Ann. Henri Poincaré 14 (2013), 1169–1192

1424-0637/13/051169-24
published online November 3, 2012
DOI 10.1007/s00023-012-0209-5

Annales Henri Poincaré

Multiple Vortices in the Aharony–Bergman–Jafferis–Maldacena Model

Shouxin Chen, Ruifeng Zhang and Meili Zhu

Abstract. Vortices in non-Abelian gauge field theory play important roles in confinement mechanism and are governed by systems of nonlinear elliptic equations of complicated structures. In this paper, we present a series of existence and uniqueness theorems for multiple vortex solutions of the BPS vortex equations, arising in the dual-layered Chern–Simons field theory developed by Aharony, Bergman, Jafferis, and Maldacena, over $\mathbb{R}^2$ and on a doubly periodic domain. In the full-plane setting, we show that the solution realizing a prescribed distribution of vortices exists and is unique. In the compact setting, we show that a solution realizing n prescribed vortices exists over a doubly periodic domain Ω if and only if the condition

$$n < \frac{\lambda|\Omega|}{2\pi}$$

holds, where $\lambda > 0$ is the Higgs coupling constant. In this case, if a solution exists, it must be unique. Our methods are based on calculus of variations.

1. Introduction

Vortices have important applications in many fundamental areas of physics. For example, in particle physics, vortices allow one to generate dually (electrically and magnetically) charged vortex-like solitons [21,32,39] known as dyons [34,42,43]; in cosmology, vortices generate topological defects know as cosmic strings [15,31] which give rise to useful mechanisms for matter formation in the early universe. Besides, both electrically and magnetically charged vortices arise in a wide range of areas in condensed-matter physics including high-temperature superconductivity [11,29], optics [8,30,33], etc.

Mathematically, Chern–Simons theories in (2 + 1)-dimensions are introduced to accommodate electricity. The equations of motions of various

Chern–Simons vortex models are hard to approach even in the radially symmetric static cases. However, since the discovery of the self-dual structure in the Abelian Chern–Simons vortex model [16,18,19] in 1990 (cf. [17]), there came a burst of fruitful works on Chern–Simons vortex equations, non-relativistic and relativistic, Abelian and non-Abelian [9,10]. For example, Aldrovandi and Schaposnik [3,28] found the non-Abelian vortex solutions when gauge field dynamics is solely governed by a Chern–Simons action and the symmetry breaking potential is six-order in order to ensure self-duality and supersymmetric extension, in the presence of a set of orientational collective coordinates. Furthermore, the existence of Chern–Simons–Higgs vortex solutions was proved in $(2+1)$-dimensions with internal collective coordinates [22]. The existence of topological solutions for relativistic Abelian Chern–Simons equations involving two Higgs particles and two gauge fields was proved through studying the full $\mathbb{R}^2$ limit of a coupled system of two nonlinear elliptic equations [25]. In 2008, Aharony et al. [2] developed the so-called ABJM theory in terms of three-dimensional Chern–Simons-matter theories with gauge groups $U(N) \times U(N)$ and $SU(N) \times SU(N)$ which have explicit $\mathcal{N} = 6$ superconformal symmetry. Before long, Auzzi and Kumar [4] find half-BPS vortex solitons, at both weak and strong couplings, in this theory.

More recently, the existence of solutions for Abelian Chern–Simons equations involving two Higgs particles and two gauge fields on a torus was proved by Lin and Prajapat [24]. Using the methods of monotone iterations, a priori estimates, degree-theory argument and constrained minimization, multiple vortex equations in $U(N)$ and $SO(2N)$ theories were discussed [14,26,27] and a series of sharp existence and uniqueness theorems were established. Lieb and Yang [23] discussed non-Abelian vortices in supersymmetric gauge field theory, over doubly periodic domains, via a highly efficient direct minimization approach. These studies unveil a broad spectrum of systems of elliptic equations with exponential nonlinearities and rich properties and structures, which present new challenges.

