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Ground States for a Stationary Mean-Field Model for a Nucleon

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Abstract. In this paper we consider a variational problem related to a model for a nucleon interacting with the ω and σ mesons in the atomic nucleus. The model is relativistic, and we study it in a nuclear physics nonrelativistic limit, which is of a very different nature than the nonrelativistic limit in the atomic physics. Ground states are shown to exist for a large class of values for the parameters of the problem, which are determined by the values of some physical constants.

1. Introduction

This article is concerned with the existence of minimizers for the energy functional

$$\mathcal{E}(\varphi) = \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma} \cdot \nabla\varphi|^2}{(1-|\varphi|^2)_+}\,dx - \frac{a}{2}\int_{\mathbb{R}^3} |\varphi|^4\,dx \tag{1.1}$$

under the L^2-normalization constraint

$$\int_{\mathbb{R}^3} |\varphi|^2\,dx = 1. \tag{1.2}$$

More precisely, for a large class of values for the parameter a, we show the existence of solutions of the following minimization problem

$$I = \inf\left\{\mathcal{E}(\varphi);\ \varphi \in X, \int_{\mathbb{R}^3} |\varphi|^2\,dx = 1\right\}, \tag{1.3}$$

where

$$X = \left\{\varphi \in L^2(\mathbb{R}^3, \mathbb{C}^2)\ ;\ \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma} \cdot \nabla\varphi|^2}{(1-|\varphi|^2)_+}\,dx < +\infty\right\}. \tag{1.4}$$

We remind that $\boldsymbol{\sigma}$ denotes the vector of Pauli matrices $(\sigma_1, \sigma_2, \sigma_3)$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Euler–Lagrange equation of the energy functional $\mathcal{E}$ under the L^2-normalization constraint is given by the second order equation

$$-\boldsymbol{\sigma}\cdot\nabla\left(\frac{\boldsymbol{\sigma}\cdot\nabla\varphi}{(1-|\varphi|^2)_+}\right) + \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+^2}\varphi - a|\varphi|^2\varphi + b\varphi = 0, \tag{1.5}$$

where b is the Lagrange multiplier associated with the L^2-constraint (1.2). Hence a solution of the minimization problem (1.3) is a solution of the equation (1.5). Moreover, Lemma 2.1 below proves that any $\varphi \in X$ satisfies $|\varphi|^2 \le 1$ a.e. in $\mathbb{R}^3$. So, a minimizer for (1.3) is actually a solution of

$$-\boldsymbol{\sigma}\cdot\nabla\left(\frac{\boldsymbol{\sigma}\cdot\nabla\varphi}{1-|\varphi|^2}\right) + \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)^2}\varphi - a|\varphi|^2\varphi + b\varphi = 0. \tag{1.6}$$

Solutions of (1.6) which are minimizers for I are called ground states.

Equation (1.6) is equivalent to the system

$$\begin{cases} i\boldsymbol{\sigma}\cdot\nabla\chi + |\chi|^2\varphi - a|\varphi|^2\varphi + b\varphi = 0, \\ -i\boldsymbol{\sigma}\cdot\nabla\varphi + (1-|\varphi|^2)\chi = 0. \end{cases} \tag{1.7}$$

As we formally derived in a previous paper [1], this system is the nuclear physics nonrelativistic limit of the σ-ω relativistic mean-field model [9,10] in the case of a single nucleon.

In [1], we proved the existence of square integrable solutions of (1.7) in the particular form

$$\begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix} = \begin{pmatrix} g(r) & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ if(r) & \begin{pmatrix} \cos\vartheta \\ \sin\vartheta e^{i\phi} \end{pmatrix} \end{pmatrix}, \tag{1.8}$$

where f and g are real valued radial functions. This ansatz corresponds to particles with minimal angular momentum, that is, $j = 1/2$ (for instance, see [8]). In this model, the equations for f and g read as follows:

$$\begin{cases} f' + \frac{2}{r}f = g(f^2 - ag^2 + b), \\ g' = f(1-g^2), \end{cases} \tag{1.9}$$

where we assumed $f(0) = 0$ in order to avoid solutions with singularities at the origin, and we showed that given $a, b > 0$ such that $a - 2b > 0$, there exists at least one nontrivial solution of (1.9) such that

$$(f(r), g(r)) \longrightarrow (0,0) \quad \text{as } r \longrightarrow +\infty. \tag{1.10}$$

In this paper, we prove the existence of solutions of the above nuclear physics nonrelativistic limit of the σ–ω relativistic mean-field model without considering any particular ansatz for the nucleon's wave function. It is not known if it is possible to use symmetrization techniques to prove symmetry of the ground state solutions, at least in some particular cases. The presence of the Pauli matrices in the kinetic energy term makes things difficult in that respect.

Note that (1.6) is the Euler–Lagrange equation of the energy functional

$$\mathcal{F}(\varphi) = \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma} \cdot \nabla\varphi|^2}{1 - |\varphi|^2}\, dx - \frac{a}{2} \int_{\mathbb{R}^3} |\varphi|^4\, dx \tag{1.11}$$

under the L^2 normalization constraint. In the Appendix, we prove that the energy functional $\mathcal{F}$ is not bounded from below. So, trying to find solutions of (1.6) which minimize the energy $\mathcal{F}$ is hopeless and the definition of ground states for (1.6) based on this functional is not clear.

In our previous work [1], we showed that for all the solutions of (1.9) which are square integrable, $g^2(r) < 1$ in $[0, +\infty)$. Hence, according to this result, we conjecture that a solution of (1.6) has to satisfy $|\varphi|^2 \leq 1$ a.e. in $\mathbb{R}^3$. As we prove in the Appendix, this assumption is also justified when we consider the intermediate model

$$\varphi = \begin{pmatrix} u \\ 0 \end{pmatrix} \tag{1.12}$$

with $u : \mathbb{R}^3 \to \mathbb{R}$ and $a > b$. Moreover, in the physical literature finite nuclei are described via functions φ such that, in the right units, $|\varphi|^2 \leq 1$ and $|\varphi|$ is rather flat near the center of the nucleus, and is equal to 0 outside it, see [2,5].

Note that if $|\varphi|^2 \leq 1$ a.e. in $\mathbb{R}^3$, then $\mathcal{F}(\varphi) = \mathcal{E}(\varphi)$, and the ground states of (1.6) can be defined without further specification as the minimizers of $\mathcal{E}$.

The main result of our paper is the following:

Theorem 1.1. *If $I < 0$ there exists a minimizer of* (1.3)*. Moreover, $I < 0$ if and only if $a > a_0$ where a_0 is a strictly positive constant. In particular, $10.96 \approx \frac{2}{S^2} < a_0 < 48.06$, where S is the best constant in the Sobolev embedding of $H^1(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$.*

Remark 1. The upper estimate for a_0 is obtained by using a particular test function and is probably not optimal. The calculation of a_0's exact value is a challenging open problem.

