma(p) = \inf \mathrm{spec}(\tilde{K}_\sigma(p))$. By [10,13],

$$\|\widetilde{\Pi}_\sigma(p)\Omega_\sigma\| \geq \frac{1}{3}, \tag{5.6}$$

for arbitrary $\sigma > 0$ and $|p| \leq 1/3$ provided that α is chosen sufficiently small. Then $\widetilde{\Phi}_\sigma(p)$ can be chosen in the following way:

$$\widetilde{\Phi}_\sigma(p) = \frac{\widetilde{\Pi}_\sigma(p)\Omega_\sigma}{\|\widetilde{\Pi}_\sigma(p)\Omega_\sigma\|}. \tag{5.7}$$

Let N denote the number operator,

$$N = \sum_\lambda \int \mathrm{d}k\, b^*_\lambda(k)\, b_\lambda(k). \tag{5.8}$$

The following proposition has been proven in [8,10,13].

Proposition 5.1. *For $\alpha \ll 1$ and $|p| \leq 1/3$, there exists a normalized vector $\Phi(p)$ in the Fock space $\mathfrak{F}$ such that $\Phi_\sigma(p) \to \Phi(p)$, strongly, as $\sigma \to 0$. The following bound holds,*

$$\|N^{\frac{1}{2}}\Phi_\sigma(p)\| \leq C\alpha^{\frac{1}{2}}\,, \tag{5.9}$$

uniformly in $\sigma \geq 0$. Moreover, For all $\delta > 0$, there exists $\alpha_\delta > 0$ and $C_\delta < \infty$ such that, for all $0 \leq \alpha \leq \alpha_\delta, 0 \leq \sigma' < \sigma \leq 1$ and $|p| \leq 1/3$,

$$\|\Phi_\sigma(p) - \Phi_{\sigma'}(p)\| \leq C_\delta\, \alpha^{\frac{1}{4}}\, \sigma^{1-\delta}, \tag{5.10}$$

$$|\nabla E_\sigma(p) - \nabla E_{\sigma'}(p)| \leq C_\delta\, \alpha^{\frac{1}{4}}\, \sigma^{1-\delta}. \tag{5.11}$$

As a consequence, we show the following corollary:

Corollary 5.2. *Let $0 < \rho \leq 1$. For all $\delta > 0$, there exists $0 < \alpha_\delta \ll 1$ such that, for all $0 \leq \alpha \leq \alpha_\delta$ and $|p| \leq 1/3$, there exists a vector $\Phi^\rho(p)$ in the Fock space such that $\Phi^\rho_\sigma(p) \to \Phi^\rho(p)$, strongly, as $\sigma \to 0$. Moreover, there exists a constant $C_\delta < \infty$ such that, for all $0 \leq \alpha \leq \alpha_\delta, 0 \leq \sigma' < \sigma \leq 1$ and $|p| \leq 1/3$,*

$$\|\Phi^\rho_\sigma(p) - \Phi^\rho_{\sigma'}(p)\| \leq C_\delta\, \alpha^{\frac{1}{4}}\, \sigma^{1-\delta}\,(1 + \alpha^{\frac{1}{2}}\ln(\rho^{-1})). \tag{5.12}$$

Proof. Using (5.2), we split

$$\begin{aligned}\Phi^\rho_\sigma(p) - \Phi^\rho_{\sigma'}(p) = &\Big(\big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^* - \big(W^{\rho,1}_{\nabla E_{\sigma'}(p)}\big)^*\Big)\Phi_\sigma(p)\\ &+\big(W^{\rho,1}_{\nabla E_{\sigma'}(p)}\big)^*\big(\Phi_\sigma(p) - \Phi_{\sigma'}(p)\big).\end{aligned} \tag{5.13}$$

By Proposition 5.1 and unitarity of $W^{\rho,1}_{\nabla E_\sigma(p)}$, the second term is estimated as

$$\left\| \left(W^{\rho,1}_{\nabla E_{\sigma'}(p)} \right)^* \left(\Phi_\sigma(p) - \Phi_{\sigma'}(p) \right) \right\| \leq C_\delta \, \alpha^{\frac{1}{4}} \, \sigma^{1-\delta}. \tag{5.14}$$

The first term in the right side of (5.13) is estimated as

$$\begin{aligned} \left\| \left(\left(W^{\rho,1}_{\nabla E_\sigma(p)} \right)^* - \left(W^{\rho,1}_{\nabla E_{\sigma'}(p)} \right)^* \right) \Phi_\sigma(p) \right\| &= \left\| \left(\mathbf{1} - W^{\rho,1}_{\nabla E_\sigma(p)} \left(W^{\rho,1}_{\nabla E_{\sigma'}(p)} \right)^* \right) \Phi_\sigma(p) \right\| \\ &\leq \left\| B(\rho) \Phi_\sigma(p) \right\|, \end{aligned} \tag{5.15}$$

by unitarity of $W^{\rho,1}_{\nabla E_\sigma(p)}$ and the spectral theorem, where

$$\begin{aligned} B(\rho) := \alpha^{\frac{1}{2}} \sum_\lambda \int_{\rho \leq |k| \leq 1} \mathrm{d}k \\ \times \Big(\frac{\nabla E_\sigma(p) \cdot \epsilon_\lambda(k) \, b_\lambda(k) - h.c.}{|k|^{1/2}(|k| - \nabla E_\sigma(p) \cdot k)} - \frac{\nabla E_{\sigma'}(p) \cdot \epsilon_\lambda(k) \, b_\lambda(k) - h.c.}{|k|^{1/2}(|k| - \nabla E_{\sigma'}(p) \cdot k)} \Big). \end{aligned} \tag{5.16}$$