In this paper, we will concentrate on the non-Abelian BPS vortex equations derived by Auzzi and Kumar [4] in a supersymmetric Chern–Simons–Higgs theory formulated by Aharony et al. [2], known as the ABJM model. Developing and extending the methods of [20,23,26,27,38,41], we obtain the existence and uniqueness of a multiple vortex solution.

The content of the rest of paper is outlined as follows. In Sect. 2, we review the multiple vortex equations in the ABJM model and compare them with those arising in the classical Abelian Chern–Simons–Higgs theory. We then state our main (sharp) existence results. In Sect. 3, we present the equations governing the multiple vortices. In Sect. 4, we prove the existence and uniqueness of a multiple vortex solution realizing an arbitrarily prescribed vortex distribution over $\mathbb{R}^2$, applying the variational method of Jaffe and Taubes [20] used for the Abelian Higgs model. In Sect. 5, we prove the existence of a multiple vortex solution over a doubly periodic domain under a necessary and sufficient condition explicitly stated in terms of some physical coupling parameters, by a multi-constrained variational approach. Furthermore, in Sect. 6, our methods are shown to be equally effective in treating the existence and uniqueness problems for the multiple vortex solution induced from independently prescribed distributions of zeros of two complex scalar fields, instead of one.

2. Vortices in the Model of Aharony, Bergman, Jafferis and Maldacena

In the classical Abelian Chern–Simons–Higgs theory [17], the Chern–Simons action density is given by

$$\mathcal{L}_{\mathrm{CS}} = \frac{\kappa}{4}\epsilon^{\mu\nu\gamma}A_\mu F_{\nu\gamma}, \tag{2.1}$$

where $\kappa > 0$ is the Chern–Simons coupling constant often referred to as the Chern–Simons level, A_μ ($\mu = 0, 1, 2$) is a real-valued gauge field defined over the Minkowski spacetime $\mathbb{R}^{2,1}$ of signature $(+ - -)$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field. Use ϕ to denote the scalar Higgs field which is complex-valued and subject to the sixth-order potential density [16,19]

$$V(\phi) = \frac{\lambda}{4}|\phi|^2(|\phi|^2 - 1)^2. \tag{2.2}$$

The gauge-covariant derivatives are then given by $D_\mu\phi = \partial_\mu\phi - \mathrm{i}A_\mu\phi$ where $\mathrm{i} = \sqrt{-1}$. The Chern–Simons–Higgs action density built over the above-described Chern–Simons electromagnetism and the Higgs scalar is written as [16,19]

$$\mathcal{L} = -\frac{\kappa}{4}\epsilon^{\mu\nu\gamma}A_\mu F_{\nu\gamma} + \frac{1}{2}D_\mu\phi\overline{D^\mu\phi} - \frac{\lambda}{4}|\phi|^2(|\phi|^2 - 1)^2. \tag{2.3}$$

In the critical BPS coupling under the condition

$$\lambda = \frac{1}{2\kappa^2}, \tag{2.4}$$

the static equations of motion of the Lagrangian (2.3) are shown to be reduced to the first-order equations [16,19]

$$D_1\phi \pm \mathrm{i}D_2\phi = 0, \tag{2.5}$$

$$F_{12} \mp \frac{1}{2\kappa^2}|\phi|^2(1 - |\phi|^2) = 0, \tag{2.6}$$

subject to the Gauss law constraint $\kappa F_{12} = A_0|\phi|^2$. It is well known that the solutions of this BPS vortex system are characterized by the locations of the zeros of the scalar field ϕ, identified as the spots where electrically and magnetically charged vortices take shape. For the system over the full plane, there exist topological [35,40] and non-topological [6,7,36] solutions; for the system over a doubly periodic domain Ω, a solution with n prescribed zeros or vortices exists if and only if the Chern–Simons coupling constant κ is no greater [5,37] than a critical level, $\kappa_{\mathrm{C}} > 0$, which satisfies the estimate

$$\kappa_{\mathrm{c}} \le \frac{1}{4}\sqrt{\frac{|\Omega|}{\pi n}}, \tag{2.7}$$

which possibly also depends on the locations of the n zeros. While the existence theory for the above-described Abelian BPS Chern–Simons–Higgs equations is much developed, there are several lasting and unsettled technical issues such as the uniqueness of solutions and the precise form of the critical Chern–Simons level κ_{c}, which remain challenging to analysts.

