Ann. Henri Poincaré 14 (2013), 1263–1285

1424-0637/13/051263-23
published online November 21, 2012
DOI 10.1007/s00023-012-0217-5

Annales Henri Poincaré

Enhanced Wegner and Minami Estimates and Eigenvalue Statistics of Random Anderson Models at Spectral Edges

François Germinet and Frédéric Klopp

Abstract. We consider the discrete Anderson model and prove enhanced Wegner and Minami estimates where the interval length is replaced by the IDS computed on the interval. We use these estimates to improve on the description of finite volume eigenvalues and eigenfunctions obtained in Germinet and Klopp (J Eur Math Soc http://arxiv.org/abs/1011.1832, 2010). As a consequence of the improved description of eigenvalues and eigenfunctions, we revisit a number of results on the spectral statistics in the localized regime obtained in Germinet and Klopp (J Eur Math Soc http://arxiv.org/abs/1011.1832, 2010) and Klopp (PTRF http://fr.arxiv.org/abs/1012.0831, 2010) and extend their domain of validity, namely:

- the local spectral statistics for the unfolded eigenvalues;
- the local asymptotic ergodicity of the unfolded eigenvalues.

In dimension 1, for the standard Anderson model, the improvement enables us to obtain the local spectral statistics at band edge, that is in the Lifshitz tail regime. In higher dimensions, this works for modified Anderson models.

1. Introduction

Anderson models are known to exhibit a region of localized states, either at the edges of the spectrum, or in a given range of energies if the disorder is large enough. Within this region of localization it is natural to study the two basic components of the spectral theory in this case: the eigenfunctions (how localized they are, where, etc.), and the eigenvalues (their multiplicity, their statistics, etc.). The localization properties of eigenfunctions in the localized phase are by now quite well understood (e.g. [1,6,8,14]). The precise description of these localization properties plays an important role in the understanding of many physical phenomena (e.g. dynamical localization, constancy of the Hall

The authors are supported by the Grant ANR-08-BLAN-0261-01. The authors would also like to thank the Centre Interfacultaire Bernoulli (EPFL, Lausanne) for its hospitality.

conductance in quantum Hall systems, Mott formula). Much less works have been devoted to the understanding of the eigenvalues statistics of the Anderson model; for random matrices many more results are available (e.g. [4,13,28]). Poisson statistics of Anderson eigenvalues have been studied in [10,23,26].

An important ingredient that enters the proof of localization is a so called Wegner estimate that controls the probability of finding an eigenvalue of the finite volume operator in a small interval of energy. As for Poisson statistics, the analysis of [10,23] relies on localization properties, the Wegner estimate and on a so called Minami estimate that enables one to control the probability of the occurrence of two eigenvalues of the finite volume operator in a small interval of energy. It is worth mentioning that simplicity of the spectrum is a direct consequence of localization properties combined with a Minami estimate [15].

Recently, in [10], the present authors introduced a refined way of describing eigenvalues and centers of localization in finite volumes, through a reduction procedure that enables one to approximate eigenvalues at finite volume by independent and identically distributed random variables. With this reduction in hand, they could in particular extend known results about Poisson statistics and obtain the first asymptotic result for the eigenlevel spacings distribution. [21] used this reduction to study the local asymptotic ergodicity of the unfolded eigenvalues.

In the present article, we introduce enhanced Wegner and Minami estimates that are valid only within the region of localization. The main novelty is that they take into account the weight that the integrated density of states (IDS) gives to intervals to estimate the probabilities of the occurrence of a single or of multiple eigenvalues in a small energy interval. These estimates enable us to revisit the reduction procedure mentioned above and get better controls. We thus remove some limitations of [10] and cover situations where the IDS gets too small for the analysis of [10] to be valid, for example, when the IDS is exponentially small in an inverse power of the length of the interval. As an application, our results enable us to prove Poisson statistics for the unfolded eigenvalues in dimension 1 at the band edges, that is where a Lifshitz tail regime occurs. To our best knowledge, this is the first such result. As another application, we provide improved large deviation estimates for the number of finite volume eigenvalues contained in suitably scaled intervals, as well as a central limit theorem for this quantity.

2. Main Results

We consider the discrete Anderson Hamiltonian

$$H_\omega := H_0 + V_\omega, \tag{2.1}$$

acting on $\ell^2(\mathbb{Z}^d)$, where

- H_0 is a convolution matrix with exponentially decaying off-diagonal coefficients i.e. exponential off-diagonal decay that is $H_0 = ((h_{k-k'}))_{k,k'\in\mathbb{Z}^d}$ such that,

– $h_{-k} = \overline{h_k}$ for $k \in \mathbb{Z}^d$ and for some $k \neq 0, h_k \neq 0$.
– there exists $c > 0$ such that, for $k \in \mathbb{Z}^d$,

$$|h_k| \leq \frac{1}{c} \mathrm{e}^{-c|k|}. \tag{2.2}$$

Define

$$h(\theta) = \sum_{k \in \mathbb{Z}^d} h_k \mathrm{e}^{\mathrm{i}k\theta} \quad \text{where } \theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d. \tag{2.3}$$

- V_ω is an Anderson potential:

$$V_\omega(x) := \sum_{j \in \mathbb{Z}^d} \omega_j \Pi_j. \tag{2.4}$$

where Π_j is the projection onto site j, and $\omega = \{\omega_j\}_{j \in \mathbb{Z}^d}$ is a family of independent identically distributed random variables whose common probability distribution μ is non-degenerate and has a bounded density g.

We denote by

- $\Sigma \subset \mathbb{R}$ the almost sure spectrum of H_ω (see e.g. [14,27]); it is known that $\Sigma = h(\mathbb{R}^d) + \operatorname{supp} g$;
- $\Sigma_{\mathrm{SDL}} \subset \Sigma \subset \mathbb{R}$ the set of energies where strong dynamical localization holds; we refer to Theorem 4.1 for a precise description of Σ_{SDL}; it is known that such a region of energies exists at least near the edges of the spectrum Σ (see e.g. [1,6–8,14]).

Recall (see [14]) the integrated density of states (IDS) may be defined as

$$N(E) = \mathbb{E}\,\mathrm{tr}(\Pi_0 \mathbf{1}_{]-\infty,E]}(H_\omega)\Pi_0) = \mathbb{E}\langle \delta_0, \mathbf{1}_{]-\infty,E]}(H_\omega)\delta_0\rangle. \tag{2.5}$$

In particular, if I is an interval, we define $N(I)$ as

$$N(I) := \mathbb{E}\,\mathrm{tr}(\Pi_0 \mathbf{1}_I(H_\omega)\Pi_0) = \mathbb{E}\langle \delta_0, \mathbf{1}_I(H_\omega)\delta_0\rangle. \tag{2.6}$$

For $L > 1$, consider $\Lambda = [-L, L]^d \cap \mathbb{Z}^d$, a cube on the lattice and let $H_\omega(\Lambda)$ be the random Hamiltonian H_ω restricted to Λ with periodic boundary conditions.

Notations: by $a \lesssim b$ we mean there exists a constant $c \in]0, \infty[$ so that $a \leq cb$. By $a \asymp b$ we mean there exists a constant $c \in]1, \infty[$ such that $c^{-1}b \leq a \leq cb$.

2.1. Improved Versions of the Wegner and Minami Estimates

We show Wegner and Minami estimates where the upper bounds keep track of the integrated density of states. In particular, they enable to take advantage of the smallness of the integrated density when this happens.

Let us first recall the usual Wegner and Minami estimates that are known to hold for H_ω (see e.g. [2,3,11,14] and references therein):

(W) $\mathbb{E}[\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda)))] \leq C|J|\,|\Lambda|$;
(M) $\mathbb{E}[\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda))) \cdot [\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda))) - 1]] \leq C(|J|\,|\Lambda|)^2$.

Our main result is

Theorem 2.1. *Fix $\xi \in (0,1)$. There exist constants $c, C \in (0, +\infty)$ such that for $L > 1$ the following holds.*

1. *Let* $I \subset \Sigma_{\mathrm{SDL}}$ *be a compact interval. Then*
$$|\mathbb{E}\,\mathrm{tr}\,\mathbf{1}_I(H_\omega(\Lambda)) - N(I)|\Lambda|| \le C\exp(-cL^\xi). \tag{2.7}$$
As a consequence, if $|N(I)| \ge C\exp(-cL^\xi)$ *we get the Wegner estimate:*
$$\mathbb{E}(\mathrm{tr}\,\mathbf{1}_I(H_\omega(\Lambda))) \le 2N(I)|\Lambda|. \tag{2.8}$$
2. (*High order Minami*) *Given* $n \ge 2$ *and* $I_1 \subset \cdots \subset I_n \subset \Sigma_{\mathrm{SDL}}$ *intervals so that* $|N(I_n)| \ge C\exp(-cL^\xi)$,
$$\mathbb{E}\left(\prod_{k=1}^{n}(\mathrm{tr}\,\mathbf{1}_{I_k}(H_\omega(\Lambda)) - k + 1)\right) \le 2\left(\prod_{k=1}^{n-1}\|\rho\|_\infty|I_k||\Lambda|\right)N(I_n)|\Lambda|. \tag{2.9}$$
In particular, for $n = 2$*, if* $|N(I)| \ge C\exp(-cL^\xi)$*, we get the Minami estimate:*
$$\mathbb{E}[\mathrm{tr}\,\mathbf{1}_I(H_\omega(\Lambda))(\mathrm{tr}\,\mathbf{1}_I(H_\omega(\Lambda)) - 1)] \le 2N(I)|I||\Lambda|^2. \tag{2.10}$$

Remark 2.1. (i) The constant 2 in (2.8), (2.9) and (2.10) can be replaced by any constant larger than 1, provided $|\Lambda|$ is large enough.

2.2. Local Spectral Statistics

We shall combine Theorem 2.1 with the description of the eigenvalues of $H_\omega(\Lambda)$ obtained in the paper [10] to obtain new results for the spectral statistics of the Anderson model. In particular, the improved Minami estimate enables us to remove the restriction on the smallness of the density of states that was imposed in [10,21] to obtain results locally in energy.