To estimate $\|B(\rho)\Phi_\sigma(p)\|$, we use the well-known fact that, for any $f \in \mathrm{L}^2(\mathbb{R}^3 \times \{+,-\})$,

$$\|a^{\#}(f)(N+1)^{-\frac{1}{2}}\| \leq \sqrt{2} \|f\|_{L^2}. \tag{5.17}$$

Clearly,

$$\begin{aligned} &\frac{\nabla E_\sigma(p) \cdot \epsilon_\lambda(k)}{|k|^{1/2}(|k| - \nabla E_\sigma(p) \cdot k)} - \frac{\nabla E_{\sigma'}(p) \cdot \epsilon_\lambda(k)}{|k|^{1/2}(|k| - \nabla E_{\sigma'}(p) \cdot k)} \\ &= \frac{(\nabla E_\sigma(p) - \nabla E_{\sigma'}(p)) \cdot \epsilon_\lambda(k)}{|k|^{1/2}(|k| - \nabla E_\sigma(p) \cdot k)} \\ &\quad + \frac{\nabla E_{\sigma'}(p) \cdot \epsilon_\lambda(k)}{|k|^{1/2}(|k| - \nabla E_\sigma(p) \cdot k)} \frac{(\nabla E_\sigma(p) - \nabla E_{\sigma'}(p)) \cdot k}{(|k| - \nabla E_{\sigma'}(p) \cdot k)}. \end{aligned} \tag{5.18}$$

Hence, by (5.11) and the facts that $|\nabla E_\sigma(p)|, |\nabla E_{\sigma'}(p)| \leq 1/2$ for α small enough (see Proposition 2.1 (2)), we obtain

$$\left| \frac{\nabla E_\sigma(p) \cdot \epsilon_\lambda(k)}{|k|^{1/2}(|k| - \nabla E_\sigma(p) \cdot k)} - \frac{\nabla E_{\sigma'}(p) \cdot \epsilon_\lambda(k)}{|k|^{1/2}(|k| - \nabla E_{\sigma'}(p) \cdot k)} \right| \leq \frac{C_\delta \, \alpha^{\frac{1}{4}} \, \sigma^{1-\delta}}{|k|^{\frac{3}{2}}}. \tag{5.19}$$

Thus, (5.16) and (5.17) yield that

$$\begin{aligned} \left\| B(\rho) \Phi_\sigma(p) \right\| &\leq C_\delta \, \alpha^{\frac{3}{4}} \, \sigma^{1-\delta} \left\| \frac{\mathbf{1}_{\rho \leq |k| \leq 1}(|k|)}{|k|^{\frac{3}{2}}} \right\|_{L^2_k} \left\| (N+1)^{\frac{1}{2}} \Phi_\sigma(p) \right\| \\ &\leq C_\delta \, \alpha^{\frac{3}{4}} \, \sigma^{1-\delta} \, \ln(\rho^{-1}). \end{aligned} \tag{5.20}$$

where we used (5.9) in the last inequality. Together with (5.13)–(5.15), this concludes the proof of Corollary 5.2. □

The following result follows from [10,13] (it is also a consequence of (2.10) in Proposition 2.1 (2)):

Proposition 5.3. *There exist $\alpha_c > 0$ and $C > 0$ such that, for all $0 \leq \alpha \leq \alpha_c$ and p, p' satisfying $|p| \leq 1/3, |p'| \leq 1/3$,*

$$\left|\nabla E_\sigma(p) - \nabla E_\sigma(p')\right| \leq C\,|p - p'|, \tag{5.21}$$

uniformly in $\sigma > 0$.

We now prove the following proposition:

Proposition 5.4. *Let $0 < \rho \leq 1$. For all $\delta > 0$, there exist $\alpha_\delta > 0$ and $C_\delta < \infty$ such that, for all $0 \leq \alpha \leq \alpha_\delta, \sigma > 0$ and $p, k \in \mathbb{R}^3$ satisfying $|p| \leq 1/3, |p+k| \leq 1/3$,*

$$\|\Phi^\rho_\sigma(p+k) - \Phi^\rho_\sigma(p)\| \leq C_\delta\,(1 + \alpha^{\frac{1}{2}} \ln(\rho^{-1}))\,|k|^{\frac{2}{3}-\delta}. \tag{5.22}$$

Proof. Step 1. We first prove that, for all $0 < \sigma < \rho \leq 1$,

$$\|\Phi^\rho_\sigma(p+k) - \Phi^\rho_\sigma(p)\| \leq C\,|k|\,(\sigma^{-\frac{1}{2}} + \alpha^{\frac{1}{2}} \ln(\rho^{-1})). \tag{5.23}$$