The main contribution of the present paper is to develop an existence theory for the recently elegantly formulated supersymmetric (non-Abelian) Chern–Simons–Higgs model by Aharony et al. [2], known also as the ABJM model, for which we show that all the unsettled issues in the classical Abelian Chern–Simons–Higgs model [16,19] disappear. In other words, for the ABJM model, we prove that, in the non-compact full-plane setting, the solution realizing a prescribed distribution of vortices exists and is unique, and that, in the compact doubly-periodic setting, we obtain an explicitly stated necessary and sufficient condition under which a unique solution realizing n prescribed vortices exists.

Recall that the ABJM model [2] is a Chern–Simons–Higgs theory within which the matter fields are four complex scalars,

$$C^I = (Q^1, Q^2, R^1, R^2), \quad I = 1, 2, 3, 4, \tag{2.8}$$

in the bifundamental matter field $(\mathbf{N}, \overline{\mathbf{N}})$ representation of the gauge group $U(N) \times U(N)$, which hosts two gauge fields, A_μ and B_μ. The Chern–Simons action associated with the two gauge group A_μ and B_μ of levels $+k$ and $-k$ is given by the Lagrangian density

$$\mathcal{L}_{\mathrm{CS}} = \frac{k}{4\pi}\epsilon^{\mu\nu\gamma}\mathrm{Tr}\left(A_\mu\partial_\nu A_\gamma + \frac{2\mathrm{i}}{3}A_\mu A_\nu A_\gamma - B_\mu\partial_\nu B_\gamma - \frac{2\mathrm{i}}{3}B_\mu B_\nu B_\gamma\right), \tag{2.9}$$

where the gauge-covariant derivatives on the bifundamental fields are defined as

$$D_\mu C^I = \partial_\mu C^I + \mathrm{i}A_\mu C^I - \mathrm{i}C^I B_\mu, \quad I = 1, 2, 3, 4. \tag{2.10}$$

The scalar potential of the mass deformed theory can be written in a compact way as in [13]

$$V = \mathrm{Tr}(M^{\alpha\dagger}M^\alpha + N^{\alpha\dagger}N^\alpha), \tag{2.11}$$

where

$$\begin{aligned} M^\alpha = {} & \rho Q^\alpha + \frac{2\pi}{k}(2Q^{[\alpha}Q^\dagger_\beta Q^{\beta]} + R^\beta R^\dagger_\beta Q^\alpha - Q^\alpha R^\dagger_\beta R^\beta \\ & +2Q^\beta R^\dagger_\beta R^\alpha - 2R^\alpha R^\dagger_\beta Q^\beta), \end{aligned} \tag{2.12}$$

$$\begin{aligned} N^\alpha = {} & -\rho R^\alpha + \frac{2\pi}{k}(2R^{[\alpha}R^\dagger_\beta R^{\beta]} + Q^\beta Q^\dagger_\beta R^\alpha - R^\alpha Q^\dagger_\beta Q^\beta \\ & +2R^\beta Q^\dagger_\beta Q^\alpha - 2Q^\alpha Q^\dagger_\beta R^\beta), \end{aligned} \tag{2.13}$$

where the Kronecker symbol $\epsilon^{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ is used to lower or raise indices, and $\rho > 0$ a massive parameter. Thus, when the spacetime metric is of the signature $(+--)$, the total (bosonic) Lagrangian density of ABJM model can be written as

$$\mathcal{L} = -\mathcal{L}_{\text{CS}} + \text{Tr}([D_\mu C^I]^\dagger [D^\mu C^I]) - V, \tag{2.14}$$

which is of a pure Chern–Simons type for the gauge field sector. As in [4], we focus on a reduced situation where (say) $R^\alpha = 0$. Then, by virtue of (2.12) and (2.13), the scalar potential density (2.11) takes the form

$$V = \text{Tr}(M^{\alpha\dagger} M^\alpha), \quad M^\alpha = \rho Q^\alpha + \frac{4\pi}{k}(Q^\alpha Q^\dagger_\beta Q^\beta - Q^\beta Q^\dagger_\beta Q^\alpha). \tag{2.15}$$