Remark 2. It is easy to prove that $I \leq 0$ for all values of a. But if $I = 0$ there will be minimizing sequences which are not relatively compact and maybe none of them is. In that case the minimum would not be achieved. It is thus an open problem to know whether I is achieved for all values of a or not.

The proof of the above theorem is an application of the concentration-compactness principle [3,4] with some new ingredients. The main new difficulty is due to the presence of the term $\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+}\, dx$ in the energy functional. As we will see below, to rule out the dichotomy case in the concentration-compactness lemma we have to choose ad-hoc cut-off functions allowing us to deal with possible singularities of the integrand. This is also necessary in order to show the localization properties of $\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+}\, dx$.

In the next section, we will establish a concentration-compactness lemma in X and then apply it to prove our main result. The Appendix contains some auxiliary results about various properties of the model problem that we consider here.

2. Proof of Theorem 1.1

To prove this theorem, we are going to apply a concentration-compactness lemma that we state below. The reader may refer to [3,4] for more details on this kind of approach. The particular shape of the energy functional, where the kinetic energy term is multiplied by a function which could present singularities as $|\varphi|$ gets close to 1 creates some complications in the use of concentration-compactness, that we deal with by using very particular cut-off functions.

Let us introduce

$$I_\nu = \inf\left\{\mathcal{E}(\varphi)\ ;\ \varphi \in X, \int_{\mathbb{R}^3} |\varphi|^2\,dx = \nu\right\} \tag{2.1}$$

where $\nu > 0$ and $I_1 = I$, and we make a few preliminary observations.

Lemma 2.1 [6]. *Let $\varphi \in X$. Then, $\varphi \in H^1(\mathbb{R}^3, \mathbb{C}^2)$ and $|\varphi|^2 \le 1$ a.e. in $\mathbb{R}^3$.*

Proof. First, by a straightforward calculation, we obtain

$$\int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx = \int_{\mathbb{R}^3} |\boldsymbol{\sigma}\cdot\nabla\varphi|^2\,dx \le \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+}\,dx < +\infty.$$

Hence, $\varphi \in H^1(\mathbb{R}^3, \mathbb{C}^2)$. Next, let $n \in \mathbb{C}^2$ such that $|n| = 1$. Note that for $\varphi \in X$, $\mathbb{1}_{\mathrm{Re}\{n\cdot\varphi\}\ge 1}(\boldsymbol{\sigma}\cdot\nabla\varphi) = 0$, a.e. in $\mathbb{R}^3$. Define the functions $f = (\mathrm{Re}\{n\cdot\varphi\}-1)_+$ and $\psi = fn$. (Note that for 2 complex vectors $A, B \in \mathbb{C}^2$, $A\cdot B$ denotes the scalar product $\Sigma_{i=1}^2 \overline{A}_i B_i$, where $\overline{z}$ stands for the complex conjugate of any complex number z).

We have $f \in H^1(\mathbb{R}^3, \mathbb{R})$ and $\psi \in H^1(\mathbb{R}^3, \mathbb{C}^2)$. Moreover, for $k = 1, 2, 3$,

$$\partial_k\psi = \partial_k f\, n \quad \text{and} \quad \partial_k f = \mathrm{Re}\{n\cdot\partial_k\varphi\}\,\mathbb{1}_{\mathrm{Re}\{n\cdot\varphi\}\ge 1} = n\cdot\partial_k\psi.$$

Hence, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla f|^2 dx\\
&\quad= \int_{\mathbb{R}^3} |\nabla\psi|^2 dx = \int_{\mathbb{R}^3} \sum_{k=1}^3 \mathrm{Re}\{\mathrm{Re}\{n\cdot\partial_k\varphi\}\, n\cdot\partial_k\psi\}\,dx\\
&\quad= \int_{\mathbb{R}^3} \sum_{k=1}^3 \mathrm{Re}\{n\cdot\partial_k\varphi\}\,\mathrm{Re}\{n\cdot\partial_k\psi\}\,dx = \int_{\mathbb{R}^3} \sum_{k=1}^3 \mathrm{Re}\{\partial_k f\, n\cdot\partial_k\varphi\}\,dx\\
&\quad= \int_{\mathbb{R}^3} \mathrm{Re}\{\nabla\psi\cdot\nabla\varphi\}\,dx = \int_{\mathbb{R}^3} \mathrm{Re}\{(\boldsymbol{\sigma}\cdot\nabla\psi)\cdot(\boldsymbol{\sigma}\cdot\nabla\varphi)\}\,dx\\
&\quad= \int_{\mathbb{R}^3} \mathrm{Re}\left\{(\boldsymbol{\sigma}\cdot\nabla\psi)\cdot\mathbb{1}_{\mathrm{Re}\{n\cdot\varphi\}\ge 1}(\boldsymbol{\sigma}\cdot\nabla\varphi)\right\}dx = 0.
\end{aligned}$$

As a consequence, $f = 0$ a.e. in $\mathbb{R}^3$ that means $\mathrm{Re}\{n\cdot\varphi\} \le 1$ a.e. for all $n \in \mathbb{C}^2$ such that $|n| = 1$. This clearly implies that $|\varphi| \le 1$ a.e. in $\mathbb{R}^3$. □

In what follows, we say that a sequence $\{\varphi_n\}_n$ is X-bounded if there exists a positive constant C independent of n such that

$$\|\varphi_n\|_{L^2}^2 + \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\,dx \le C. \tag{2.2}$$

Lemma 2.2. *Let $\{\varphi_n\}_n$ be a minimizing sequence of (2.1), then $\{\varphi_n\}_n$ is X-bounded, bounded in $H^1(\mathbb{R}^3)$ and $I_\nu > -\infty$.*

Proof. Indeed, since $\{\varphi_n\}_n$ is a minimizing sequence, there exists a constant C such that

$$\begin{aligned} C \ge \mathcal{E}(\varphi_n) &\ge \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\,dx - \frac{a}{2}\nu \ge \int_{\mathbb{R}^3} |\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2\,dx - \frac{a}{2}\nu \\ &= \int_{\mathbb{R}^3} |\nabla\varphi_n|^2\,dx - \frac{a}{2}\nu \ge -\frac{a}{2}\nu. \end{aligned}$$

As a conclusion, $\|\varphi_n\|_{H^1}$ is bounded independently of n and I_ν is bounded from below. $\square$

Lemma 2.3. *For all $\nu \in (0,1), I_\nu \le 0$. Moreover, the strict inequality $I < 0$ is equivalent to the strict concentration-compactness inequalities*

$$I < I_\nu + I_{1-\nu}, \quad \forall \nu \in (0,1). \tag{2.3}$$

Proof. Indeed, let $\varphi \in \mathcal{D}(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3}|\varphi|^2 = \nu$ and $\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+}\,dx < +\infty$, and let $\varphi_\gamma(x) = \gamma^{-3/2}\varphi(\gamma^{-1}x)$ for $\gamma > 1$. Then

$$I_\nu \le \mathcal{E}(\varphi_\gamma) = \frac{1}{\gamma^2}\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{\left(1-\frac{1}{\gamma^3}|\varphi|^2\right)_+}\,dx - \frac{1}{\gamma^3}\frac{a}{2}\int_{\mathbb{R}^3}|\varphi|^4\,dx,$$

and letting $\gamma \to +\infty$, we prove $I_\nu \le 0$.