We decompose

$$\begin{aligned}\Phi^\rho_\sigma(p+k) - \Phi^\rho_\sigma(p) &= \big(W^{\rho,1}_{\nabla E_\sigma(p+k)}\big)^*\,\Phi_\sigma(p+k)\; - \;\big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^*\,\Phi_\sigma(p)\\ &= \Big(\big(W^{\rho,1}_{\nabla E_\sigma(p+k)}\big)^* - \big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^*\Big)\,\Phi_\sigma(p)\\ &\quad + \big(W^{\rho,1}_{\nabla E_\sigma(p+k)}\big)^*\,\big(\Phi_\sigma(p+k) - \Phi_\sigma(p)\big).\end{aligned} \tag{5.24}$$

To estimate the first term in the right side of (5.24), we proceed as in the proof of Corollary 5.2. Namely, we have that

$$\begin{aligned}&\Big\|\Big(\big(W^{\rho,1}_{\nabla E_\sigma(p+k)}\big)^* - \big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^*\Big)\Phi_\sigma(p)\Big\|\\ &\quad = \Big\|\Big(\mathbf{1} - W^{\rho,1}_{\nabla E_\sigma(p+k)}\big(W^{\rho,1}_{\nabla E_\sigma(p)}\big)^*\Big)\Phi_\sigma(p)\Big\|\\ &\quad \leq \big\|C(\rho)\Phi_\sigma(p)\big\|,\end{aligned} \tag{5.25}$$

by the spectral theorem, where

$$C(\rho) := \alpha^{\frac{1}{2}} \sum_\lambda \int_{\rho \leq |\tilde{k}| \leq 1} \mathrm{d}\tilde{k} \times \left(\frac{\nabla E_\sigma(p+k)\cdot\epsilon_\lambda(\tilde{k})\,b_\lambda(\tilde{k}) - h.c.}{|\tilde{k}|^{1/2}(|\tilde{k}| - \nabla E_\sigma(p+k)\cdot\tilde{k})} - \frac{\nabla E_\sigma(p)\cdot\epsilon_\lambda(\tilde{k})\,b_\lambda(\tilde{k}) - h.c.}{|\tilde{k}|^{1/2}(|\tilde{k}| - \nabla E_\sigma(p)\cdot\tilde{k})}\right).$$

Using Proposition 5.3, one verifies that

$$\Big|\frac{\nabla E_\sigma(p+k)\cdot\epsilon_\lambda(\tilde{k})}{|\tilde{k}|^{1/2}(|\tilde{k}| - \nabla E_\sigma(p+k)\cdot\tilde{k})} - \frac{\nabla E_\sigma(p)\cdot\epsilon_\lambda(\tilde{k})}{|\tilde{k}|^{1/2}(|\tilde{k}| - \nabla E_\sigma(p)\cdot\tilde{k})}\Big| \leq \frac{C\,|k|}{|\tilde{k}|^{\frac{3}{2}}}. \tag{5.26}$$

Hence (5.17) implies that

$$\begin{aligned}\big\|C(\rho)\Phi_\sigma(p)\big\| &\leq C\,\alpha^{\frac{1}{2}}\,|k|\,\Big\|\frac{\mathbf{1}_{\rho\leq|\tilde{k}|\leq 1}(\tilde{k})}{|\tilde{k}|^{\frac{3}{2}}}\Big\|_{L^2_{\tilde{k}}}\big\|(N+1)^{\frac{1}{2}}\Phi_\sigma(p)\big\|\\ &\leq C\,\alpha^{\frac{1}{2}}\,|k|\,\ln(\rho^{-1}),\end{aligned} \tag{5.27}$$

where we used (5.9) in the last inequality. Equations (5.25) and (5.27) yield

$$\left\| \left(\left(W^{\rho,1}_{\nabla E_\sigma(p+k)}\right)^* - \left(W^{\rho,1}_{\nabla E_\sigma(p)}\right)^* \right) \Phi_\sigma(p) \right\| \le C\, \alpha^{\frac{1}{2}}\, |k|\, \ln(\rho^{-1}). \tag{5.28}$$

It remains to estimate the second term in the right side of (5.24). By unitarity of $W^{\rho,1}_{\nabla E_\sigma(p+k)}$, it suffices to estimate $\|\Phi_\sigma(p+k) - \Phi_\sigma(p)\|$. Using (5.6) and the relation