The equations of motion of the Lagrangian (2.14) are rather complicated. However, in the static limit, Auzzi and Kumar [4] showed that these equations may be reduced into the following first-order BPS system of equations

$$D_0 Q^1 - \mathrm{i} W^1 = 0, \quad D_1 Q^2 - \mathrm{i} D_2 Q^2 = 0, \tag{2.16}$$

$$D_1 Q^1 = 0, \quad D_2 Q^1 = 0, \quad D_0 Q^2 = 0, \quad W^2 = 0, \tag{2.17}$$

coupled with the Gauss law constraints which are the temporal components of the Chern–Simons equations

$$\frac{k}{4\pi}\epsilon^{\mu\nu\gamma} F^{(A)}_{\nu\gamma} = \mathrm{i}(Q^\alpha [D^\mu Q^\alpha]^\dagger - [D^\mu Q^\alpha] Q^{\alpha\dagger}), \tag{2.18}$$

$$\frac{k}{4\pi}\epsilon^{\mu\nu\gamma} F^{(B)}_{\nu\gamma} = \mathrm{i}([D^\mu Q^\alpha]^\dagger Q^\alpha - Q^{\alpha\dagger}[D^\mu Q^\alpha]), \tag{2.19}$$

where

$$\begin{aligned} F^{(A)}_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}[A_\mu, A_\nu], \\ F^{(B)}_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu + \mathrm{i}[B_\mu, B_\nu], \\ W^1 &= \rho Q^1 + \frac{2\pi}{k}(Q^1 Q^{2\dagger} Q^2 - Q^2 Q^{2\dagger} Q^1), \\ W^2 &= \rho Q^2 + \frac{2\pi}{k}(Q^2 Q^{1\dagger} Q^1 - Q^1 Q^{1\dagger} Q^2), \end{aligned}$$

provided that [4] one takes that Q^1 assumes its vacuum expectation value

$$Q^1 = \sqrt{\frac{\rho k}{2\pi}}\,\text{diag}\left(0, 1, \ldots, \sqrt{N-2}, \sqrt{N-1}\right), \tag{2.20}$$

the non-trivial entries of Q^2 are given by $(N-1)$ complex scalar fields χ and ϕ_ℓ $(\ell = 1, \ldots, N-2)$ according to

$$Q^2_{N,N-1} = \sqrt{\frac{\rho k}{2\pi}}\chi, \quad Q^2_{N-\ell,N-\ell-1} = \sqrt{\frac{\rho k}{2\pi}}\phi_\ell, \tag{2.21}$$

and the spatial components of the gauge fields A_j and B_j $(j = 1, 2)$ are expressed in terms of $(N-1)$ real-valued vector potentials $a^\ell = (a^\ell_j)$ and $b = (b_j)$ $(j = 1, 2; \ell = 1, \ldots, N-2)$ satisfying

$$A_j = B_j = \text{diag}(0, a^{N-2}_j, \ldots, a^1_j, b_j), \quad j = 1, 2. \tag{2.22}$$

We now consider the solution for the $N = 3$ case. Our ansatz is taken to be ([4])

$$
\begin{aligned}
Q^1 &= \sqrt{\frac{\rho k}{2\pi}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}, \\
Q^2 &= \sqrt{\frac{\rho k}{2\pi}} \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\chi & 0 & 0 \\ 0 & \phi & 0 \end{pmatrix}, \\
A_j &= B_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_j & 0 \\ 0 & 0 & b_j \end{pmatrix}, \quad j = 1, 2,
\end{aligned}
\tag{2.23}
$$

where χ is a real-valued scalar field, ϕ is a complex-valued scalar field, and a_j and b_j are two real-valued gauge potential vector fields.