By a scaling argument, we obtain

$$I_{\vartheta\nu} \le \inf\left\{\vartheta^{1/3}\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+}\,dx - \frac{\vartheta\, a}{2}\int_{\mathbb{R}^3}|\varphi|^4\,dx \,\middle|\, \varphi\in X, \int_{\mathbb{R}^3}|\varphi|^2\,dx = \nu\right\},$$

and, if $I_\nu < 0$, we may restrict the infimum I_ν to elements φ satisfying

$$K(\varphi) = \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi|^2}{(1-|\varphi|^2)_+}\,dx \ge \delta > 0,$$

for some $\delta > 0$. Indeed, if there is a minimizing sequence $\{\varphi_n\}_n$ of I_ν such that $K(\varphi_n) \underset{n}{\longrightarrow} 0$, then, by Sobolev embeddings, $\varphi_n \underset{n}{\longrightarrow} 0$ in $L^p(\mathbb{R}^3)$ for $2 < p \le 6$ and $I_\nu \ge 0$. As a conclusion, if $I_\nu < 0$, then, for all $\vartheta > 1$ and for all $\nu > 0$,

$$I_{\vartheta\nu} < \vartheta \inf\left\{\mathcal{E}(\varphi) \,\middle|\, \varphi\in X, K(\varphi) > 0, \int_{\mathbb{R}^3}|\varphi|^2\,dx = \nu\right\} = \vartheta I_\nu. \tag{2.4}$$

Hence, a straightforward argument (see lemma II.1 of [3]) proves that (2.3) is equivalent to $I < 0$. $\square$

In order to prove Theorem 1.1 we need to analyse the possible behaviour of minimizing sequences for I. This is done in the following lemma.

Lemma 2.4. *Let $\{\varphi_n\}_n$ be a X-bounded sequence such that $\int_{\mathbb{R}^3} |\varphi_n|^2\,dx = 1$ for all $n \geq 0$. Then there exists a subsequence that we still denote by $\{\varphi_n\}_n$ such that one of the following properties holds:*

1. *Compactness up to a translation: there exists a sequence $\{y_n\}_n \subset \mathbb{R}^3$ such that, for every $\varepsilon > 0$, there exists $0 < R < \infty$ with*
$$\int_{B(y_n,R)} |\varphi_n|^2\,dx \geq 1 - \varepsilon;$$
2. *Vanishing: for all $0 < R < \infty$*
$$\sup_{y\in\mathbb{R}^3} \int_{B(y,R)} |\varphi_n|^2\,dx \underset{n}{\longrightarrow} 0;$$
3. *Dichotomy: there exist $\alpha \in (0,1)$ and $n_0 \geq 0$ such that there exist two X-bounded sequences, $\{\varphi_1^n\}_{n\geq n_0}$ and $\{\varphi_2^n\}_{n\geq n_0}$, satisfying the following properties:*
$$\|\varphi_n - (\varphi_1^n + \varphi_2^n)\|_{L^p} \underset{n}{\longrightarrow} 0, \quad \textit{for } 2 \leq p < 6, \tag{2.5}$$
and
$$\int_{\mathbb{R}^3} |\varphi_1^n|^2\,dx \underset{n}{\longrightarrow} \alpha \quad \textit{and} \quad \int_{\mathbb{R}^3} |\varphi_2^n|^2\,dx \underset{n}{\longrightarrow} 1 - \alpha, \tag{2.6}$$
$$\mathrm{dist}(\mathrm{supp}\,\varphi_1^n, \mathrm{supp}\,\varphi_2^n) \underset{n}{\longrightarrow} +\infty. \tag{2.7}$$
Moreover, in this case we have that
$$\liminf_{n\to+\infty} \mathcal{E}(\varphi_n) - \mathcal{E}(\varphi_1^n) - \mathcal{E}(\varphi_2^n) \geq 0, \tag{2.8}$$
which implies $I \geq I_\alpha + I_{1-\alpha}$.

Proof of Lemma 2.4. Let $\{\varphi_n\}_n$ be a X-bounded sequence such that $\int_{\mathbb{R}^3} |\varphi_n|^2\,dx = \nu$ for all $n \geq 0$. We remind that X-bounded means that there exists $C > 0$ such that
$$\|\varphi_n\|_{L^2}^2 + \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\,dx \leq C.$$

Moreover, thanks to Lemma 2.1, if $\{\varphi_n\}_n$ is a X-bounded sequence then $\{\varphi_n\}_n$ is bounded in L^∞ (by the constant 1) and in $H^1(\mathbb{R}^3)$. Then, along the lines of

[3], we introduce the so-called Lévy concentration functions

$$Q_n(R) = \sup_{y\in\mathbb{R}^3} \int_{|x-y|<R} |\varphi_n|^2\, dx, \tag{2.9}$$

$$K_n(R) = \sup_{y\in\mathbb{R}^3} \int_{|x-y|<R} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\, dx \tag{2.10}$$

for $R > 0$. Note that Q_n and K_n are continuous non-decreasing functions on $[0, +\infty)$, such that for all $n \geq 0$ and for all $R > 0$

$$Q_n(R) + K_n(R) \leq C$$

since $\{\varphi_n\}_n$ is X-bounded. Then, up to a subsequence, we have for all $R > 0$

$$Q_n(R) \underset{n}{\longrightarrow} Q(R), \tag{2.11}$$

$$K_n(R) \underset{n}{\longrightarrow} K(R), \tag{2.12}$$

where Q and K are nonnegative, non-decreasing functions. Clearly, we have that

$$\alpha = \lim_{R\to+\infty} Q(R) \in [0, 1],$$

and we denote $l = \lim_{R\to+\infty} K(R)$.