$$\begin{aligned}
&\big\|(\widetilde{\Pi}_\sigma(p) - \widetilde{\Pi}_\sigma(p+k))\varphi\big\|^2 \\
&\quad = \langle \varphi, (\widetilde{\Pi}_\sigma(p+k) + \widetilde{\Pi}_\sigma(p) - \widetilde{\Pi}_\sigma(p)\widetilde{\Pi}_\sigma(p+k) - \widetilde{\Pi}_\sigma(p+k)\widetilde{\Pi}_\sigma(p))\varphi\rangle \\
&\quad = \langle \varphi, (\widetilde{\Pi}^\perp_\sigma(p+k)\widetilde{\Pi}_\sigma(p) + \widetilde{\Pi}^\perp_\sigma(p)\widetilde{\Pi}_\sigma(p+k))\varphi\rangle, \\
&\quad = \langle \varphi, (\widetilde{\Pi}_\sigma(p)\widetilde{\Pi}^\perp_\sigma(p+k)\widetilde{\Pi}_\sigma(p) + \widetilde{\Pi}^\perp_\sigma(p)\widetilde{\Pi}_\sigma(p+k)\widetilde{\Pi}^\perp_\sigma(p))\varphi\rangle, \\
&\quad = \|\widetilde{\Pi}^\perp_\sigma(p+k)\widetilde{\Pi}_\sigma(p)\varphi\|^2 + \|\widetilde{\Pi}_\sigma(p+k)\widetilde{\Pi}^\perp_\sigma(p)\varphi\|^2,
\end{aligned}$$

for any $\varphi \in \mathfrak{F}_\sigma$, where $\widetilde{\Pi}^\perp_\sigma(p) := I - \widetilde{\Pi}_\sigma(p)$, we obtain that

$$\begin{aligned}
\|\Phi_\sigma(p+k) - \Phi_\sigma(p)\| &= \|\widetilde{\Phi}_\sigma(p+k) - \widetilde{\Phi}_\sigma(p)\| \\
&\le \frac{2}{\|\widetilde{\Pi}_\sigma(p)\Omega_\sigma\|} \big\|(\widetilde{\Pi}_\sigma(p) - \widetilde{\Pi}_\sigma(p+k))\Omega_\sigma\big\| \\
&\le 6\big\|\widetilde{\Pi}_\sigma(p) - \widetilde{\Pi}_\sigma(p+k)\big\| \\
&\le 6(\big\|\widetilde{\Pi}^\perp_\sigma(p+k)\widetilde{\Pi}_\sigma(p)\| + \|\widetilde{\Pi}^\perp_\sigma(p)\widetilde{\Pi}_\sigma(p+k)\big\|) \\
&\le 6(\|\widetilde{\Pi}^\perp_\sigma(p+k)\widetilde{\Phi}_\sigma(p)\| + \|\widetilde{\Pi}^\perp_\sigma(p)\widetilde{\Phi}_\sigma(p+k)\|).
\end{aligned} \tag{5.29}$$

Since there is an energy gap of size $\eta\sigma$ above $E_\sigma(p+k)$ in the spectrum of the operator $\widetilde{K}_\sigma(p+k)$, we can estimate

$$\widetilde{\Pi}^\perp_\sigma(p+k) \le \frac{1}{\eta\sigma}\left(\widetilde{K}_\sigma(p+k) - E_\sigma(p+k)\right),$$

and hence

$$\|\widetilde{\Pi}^\perp_\sigma(p+k)\widetilde{\Phi}_\sigma(p)\| \le \frac{2}{\eta^{1/2}\sigma^{1/2}} \left\|\left(\widetilde{K}_\sigma(p+k) - E_\sigma(p+k)\right)^{1/2}\widetilde{\Phi}_\sigma(p)\right\|. \tag{5.30}$$

We have by (2.15), (5.1), the definition after (5.3) and (2.6)

$$\widetilde{K}_\sigma(p+k) = \widetilde{K}_\sigma(p) + k\cdot\nabla_p\widetilde{K}_\sigma(p) + k^2/2, \tag{5.31}$$

where $\nabla_p\widetilde{K}_\sigma(p) := W^1_{\nabla E_\sigma(p)}\, \nabla_p H_\sigma(p)\, (W^1_{\nabla E_\sigma(p)})^*$, with $\nabla_p H_\sigma(p) := p - P_f - \alpha^{\frac{1}{2}} A_\sigma$. Using this expansion and the Feynman-Hellman formula,

$$\langle \tilde{\Phi}_\sigma(p), \nabla_p\widetilde{K}_\sigma(p)\tilde{\Phi}_\sigma(p)\rangle = \nabla E_\sigma(p)\,, \tag{5.32}$$

together with the mean-value theorem and Proposition 5.3, we have that (see also [7, Lemma 3.6])

$$\begin{aligned}
&\big\|(\widetilde{K}_\sigma(p+k) - E_\sigma(p+k))^{\frac{1}{2}}\widetilde{\Phi}_\sigma(p)\big\|^2 \\
&\quad = \big\langle \widetilde{\Phi}_\sigma(p), (\widetilde{K}_\sigma(p+k) - E_\sigma(p+k))\widetilde{\Phi}_\sigma(p)\big\rangle \\
&\quad = \big\langle \widetilde{\Phi}_\sigma(p), (\widetilde{K}_\sigma(p) + k\cdot(\nabla_p \widetilde{K}_\sigma(p)) + k^2/2 - E_\sigma(p+k))\widetilde{\Phi}_\sigma(p)\big\rangle \\
&\quad = E_\sigma(p) - E_\sigma(p+k) + k\cdot(\nabla_p E_\sigma(p)) + k^2/2 \\
&\quad = \frac{1}{2}k^2 + \int_0^1 k\cdot\big[\nabla_p E_\sigma(p) - \nabla_p E_\sigma(p+\tau k)\big]\,\mathrm{d}\tau \\
&\quad \le C\,k^2.
\end{aligned} \tag{5.33}$$