Consider the eigenvalues of $H_\omega(\Lambda)$ ordered increasingly and repeated according to multiplicity and denote them by
$$E_1(\omega,\Lambda) \le E_2(\omega,\Lambda) \le \cdots \le E_{|\Lambda|}(\omega,\Lambda).$$
Following [24,25], define the *unfolded eigenvalues* as
$$0 \le N(E_1(\omega,\Lambda)) \le N(E_2(\omega,\Lambda)) \le \cdots \le N(E_{|\Lambda|}(\omega,\Lambda)) \le 1.$$
Let E_0 be an energy in Σ_{SDL}. The *unfolded local level statistics* near E_0 is the point process defined by
$$\Xi(\xi; E_0,\omega,\Lambda) = \sum_{j\ge 1}\delta_{\xi_j(E_0,\omega,\Lambda)}(\xi), \tag{2.11}$$
where
$$\xi_j(E_0,\omega,\Lambda) = |\Lambda|(N(E_j(\omega,\Lambda)) - N(E_0)). \tag{2.12}$$

Remark 2.2. It is convenient to consider the unfolded (or renormalized) levels rather than the levels themselves as these, independent of the system, are random variables distributed over $[0,1]$ (see [25]). In particular, one may hope that they follow a distribution independent of the density of states. This is in fact part of the content of Theorem 2.2.

The unfolded local level statistics are described by

Theorem 2.2. *Pick E_0 be an energy in Σ_{SDL} such that the integrated density of states satisfies, for some $\rho \in (0, 1/d), \exists\, a_0 > 0$ s.t. $\forall a \in (-a_0, a_0) \cap (\Sigma - E_0)$,*

$$|N(E_0 + a) - N(E_0)| \geq e^{-|a|^{-\rho}}. \tag{2.13}$$

When $|\Lambda| \to +\infty$, the point process $\Xi(E_0, \omega, \Lambda)$ converges weakly to

- *a Poisson point process on the real line with intensity 1 if $E_0 \in \overset{\circ}{\Sigma}$, the interior of Σ.*
- *a Poisson point process on the half line with intensity 1 if $E_0 \in \partial\Sigma$, the half-line being $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) if $(E_0 - \varepsilon, E_0) \cap \Sigma = \emptyset$ (resp. $(E_0, E_0 + \varepsilon) \cap \Sigma = \emptyset$) for some $\varepsilon > 0$.*

The main improvement over [10, Theorem 1.2] is that the decay in assumption (2.13) can be taken exponential (compare with [10, (1.12)]); it does not depend anymore on the Minami estimate.

In [10], we also state and prove stronger uniform results for the convergence to Poisson of the local unfolded statistics (see [10, Theorems 1.3 and 1.6]). In the present case, these results still hold under assumption (2.13). Moreover, the size of intervals over which the uniform convergence of Poisson statistics is proved in [10] can be notably improved thanks to the improved Wegner and Minami estimates of Theorem 2.1 if the density of states is zero at the point E_0. We refer to Remark 4.1. for further precisions.

2.3. Local Spectral Statistics at Spectral Edges

In dimension one, for any H_0 (thus, in particular, for the free Laplace operator), the condition (2.13) is satisfied at all the spectral edges, i.e. in the Lifshitz tails region as the Lifshitz exponent is 1/2 if the density g does not decay too fast at the edges of its support (see [17]). So, we get the Poisson behavior for the unfolded eigenvalues at all the spectral edges, namely,

Theorem 2.3. *Assume $d = 1$. Let $E_0 \in \partial\Sigma$.*

When $|\Lambda| \to +\infty$, the point process $\Xi(E_0, \omega, \Lambda)$ converges weakly to a Poisson point process on the half line with intensity 1 if $E_0 \in \partial\Sigma$, the half-line being $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) if $(E_0 - \varepsilon, E_0) \cap \Sigma = \emptyset$ (resp. $(E_0, E_0 + \varepsilon) \cap \Sigma = \emptyset$) for some $\varepsilon > 0$.

To the best of our knowledge, this is the first proof of Poisson asymptotics for the unfolded eigenvalues at the spectral edges. In [12,28], for fixed k, the authors studied the joint law of the first k eigenvalues of special one-dimensional random (continuous) models; for these models, the density of states $N(E)$ can be computed explicitly.

In higher dimensions, if H_0 is the free Laplace operator, the Lifshitz exponent at spectral edges is usually $d/2$, even more so at the bottom of the spectrum (see e.g. [16,18,19]); so condition (2.13) is not satisfied in this case. Nevertheless, it may be satisfied if H_0 is not the free Laplace operator as we shall see now.

By assumption, the function h defined in (2.3) is real analytic on $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$. Under some additional assumptions on h near its, say, minimum, one can show that condition (2.13) is satisfied near the infimum of the almost sure spectrum of H_ω (see e.g. [17,20])

Theorem 2.4. *Assume that* $\min_{\theta\in\mathbb{T}^d} h(\theta) = 0$*, that* $h^{-1}(0)$ *is discrete and that, for* $\theta_0 \in h^{-1}(0)$*, there exists* $\alpha > d^2$ *such that, for* θ *close to* θ_0*, one has* $h(\theta) \le |\theta - \theta_0|^\alpha$*.*

Let $E_- = \inf\Sigma$ *where* Σ *is the almost sure spectrum of* H_ω*.*

When $|\Lambda| \to +\infty$*, the point process* $\Xi(E_-,\omega,\Lambda)$ *converges weakly to a Poisson point process on the half line* $\mathbb{R}^+$ *with intensity* 1*.*

2.4. Ergodicity of the Local Eigenvalue Distribution

The local results of [21] can also be improved along the same lines. For $J = [a,b]$ a compact interval such that $N(b) - N(a) = N(J) > 0$ and a fixed configuration ω, consider the point process

$$\Xi_J(\omega,t,\Lambda) = \sum_{E_n(\omega,\Lambda)\in J} \delta_{N(J)|\Lambda|[N_J(E_n(\omega,\Lambda))-t]}, \tag{2.14}$$

under the uniform distribution in $[0,1]$ in t; here we have set

$$N_J(\cdot) := \frac{N(\cdot) - N(a)}{N(b) - N(a)}. \tag{2.15}$$

Theorem 2.5. *Pick* $E_0 \in \Sigma_{\mathrm{SDL}}$*.*

Fix $(I_\Lambda)_\Lambda$ *a decreasing sequence of intervals such that* $\sup_{I_\Lambda} |x| \to_{|\Lambda|\to+\infty} 0$*. Assume that, for some* $\delta \in (0,1)$*, one has*

$$|\Lambda|^\delta \cdot N(E_0 + I_\Lambda) \to +\infty, \tag{2.16}$$

and

$$\text{if } \ell' = o(L) \text{ then } \frac{N(E_0 + I_{\Lambda_{L+\ell'}})}{N(E_0 + I_{\Lambda_L})} \underset{|\Lambda|\to+\infty}{\to} 1. \tag{2.17}$$

Then, ω*-almost surely, the probability law of the point process* $\Xi_{E_0+I_\Lambda}(\omega,\cdot,\Lambda)$ *under the uniform distribution* $\mathbf{1}_{[0,1]}(t)dt$ *converges to the law of the Poisson point process on the real line with intensity* 1*.*

The main improvement over [21, Theorem 1.5] is that there is no restriction anymore on the relative sizes of $N(E_0 + I_\Lambda)$ and $|I_\Lambda|$, respectively, the density of states measure and the length of $E_0 + I_\Lambda$ (compare (2.16) and [21, (1.9)]).

2.5. Eigenvalue Spacings Statistics

As a consequence of Theorem 2.5, using the results of [24], we obtain the following result which improves upon [10, Theorem 1.5] and [21, Theorem 1.5] in the sense that we cover a larger region of energies and the required lower bound on the IDS is relaxed.

Theorem 2.6. *Fix* $E_0 \in \Sigma_{\mathrm{SDL}}$ *and* $(I_\Lambda)_\Lambda$ *a decreasing sequence of intervals satisfying* (2.16) *and* (2.17)*.*

Define

$$\delta N_j(\omega,\Lambda) = |\Lambda|(N(E_{j+1}(\omega,\Lambda)) - N(E_j(\omega,\Lambda))) \ge 0. \tag{2.18}$$

Define the empirical distribution of these spacings to be the random numbers, for $x \geq 0$

$$DLS(x; E_0 + I_\Lambda, \omega, \Lambda) = \frac{\#\{j;\ E_j(\omega,\Lambda) \in E_0 + I_\Lambda,\ \delta N_j(\omega,\Lambda) \geq x\}}{N(E_0 + I_\Lambda, \Lambda, \omega)}, \quad (2.19)$$

where $N(E_0 + I_\Lambda, \Lambda, \omega)$ *is the random number of eigenvalues of* $H_\omega(\Lambda)$ *in* $E_0 + I_\Lambda$*.*

Then, with probability 1*, as* $|\Lambda| \to +\infty$, $DLS(x; E_0 + I_\Lambda, \omega, \Lambda)$ *converges uniformly to the distribution* $x \mapsto e^{-x}$*, that is, with probability* 1*,*

$$\sup_{x \geq 0} |DLS(x; E_0 + I_\Lambda, \omega, \Lambda) - e^{-x}| \underset{|\Lambda| \to +\infty}{\to} 0. \quad (2.20)$$

Spacings statistics over intervals of macroscopic size are also available (see [10, Theorem 1.6] and [21, Theorem 1.2]) in the present context.

2.6. A Large Deviation and a Central Limit Theorem for the Eigenvalue Counting Function

Finally, in some regimes, we also can improve upon the large deviation estimate obtained for the eigenvalue counting function in [10, Theorem 1.3] for which we also prove a central limit theorem in

Theorem 2.7. *For* $L > 1$*, let* $\Lambda = \Lambda_L$*. Pick a sequence of compact intervals* $I_\Lambda \subset \Sigma_{\mathrm{SDL}}$ *so that, for some* $1 \leq \beta \leq \beta' < \alpha' \leq \alpha < \infty$*, for all* L*, one has*

$$|I_\Lambda|^{-\alpha'} \lesssim |\Lambda| \lesssim |I_\Lambda|^{-\alpha} \quad \text{and} \quad |I_\Lambda|^{\beta'} \lesssim N(I_\Lambda) \lesssim |I_\Lambda|^{\beta}. \quad (2.21)$$

Set $\nu_0 = \frac{1}{\alpha - \beta} \min(\alpha' - \beta', \frac{1}{d+1})$*.*

1. **Large deviation estimate.** *For* $\varepsilon > 0$ *small enough (depending on* ν_0*), we have*

$$\mathbb{P}\left\{ |\operatorname{tr} \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) - N(I_\Lambda)|\Lambda|| \geq (N(I_\Lambda)|\Lambda|)^{\max(\frac{1}{2}, 1-\nu_0)+\varepsilon} \right\}$$
$$\leq \exp(-(N(I_\Lambda)|\Lambda|)^{\varepsilon}).$$

2. **Central limit theorem.** *Assume* $\nu_0 > \frac{1}{2}$*. Then the random variable*

$$\frac{\operatorname{tr} \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) - N(I_\Lambda)|\Lambda|}{(N(I_\Lambda)|\Lambda|)^{\frac{1}{2}}}$$

converges in law to the standard Normal distribution.