If $\alpha = 0$, then the situation (2) of the lemma arises as a direct consequence of definition (2.9). If $\alpha = 1$, then (1) follows, see [3] for details. Assume that $\alpha \in (0, 1)$, we have to show that (3) holds.

First of all, consider $\varepsilon > 0$, small, and $R_\varepsilon > 0$ such that $Q(R_\varepsilon) = \alpha - \varepsilon$ and $K(R_\varepsilon) \leq l - \varepsilon$. Then, for n large enough,

$$Q_n(R_\varepsilon) - Q(R_\varepsilon) < 1/n, \quad K_n(R_\varepsilon) - K(R_\varepsilon) < 1/n,$$

and by definition of the Lévy functions Q_n and K_n, extracting subsequences if necessary, there exists $y_n \in \mathbb{R}^3$ such that

$$\left| \int_{|x-y_n|<R_\varepsilon} |\varphi_n|^2\, dx - Q_n(R_\varepsilon) \right| \leq \frac{1}{n},$$

$$\left| \int_{|x-y_n|<R_\varepsilon} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\, dx - K_n(R_\varepsilon) \right| \leq \frac{1}{n}.$$

Next define $R_n > R_\varepsilon$ such that

$$\int_{R_\varepsilon<|x-y_n|<R_n} |\varphi_n|^2\, dx = \frac{3}{n} + \varepsilon.$$

Necessarily, $R_n \to +\infty$ as $n \to +\infty$. Indeed, if $R_n \le M$ for some $M > 0$, then $Q(M) > \alpha$, which is impossible. We then deduce that for n large enough,

$$\int_{\frac{R_n}{8} \le |x-y_n| \le R_n} |\varphi_n|^2\, dx \le \frac{3}{n} + \varepsilon.$$

Let ξ, ζ be cut-off functions: $\xi, \zeta \in \mathcal{D}(\mathbb{R}^3)$ such that

$$\xi(x) = \begin{cases} 1 & |x| \le 1 \\ 1 - \exp\left(1 - \dfrac{1}{1 - \exp\left(1 - \frac{1}{2-|x|}\right)}\right) & 1 < |x| < 2 \\ 0 & |x| \ge 2 \end{cases}$$

$$\zeta(x) = \begin{cases} 0 & |x| \le 1 \\ \exp\left(1 - \dfrac{1}{1 - \exp\left(1 - \frac{1}{2-|x|}\right)}\right) & 1 < |x| < 2 \\ 1 & |x| \ge 2 \end{cases},$$

and let ξ_μ, ζ_μ denote $\xi\left(\frac{\cdot}{\mu}\right)$, $\zeta\left(\frac{\cdot}{\mu}\right)$. We define

$$\varphi_1^n(\cdot) = \xi_{\frac{R_n}{8}}(\cdot - y_n)\varphi_n(\cdot) = \xi_{\frac{R_n}{8}, y_n}(\cdot)\varphi_n(\cdot) \tag{2.13}$$

$$\varphi_2^n(\cdot) = \zeta_{\frac{R_n}{2}}(\cdot - y_n)\varphi_n(\cdot) = \zeta_{\frac{R_n}{2}, y_n}(\cdot)\varphi_n(\cdot) \tag{2.14}$$

with $R_n \to +\infty$. The limit (2.7) follows easily from these definitions. Furthermore, (2.5) and (2.6) are obtained in the following way:

$$\begin{aligned} &\lim_{n\to+\infty} \int_{\mathbb{R}^3} |\varphi_n - (\varphi_1^n + \varphi_2^n)|^2\, dx \\ &= \lim_{n\to+\infty} \int_{\frac{R_n}{8} \le |x-y_n| \le R_n} |(1 - \xi_{\frac{R_n}{8}} - \zeta_{\frac{R_n}{2}})\varphi_n|^2\, dx \\ &\le \lim_{n\to+\infty} \int_{\frac{R_n}{8} \le |x-y_n| \le R_n} |\varphi_n|^2\, dx \le \varepsilon. \end{aligned}$$

Now by taking a sequence of ε tending to 0, and by taking a diagonal sequence of the functions φ_n, and calling it by the same name, we find

$$\int_{\frac{R_n}{8} \le |x-y_n| \le R_n} |\varphi_n|^2\, dx \underset{n}{\longrightarrow} 0,$$

and, since $\{\varphi_1^n\}_n$ and $\{\varphi_2^n\}_n$ are bounded in $H^1(\mathbb{R}^3)$, we also obtain

$$\lim_{n\to+\infty} \|\varphi_n - (\varphi_1^n + \varphi_2^n)\|_{L^p} \underset{n}{\longrightarrow} 0,$$

for $2 \leq p < 6$. Next, we have to prove that $\{\varphi_1^n\}_{n\geq n_0}$ and $\{\varphi_2^n\}_{n\geq n_0}$ are X-bounded. To this purpose, we show that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_1^n|^2}{(1-|\varphi_1^n|^2)_+}\,dx - \int_{\mathbb{R}^3} \frac{\xi^2_{\frac{R_n}{8},y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2)_+}\,dx = 0 \tag{2.15}$$

and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_2^n|^2}{(1-|\varphi_2^n|^2)_+}\,dx - \int_{\mathbb{R}^3} \frac{\zeta^2_{\frac{R_n}{2},y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_2^n|^2)_+}\,dx = 0. \tag{2.16}$$

Indeed, if (2.15) and (2.16) hold, we obtain that for all $\varepsilon > 0$, there exists $n_0 \geq 0$ such that for all $n \geq n_0$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_1^n|^2}{(1-|\varphi_1^n|^2)_+}\,dx &\leq \int_{\mathbb{R}^3} \frac{\xi^2_{\frac{R_n}{8},y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2)_+}\,dx + o(1)_{n\to+\infty}\\ &\leq \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\,dx + o(1)_{n\to+\infty} \leq C + o(1)_{n\to+\infty},\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_2^n|^2}{(1-|\varphi_2^n|^2)_+}\,dx &\leq \int_{\mathbb{R}^3} \frac{\zeta^2_{\frac{R_n}{2},y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_2^n|^2)_+}\,dx + o(1)_{n\to+\infty}\\ &\leq \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\,dx + o(1)_{n\to+\infty} \leq C + o(1)_{n\to+\infty}.\end{aligned}$$