Hence,

$$\big\|\big(\widetilde{K}_\sigma(p+k) - E_\sigma(p+k)\big)^{\frac{1}{2}}\tilde{\Phi}_\sigma(p)\big\| \le C\,|k|. \tag{5.34}$$

Combining (5.30) and (5.34), we obtain that

$$\|\widetilde{\Pi}_\sigma^\perp(p+k)\widetilde{\Phi}_\sigma(p)\| \le C\,|k|\,\sigma^{-\frac{1}{2}}. \tag{5.35}$$

Proceeding in the same way, it follows likewise that

$$\|\widetilde{\Pi}_\sigma^\perp(p)\widetilde{\Phi}_\sigma(p+k)\| \le C\,|k|\,\sigma^{-\frac{1}{2}}\,, \tag{5.36}$$

and hence, by (5.29), (5.23) follows.

Step 2. We now prove that $\|\Phi_\sigma^\rho(p+k) - \Phi_\sigma^\rho(p)\| \le C_\delta\,(1+\alpha^{\frac{1}{2}}\ln(\rho^{-1}))\,|k|^{\frac{2}{3}-\delta}$ (with $C_\delta < \infty$ for $\delta > 0$).

Suppose first that $\sigma \ge |k|^{2/3}$. Then by Step 1, we have that

$$\begin{aligned}
\|\Phi_\sigma^\rho(p+k) - \Phi_\sigma^\rho(p)\| &\le C\,|k|\,\big(|k|^{-\frac{1}{3}} + \alpha^{\frac{1}{2}}\,\ln(\rho^{-1})\big) \\
&= C\,|k|^{\frac{2}{3}} + C\,\alpha^{\frac{1}{2}}\,\ln(\rho^{-1})\,|k|.
\end{aligned} \tag{5.37}$$

Conversely, assume that $\sigma \le |k|^{2/3}$. We write

$$\begin{aligned}
&\|\Phi_\sigma^\rho(p+k) - \Phi_\sigma^\rho(p)\| \\
&\quad \le \|\Phi_\sigma^\rho(p+k) - \Phi^\rho(p+k)\| + \|\Phi^\rho(p+k) - \Phi^\rho_{|k|^{2/3}}(p+k)\| \\
&\qquad + \|\Phi_\sigma^\rho(p) - \Phi^\rho(p)\| + \|\Phi^\rho(p) - \Phi^\rho_{|k|^{2/3}}(p)\| \\
&\qquad + \|\Phi^\rho_{|k|^{2/3}}(p+k) - \Phi^\rho_{|k|^{2/3}}(p)\|.
\end{aligned} \tag{5.38}$$

By Corollary 5.2, the first two lines are bounded by

$$\begin{aligned}
&\|\Phi_\sigma^\rho(p+k) - \Phi^\rho(p+k)\| + \|\Phi^\rho(p+k) - \Phi^\rho_{|k|^{2/3}}(p+k)\| \\
&\quad + \|\Phi_\sigma^\rho(p) - \Phi^\rho(p)\| + \|\Phi^\rho(p) - \Phi^\rho_{|k|^{2/3}}(p)\| \\
&\le C_\delta\,\alpha^{\frac{1}{4}}\,\big(1 + \alpha^{\frac{1}{2}}\,\ln(\rho^{-1})\big)\,|k|^{\frac{2}{3}(1-\delta)}\,,
\end{aligned} \tag{5.39}$$

whereas by Step 1, the last term is bounded by $C\,|k|^{\frac{2}{3}} + C\,\alpha^{\frac{1}{2}}\,\ln(\rho^{-1})\,|k|$. Setting $\delta' = 2\delta/3$ and changing notations concludes the proof of the proposition. □

Acknowledgements

The authors are very grateful to Herbert Spohn and Stefan Teufel for pointing out a somewhat serious problem in an earlier version of this paper and for suggesting to us a solution. J.Fa., I.M.S., and T.C. are grateful to J.Fr. for hospitality at ETH Zürich. T.C. thanks I.M.S. for hospitality at the University of Toronto. The research of I.M.S. has been supported by NSERC under Grant NA 7901. T.C. has been supported by the NSF under grants DMS-0940145, DMS-1009448, and DMS-1151414 (CAREER).

Appendix A. Proof of Estimate (2.30)

In this Appendix, we prove (2.30). It asserts that

$$\|(K_\sigma^\rho(p) - H_\sigma(p))\Phi_\sigma^\rho(p)\|_{\mathfrak{F}} \le C\,\alpha^{\frac{1}{2}}\,\rho^{\frac{1}{2}}\,|p|\,, \tag{A.1}$$

for all $p \in \mathcal{S}$, for a constant $C < \infty$ independent of α, σ, and ρ, where $0 < \sigma < \rho \le 1$.