Define $a_{jk} = \partial_j a_k - \partial_k a_j$ and set $\lambda = 4\rho^2$. Then the BPS vortex equations (2.16) and (2.17) without assuming radial symmetry are reduced into

$$(\partial_1 + \mathrm{i}\partial_2)\chi = \mathrm{i}(a_1 + \mathrm{i}a_2)\chi, \tag{2.24}$$

$$(\partial_1 + \mathrm{i}\partial_2)\phi = -\mathrm{i}([a_1 + \mathrm{i}a_2] - [b_1 + \mathrm{i}b_2])\phi. \tag{2.25}$$

$$a_{12} = -\frac{\lambda}{2}(2\chi^2 - |\phi|^2 - 1), \tag{2.26}$$

$$b_{12} = -\lambda(|\phi|^2 - 1). \tag{2.27}$$

We shall look for solutions of these equations so that χ never vanishes but ϕ vanishes exactly at the finite set of points

$$Z = \{p_1, p_2, \ldots, p_n\}. \tag{2.28}$$

A solution is called an n-vortex solution as in the Abelian Higgs situation [20, 41]. Our main existence theorem for the ABJM multiple vortices may be stated as follows.

Theorem 2.1. *For the BPS multiple vortex equations* (2.24)–(2.27) *arising in the ABJM Chern–Simons–Higgs model expressed in terms of the gauge fields a_j, b_j and scalar fields χ, ϕ so that χ is real-valued and ϕ is complex-valued with the prescribed set of zeros of ϕ given in* (2.28)*, the existence and uniqueness of a finite-energy solution is always ensured over the full plane $\mathbb{R}^2$ which also satisfies the asymptotic condition $\chi \to 1, |\phi| \to 1$ as $|x| \to \infty$ exponentially fast. Furthermore, the existence and uniqueness of an n-vortex solution over a doubly periodic domain Ω is ensured under the necessary and sufficient condition*

$$n < \frac{\lambda|\Omega|}{2\pi}. \tag{2.29}$$

Besides, the associated fluxes over $\mathbb{R}^2$ or Ω have the values

$$\int a_{12}\,\mathrm{d}x = 0, \quad \int b_{12}\,\mathrm{d}x = 2\pi n. \tag{2.30}$$

It should be noted that, although the gauge field a_j generates only zero flux, it cannot be gauged away to make a_{12} vanish everywhere, unless there is no vortex present or $n = 0$. This point will become apparent in the next section.

3. Governing System of Elliptic Equations

To facilitate our computation, it will be convenient to adopt the complexified derivatives

$$\partial = \frac{1}{2}(\partial_1 - \mathrm{i}\partial_2), \quad \overline{\partial} = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2), \tag{3.1}$$

and the notation

$$a = a_1 + \mathrm{i}a_2, \quad b = b_1 + \mathrm{i}b_2. \tag{3.2}$$

As a consequence, away from Z, Eqs. (2.24) and (2.25) become

$$\overline{\partial} \ln \chi = -\frac{1}{2}\mathrm{i}a, \quad \overline{\partial} \ln \phi = -\frac{1}{2}\mathrm{i}(a - b), \tag{3.3}$$

which allow us to solve for a, b to get

$$a = 2\mathrm{i}\overline{\partial} \ln \chi, \quad a - b = 2\mathrm{i}\overline{\partial} \ln \phi. \tag{3.4}$$

Using

$$a_{12} = -\mathrm{i}(\partial a - \overline{\partial}\overline{a}), \tag{3.5}$$

(2.26), (2.27), (3.4), and the fact that $\partial\overline{\partial} = \overline{\partial}\partial = \frac{1}{4}\Delta$, we have

$$a_{12} = -\Delta \ln \chi. \tag{3.6}$$

Likewise, we have, away from Z, the relation

$$b_{12} = a_{12} - \frac{1}{2}\Delta \ln |\phi|^2 = -\frac{1}{2}\Delta(\ln \chi^2 + \ln |\phi|^2). \tag{3.7}$$

Set $u = \ln \chi^2$ and $v = \ln |\phi|^2$ and note that $|\phi|$ behaves like $|x - p_s|$ for x near p_s $(s = 1, \ldots, n)$. We see that u and v satisfy the equations

$$\Delta u = \lambda(2e^u - e^v - 1), \tag{3.8}$$

$$\Delta u + \Delta v = 2\lambda(e^v - 1) + 4\pi \sum_{s=1}^{n} \delta_{p_s}(x), \tag{3.9}$$

where we have included our consideration of the zero set Z of ϕ as given in (2.28).