To prove (2.15) we proceed as follows. We remark that

$$\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_1^n|^2}{(1-|\varphi_1^n|^2)_+}\,dx - \int_{\mathbb{R}^3} \frac{\xi^2_{\frac{R_n}{8},y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2)_+}\,dx = A_n + B_n,$$

where

$$\begin{aligned}A_n &:= \int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot(\nabla\xi_{\frac{R_n}{8},y_n})\varphi_n|^2}{(1-|\varphi_1^n|^2)_+}\,dx\\ &= \int_{\frac{R_n}{8}\leq|x-y_n|\leq\frac{R_n}{4}} \frac{|\boldsymbol{\sigma}\cdot(\nabla\xi_{\frac{R_n}{8},y_n})\varphi_n|^2}{(1-|\varphi_1^n|^2)_+}\,dx\\ &\leq \int_{\frac{R_n}{8}\leq|x-y_n|\leq\frac{R_n}{4}} \frac{|\boldsymbol{\sigma}\cdot(\nabla\xi_{\frac{R_n}{8},y_n})\varphi_n|^2}{1-\xi^2_{\frac{R_n}{8},y_n}}\,dx := C_n,\end{aligned}$$

and

$$|B_n| \le 2(C_n)^{\frac{1}{2}} \left(\int\limits_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma} \cdot \nabla \varphi_n|^2}{(1-|\varphi_n|^2)_+}\, dx \right)^{\frac{1}{2}}.$$

Let us now prove that C_n tends to 0 as n goes to $+\infty$. Using spherical coordinates, we obtain

$$\begin{aligned} C_n &\le \int\limits_{\frac{R_n}{8}}^{\frac{R_n}{4}} \int\limits_0^{\pi} \int\limits_0^{2\pi} \frac{|(\sigma \cdot \boldsymbol{e}_r)\, \varphi_n(s,\theta,\phi)|^2 \left(\xi'_{\frac{R_n}{8}}(s)\right)^2}{1-\xi^2_{\frac{R_n}{8}}(s)}\, s^2 \sin\theta\, ds\, d\theta\, d\phi \\ &\le \frac{64}{R_n^2} \int\limits_{\frac{R_n}{8}}^{\frac{R_n}{4}} \int\limits_0^{\pi} \int\limits_0^{2\pi} \frac{|\varphi_n(s,\theta,\phi)|^2 \left(\xi'\left(\frac{8}{R_n}s\right)\right)^2}{1-\xi^2\left(\frac{8}{R_n}s\right)}\, s^2 \sin\theta\, ds\, d\theta\, d\phi \\ &\le \frac{64}{R_n^2} \max_{1\le r\le 2} \frac{(\xi'(r))^2}{1-\xi^2(r)} \\ &\quad \times \int\limits_0^{+\infty} \int\limits_0^{\pi} \int\limits_0^{2\pi} |\varphi_n(s,\theta,\phi)|^2\, s^2 \sin\theta\, ds\, d\theta\, d\phi = O\left(\frac{1}{R_n^2}\right) \end{aligned}$$

since $\max_{1\le r\le 2} \frac{(\xi'(r))^2}{1-\xi^2(r)} \le C$. Indeed, since $\xi^2(r) = 1$ if and only if $r=1$, $\frac{(\xi'(r))^2}{1-\xi^2(r)}$ is a continuous function on $(1,2)$. Moreover, by a straightforward calculation, we obtain $\lim_{r\to 1^+} \frac{(\xi'(r))^2}{1-\xi^2(r)} = 0 = \lim_{r\to 2^-} \frac{(\xi'(r))^2}{1-\xi^2(r)}$. Hence, we can conclude, that $\frac{(\xi'(r))^2}{1-\xi^2(r)}$ is bounded in $[1,2]$. As a conclusion, since $R_n \to +\infty$, we obtain

$$\lim_{n\to+\infty} \int\limits_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma} \cdot \nabla \varphi_1^n|^2}{(1-|\varphi_1^n|^2)_+}\, dx - \int\limits_{\mathbb{R}^3} \frac{\xi^2_{\frac{R_n}{8},y_n} |\boldsymbol{\sigma} \cdot \nabla \varphi_n|^2}{(1-|\varphi_1^n|^2)_+}\, dx = 0.$$

With the same argument, we prove (2.16).

Finally, it remains to show that

$$\liminf_{n\to+\infty} \mathcal{E}(\varphi_n) - \mathcal{E}(\varphi_1^n) - \mathcal{E}(\varphi_2^n) \ge 0.$$

First of all, using the definitions (2.13) and (2.14), we obtain

$$\lim_{n\to+\infty} \int\limits_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma} \cdot \nabla \varphi_n|^2}{(1-|\varphi_n|^2)_+}\, dx \ge \lim_{n\to+\infty} \int\limits_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma} \cdot \nabla \varphi_n|^2}{(1-|\varphi_1^n|^2-|\varphi_2^n|^2)_+}\, dx.$$

Next, we remark that

$$\int\limits_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma} \cdot \nabla \varphi_n|^2}{(1-|\varphi_1^n|^2-|\varphi_2^n|^2)_+}\, dx - \int\limits_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma} \cdot \nabla \varphi_1^n|^2}{(1-|\varphi_1^n|^2)_+}\, dx - \int\limits_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma} \cdot \nabla \varphi_2^n|^2}{(1-|\varphi_2^n|^2)_+}\, dx$$

$$= \int\limits_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2-|\varphi_2^n|^2)_+}\,dx - \int\limits_{\mathbb{R}^3} \frac{\xi^2_{\frac{R_n}{8},y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2)_+}\,dx$$
$$- \int\limits_{\mathbb{R}^3} \frac{\zeta^2_{\frac{R_n}{2},y_n}|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_2^n|^2)_+}\,dx + o(1)_{n\to\infty}$$
$$= \int\limits_{\mathbb{R}^3} \frac{(1-\xi^2_{\frac{R_n}{8},y_n}-\zeta^2_{\frac{R_n}{2},y_n})|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_1^n|^2-|\varphi_2^n|^2)_+}\,dx + o(1)_{n\to\infty}$$
$$\geq o(1)_{n\to\infty}.$$

As a conclusion,

$$\lim_{n\to+\infty}\int\limits_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_n|^2}{(1-|\varphi_n|^2)_+}\,dx \geq \lim_{n\to+\infty}\int\limits_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_1^n|^2}{(1-|\varphi_1^n|^2)_+}\,dx + \lim_{n\to+\infty}\int\limits_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\varphi_2^n|^2}{(1-|\varphi_2^n|^2)_+}\,dx,$$

and, using (2.5) and the localization properties of φ_1^n and φ_2^n, we have

$$I = \lim_{n\to+\infty}\mathcal{E}(\varphi_n) \geq \liminf_{n\to+\infty}\mathcal{E}(\varphi_1^n) + \liminf_{n\to+\infty}\mathcal{E}(\varphi_2^n) \geq I_\alpha + I_{1-\alpha}.$$

□

Proof of Theorem 1.1. Assume that $I < 0$. By Lemma 2.2, any minimizing sequence $\{\varphi_n\}_n$ is X-bounded, and then we can use Lemma 2.4 to it. It is easy to rule out vanishing and dichotomy whenever $I < 0$.