To begin with, let

$$v_\lambda^\sharp(k) := \alpha^{\frac{1}{2}}\,\mathbf{1}_{\sigma\le|k|\le\rho}(|k|)\,\frac{\nabla E_\sigma(p)\cdot\epsilon_\lambda^\sharp(k)}{|k|^{1/2}(|k| - \nabla E_\sigma(p)\cdot k)}\,, \tag{A.2}$$

(scalar-valued) and

$$w_\lambda^\sharp(k) := \alpha^{\frac{1}{2}}\,\mathbf{1}_{\sigma\le|k|\le 1}(|k|)\,\frac{\epsilon_\lambda^\sharp(k)}{|k|^{1/2}} \tag{A.3}$$

(vector-valued). We note that

$$|v_\lambda(k)| \le C\,\alpha^{\frac{1}{2}}\,|p|\,\frac{\mathbf{1}_{\sigma\le|k|\le\rho}(|k|)}{|k|^{\frac{3}{2}}} \tag{A.4}$$

and

$$|w_\lambda(k)| \le C\,\alpha^{\frac{1}{2}}\,\frac{\mathbf{1}_{\sigma\le|k|\le 1}(|k|)}{|k|^{\frac{1}{2}}} \tag{A.5}$$

where we have used that $|\nabla E_\sigma(p)| \le C\,|p|$, uniformly in the infrared cutoff $0 \le \sigma \le 1$.

Using that

$$W_{\nabla E_\sigma(p)}^{\sigma,\rho}\,b_\lambda^\sharp(k)\,(W_{\nabla E_\sigma(p)}^{\sigma,\rho})^* = b_\lambda^\sharp(k) + v_\lambda^\sharp(k)\,, \tag{A.6}$$

a straightforward calculation yields

$$\begin{aligned} K_\sigma^\rho(p) - H_\sigma(p) &= W_{\nabla E_\sigma(p)}^{\sigma,\rho}\,H_\sigma(p)\,(W_{\nabla E_\sigma(p)}^{\sigma,\rho})^* - H_\sigma(p) \\ &= 2V(p)\cdot(\nabla_p H_\sigma(p)) + V^2(p) + Y(p)\,, \end{aligned} \tag{A.7}$$

where

$$\nabla_p H_\sigma(p) = p - P_f - \alpha^{\frac{1}{2}}\,A_\sigma\,, \tag{A.8}$$

with

$$A_\sigma = \sum_\lambda \left(b_\lambda(w_\lambda) + b_\lambda^*(w_\lambda) \right), \tag{A.9}$$

and

$$V(p) := \sum_\lambda \Big[b_\lambda(kv_\lambda) + b_\lambda^*(kv_\lambda) + 2Re(w_\lambda, v_\lambda) + (v_\lambda, kv_\lambda) \Big], \tag{A.10}$$

(vector-valued operator) and

$$Y(p) := \sum_\lambda \Big[b_\lambda\big((k^2+|k|)v_\lambda\big) + b_\lambda^*\big((k^2+|k|)v_\lambda\big) + (v_\lambda, |k|v_\lambda) \\ +2Re(k \cdot w_\lambda, v_\lambda) \Big], \tag{A.11}$$

(scalar-valued operator). Note that both $V(p)$ and $Y(p)$ are proportional to $|\nabla E_\sigma(p)|$ since all terms are of first or higher order in v_λ (which is proportional to $|\nabla E_\sigma(p)| \le C|p|$).

Using Lemma 3.2 and (A.4), we observe that

$$\|V(p)(H_f+1)^{-1/2}\| \le 2\big\|(|k|+|k|^2)^{\frac{1}{2}} v_\lambda\big\|_{L^2} + \big\||k|^{\frac{1}{2}} v_\lambda\big\|_{L^2}^2 + \big\|w_\lambda\, v_\lambda\big\|_{L^1} \\ \le C\,\alpha^{1/2}\,|p|\,\rho^{1/2}, \tag{A.12}$$

and similarly

$$\|V(p)^2(H_f+1)^{-1}\| \le C\,\alpha\,|p|^2\,\rho, \tag{A.13}$$

$$\|Y(p)(H_f+1)^{-1/2}\| \le C\,\alpha^{1/2}\,|p|\,\rho. \tag{A.14}$$

Next we note that for any normalized vector $\Phi \in D(H(p))$, we have the estimate

$$\begin{aligned} &\Big\| \left(\frac{1}{2}(p-P_f)^2 + H_f + 1 \right) \Phi \Big\| \\ &\quad \le \big\|(H_\sigma(p)+1)\,\Phi\big\| + \alpha^{1/2}\big\|A_\sigma \cdot \nabla H_\sigma(p)\,\Phi\big\| + \alpha\big\|A_\sigma^2\,\Phi\big\| \\ &\quad \le \big\|(H_\sigma(p)+1)\,\Phi\big\| + C\,\alpha^{1/2}\big\|(H_f+1)^{1/2}\nabla H_\sigma(p)\,\Phi\big\| \\ &\qquad + C\,\alpha\big\|(H_f+1)\,\Phi\big\|. \end{aligned} \tag{A.15}$$