We first point out that only the size of the intervals I_Λ matters, more precisely, their relative size compared to the volume and the density of states [see (2.21)]. In particular, the intervals I_Λ need not be centered at a given point. Let us also note that, by the standard Wegner estimate (W), one can always pick $\beta = 1$.

Let us close this section with a brief outline of the paper. In Sect. 3, we prove the enhanced Wegner and Minami estimates, namely Theorem 2.1. Then, we turn to the proofs of the results on spectral statistics. In Sect. 4, we prove

- three different theorems of approximation of the eigenvalues of $H_\omega(\Lambda)$ by eigenvalues on smaller cubes as in [10, Theorems 1.1 and 1.2] (each theorem being optimized according to a given point of view),
- the distribution functions of these approximated eigenvalues as in [10, Lemma 2.1].

In Sect. 5, we derive the spectral statistics theorem per se, namely Theorems 2.2, 2.5, 2.6 and 2.7.

3. The Proofs of the Enhanced Wegner and Minami Estimates

We start with the proof of Theorem 2.1. Then, we use it to derive the distribution of the "unique" eigenvalue of $H_\omega(\Lambda)$ in I when $N(I)|\Lambda|$ is small.

3.1. Proof of the Improved Wegner and Minami Estimates

Proof of Theorem 2.1. The proof of point (1) is analogous to that of [10, Lemma 2.2]. The main gain is obtained by the use of covariance which relies on the specific approximations we take for the finite volume Hamiltonians.

To that end, note that by covariance

$$\mathbb{E}[\mathrm{tr}\,\mathbf{1}_{I_\Lambda}(H_\omega(\Lambda))] = |\Lambda|\,\mathbb{E}[\mathrm{tr}(\chi_0\mathbf{1}_{I_\Lambda}(H_\omega(\Lambda))\chi_0)]. \tag{3.1}$$

Recall (2.5): the increase of the integrated density of states of H_ω on I_Λ is given by

$$N(I_\Lambda) = N(b_\Lambda) - N(a_\Lambda) = \mathbb{E}[\mathrm{tr}(\chi_0\mathbf{1}_{I_\Lambda}(H_\omega)\chi_0)].$$

To control $|\mathbb{E}[\mathrm{tr}(\chi_0\mathbf{1}_{I_\Lambda}(H_\omega(\Lambda))\chi_0)] - \mathbb{E}[\mathrm{tr}(\chi_0\mathbf{1}_{I_\Lambda}(H_\omega)\chi_0)]|$, we use localization estimates after having smoothed out the characteristic function. Let f_δ be a $\mathcal{C}^\infty$ and compactly supported function such that $f_\delta = 1$ in I_Λ, and $f_\delta = 0$ outside a neighborhood of length δ of I_Λ (δ is small enough so that the support of f_δ lies in Σ_{SDL}). Note that, by Wegner's estimate (W) and the Lipschitz continuity of N,

$$\begin{aligned}&|\mathbb{E}[\mathrm{tr}(\chi_0[\mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) - f_\delta(H_\omega(\Lambda))]\chi_0)]|\\&\quad+|\mathbb{E}[\mathrm{tr}(\chi_0(\mathbf{1}_{I_\Lambda}(H_\omega) - f_\delta(H_\omega))\chi_0)]| \le C\delta.\end{aligned} \tag{3.2}$$

To estimate $|\mathbb{E}[\mathrm{tr}(\chi_0(f_\delta(H_\omega(\Lambda)) - f_\delta(H_\omega))\chi_0)]|$, we use a Helffer–Sjöstrand formula to represent $f_\delta(H_\omega(\Lambda))$ and $f_\delta(H_\omega)$. As the support of f_δ lies in the localization region and as f is of $\mathcal{C}^\infty$ regularity, the exponential decay estimate for the resolvents of $H_\omega(\Lambda)$ and H_ω imply that, for $\xi' \in (\xi, 1)$, there exists $C > 0$ such that, for Λ sufficiently large (see e.g. the computation of [9, (8.9)]), one has,

$$|\mathbb{E}[\mathrm{tr}(\chi_0(f_\delta(H_\omega(\Lambda)) - f_\delta(H_\omega))\chi_0)]| \le \delta^{-C}\mathrm{e}^{-\ell^{\xi'}}.$$

We set $\delta = \mathrm{e}^{-\ell^{\xi''}}$ with $\xi'' \in (\xi, \xi')$. Plugging this into (3.2) and (3.1) yields (2.7) and completes the proof of (2.7). We turn to point (2) and shall take advantage of the strategy introduced in [3] to prove a Minami estimate from spectral averaging and Wegner estimate. We adapt [3, Theorem 2.3] to get Minami's estimate in its generalized form. The proof is the same except at the very last step, [3, (4.17)], where the estimate (2.8) is used in replacement of the usual Wegner bound. □

Remark 3.1. The periodic boundary conditions imposed upon the restrictions $H_\omega(\Lambda)$ are important as they enable us to preserve covariance.

3.2. The Distribution of the "Local" Eigenvalues

Consider a cube Λ of side length ℓ and an interval $I_\Lambda = [a_\Lambda, b_\Lambda] \subset I$ (i.e. I_Λ is contained in the localization region). Consider the following random variables:

- $X = X(\Lambda, I_\Lambda)$ is the Bernoulli random variable
$$X = \mathbf{1}_{H_\omega(\Lambda) \text{ has exactly one eigenvalue in } I_\Lambda};$$
- $\tilde{E} = \tilde{E}(\Lambda, I_\Lambda)$ is the eigenvalue of $H_\omega(\Lambda)$ in I_Λ conditioned on $X = 1$;
- $\tilde{\xi} = \tilde{\xi}(\Lambda, I_\Lambda) = (\tilde{E}(\Lambda, I_\Lambda) - a_\Lambda)/|I_\Lambda|$.

Clearly $\tilde{\xi}$ is valued in $[0, 1]$; let $\tilde{\Xi}$ be its distribution function.

In the present section, we will describe the distribution of these random variables as $|\Lambda| \to +\infty$ and $|I_\Lambda| \to 0$. We prove

Lemma 3.1. *For any $\nu \in (0, 1)$ and K compact interval in Σ_{SDL}, there exists $C > 1$ such that, for $\Lambda = \Lambda_\ell$ and $I_\Lambda \subset K$ such that $N(I_\Lambda) \geq e^{-\ell^\nu/C}$, one has*

$$\mathbb{P}(X = 1) = N(I_\Lambda)|\Lambda|(1 + O(|I_\Lambda||\Lambda|) + O(\mathrm{e}^{-\ell^\nu})), \tag{3.3}$$

where $O(\cdot)$ are locally uniform in Σ_{SDL}.

Moreover, for $(x, y) \in [0, 1]$, one has

$$\begin{aligned}
&(\tilde{\Xi}(x) - \tilde{\Xi}(y))\, P(X = 1) \\
&\quad = [N(a_\Lambda + x|I_\Lambda|) - N(a_\Lambda + y|I_\Lambda|)]|\Lambda|\,(1 + O(|x - y||I_\Lambda||\Lambda|) \\
&\qquad + O((|x - y|)^{-C}\mathrm{e}^{-\ell^\nu/C})
\end{aligned} \tag{3.4}$$

where $O(\cdot)$ are locally uniform in Σ_{SDL}.

This lemma is to be compared with [10, Lemma 2.1]; it gives a fairly good description of the random variables X and $\tilde{\xi}$ if $|I_\Lambda||\Lambda| \ll 1$.

Proof of Lemma 3.1. We follow the proof of [10, Lemma 2.1]. Using (2.10), the estimate [10, (2.5)] becomes

$$0 \leq \mathbb{E}(\mathrm{tr}[\mathbf{1}_{I_\Lambda}(H_\omega(\Lambda))]) - \mathbb{P}(X = 1) \leq CN(I_\Lambda)|I_\Lambda||\Lambda|^2.$$

Thus, the proof of [10, Lemma 2.1] yields

$$\mathbb{P}(X = 1) = N(I_\Lambda)|\Lambda|(1 + O(|I_\Lambda||\Lambda|) + O(|I_\Lambda|^{-C}\mathrm{e}^{-\ell^\nu/C}). \tag{3.5}$$

Recall that, by Wegner's estimate, $N(I_\Lambda)|\Lambda| \leq C|I_\Lambda||\Lambda|$. Thus, as $N(I_\Lambda) \geq \mathrm{e}^{-\ell^\nu/C}$, enlarging possibly C, one obtains (3.3).

Replacing I_Λ with the interval $I_{x,y,\Lambda}$ in the estimation of $P(X = 1)$ yields the proof of (3.4).

The proof of Lemma 3.1 is complete. □

4. Box Reduction and Description of the Eigenvalues

After giving a precise description of $\Sigma_{\rm SDL}$, we shall state three different reductions that improve on the ones given in [10] thanks to Theorem 2.1. Each of them is optimized according to a given point of view, depending on the application it will be used for (Poisson statistics, local ergodicity of eigenvalues and level spacings statistics, or deviation estimates and CLT).

4.1. The Strong Dynamical Localization Regime

Before turning to the local description of the eigenvalues, let us underline that we will give this description in the discrete setting. In particular, we will use the fact that in the localization region $\Sigma_{\rm SDL}$, we have true exponential decay.