Vanishing cannot occur. Indeed, if vanishing occurs, then, up to a subsequence, $\forall R < +\infty$ we have

$$\lim_{n\to+\infty}\sup_{y\in\mathbb{R}^3}\int\limits_{B(y,R)} |\varphi_n|^2 = 0. \tag{2.17}$$

This implies that φ_n converges strongly in $L^p(\mathbb{R}^3)$ for $2 < p < 6$ and, as a consequence, $I \geq 0$. Clearly, this contradicts $I < 0$.
Moreover, if dichotomy occurs, we have

$$I = \lim_{n\to+\infty}\mathcal{E}(\varphi_n) \geq \liminf_{n\to+\infty}\mathcal{E}(\varphi_1^n) + \liminf_{n\to+\infty}\mathcal{E}(\varphi_2^n) \geq I_\alpha + I_{1-\alpha}$$

which contradicts Lemma 2.3, since $I < 0$.

Hence, for n large enough, there exists $\{y_n\}_n \subset \mathbb{R}^3$ such that $\forall \varepsilon > 0$, $\exists R < +\infty$,

$$\int\limits_{B(y_n,R)} |\varphi_n|^2 \geq 1-\varepsilon.$$

We denote by $\tilde{\varphi}_n(\cdot) = \varphi_n(\cdot + y_n)$. Since $\{\tilde{\varphi}_n\}_n$ is bounded in H^1, $\{\tilde{\varphi}_n\}_n$ converges weakly in H^1, almost everywhere on $\mathbb{R}^3$ and in L^p_{loc} for $2 \leq p < 6$ to some $\tilde{\varphi}$. In particular, as a consequence of weak convergence in H^1, $\boldsymbol{\sigma}\cdot\nabla\tilde{\varphi}_n$

converges weakly to $\boldsymbol{\sigma} \cdot \nabla \tilde{\varphi}$ in L^2. Moreover, thanks to the concentration-compactness argument, $\{\tilde{\varphi}_n\}_n$ converges strongly in L^2 and in L^p for $2 \le p < 6$.

Lemma 2.5. *Let $\{f_n\}_n$ and $\{g_n\}_n$ be two sequences of functions such that $f_n : \mathbb{R}^3 \to \mathbb{R}_+, g_n : \mathbb{R}^3 \to \mathbb{C}^2$, f_n converges to f a.e., g_n converges weakly to g in L^2 and there exists a constant C, that does not depend on n, such that $\int_{\mathbb{R}^3} f_n |g_n|^2 \, dx \le C$. Then*

$$\int_{\mathbb{R}^3} f|g|^2 \, dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^3} f_n |g_n|^2 \, dx.$$

□

Proof. Given a function $h : \mathbb{R}^3 \to \mathbb{R}_+$, let T_k be the function defined by

$$T_k(h)(x) = \begin{cases} h(x) & \text{if } h(x) \le k \\ k & \text{if } h(x) > k \end{cases}$$

for all $k \in [0, \infty)$. Hence, the following properties are satisfied for all $k \in [0, \infty)$:

$$T_k(f_n) \underset{n}{\longrightarrow} T_k(f) \quad \text{a.e. in } \mathbb{R}^3, \tag{2.18}$$

$$T_k(f_n)|g|^2 \underset{n}{\longrightarrow} T_k(f)|g|^2 \quad \text{in } L^1, \tag{2.19}$$

$$T_k(f_n) g \underset{n}{\rightharpoonup} T_k(f) g \quad \text{in } L^2, \tag{2.20}$$

$$\|T_k(f_n) g\|_{L^2} \underset{n}{\longrightarrow} \|T_k(f) g\|_{L^2}, \tag{2.21}$$

where to obtain (2.19) and (2.21), we use Lebesgue's dominated convergence theorem. Moreover, as a consequence of (2.20) and (2.21), we have

$$T_k(f_n) g \underset{n}{\longrightarrow} T_k(f) g \quad \text{in } L^2. \tag{2.22}$$

Next, we have

$$\begin{aligned}
0 \le \liminf_{n\to+\infty} \int_{\mathbb{R}^3} T_k(f_n)|g_n - g|^2 \, dx &= \liminf_{n\to+\infty} \int_{\mathbb{R}^3} T_k(f_n)|g_n|^2 \, dx \\
&\quad + \liminf_{n\to+\infty} \int_{\mathbb{R}^3} T_k(f_n)|g|^2 \, dx \\
&\quad - \liminf_{n\to+\infty} \left(\int_{\mathbb{R}^3} T_k(f_n)\, \overline{g}_n \cdot g \, dx + \int_{\mathbb{R}^3} T_k(f_n)\, g_n \cdot \overline{g} \, dx \right) \\
&= \liminf_{n\to+\infty} \int_{\mathbb{R}^3} T_k(f_n)|g_n|^2 \, dx + \int_{\mathbb{R}^3} T_k(f)|g|^2 \, dx - 2 \int_{\mathbb{R}^3} T_k(f)|g|^2 \, dx.
\end{aligned}$$

thanks to (2.19), (2.22) and the fact that g_n converges weakly to g in L^2. As a consequence,

$$\int_{\mathbb{R}^3} T_k(f)|g|^2 \, dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^3} T_k(f_n)|g_n|^2 \, dx. \tag{2.23}$$

Since

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^3} T_k(f_n)|g_n|^2\,dx \le \liminf_{n\to+\infty}\int_{\mathbb{R}^3} f_n|g_n|^2\,dx \le C,$$

we can pass to the limit for k that goes to $+\infty$ in (2.23) and we obtain

$$\int_{\mathbb{R}^3} f|g|^2\,dx \le \liminf_{n\to+\infty}\int_{\mathbb{R}^3} f_n|g_n|^2\,dx.$$

□

By applying Lemma 2.5 to $f_n = \frac{1}{(1-|\tilde{\varphi}_n|^2)_+}$ and $g_n = |\boldsymbol{\sigma}\cdot\nabla\tilde{\varphi}_n|$, we obtain

$$\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\tilde{\varphi}|^2}{(1-|\tilde{\varphi}|^2)_+}\,dx \le \liminf_{n\to+\infty}\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\tilde{\varphi}_n|^2}{(1-|\tilde{\varphi}_n|^2)_+}\,dx.$$

Hence, $\tilde{\varphi}\in X$, $\int_{\mathbb{R}^3}|\tilde{\varphi}|^2\,dx = 1$, and

$$\mathcal{E}(\tilde{\varphi}) \le \liminf_{n\to+\infty}\mathcal{E}(\tilde{\varphi}_n) \le \mathcal{E}(\tilde{\varphi}).$$

As a conclusion, the minimum of I is achieved by $\tilde{\varphi}$.