Since furthermore P_f and H_f commute, we have that $(H_f+1)^2 \le (\frac{1}{2}(p - P_f)^2 + H_f + 1)^2$ and hence

$$\big\|(H_f+1)\,\Phi\big\| \le 2\big\|(H_\sigma(p)+1)\,\Phi\big\| + C\,\alpha^{1/2}\big\|(H_f+1)^{1/2}\nabla H_\sigma(p)\,\Phi\big\|, \tag{A.16}$$

provided $\alpha > 0$ is sufficiently small. Now, we observe that

$$\big\|[H_f\,,\,\nabla H_\sigma(p)]\,(H_f+1)^{-1/2}\big\| = \alpha^{1/2}\big\|[H_f\,,\,A_\sigma]\,(H_f+1)^{-1/2}\big\| \le C\,\alpha^{1/2}, \tag{A.17}$$

which implies that

$$\begin{aligned} &\|(H_f+1)^{1/2}\nabla H_\sigma(p)\,\Phi\|^2 \\ &\quad = \big\langle \nabla H_\sigma(p)\Phi\cdot\,,\,(H_f+1)\,\nabla H_\sigma(p)\,\Phi\big\rangle \\ &\quad = \big\langle \nabla H_\sigma(p)^2\Phi\,,\,(H_f+1)\,\Phi\big\rangle - \big\langle [H_f\,,\,\nabla H_\sigma(p)]\,\Phi\cdot\,,\,\nabla H_\sigma(p)\,\Phi\big\rangle \\ &\quad \le C\big\|H_\sigma(p)\,\Phi\big\|\;\big\|(H_f+1)\,\Phi\big\| + C\,\alpha^{1/2}\|(H_f+1)^{1/2}\nabla H_\sigma(p)\,\Phi\|. \end{aligned} \tag{A.18}$$

Hence, for sufficiently small $\alpha > 0$, we have that

$$\|(H_f+1)^{1/2}\nabla H_\sigma(p)\,\Phi\| \le C\,\|H_\sigma(p)\,\Phi\|^{1/2}\,\|(H_f+1)\,\Phi\|^{1/2}. \tag{A.19}$$

Inserting this estimate into (A.16), we obtain for all normalized Φ that

$$\big\|(H_f+1)\,\Phi\big\| \le C\big\|(H_\sigma(p)+1)\,\Phi\big\|\,, \tag{A.20}$$

and, additionally using (A.19), that

$$\|(H_f+1)^{1/2}\nabla H_\sigma(p)\,\Phi\| \le C\big\|(H_\sigma(p)+1)\,\Phi\big\|\,, \tag{A.21}$$

provided $\alpha > 0$ is sufficiently small.

We arrive at the assertion by applying Estimates (A.12), (A.13), (A.14), (A.20), and (A.21),

$$\begin{aligned} &\big\|\big(K^\rho_\sigma(p) - H_\sigma(p)\big)\,\Phi^\rho_\sigma(p)\big\| \\ &\quad \le 2\|V(p)\cdot\nabla_p H_\sigma(p)\,\Phi^\rho_\sigma(p)\| + \|V(p)^2\,\Phi^\rho_\sigma(p)\| + \|Y(p)\,\Phi^\rho_\sigma(p)\| \\ &\quad \le 2\|V(p)(H_f+1)^{-1/2}\|\;\|(H_f+1)^{1/2}\nabla_p H_\sigma(p)\,\Phi^\rho_\sigma(p)\| \\ &\qquad + \|V(p)^2(H_f+1)^{-1}\|\;\|(H_f+1)\,\Phi^\rho_\sigma(p)\| \\ &\qquad + \|Y(p)(H_f+1)^{-1/2}\|\;\|(H_f+1)\,\Phi^\rho_\sigma(p)\| \\ &\quad \le C\,\alpha^{1/2}\,|p|\,\rho^{1/2}\,\big\|(H_\sigma(p)+1)\,\Phi^\rho_\sigma(p)\big\| \\ &\quad \le C'\,\alpha^{1/2}\,|p|\,\rho^{1/2}\,, \end{aligned} \tag{A.22}$$

which is Inequality (A.1) or (2.30), respectively. □

References

[1] Bach, V., Chen, T., Fröhlich, J., Sigal, I.M.: Smooth Feshbach map and oprator-theoretic renormalization group methods. J. Funct. Anal. **203**(1), 44–92 (2003)

[2] Bach, V., Chen, T., Fröhlich, J., Sigal, I.M.: The renormalized electron mass in non-relativistic QED. J. Funct. Anal. **243**(2), 426–535 (2007)

[3] Bach, V., Fröhlich, J., Pizzo, A.: Infrared-finite algorithms in QED: the groundstate of an atom interacting with the quantized radiation field. Commun. Math. Phys. **264**, 145–165 (2006)

[4] Bach, V., Fröhlich, J., Sigal, I.M.: Quantum electrodynamics of confined nonrelativistic particles. Adv. Math. **137**(2), 299–395 (1998)

[5] Bach, V., Fröhlich, J., Sigal, I.M.: Renormalization group analysis of spectral problems in quantum field theory. Adv. Math. **137**, 205–298 (1998)