We first recall

Theorem 4.1 [10]. *Let $I \subset \Sigma$ be a compact interval and assume that Wegner's estimate* **(W)** *holds in I. For L given, consider $\Lambda = \Lambda_L(0)$ a cube of side length L centered at 0, and denote by $\varphi_{\omega,\Lambda,j}$, $j = 1, \ldots, \mathrm{tr}\mathbf{1}_I(H_\omega(\Lambda))$, the normalized eigenvectors of $H_\omega(\Lambda)$ with corresponding eigenvalue in I. The following are equivalent*

1. $I \subset \Sigma_{\rm SDL}$
2. *For all $E \in I$, there exists $\theta > 3d - 1$,*

$$\limsup_{L\to\infty} \mathbb{P}\left\{\forall x, y \in \Lambda, |x-y| \geq \frac{L}{2}, \|\chi_x(H_\omega(\Lambda)-E)^{-1}\chi_y\| \leq L^{-\theta}\right\} = 1. \tag{4.1}$$

3. *There exists $\xi > 0$,*

$$\sup_{y\in\Lambda} \mathbb{E}\left\{\sum_{x\in\Lambda} \mathrm{e}^{\xi|x-y|} \sup_j \|\varphi_{\omega,\Lambda,j}\|_x \|\varphi_{\omega,\Lambda,j}\|_y\right\} < \infty. \tag{4.2}$$

4. *There exists $\xi > 0$,*

$$\sup_{y\in\Lambda} \mathbb{E}\left\{\sum_{x\in\Lambda} \mathrm{e}^{\xi|x-y|} \sup_j \|\varphi_{\omega,\Lambda,j}\|_x \|\varphi_{\omega,\Lambda,j}\|_y\right\} < \infty. \tag{4.3}$$

5. *There exists $\xi > 0$,*

$$\sup_{y\in\Lambda} \mathbb{E}\left\{\sum_{x\in\Lambda} \mathrm{e}^{\xi|x-y|} \sup_{\substack{\mathrm{supp}\, f\subset I\\ |f|\leq 1}} \|\chi_x f(H_\omega(\Lambda))\chi_y\|_2\right\} < \infty. \tag{4.4}$$

6. *There exists $\xi > 0$,*

$$\sup_{y\in\Lambda} \mathbb{E}\left\{\sum_{x\in\Lambda} \mathrm{e}^{\xi|x-y|} \sup_{\substack{\mathrm{supp}\, f\subset I\\ |f|\leq 1}} \|\chi_x f(H_\omega(\Lambda))\chi_y\|_2\right\} < \infty. \tag{4.5}$$

7. *There exists* $\xi > 0$,

$$\sup_{y\in\Lambda}\sup_{\substack{\operatorname{supp} f\subset I\\ |f|\leq 1}} \mathbb{E}\left\{\sum_{x\in\Lambda} \mathrm{e}^{\xi|x-y|}\|\chi_x f(H_\omega(\Lambda))\chi_y\|_2\right\} < \infty. \tag{4.6}$$

8. (*SUDEC for finite volume with polynomial probability*) *There exists* $\xi > 0$ *such that for all* $p > d$*, there is* $q = q_{p,d}$ *so that for any* L *large enough, the following holds with probability at least* $1 - L^{-p}$*: for any eigenvector* $\varphi_{\omega,\Lambda,j}$ *of* $H_{\omega,\Lambda}$*, with energy in* I*, for any* $(x, y) \in \Lambda^2$*, one has*

$$\|\varphi_{\omega,\Lambda,j}\|_x\|\varphi_{\omega,\Lambda,j}\|_y \leq L^q\mathrm{e}^{-\xi|x-y|}. \tag{4.7}$$

9. (*SULE for finite volume with polynomial probability*) *There exists* $\xi > 0$ *such that for all* $p > d$*, there is* $q = q_{p,d}$ *so that, for any* L *large enough, the following holds with probability at least* $1 - L^{-p}$*: for any eigenvector* $\varphi_{\omega,\Lambda,j}$ *of* $H_{\omega,\Lambda}$*, with energy in* I*, there exists a center of localization, that is, a point* $x_{\omega,\Lambda,j} \in \Lambda$*, so that for any* $x \in \Lambda$*, one has*

$$\|\varphi_{\omega,\Lambda,j}\|_x \leq L^q\mathrm{e}^{-\xi|x-x_{\omega,\Lambda,j}|}. \tag{4.8}$$

10. (*SUDEC for finite volume with sub-exponential probability*) *There exists* $\xi > 0$ *such that for all* $\nu \in (0, 1)$*, for any* L *large enough, the following holds with probability at least* $1 - \mathrm{e}^{-L^\nu}$*: for any eigenvector* $\varphi_{\omega,\Lambda,j}$ *of* $H_{\omega,\Lambda}$*, with energy in* I*, for any* $(x, y) \in \Lambda^2$*, one has*

$$\|\varphi_{\omega,\Lambda,j}\|_x\|\varphi_{\omega,\Lambda,j}\|_y \leq \mathrm{e}^{2L^\nu}\mathrm{e}^{-\xi|x-y|}. \tag{4.9}$$

11. (*SULE for finite volume with sub-exponential probability*) *There exists* $\xi > 0$ *such that for all* $\nu \in (0, 1)$*, for any* L *large enough, the following holds with probability at least* $1 - \mathrm{e}^{-L^\nu}$*: for any eigenvector* $\varphi_{\omega,\Lambda,j}$ *of* $H_{\omega,\Lambda}$*, with energy in* I*, there is a center of localization* $x_{\omega,\Lambda,j} \in \Lambda$*, so that for any* $x \in \Lambda$*, one has*

$$\|\varphi_{\omega,\Lambda,j}\|_x \leq \mathrm{e}^{2L^\nu}\mathrm{e}^{-\xi|x-x_{\omega,\Lambda,j}|}. \tag{4.10}$$

Moreover one can pick $q = p + d$ *in* (8) *and* $q = p + \frac{3}{2}d$ *in* (9).

Let us note here that the centers of localizations [defined in point (9) or (11)] are not unique for a given eigenfunction; nevertheless, one can easily show that all the centers of localization of a given eigenfunction are contained in a ball of radius at most $C\log L$ (resp. CL^ν) in the sense of point (9) [resp. (11)] of Theorem 4.1.

4.2. Controlling all the Eigenvalues

Assume E_0 is such that (2.13) holds for some $\rho \in (0, 1/d)$. Let us first explain why the restriction $\rho < 1/d$ is necessary. For an interval I_Λ to contain a large number of eigenvalues of $H_\omega(\Lambda)$ (at least in expectation), one needs that $N(I_\Lambda)|\Lambda|$ be large. On the other hand, as we shall see in the proof of the next result, we also need $|I_\Lambda|(\log|\Lambda|)^d$ to be small. This second restriction is essentially enforced by the localization of the eigenfunctions in region of (linear) size $\log|\Lambda|$. These two requirements can only be met if (2.13) holds.

We prove

Theorem 4.2. *Assume E_0 is such that (2.13) holds for some $\rho \in (0, 1/d)$. Fix $0 < d\rho < \rho' < \rho'' < 1$ and $0 < \alpha < \min(d/\rho' - d/\rho'', 1/\rho - d/\rho')$. Pick I_Λ centered at E_0 such that $N(I_\Lambda)|\Lambda| = \log^\alpha |\Lambda|$. For L sufficiently large and $\ell = \log^{1/\rho'} L$, we have a decomposition of Λ into disjoint cubes of the form $\Lambda_\ell(\gamma_j) := \gamma_j + [0, \ell]^d$:*

- $\cup_j \Lambda_\ell(\gamma_j) \subset \Lambda_L$,
- $\operatorname{dist}(\Lambda_\ell(\gamma_j), \Lambda_\ell(\gamma_k)) \geq \log^{1/\rho''} |\Lambda|$ *if* $j \neq k$,
- $\operatorname{dist}(\Lambda_\ell(\gamma_j), \partial\Lambda) \geq \log^{1/\rho''} |\Lambda|$
- $|\Lambda_L \backslash \cup_j \Lambda_\ell(\gamma_j)| \lesssim |\Lambda| \log^{d/\rho'' - d/\rho'} |\Lambda|$,

such that, there exists a set of configurations $\mathcal{Z}_\Lambda$ s.t.:

- $\mathbb{P}(\mathcal{Z}_\Lambda) \geq 1 - (\log L)^{-(\min(d/\rho' - d/\rho'', 1/\rho - d/\rho') - \alpha)/2}$,
- *for $\omega \in \mathcal{Z}_\Lambda$, each center of localization associated to $H_\omega(\Lambda)$ belong to some $\Lambda_\ell(\gamma_j)$ and each box $\Lambda_\ell(\gamma_j)$ satisfies:*
    1. *the Hamiltonian $H_\omega(\Lambda_\ell(\gamma_j))$ has at most one eigenvalue in I_Λ, say, $E(\omega, \Lambda_\ell(\gamma_j))$;*
    2. *$\Lambda_\ell(\gamma_j)$ contains at most one center of localization, say $x_{k_j}(\omega, L)$, of an eigenvalue of $H_\omega(\Lambda)$ in I_Λ, say $E_{k_j}(\omega, L)$;*
    3. *$\Lambda_\ell(\gamma_j)$ contains a center $x_{k_j}(\omega, L)$ if and only if $\sigma(H_\omega(\Lambda_\ell(\gamma_j))) \cap I_\Lambda \neq \emptyset$; in which case, one has, with $\ell' = \log^{-1/\rho''} L$,*

$$|E_{k_j}(\omega, L) - E(\omega, \Lambda_\ell(\gamma_j))| \leq e^{-\ell'} \quad \text{and} \quad \operatorname{dist}(x_{k_j}(\omega, L), \Lambda_L \backslash \Lambda_\ell(\gamma_j)) \geq \ell'. \tag{4.11}$$

In particular, if $\omega \in \mathcal{Z}_\Lambda$, all the eigenvalues of $H_\omega(\Lambda)$ are described by (4.11).

Remark 4.1. (i) In Theorem 4.2, the condition $N(I_\Lambda)|\Lambda| = \log^\alpha |\Lambda|$ does not, in general, provide the largest possible interval I_Λ where our analysis works. It is chosen so as to work in all regimes provided (2.13) holds; it is optimal only in regimes where the integrated density of states $N(I_\Lambda)$ is exponentially small in $|I_\Lambda|^{-1}$. In other regimes, one may actually take $N(I_\Lambda)|\Lambda|$ larger. Note that, as in Theorem 4.2, one has $|I_\Lambda|\ell^d \ll 1$, Lemma 3.1 gives a precise description of:

- the probability distribution of the γ's for which $H_\omega(\Lambda_\ell(\gamma))$ has exactly one eigenvalue in I_Λ,
- the distribution of this eigenvalue when this is the case.