Finally, it remains to prove that there exists $a_0 > 0$ such that for all $a > a_0$ we have $I < 0$. We do it in the lemma below.

Lemma 2.6. *There exists a strictly positive constant a_0 such that $I < 0$ if and only if $a > a_0$. In particular, $10.96 \approx \frac{2}{S^2} < a_0 < 48.06$, where S is the best constant in the Sobolev embedding of $H^1(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$.*

Proof. It is clear that $I < 0$ for a large enough. Since I is non-increasing with respect to a, we may denote by a_0 the least positive constant such that $I < 0$ for $a > a_0$. We have to prove that $a_0 > 0$ or in other words $I = 0$ for a small enough. Using Sobolev and Hölder inequalities, we find, for $\varphi\in X$ such that $\int_{\mathbb{R}^3}|\varphi|^2\,dx = 1$,

$$\mathcal{E}(\varphi) \ge \frac{1}{S^2}\left(\int_{\mathbb{R}^3}|\varphi|^6\,dx\right)^{1/3} - \frac{a}{2}\left(\int_{\mathbb{R}^3}|\varphi|^6\,dx\right)^{1/3}.$$

Hence, if $a \le \frac{2}{S^2}, I = 0$. This implies $a_0 > \frac{2}{S^2}$. According to [7] the best constant for the Sobolev inequality

$$\|u\|_{L^q(\mathbb{R}^m)} \le C\|\nabla u\|_{L^p(\mathbb{R}^m)}$$

with $1 < p < m$ and $q = \frac{mp}{(m-p)}$ is given by

$$C = \pi^{-1/2}m^{-1/p}\left(\frac{p-1}{m-p}\right)^{1-1/p}\left(\frac{\Gamma(1+m/2)\Gamma(m)}{\Gamma(m/p)\Gamma(1+m-m/p)}\right)^{1/m}.$$

In particular,

$$S = \frac{1}{\sqrt{3\pi}}\left(\frac{4}{\sqrt{\pi}}\right)^{1/3},$$

and

$$\frac{2}{S^2} = \frac{3\pi^{4/3}}{2^{1/3}} \approx 10.96.$$

To obtain an upper estimate for a_0, we consider the following test function

$$\bar{\varphi}(x) = \begin{pmatrix} \bar{f}_R(|x|) \\ 0 \end{pmatrix}$$

where $\bar{f}_R(|x|) = \bar{f}\left(\frac{|x|}{R}\right)$,

$$\bar{f}(|x|) = \begin{cases} \cos(|x|) & |x| \leq \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$$

and $R \in (0,1)$ is such that $\int |\bar{f}_R|^2\, dx = 1$. This implies

$$R = \left(\frac{2}{\pi}\right)^{2/3} \left(\frac{3}{\pi^2 - 6}\right)^{1/3}.$$

Next, we denote by $\bar{a}$ the positive constant such that $\mathcal{E}(\varphi) = 0$. By definition,

$$\bar{a} = \frac{2\int_{\mathbb{R}^3} \frac{|\boldsymbol{\sigma}\cdot\nabla\bar{\varphi}|^2}{(1-|\bar{\varphi}|^2)_+}\, dx}{\int |\bar{\varphi}|^4} = \frac{2\int_{\mathbb{R}^3} \frac{|\nabla \bar{f}_R|^2}{1-|\bar{f}_R|^2}\, dx}{\int |\bar{f}_R|^4},$$

and, by a straightforward calculation, we obtain

$$\int_{\mathbb{R}^3} \frac{|\nabla \bar{f}_R|^2}{1-|\bar{f}_R|^2}\, dx = \frac{\pi^4}{6} R = \frac{\pi^{10/3}}{3^{2/3}(2(\pi^2-6))^{1/3}},$$

$$\int |\bar{f}_R|^4 = \frac{\pi^2(2\pi^2-15)}{32} R^3 = \frac{3(2\pi^2-15)}{8(\pi^2-6)}.$$

As a consequence,

$$\bar{a} = \frac{8\pi^{10/3}\left(\frac{2}{3}(\pi^2-6)\right)^{2/3}}{3(2\pi^2-15)} \approx 48.06.$$

Since the energy functional $\mathcal{E}$ is decreasing in a, if $a > \bar{a}$ then $I \leq \mathcal{E}(\bar{\varphi}) < 0$. As a conclusion, $a_0 \leq \bar{a} + \varepsilon$ for all $\varepsilon > 0$. $\square$

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Appendix A.

A.1.

We begin this section by proving that if (φ, χ) is a solution of (1.7) with $\varphi \in H^1(\mathbb{R}^3)$ of the form (1.12), then $|\varphi|^2 \leq 1$ a.e. in $\mathbb{R}^3$. As we saw before, φ is a solution of

$$-\boldsymbol{\sigma} \cdot \nabla \left(\frac{\boldsymbol{\sigma} \cdot \nabla \varphi}{1 - |\varphi|^2} \right) + \frac{|\boldsymbol{\sigma} \cdot \nabla \varphi|^2}{(1 - |\varphi|^2)^2} \varphi - a|\varphi|^2 \varphi + b\varphi = 0, \tag{A.1}$$

or equivalently,

$$\frac{\Delta \varphi}{|\varphi|^2 - 1} - \frac{|\boldsymbol{\sigma} \cdot \nabla \varphi|^2}{(|\varphi|^2 - 1)^2} \varphi - a|\varphi|^2 \varphi + b\varphi = 0,$$

or still,

$$\Delta \varphi - \frac{|\nabla \varphi|^2}{|\varphi|^2 - 1} \varphi - (a|\varphi|^2 \varphi - b\varphi)(|\varphi|^2 - 1) = 0,$$

because for functions φ of the form (1.12),

$$|\sigma \cdot \nabla \varphi|^2 = |\nabla \varphi|^2 \quad \text{and} \quad \sigma \cdot (\nabla \varphi \wedge \nabla \varphi) = 0 \quad \text{a. e.}$$

For any $K > 1$, we define the truncation function $T_K(s)$ by $T_K(s) = s$ if $1 < s < K$, and $T_K(s) = 0$ otherwise. Multiplying the above equation by $\varphi\, T_K(|\varphi|^2) \in L^2(\mathbb{R}^3)$, we obtain

$$\begin{aligned} &- \int_{\mathbb{R}^3} |\nabla \varphi|^2\, T_K(|\varphi|^2) - \int_{\mathbb{R}^3} (\nabla \varphi \cdot \varphi)\, \nabla T_K(|\varphi|^2) - \int_{\mathbb{R}^3} \frac{|\nabla \varphi|^2}{|\varphi|^2 - 1} |\varphi|^2\, T_K(|\varphi|^2) \\ &\quad - \int_{\mathbb{R}^3} (a|\varphi|^2 - b)(|\varphi|^2 - 1)|\varphi|^2\, T_K(|\varphi|^2) = 0. \end{aligned} \tag{A.2}$$