[6] Chen, T.: Infrared renormalization in non-relativistic QED and scaling criticality. J. Funct. Anal. **254**(10), 2555–2647 (2008)

[7] Chen, T., Faupin, J., Fröhlich, J., Sigal, I.M.: Local decay in non-relativistic QED. Commun. Math. Phys. **309**, 543–583 (2012)

[8] Chen, T., Fröhlich, J.: Coherent infrared representations in non-relativistic QED. Spectral theory and mathematical physics: a festschrift in honor of Barry Simon's 60th Birthday, Proc. Symp. Pure Math., AMS (2007)

[9] Chen, T., Fröhlich, J., Pizzo, A.: Infraparticle scattering states in non-relativistic QED—I. The Bloch-Nordsieck paradigm. Commun. Math. Phys. **294**(3), 761–825 (2010)

[10] Chen, T., Fröhlich, J., Pizzo, A.: Infraparticle scattering states in non-relativistic QED—II. Mass shell properties. J. Math. Phys. **50**(1), 012103 (2009)

[11] Fröhlich, J.: On the infrared problem in a model of scalar electrons and massless, scalar bosons. Ann. Inst. Henri Poincaré, Section Physique Théorique **19**(1), 1–103 (1973)

[12] Fröhlich, J.: Existence of dressed one electron states in a class of persistent models. Fortschritte der Physik **22**, 159–198 (1974)

[13] Fröhlich, J., Pizzo, A.: The renormalized electron mass in nonrelativistic QED. Commun. Math. Phys. **294**, 439–470 (2010)

[14] Griesemer, M., Lieb, E.H., Loss, M.: Ground states in non-relativistic quantum electrodynamics. Invent. Math. **145**(3), 557–595 (2001)

[15] Hainzl, C., Seiringer, R.: Mass renormalization and energy level shift in non-relativistic QED. Adv. Theor. Math. Phys. **6**(5), 847–871 (2003)

[16] Hiroshima, F., Ito, K.R.: Mass renormalization in non-relativistic quantum electrodynamics with spin 1/2. Rev. Math. Phys. **19**, 405–454 (2007)

[17] Hiroshima, F., Spohn, H.: Mass renormalization in nonrelativistic quantum electrodynamics. J. Math. Phys. **46**(4), 042302 (2005)

[18] Kato, T.: Perturbation theory for linear operators. Springer, New-York (1966)

[19] Lieb, E.H., Loss, M.: Self-energy of electrons in non-perturbative QED. Conférence Moshé Flato 1999, Vol. I (Dijon), 327–344, Math. Phys. Stud., 21

[20] Lieb, E.H., Loss, M.: A bound on binding energies and mass renormalization in models of quantum electrodynamics. J. Stat. Phys. **108**, 1057–1069 (2002)

[21] Lieb, E.H., Loss, M.: The thermodynamic limit for matter interacting with Coulomb forces and with the quantized electromagnetic field. I. The lower bound. Commun. Math. Phys. **258**(3), 675–695 (2005)

[22] Pizzo, A.: One-particle (improper) states in Nelson's massless model. Ann. Henri Poincaré **4**(3), 439–486 (2003)

[23] Pizzo, A.: Scattering of an infraparticle: the one particle sector in Nelson's massless model. Ann. Henri Poincaré **6**(3), 553–606 (2005)

[24] Reed, M., Simon, B.: Methods of modern mathematical physics IV. Analysis of operators. Academic Press, New York (1978)

[25] Spohn, H.: Dynamics of charged particles and their radiation field. Cambridge University Press, Cambridge (2004)

[26] Spohn, H., Teufel, S.: Semiclassical motion of dressed electrons. Rev. Math. Phys. **14**(1), 1–28 (2002)

[27] Tenuta, L., Teufel, S.: Effective dynamics of particles coupled to a quantized scalar field. Commun. Math. Phys. **280**, 751–805 (2008)

Volker Bach
Institut fuer Analysis und Algebra
Carl-Friedrich-Gauss-Fakultaet
Technische Universitaet Braunschweig
38106 Braunschweig, Germany
e-mail: v.bach@tu-bs.de

Thomas Chen
Department of Mathematics
University of Texas at Austin
Austin, TX 78712, USA
e-mail: tc@math.utexas.edu

Jérémy Faupin
Institut de Mathématiques de Bordeaux
UMR-CNRS 5251
Université de Bordeaux 1
351 cours de la libération
33405 Talence Cedex, France
e-mail: jeremy.faupin@math.u-bordeaux1.fr

Jürg Fröhlich
Institut für Theoretische Physik
ETH Hönggerberg
8093 Zürich, Switzerland

and

Present address:
School of Mathematics
The Institute for Advanced Study
Princeton, NJ 08540, USA
e-mail: juerg@phys.ethz.ch

Israel Michael Sigal
Department of Mathematics
University of Toronto
Toronto, ON M5S 2E4, Canada
e-mail: im.sigal@utoronto.ca

Communicated by Abdelmalek Abdesselam.

Received: February 16, 2012.

Accepted: November 11, 2012.