(ii) If the integrated density of states $N(I_\Lambda)$ is exponentially small in $|I_\Lambda|^{-1}$ then, as pointed out in (4.12) below, the typical size of intervals where we can control all the eigenvalues, and thus, prove Poisson convergence, is of order an inverse power of $\log|\Lambda|$. This should be compared to [10] where the admissible size for $|I_\Lambda|$ was of order $|\Lambda|^{-\alpha}$, with $\alpha > (1+(d+1)^{-1})^{-1}$ (see [10, (1.43)] with $\rho = 1$ and $\rho' = 0$). If now we have $N(I_\Lambda) \asymp |I_\Lambda|^{1+\rho'}, \rho' \geq 0$, then the admissible size for $|I_\Lambda|$ is of order $|\Lambda|^{-\alpha}$, with $\alpha > (1 + \rho' + (d+1)^{-1})^{-1}$, with no restriction on ρ' (compare to [10, (1.44)]). In particular α can be close to zero if ρ' is large. These improvements are direct consequences of the improved Wegner and Minami estimates of Theorem 2.1 above.

Proof of Theorem 4.2. As $N(I_\Lambda)|\Lambda| = \log^\alpha |\Lambda|$, assumption (2.13) yields, for $|\Lambda|$ sufficiently large,

$$|I_\Lambda| \leq 2\log^{-1/\rho}|\Lambda|. \tag{4.12}$$

We follow the proof of [10, Theorem 1.1]. Let $\mathcal{S}_{\ell,L}$ be the set of boxes $\Lambda_\ell(\gamma_j) \subset \Lambda$ containing at least two centers of localization of $H_{\omega,L}$ (see the proof of [10, Theorem 1.1] for more details). First, we note that, by the localization property (Loc), we need ℓ to be larger than $C\log^{1/\rho'} L$ (for some large C) to get (4.11). With the choices made in Theorem 4.2, the estimate (3.1) in [10] becomes

$$\begin{aligned}\mathbb{P}(\#\mathcal{S}_{\ell,L} \geq 1) &\lesssim \frac{L^d}{\log^{d/\rho'} L} N(I_\Lambda)|I_\Lambda|(\log L)^{2d/\rho'} \lesssim L^d(\log^{d/\rho'} L)N(I_\Lambda)|I_\Lambda| \\ &\lesssim \log^{\alpha-(1/\rho-d/\rho')} L \end{aligned} \tag{4.13}$$

by our choice of I_Λ.

Let Υ be the complement of the union of the boxes $(\Lambda_{\ell-\ell'}(\gamma_j))_j$ (see the proof of [10, Theorem 1.1] for more details). In the same way as above, the estimate (3.3) in [10] becomes

$$\begin{aligned}&\mathbb{P}(H_\omega(\Lambda) \text{ has a localization center in } \Upsilon) \\ &\quad\lesssim |\Upsilon|N(I_\Lambda) \\ &\quad\lesssim |I_\Lambda|^{-1}N(I_\Lambda)|I_\Lambda||\Lambda|(\log^{d/\rho''-d/\rho'}|\Lambda|) \\ &\quad\lesssim \log^{\alpha-d(1/\rho'-1/\rho'')}|\Lambda| \end{aligned} \tag{4.14}$$

by our choice of I_Λ.

This completes the proof of Theorem 4.2. □

4.3. Controlling Most Eigenvalues

We will give two versions of this reduction. In Theorem 4.3, the first version, we consider energy intervals where the density of states is not too small: it can be polynomially small to any order but not smaller. In this region, we give a version of the reduction that minimizes the estimate on the probability of the bad set (where our description does not work) as well as the number of eigenvalues that are not described by our scheme. This version is used in the proof of Theorem 2.7.

In Theorem 4.4, the second version of the reduction theorem, we want to allow exponentially small density of states as in Theorem 4.2. The control will still be obtained with a good probability but we do not control as many eigenvalues. This version is used in the proofs of Theorems 2.5 and 2.6.

This reduction goes back to the result obtained in [10, Theorem 1.2] and improves upon it. We follow the proof of that result and only indicate the differences.

Theorem 4.3. *Set $\ell' = R\log|\Lambda|$ with R large and consider intervals $I_\Lambda \subset \Sigma_{\rm SDL}$. Assume that for some $1 \leq \beta \leq \beta' < \alpha' \leq \alpha < \infty$, for $|\Lambda|$ large, we have*

$$|I_\Lambda|^{-\alpha'} \lesssim |\Lambda| \lesssim |I_\Lambda|^{-\alpha} \quad \text{and} \quad |I_\Lambda|^{\beta'} \lesssim N(I_\Lambda) \lesssim |I_\Lambda|^{\beta}$$

Set

$$\delta_0 = (\alpha-\beta)^{-1} > 0, \quad \zeta = \frac{\alpha-\beta}{\alpha'-\beta'} \geq 1, \quad \nu_0 = \min\left(\zeta^{-1}, \frac{\delta_0}{d+1}\right) \leq 1. \tag{4.15}$$

Note that $\nu_0\zeta \leq 1$.

For any $\nu < \nu_0$ and $\kappa \in (0,1)$, there exist

- *a decomposition of Λ_L into $\mathcal{O}(|\Lambda|/\ell_\Lambda^d)$ disjoint cubes of the form $\Lambda_\ell(\gamma_j) := \gamma_j + [0,\ell]^d$, where $\ell \sim (|I_\Lambda|\ell')^{-\frac{1}{d+1}}$, so that:*
  - $\cup_j \Lambda_\ell(\gamma_j) \subset \Lambda_L$,
  - $\mathrm{dist}(\Lambda_\ell(\gamma_j), \Lambda_\ell(\gamma_k)) \geq \ell'$ *if* $j \neq k$,
  - $\mathrm{dist}(\Lambda_\ell(\gamma_j), \partial\Lambda) \geq \ell'$
  - $|\Lambda_L \backslash \cup_j \Lambda_\ell(\gamma_j)| \lesssim |\Lambda|\ell'/\ell$,
- *a set of configurations $\mathcal{Z}_\Lambda$ satisfying*

$$\mathbb{P}(\mathcal{Z}_\Lambda) \geq 1 - \exp(-(N(I_\Lambda)|\Lambda|)^{\delta_{\kappa,\nu}}/C),$$

  with $\delta_{\kappa,\nu} = \min(1-\nu\zeta, \kappa\nu) \in]0, \frac{1}{2}[$, such that, for $|\Lambda|$ sufficiently large (depending only on $\beta, \beta', \alpha, \alpha', \nu$ and κ),
- *for all $\omega \in \mathcal{Z}_\Lambda$, there exist at least $\frac{|\Lambda|}{\ell^d}(1+o(1))$ disjoint boxes $\Lambda_\ell(\gamma_j)$ satisfying the properties (1), (2) and (3) described in Theorem 4.2 with $\ell' = R\log|\Lambda|$,*
- *the number of eigenvalues of $H_{\omega,L}$ that are not described by the above picture is bounded by $C(N(I_\Lambda)|\Lambda|)^{\gamma_{\kappa,\nu}}$, with $\gamma_{\kappa,\nu} = 1-(1-\kappa)\nu \in]0,1[$.*

Particular cases:

- *If $|I_\Lambda| \asymp |\Lambda|^{-\alpha^{-1}}$ and $N(I_\Lambda) \asymp |I_\Lambda|^{\beta}$, then $\zeta = 1$.*
- *If $n(E) > 0$ and I_Λ's are centered at E, then $\beta = \beta' = 1$.*

Proof of Theorem 4.3. We proceed as in [10, Proof of Theorem 1.2] but take advantage of Theorem 2.1 above.

By assumption, we have

$$|I_\Lambda|^{-(\alpha'-\beta')} \lesssim N(I_\Lambda)|\Lambda| \lesssim |I_\Lambda|^{-(\alpha-\beta)}. \tag{4.16}$$

or equivalently, with notations defined in (4.15),

$$(N(I_\Lambda)|\Lambda|)^{-\zeta\delta_0} \lesssim |I_\Lambda| \lesssim (N(I_\Lambda)|\Lambda|)^{-\delta_0}. \tag{4.17}$$

First, use in the reduction procedure (2.8) and (2.10) as the Wegner and Minami estimates. As in [10], we consider a collection of $\mathcal{O}(|\Lambda|\ell^{-d})$ boxes $\Lambda_\ell(\gamma_j)$ two by two distant by at least ℓ', and such that $|\Lambda \backslash \cup_j \Lambda_\ell(\gamma_j)| \lesssim |\Lambda|\ell'/\ell$. As above, let $\mathcal{S}_{\ell,L}$ be the set of boxes $\Lambda_\ell(\gamma_j) \subset \Lambda$ containing at least two centers of localization of $H_{\omega,L}$ (see the proof of [10, Theorem 1.2] for more details). Set

$$K := 4\mathrm{e}N(I_\Lambda)|\Lambda||I_\Lambda|\ell^d \tag{4.18}$$

Then, by (2.10),

$$\mathbb{P}(\#\mathcal{S}_{\ell,L} \geq K) \lesssim 2^{-K}. \tag{4.19}$$

Next, cover $\Lambda \backslash \cup_j \Lambda_\ell(\gamma_j)$ with a "partition" of boxes $\Lambda_{\ell'}(\gamma'_j)$ and let $\mathcal{S}'_{\ell,L}$ be the set of boxes $\Lambda_{\ell'}(\gamma_j) \subset \Lambda$ containing at least one center of localization of $H_{\omega,L}$ (see the proof of [10, Theorem 1.2] for more details). Set

$$K' := 2^{d+1} N(I_\Lambda)|\Lambda| \frac{\ell'}{\ell}. \tag{4.20}$$

It follows from (2.8) that

$$\mathbb{P}(\#\mathcal{S}'_{\ell,L} \geq K') \lesssim 2^{-K'}. \tag{4.21}$$

To evaluate the number of eigenvalues of $H_\omega(\Lambda)$ we may miss because of this reduction, we need to control the number of centers $x_k(\omega,\Lambda)$ that may fall into K boxes of $\mathcal{S}_{\ell,L}$ and K' boxes of $\mathcal{S}'_{\ell,L}$. In [10] we used the crude deterministic bound given by the volume of the considered boxes. Here we estimate this number using the high order Minami estimate (2.9). Given an integer $r \geq 1$, it follows from (2.9) that

$$\begin{aligned}\mathbb{P}\left\{\begin{array}{r}\exists \text{ a box } \Lambda_\ell(\gamma_j) \text{ s.t.}\\ \operatorname{tr} \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda_\ell(\gamma_j))) \geq r\end{array}\right\} &\lesssim \frac{|\Lambda|}{\ell^d}\mathbb{P}\{\operatorname{tr} \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda_\ell)) \geq r\}\\ &\lesssim \frac{1}{r!}N(I_\Lambda)|\Lambda|(C|I_\Lambda|\ell^d)^{r-1}\\ &\lesssim N(I_\Lambda)|\Lambda|\left(\frac{C|I_\Lambda|\ell^d}{r}\right)^{r-1}.\end{aligned} \tag{4.22}$$

In the same way, we have, for $r' \geq 1$ an integer,

$$\begin{aligned}\mathbb{P}\left\{\begin{array}{r}\exists \text{ a box } \Lambda_{\ell'}(\gamma'_j) \text{ s.t.}\\ \operatorname{tr} \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda_{\ell'}(\gamma'_j))) \geq r'\end{array}\right\} &\lesssim \frac{|\Lambda|}{\ell^d}\frac{\ell'}{\ell}\mathbb{P}\{\operatorname{tr} \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda_{\ell'})) \geq r'\}\\ &\leq N(I_\Lambda)|\Lambda|\left(\frac{\ell'}{\ell}\right)^{d+1}\left(\frac{C|I_\Lambda|(\ell')^d}{r'}\right)^{r'/C-1}.\end{aligned} \tag{4.23}$$

The additional constant C in the exponent in the right hand side of (4.23) comes form the fact that the Hamiltonians $(H_\omega(\Lambda_{\ell'})_{\Lambda_{\ell'}\in\mathcal{S}'_{\ell,L}}$ need not be independent, but there are finitely many subfamilies of independent Hamiltonians.