Moreover, for all $K > 1$,

$$\nabla T_K(|\varphi|^2) = \begin{cases} 2\varphi \cdot \nabla \varphi & 1 < |\varphi|^2 < K \\ 0 & |\varphi|^2 \leq 1 \text{ or } |\varphi|^2 \geq K \end{cases}$$

Therefore, if $a - b > 0$ the l.h.s. of (A.2) is negative and this implies that either $|\varphi|^2 \leq 1$ or $|\varphi|^2 \geq K$ a.e. As a conclusion, taking the limit $K \to +\infty$, if $a - b > 0$ then any solution φ of (A.1) of the form (1.12) satisfies $|\varphi|^2 \leq 1$ a.e. in $\mathbb{R}^3$, and in the equation (A.1) we can replace the term $(1 - |\varphi|^2)$ by $(1 - |\varphi|^2)_+$ without changing its solution set. The same happens for solutions of the form (1.8).

A.2.

Let us next prove that the functional $\mathcal{F}$, defined by (1.11), is not bounded from below. Consider the function ξ introduced in the proof of Lemma 2.4. Let us denote $A := \int_{\mathbb{R}^3} |\xi(x)|^2\, dx$.

Then, let us define the radially symmetric function

$$f(r) = \begin{cases} e^{(r-\sqrt{\ln 2})^2}, & 0 \le r < \sqrt{\ln 2}, \\ \bar{\xi}(r+1-\sqrt{\ln 2}), & r \ge \sqrt{\ln 2}, \end{cases}$$

where $\bar{\xi}(|x|) = \xi(x)$ for all x, and take $a := \int_{\mathbb{R}^3} f(|x|)^2\, dx$. Note that $\operatorname{supp}(f) \subset [0, 1+\sqrt{\ln 2}]$ and $\max_{0\le r\le 1+\sqrt{\ln 2}} \frac{(f'(r))^2}{1-f^2(r)} \le C$, for some constant $C > 0$.

Next, for all integers $n > 0$, define the rescaled functions $\xi_n(x) := n^{3/2}\xi(nx)$. This change of variables leaves invariant the $L^2(\mathbb{R}^3)$ norm. Then for n large, consider the function

$$g_n(x) := \max_{\mathbb{R}^3}\{\xi_n(x), f(|x|)\}.$$

Note that the measure of the set $\{x \in \mathbb{R}^3 \; ; \; g_n = \xi_n\}$ tends to 0 as n goes to $+\infty$. This function satisfies $\int_{\mathbb{R}^3} |g_n(x)|^2\, dx = A + a + o(1)$, as n goes to $+\infty$. In order to normalize it in the L^2 norm, let us finally define the rescaled function $g_n^R(x) := g_n\left(\frac{x}{R}\right), R > 0$ and choose R_n such that $\int_{\mathbb{R}^3} |g_n^{R_n}(x)|^2\, dx = 1$. As n goes to $+\infty, R_n \to \bar{R} := (A+a)^{-1/3} > 0$. We compute now the energy $\mathcal{F}$ of the vector function $\varphi_n^{R_n}$ defined by

$$\varphi_n^{R_n}(x) = \begin{pmatrix} g_n^{R_n}(x) \\ 0 \end{pmatrix}.$$

We find

$$\begin{aligned} \frac{\mathcal{F}(\varphi_n^{R_n})}{4\pi} &= \int\limits_{\xi_n^{R_n} \ge f^{R_n}} \frac{((\xi_n^{R_n})'(r))^2}{1-(\xi_n^{R_n}(r))^2}\, r^2 dr - \frac{a\, n^3\, R_n^3}{2} \int\limits_{\xi_n^{R_n} \ge f^{R_n}} (\xi(r))^4\, r^2 dr \\ &\quad + R_n \int\limits_{\xi_n^{R_n} \le f^{R_n}} \frac{(f'(r))^2}{1-f^2(r)}\, r^2 dr - \frac{a\, R_n^3}{2} \int\limits_{\xi_n^{R_n} \le f^{R_n}} f(r)^4\, r^2 dr \\ &\le -\frac{a\, n^3\, R_n^3}{2} \int_0^{+\infty} (\xi(r))^4\, r^2 dr + R_n \int_0^{+\infty} \frac{(f'(r))^2}{1-f^2(r)}\, r^2 dr \\ &\quad - \frac{a\, R_n^3}{2} \int_0^{+\infty} f(r)^4\, r^2 dr + o(n^3), \end{aligned}$$

because whenever $\xi_n^{R_n} \ge f^{R_n}, (\xi_n^{R_n})^2 > 1$ and because the sequence $\{R_n\}_n$ is bounded. This clearly shows that $\mathcal{F}$ is unbounded from below.

References

[1] Esteban, M.J., Rota Nodari, S.: Symmetric ground states for a stationary relativistic mean-field model for nucleons in the non relativistic limit. Rev. Math. Phys. **24**(10):1250025 (2012)

[2] Greiner, W., Maruhn, J.: Nuclear Models. Springer, Berlin (1996)

[3] Lions, P.L.: The concentration-compactness principle in the calculus of variations. The locally compact case. I. Ann. Inst. H. Poincaré Anal. Non Linéaire **1**(2), 109–145 (1984)

[4] Lions, P.L.: The concentration-compactness principle in the calculus of variations. The locally compact case. II. Ann. Inst. H. Poincaré Anal. Non Linéaire **1**(4), 223–283 (1984)

[5] Ring, P.: Relativistic mean field theory in finite nuclei. Prog. Part. Nucl. Phys. **37**, 193–236 (1996)

[6] Séré, É.: Private Communication, 2012

[7] Talenti, G.: Best constant in Sobolev inequality. Annali di Matematica Pura ed Applicata **110**, 353–372 (1976). doi:10.1007/BF02418013. ISSN: 0373-3114

[8] Thaller, B.: The Dirac equation. Springer, Berlin (1992)

[9] Walecka, J.D.: A theory of highly condensed matter. Ann. Phys. **83**(2), 491–529 (1974)

[10] Walecka, J.D.: Theoretical Nuclear and Subnuclear Physics, 2nd edn. Imperial College Press and World Scientific Publishing Co. Pte. Ltd., London (2004)

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