We pick $r \asymp r' \asymp (N(I_\Lambda)|\Lambda|)^\delta$, with $\delta > 0$ to be chosen later. We thus end up with

$$(4.22), (4.23) \lesssim \exp(-(N(I_\Lambda)|\Lambda|)^\delta). \tag{4.24}$$

Hence, with a probability at least $1 - \exp(-(N(I_\Lambda)|\Lambda|)^\delta)$, the number of eigenvalues of $H_\omega(\Lambda)$ we miss with our reduction is bounded by

$$C(K + K')(N(I_\Lambda)|\Lambda|)^\delta. \tag{4.25}$$

We now optimize in ℓ by requiring $K \sim K'$, that is

$$|I_\Lambda|\ell^d = \frac{\ell'}{\ell}. \tag{4.26}$$

In other terms we choose $\ell = (|I_\Lambda|\ell')^{-\frac{1}{d+1}}$. So that any K, K' satisfying

$$K \sim K' \gtrsim K_0 := (N(I_\Lambda)|\Lambda|)|I_\Lambda|^{\frac{1}{d+1}}(\ell')^{1+\frac{1}{d+1}}, \tag{4.27}$$

is good enough. To ensure that K and K' grow fast enough [in order to get a fast decaying probability in (4.19) and (4.21)], we actually enlarge them and set for any $\nu < \nu_0$ (note that $\log(N(I_\Lambda)|\Lambda|) \asymp \log|\Lambda|$),

$$K \sim K' \sim (N(I_\Lambda)|\Lambda|)|I_\Lambda|^{\nu\delta_0^{-1}}. \tag{4.28}$$

Taking (4.17) into account, we have the following lower and upper bound,

$$(N(I_\Lambda)|\Lambda|)^{1-\nu\zeta} \lesssim K \sim K' \lesssim (N(I_\Lambda)|\Lambda|)^{1-\nu}. \tag{4.29}$$

Next, let $\kappa \in]0,1[$ be given, and fix δ above so that $\delta = \kappa\nu$. It follows from (4.19), (4.21), (4.24), (4.25) and (4.29) that the number of missing eigenvalues is bounded by $1 - \exp(-(N(I_\Lambda)|\Lambda|)^{1-\zeta\nu}) - \exp(-(N(I_\Lambda)|\Lambda|)^{\kappa\nu})$. At last, to see that $\delta_{\kappa,\nu} < \frac{1}{2}$, note that $\min(1-\zeta\nu, \kappa\nu) \leq (1+\zeta\kappa^{-1})^{-1} < \frac{1}{2}$, since $\kappa < 1$ and $\zeta \geq 1$. The theorem follows. □

We now turn to the second version of our reduction theorem. One has

Theorem 4.4. *Pick $\rho \in (0, 1/d)$. For a cube $\Lambda = \Lambda_L$, consider an interval $I_\Lambda = [a_\Lambda, b_\Lambda] \subset I, I$ a fixed compact in Σ_{SDL}. Pick $\rho < \rho' < \rho'' < 1/d$.*

There exists $\alpha_0 > 0$ such that, for $\alpha_0 \leq \alpha < \alpha'$, if (I_Λ) satisfies

$$\log^\alpha|\Lambda| \leq N(I_\Lambda)|\Lambda| \leq \log^{\alpha'}|\Lambda| \quad \textit{and} \quad N(I_\Lambda)e^{|I_\Lambda|^{-\rho}} \geq 1. \tag{4.30}$$

then, picking $(\tilde\ell_\Lambda, \ell'_\Lambda)$ such that

$$\tilde\ell_\Lambda^d|I_\Lambda| \asymp \log^{1/\rho'-1/\rho}|\Lambda| \quad \textit{and} \quad (\ell'_\Lambda)^d \asymp \tilde\ell_\Lambda^d \log^{1/\rho''-1/\rho'}|\Lambda|, \tag{4.31}$$

there exist

- *a decomposition of Λ_L into disjoint cubes of the form $\Lambda_{\ell_\Lambda}(\gamma_j) := \gamma_j + [0,\ell_\Lambda]^d$ where $\ell_\Lambda = \tilde\ell_\Lambda(1+o(1))$ such that*
  - $\cup_j \Lambda_{\ell_\Lambda}(\gamma_j) \subset \Lambda_L$,
  - $\mathrm{dist}(\Lambda_{\ell_\Lambda}(\gamma_j), \Lambda_{\ell_\Lambda}(\gamma_k)) \geq \ell'_\Lambda$ *if* $j \neq k$,
  - $\mathrm{dist}(\Lambda_{\ell_\Lambda}(\gamma_j), \partial\Lambda) \geq \ell'_\Lambda$
  - $|\Lambda_L \setminus \cup_j \Lambda_{\ell_\Lambda}(\gamma_j)| \lesssim |\Lambda|\ell'_\Lambda/\ell_\Lambda$,
- *a set of configurations $\mathcal{Z}_\Lambda$ satisfying $\mathbb{P}(\mathcal{Z}_\Lambda) \geq 1 - e^{-\log^{\alpha-2}|\Lambda|}$,*

so that, for L sufficiently large (depending only on $(\alpha, \alpha', \rho, \rho', \rho'')$), one has

- *for $\omega \in \mathcal{Z}_\Lambda$, there exist at least $\frac{|\Lambda|}{\ell_\Lambda^d}(1 + O(\log^{1/\rho'-1/\rho}|\Lambda|))$ disjoint boxes $\Lambda_{\ell_\Lambda}(\gamma_j)$ satisfying the properties* (1), (2) *and* (3) *described in Theorem 4.2 where ℓ'_Λ in* (4.11) *satisfies* (4.31)*;*
- *the number of eigenvalues of $H_\omega(\Lambda)$ that are not described above is bounded by*

$$CN(I_\Lambda)|\Lambda|\,[\log^{1/\rho'-1/\rho}|\Lambda| + \log^{1/(d\rho'')-1/(d\rho')}|\Lambda|].$$

In Theorem 4.4, our choice of parameters is made so as to allow as small as possible a density of states [see the second condition in (4.30)]. The price to pay for this is that the width of the interval I_Λ may not be optimal in all regimes (compare the first condition in (4.30) with the width of the intervals treated in [10] when $N(I_\Lambda) \gtrsim |I_\Lambda|^{1+\rho}$). It is nevertheless essentially optimal when $N(I_\Lambda) \asymp \mathrm{e}^{-|I_\Lambda|^{-\rho}}$.

We also note that, in the proof of Theorem 4.4, the choice of $\ell = \ell_\Lambda$ guarantees that $|I_\Lambda|\ell^d \ll 1$ [see (4.32)]; thus, Lemma 3.1 gives a precise description of:

- the probability distribution of the γ's for which $H_\omega(\Lambda_\ell(\gamma))$ has exactly one eigenvalue in I_Λ,
- the distribution of this eigenvalue when this is the case

Proof of Theorem 4.4. We follow the proof of Theorem 4.3. First, note that, as in the proof of Theorem 4.2, assumption (4.30) implies that

$$|I_\Lambda| \lesssim \log^{-1/\rho}|\Lambda|, \quad \log^{1/(d\rho')}|\Lambda| \lesssim \ell_\Lambda, \quad \log^{1/(d\rho'')}|\Lambda| \lesssim \ell'_\Lambda \tag{4.32}$$

With our choice of ℓ_Λ and ℓ'_Λ, one estimates K defined in (4.18) and K' defined in (4.20) by

$$\begin{aligned}\log^{\alpha+1/\rho'-1/\rho}|\Lambda| \lesssim K \lesssim \log^{\alpha'+1/\rho'-1/\rho}|\Lambda|\\ \text{and} \quad \log^{\alpha+1/(d\rho')-1/(d\rho)}|\Lambda| \lesssim K' \lesssim \log^{\alpha'+1/(d\rho')-1/(d\rho)}|\Lambda|.\end{aligned} \tag{4.33}$$

In the present case, the computations done in (4.22) and (4.23) give a too gross estimate of the number of eigenvalues, or, equivalently, of the number of localization centers, on which one cannot get a precise control if one wants to keep a good probability estimate; this comes from the fact that the quantities $N(I_\Lambda)|\Lambda||I_\Lambda|, \ell, \ell'$ may be powers of $\log|\Lambda|$. We need to study more carefully the number of eigenvalues missed by the description constructed in the proof of Theorem 4.3. Therefore, we follow the ideas used in the proof of [22, Theorem 4.1].

Let $\Gamma_{\ell,L}$ be the set of cubes $\{\Lambda_\ell(\gamma_j);\ j\}$ of the decomposition introduced in the proof of Theorem 4.3. We "partition" $\Gamma_{\ell,L}$ into 2^d sets such that, any two cubes $\Lambda_\ell(\gamma)$ and $\Lambda_\ell(\gamma')$ in each set, the Hamiltonians $H_\omega(\Lambda_\ell(\gamma))$ and $H_\omega(\Lambda_\ell(\gamma'))$ are independent. Let these sets be $(\Gamma_{\ell,L,j})_{1\leq j\leq 2^d}$ and their cardinality be $\tilde N_j := \#\Gamma_{\ell,L,j}$. One has $\tilde N_j \asymp |\Lambda|\ell^{-d}$.

For $\Lambda_\ell(\gamma_k) \in \Gamma_{\ell,L,j}$, set $X_{j,k} = \operatorname{tr} \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda_\ell(\gamma_k)))$. These variables are i.i.d. and their common distribution is described by 2.1.

We now want to estimate the maximal number of localization centers of $H_\omega(\Lambda)$ contained in boxes of $(\Gamma_{\ell,L})_{1\leq j\leq 2^d}$ that each contain at least two centers (these are the boxes of $\mathcal{S}_{\ell,L}$ in the notations of the proof of Theorem 4.3). We want to show that this number is, with a probability close to 1, bounded by CK where K is defined in (4.18) and $C > 0$ is a constant to be chosen.

Let $\mathbb{P}_r$ be the probability to have $2^{d+1}r$ localization centers of $H_\omega(\Lambda)$ in cubes containing at least two centers. We compute

$$\begin{aligned}\mathbb{P}_r &\le \sum_{j=1}^{2^d}\sum_{n_j=1}^{r}\binom{\tilde N_j}{n_j}\mathbb{P}\left(\min_{1\le k\le n_j} X_{j,k}\ge 2 \text{ and } \sum_{k=1}^{n_j} X_{j,k}\ge 2r\right)\\ &\le \sum_{j=1}^{2^d}\sum_{n_j=1}^{r}\binom{\tilde N_j}{n_j}\sum_{\substack{l_1+\cdots+l_{n_j}\ge 2r\\ \forall 1\le k\le n_j,\ l_k\ge 2}}\prod_{k=1}^{n_j}\mathbb{P}(X_{j,k}=l_k)\\ &\le \sum_{j=1}^{2^d}\sum_{n_j=1}^{r}\binom{\tilde N_j}{n_j}\sum_{\substack{l_1+\cdots+l_{n_j}\ge 2(r-n_j)\\ \forall 1\le k\le n_j,\ l_k\ge 0}}\prod_{k=1}^{n_j}\mathbb{P}(X_{j,k}=l_k+2).\end{aligned}$$

Then, using (2.9) and the independence of the local Hamiltonians associated to boxes in $\Gamma_{\ell,L,j}$, we obtain

$$\begin{aligned}\mathbb{P}_r &\lesssim \sum_{j=1}^{2^d}\sum_{n_j=1}^{r}\binom{\tilde N_j}{n_j}\sum_{\substack{l_1+\cdots+l_{n_j}\ge 2(r-n)\\ \forall 1\le k\le n_j,\ l_k\ge 0}}\frac{(CN(I_\Lambda)\ell^d)^{n_j}(|I_\Lambda|\ell^d)^{2r-n_j}}{(l_1+1)!\cdots(l_{n_j}+1)!}\\ &\lesssim \sum_{j=1}^{2^d}\sum_{n_j=1}^{r}\frac{1}{n_j!}(\tilde N_j N(I_\Lambda)\ell^d)^{n_j}(|I_\Lambda|\ell^d)^{2r-n_j}\\ &\lesssim \sum_{n=1}^{r}\frac{1}{n!}(|\Lambda|N(I_\Lambda))^n(|I_\Lambda|\ell^d)^{2r-n}\\ &\lesssim (K^\eta|I_\Lambda|\ell^d)^r+\left(\frac{CK}{\eta r}\right)^r.\end{aligned}$$

In the last estimate, we have cut the previous sum into two parts, the first when n runs from 0 to ηr and the second from ηr to r. We now choose $\eta>0$ so that $\eta\alpha'<1/\rho-1/\rho'$ [for α' in (4.30), see also (4.33)]. Recall that $|I_\Lambda|\ell^d$ is small. Thus, setting $r=\eta^{-1}CK$ for some large $C>0$ and using (4.33), we obtain

$$\mathbb{P}_r\le \mathrm{e}^{-K/C}\le \mathrm{e}^{-\log^{\alpha-1}|\Lambda|/C}. \tag{4.34}$$

Now, as in the proof of Theorem 4.3, cover $\Lambda\setminus\cup_j\Lambda_\ell(\gamma_j)$ with a partition of boxes $\Lambda_{\ell'}(\gamma_j')$ (see the proof of [10, Theorem 1.2] for more details). In the same way as above, one estimates the maximal number of centers of localization contained in the union of the boxes $\Lambda_{\ell'}(\gamma_j)\subset\Lambda$. Let $\mathbb{P}_r'$ be the probability that this number exceeds r. Then, for $r=CK'$ (for some constant $C>0$) where K' is defined in (4.20), as above, we prove

$$\mathbb{P}_r'\le \mathrm{e}^{-K'/C}\le \mathrm{e}^{-\log^{\alpha-1}|\Lambda|/C}. \tag{4.35}$$

Summing (4.34) and (4.35), taking into account (4.33) and (4.32), we obtain that, for α sufficiently large and properly chosen $(\rho',\rho'')\in(d\rho,1)^2$, with probability at least $1-\mathrm{e}^{-\log^2|\Lambda|}$, there are at most

$$C(K+K')\lesssim N(I_\Lambda)|\Lambda|\,[\log^{1/\rho'-1/\rho}|\Lambda|+\log^{1/(d\rho'')-1/(d\rho')}|\Lambda|].$$

eigenvalues that are not accounted for by the description given in Theorem 4.4. This completes the proof of Theorem 4.4. □

5. Applications to Eigenvalue Statistics: Proofs

5.1. The Proof of Theorem 2.7

First note that under the assumptions of Theorem 2.7, the assumptions of Theorem 4.3 are fulfilled. We will use this decomposition.

Let $X = X(\Lambda_\ell, I_\Lambda)$ the Bernoulli random variable equal to 1 if H_{ω,Λ_ℓ} has an eigenvalue in I_Λ and zero otherwise. Recall that the distribution of this random variable is described by (3.3) in Lemma 3.1.

To prove Theorem 2.7, we consider the collection of Bernoulli random variables $X_j := X(\Lambda_\ell(\gamma_j), I_\Lambda), j = 1, \dots, \tilde{N}$ defined by

- $X(\Lambda_\ell(\gamma), I_\Lambda)$ is defined in Sect. 3.2
- the boxes $(\Lambda_\ell(\gamma_j))_{1\leq j\leq \tilde{N}}$ are given by Theorem 4.3 and $\tilde{N} \asymp |\Lambda|/\ell^d$.

Thus, the random variables $(X_j)_j$ are i.i.d.

We have

$$\begin{aligned}&|\operatorname{tr} \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) - N(I_\Lambda)|\Lambda||\\ &\quad\leq \left|\operatorname{tr} \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) - \sum_{j=1}^{\tilde{N}} X_j\right| + \left|\sum_{j=1}^{\tilde{N}} X_j - N(I_\Lambda)|\Lambda|\right| \end{aligned} \tag{5.1}$$

By Theorem 4.3, with a probability $\geq 1 - \exp(-c(N(I_\Lambda)|\Lambda|)^{\delta_{\kappa,\nu}})$ we have

$$\left|\operatorname{tr} \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) - \sum_{j=1}^{\tilde{N}} X_j\right| \lesssim (N(I_\Lambda)|\Lambda|)^{\gamma_{\kappa,\nu}}. \tag{5.2}$$

Next, the large deviation principle for i.i.d. (0, 1)-Bernoulli variables with expectation p gives, for $\delta \in]\frac{1}{2}, 1[$, yields (see e.g. [5]),

$$\mathbb{P}\left(\left|\sum_{j=1}^{\tilde{N}} X_j - p\tilde{N}\right| \geq (p\tilde{N})^\delta\right) \leq C \exp(-c_p(p\tilde{N})^{2\delta-1}), \tag{5.3}$$

where the constant c_p is uniformly bounded as $p \downarrow 0$. We apply the latter with $p = \mathbb{P}(X = 1)$. On the account of Lemma 3.1, (4.26) and (4.29), we have

$$|p\tilde{N} - N(I_\Lambda)|\Lambda|| \lesssim (N(I_\Lambda)|\Lambda|)^{1-\nu}. \tag{5.4}$$

Combining (5.1), (5.2), (5.3) and (5.4) we obtain, for any $\delta' \in]0, 1[$,

$$\begin{aligned}&\mathbb{P}\{|\operatorname{tr} \mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) - N(I_\Lambda)|\Lambda||\\ &\quad\leq (N(I_\Lambda)|\Lambda|)^{\gamma_{\kappa,\nu}} + (N(I_\Lambda)|\Lambda|)^{1-\nu} + (p\tilde{N})^{\frac{1}{2}+\frac{1}{2}\delta'}\}\\ &\quad\geq 1 - \exp(-(N(I_\Lambda)|\Lambda|)^{\delta_{\kappa,\nu}}) - C\exp(-c_p(p\tilde{N})^{\delta'}). \end{aligned} \tag{5.5}$$

Since $\gamma_{\kappa,\nu} > 1-\nu$, and choosing $\delta' = \delta_{\kappa,\nu}$, we get

$$\mathbb{P}\left\{|\mathrm{tr}\,\mathbf{1}_{I_\Lambda}(H_\omega(\Lambda)) - N(I_\Lambda)|\Lambda|| \le 3(N(I_\Lambda)|\Lambda|)^{\max(\gamma_{\kappa,\nu},\frac12+\frac12\delta_{\kappa,\nu})}\right\}$$
$$\ge 1 - 2\exp(-(N(I_\Lambda)|\Lambda|)^{\delta_{\kappa,\nu}}). \tag{5.6}$$

Taking κ sufficiently small, we get $\delta_{\kappa,\nu} = \kappa\nu$. The result follows with $\varepsilon = \kappa\nu$.

We turn to the proof of point (2) which follows from the previous analysis and the central limit theorem for Bernoulli random variables, provided $\nu > \frac12$.

This completes the proof of Theorem 2.7.

5.2. The Proofs of Theorems 2.2, 2.5 and 2.6

As already stated above, one can follow verbatim the proofs of the corresponding results in [10,21]. Let us just make a few comments on those proofs.

5.2.1. The Proof of Theorem 2.2. The corresponding result is [10, Theorem 1.2]. In [10], we also have the stronger Theorems 1.3 and 1.6 that also have their analogues in the present setting. Comparing (2.13) with the condition [10, (1.12)], we see that we have now only a local condition at E_0 only. This gain is obtained thanks to Lemma 3.1 and Theorem 4.2.

5.2.2. The Proofs of Theorem 2.5 and 2.6. The result corresponding to Theorem 2.5 is [21, Theorem 1.4]. Theorem 2.6 is a then a consequence of Theorem 2.5, see e.g. [21,25]. Under uniform assumptions of the type $N(J)\mathrm{e}^{|J|^{-\rho}} \ge 1$ for some $\rho \in (0,1)$ and for all $J \subset E_0 + I_\Lambda$, one may as well follow the method developed in [10].

As we shall see, to prove Theorem 2.5 in the present case is easier than in [21, Theorem1.4]. The basic idea of the proof of asymptotic ergodicity in [21, Theorem 1.4] is, for a given interval $E_0 + I_\Lambda$, to split it into smaller intervals such that, on most of these intervals, the assumptions of a reduction of the same type as Theorem 4.4 is valid plus a remaining set of energies that only contains a negligible fraction of the eigenvalues in $E_0 + I_\Lambda$. Then, one proves asymptotic ergodicity for each of the small intervals. Therefore, one needs to use an analogue of Lemma 3.1 to control the eigenvalues. This imposes further restrictions on how one has to choose the small intervals. In the present case, thanks to the improvement obtained in Lemma 3.1 over its analogues in [10, Lemma 2.2] and [21, Lemma 2.2], the way to split the interval $E_0 + I_\Lambda$ will be much simpler (as a comparison of what follows with the discussions following [21, Theorem 2.1 and Lemma 2.2] and those in [21, Section 3.2.1] will immediately show).

We will not give a complete proof of Theorem 2.5 but only indicate the changes to be made in the proof of [21, Theorem 1.4].

Pick $\alpha_0 < \alpha < \alpha'' < \alpha'$ (where α_0 is given by Theorem 4.4). Pick $\mu > 0$. Now, partition I_Λ into intervals $(I_j)_{j\in J}$ such that $N(I_j) \asymp |\Lambda|^{-1}\log^{\alpha''}|\Lambda|$. Define

$$B = \{j \in J;\ N(I_j) \le |I_j|^{\mu}\}. \tag{5.7}$$

Then, one clearly has

$$|I_\Lambda| \geq \sum_{j\in B} |I_j| \geq \sum_{j\in B} N(I_j)^{1/\mu} \asymp \#B|\Lambda|^{-1/\mu} \log^{\alpha''/\mu} |\Lambda|$$

thus, $\#B \lesssim |\Lambda|^{1/\mu}$ and

$$N\left(\bigcup_{j\in B} I_j\right) \lesssim |\Lambda|^{(1-\mu)/\mu} \log^{\alpha''} |\Lambda|.$$

The number of eigenvalues expected in $E_0 + I_\Lambda$ is of order $N(E_0 + I_\Lambda)|\Lambda|$, thus, by (2.16), larger than $|\Lambda|^{1-\delta}$. Pick $\nu > 0$. By the enhanced Wegner's estimate (2.8) and Markov's inequality, we know that, with probability at least $1 - |\Lambda|^{-\nu}$, the number of eigenvalues in $\bigcup_{j\in B} I_j$ is bounded by $|\Lambda|^{\nu+1/\mu} \log^{\alpha''} |\Lambda|$. We now pick ν and μ^{-1} small so that $\nu + 1/\mu < 1 - \delta$.

For $j \notin B$, one has $N(I_j) \geq |I_j|^\mu$ and, thus, one can apply Theorem 4.4 to I_j for $j \notin B$. So, in each I_j, we control $N(I_j)|\Lambda|(1+o(1))$ eigenvalues (the error is uniform in j by Theorem 4.4); thus, the total number of eigenvalues we control exceeds $\sum_{j\notin B} N(I_j)|\Lambda|(1+o(1))$ that is, exceeds $N(E_0+I_\Lambda)|\Lambda|(1+o(1)) \geq |\Lambda|^{1-\delta}$ as announced above.

For $j \notin B$, we moreover want to be able to apply Lemma 3.1 to control the eigenvalues "in" the cubes $\Lambda_{\ell_\Lambda}(\gamma)$ constructed in Theorem 4.4 and the interval I_j. Therefore, we need to check that $|I_j||\Lambda_{\ell_\Lambda}(\gamma)| \ll 1$ (see (3.3) and (3.4)). This is guaranteed by (4.31) in Theorem 4.4.

Now thanks to Theorem 4.4 and Lemma 3.1, in each I_j for $j \notin B$, we reason as in the proof of [21, Theorem 1.4], or more precisely, as in the proof of [21, Lemma 3.2] to obtain the asymptotic ergodicity.

Remark 5.1. One can actually prove Theorem 2.5 on intervals to which the IDS gives a smaller weight, that is, relax assumption (2.16) into $|\Lambda| \cdot \log^{-\beta} |\Lambda| \cdot N(E_0 + I_\Lambda) \to +\infty$ for not too small β (e.g. for not too negative β). Then, the condition $N(I_j) \leq |I_j|^\mu$ defining B will have to be replaced with conditions of the type $N(I_j) \leq e^{-|I_j|^{-\rho}}$ (ρ will now be in $(0, 1)$).

Moreover, if β is not sufficiently large, a restriction analogous to [21, (1.10) in Theorem 1.4] will come up again.

References

[1] Aizenman, M., Schenker, J.H., Friedrich, R.M., Hundertmark, D.: Finite-volume fractional-moment criteria for Anderson localization. Commun. Math. Phys. **224**(1), 219–253 (2001). Dedicated to Joel L. Lebowitz

[2] Bellissard, J.V., Hislop, P.D., Stolz, G.: Correlation estimates in the Anderson model. J. Stat. Phys. **129**(4), 649–662 (2007)

[3] Combes, J.-M., Germinet, F., Klein, A.: Generalized eigenvalue-counting estimates for the Anderson model. J. Stat. Phys. **135**(2), 201–216 (2009)

[4] Costin, O., Lebowitz, J.L.: Gaussian fluctuation in random matrices. Phys. Rev. Lett. **75**(1), 69–72 (1995)

[5] Durrett, R.: Probability: theory and examples second edition. Duxbury Press, Belmont (1996)

[6] Germinet, F., Klein, A.: Bootstrap multiscale analysis and localization in random media. Commun. Math. Phys. **222**(2), 415–448 (2001)

[7] Germinet, F., Klein, A.: A characterization of the Anderson metal-insulator transport transition. Duke Math. J. **124**(2), 309–350 (2004)

[8] Germinet, F., Klein, A.: New characterizations of the region of complete localization for random Schrödinger operators. J. Stat. Phys. **122**(1), 73–94 (2006)

[9] Germinet, F., Klein, A.: A comprehensive proof of localization for continuous Anderson models with singular random potentials. J. Eur. Math. Soc. (to appear)

[10] Germinet, F., Klopp, F.: Spectral statistics for random Schrödinger operators in the localized regime. J. Eur. Math. Soc. (2010). http://arxiv.org/abs/1011.1832

[11] Graf, G.M., Vaghi, A.: A remark on the estimate of a determinant by Minami. Lett. Math. Phys. **79**(1), 17–22 (2007)

[12] Grenkova, L.N., Molchanov, S.A., Sudarev, Yu.N.: The structure of the edge of the multidimensional Anderson model spectrum. Teoret. Mat. Fiz. **85**(1), 32–40 (1990)

[13] Gustavsson, J.: Gaussian fluctuations of eigenvalues in the GUE. Ann. Inst. H. Poincaré Probab. Statist. **41**(2), 151–178 (2005)

[14] Kirsch, W.: An invitation to random Schrödinger operators. In: Random Schrödinger Operators. Panor. Synthèses, vol. 25, pp. 1–119. Soc. Math. France, Paris, 2008. With an appendix by Frédéric Klopp

[15] Klein, A., Molchanov, S.: Simplicity of eigenvalues in the Anderson model. J. Stat. Phys. **122**(1), 95–99 (2006)

[16] Kirsch, W., Metzger, B.: The integrated density of states for random Schrödinger operators. In: Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday. Proceedings of Symposia in Pure Mathematics, vol. 76, pp. 649–696. American Mathematical Society, Providence, RI (2007)

[17] Klopp, F.: Band edge behavior of the integrated density of states of random Jacobi matrices in dimension 1. J. Stat. Phys. **90**(3–4), 927–947 (1998)

[18] Klopp, F.: Internal Lifshits tails for random perturbations of periodic Schrödinger operators. Duke Math. J. **98**(2), 335–396 (1999)

[19] Klopp, F.: Correction to: "Internal Lifshits tails for random perturbations of periodic Schrödinger operators" [18]. Duke Math. J. **109**(2), 411–412 (2001)

[20] Klopp, F.: Weak disorder localization and Lifshitz tails. Commun. Math. Phys. **232**(1), 125–155 (2002)

[21] Klopp, F.: Asymptotic ergodicity of the eigenvalues of random operators in the localized phase. PTRF (2010). http://fr.arxiv.org/abs/1012.0831

[22] Klopp, F.: Universal joint asymptotic ergodicity of the eigenvalues and localization centers of random operators in the localized phase (2011, in preparation)

[23] Minami, N.: Local fluctuation of the spectrum of a multidimensional Anderson tight binding model. Commun. Math. Phys. **177**(3), 709–725 (1996)

[24] Minami, N.: Theory of point processes and some basic notions in energy level statistics. In: Probability and mathematical physics. CRM Proceedings and Lecture Notes, vol. 42, pp. 353–398. American Mathematical Society, Providence, RI (2007)

[25] Minami, N.: Energy level statistics: a formulation and some examples. In: Spectra of Random Operators and Related Topics, pp. 79–97. RIMS, Kyoto (2011)

[26] Molchanov, S.A.: The local structure of the spectrum of a random one-dimensional Schrödinger operator. Trudy Sem. Petrovsk. **8**, 195–210 (1982)

[27] Pastur, L., Figotin, A.: Spectra of random and almost-periodic operators. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 297. Springer, Berlin (1992)

[28] Texier, C.: Individual energy level distributions for one-dimensional diagonal and off-diagonal disorder. J. Phys. A **33**(35), 6095–6128 (2000)

François Germinet
IUF, Département de Mathématiques
Université de Cergy-Pontoise, CNRS UMR 8088
95000 Cergy-Pontoise, France
e-mail: francois.germinet@u-cergy.fr

Frédéric Klopp
Institut de Mathématiques de Jussieu
Université Pierre et Marie Curie
Case 186, 4 place Jussieu
75252 Paris Cedex 05, France
e-mail: klopp@math.jussieu.fr

Communicated by Jean Bellissard.

Received: March 4, 2012.

Accepted: October 16, 2012.