If the gauge field a_j may be gauged away, then in view of (2.26) the associated curvature a_{12} vanishes everywhere which results in

$$2\chi^2 = |\phi|^2 + 1. \tag{3.10}$$

Using this result in (3.8) (with $\chi^2 = e^u, |\phi|^2 = e^v$), we see that u is harmonic, which leads to the conclusion that u, thus χ, is a constant. Hence, $|\phi|^2$ or v is a constant. This is impossible by virtue of (3.9) when vortices are present.

In the subsequent sections, we study the existence and uniqueness of solutions of the system of Eqs. (3.8) and (3.9).

4. Solution on Full Plane

In this section, we prove the existence and uniqueness of the solution of the system of Eqs. (3.8) and (3.9) over $\mathbb{R}^2$ satisfying the boundary condition

$$u \to 0, \quad v \to 0 \quad \text{as } |x| \to \infty. \tag{4.1}$$

To proceed further, we introduce the background function [20]

$$v_0(x) = -\sum_{s=1}^{n} \ln(1+\tau|x-p_s|^{-2}), \quad \tau > 0. \tag{4.2}$$

Then, we have

$$\Delta v_0 = -h(x) + 4\pi \sum_{s=1}^{n} \delta_{p_s}(x), \quad h(x) = 4\sum_{s=1}^{n} \frac{\tau}{(\tau+|x-p_s|^2)^2}. \tag{4.3}$$

Using the substitution $v = v_0 + w$, we have

$$\Delta u = \lambda(2e^u - e^{v_0+w} - 1), \tag{4.4}$$

$$\Delta(u+w) = 2\lambda(e^{v_0+w} - 1) + h(x). \tag{4.5}$$

Taking $f = u + w$, we change (4.4) and (4.5) into

$$\Delta u = \lambda(2e^u - e^{v_0+f-u} - 1), \tag{4.6}$$

$$\Delta f = 2\lambda(e^{v_0+f-u} - 1) + h(x). \tag{4.7}$$

It is clear that (4.6) and (4.7) are the Euler–Lagrange equations of the action functional

$$I(u,f) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2\lambda}|\nabla u|^2 + \frac{1}{4\lambda}|\nabla f|^2 + (2(e^u-1)-u) + (e^{v_0+f-u} - e^{v_0}) + \left(\frac{h}{2\lambda} - 1\right) f \right\} \mathrm{d}x. \tag{4.8}$$

It is clear that the functional I is a C^1-functional for $u, f \in W^{1,2}(\mathbb{R}^2)$ and its Fréchet derivative satisfies

$$DI(u,f)(u,f) = \int_{\mathbb{R}^2} \left\{ \frac{1}{\lambda}|\nabla u|^2 + \frac{1}{2\lambda}|\nabla f|^2 + e^{v_0}(e^{f-u}-1)(f-u) + 2(e^u-1)u + (e^{v_0}-1)(f-u) + \frac{h}{2\lambda} f \right\} \mathrm{d}x. \tag{4.9}$$

Since

$$|\nabla u|^2 + |\nabla f|^2 = 2|\nabla u|^2 + |\nabla w|^2 + 2(\nabla u, \nabla w), \tag{4.10}$$

Hence

$$|\nabla u|^2 + |\nabla f|^2 \leq 3|\nabla u|^2 + 2|\nabla w|^2 \leq 3(|\nabla u|^2 + |\nabla w|^2). \tag{4.11}$$

On the other hand, we have

$$|\nabla u|^2+|\nabla f|^2 \geq 2|\nabla u|^2+|\nabla w|^2-2|(\nabla u,\nabla w)| \\ \geq \left(2-\frac{1}{\varepsilon}\right)|\nabla u|^2+(1-\varepsilon)|\nabla w|^2, \quad (4.12)$$

for any $\varepsilon \in (\frac{1}{2},1)$.

Taking $\varepsilon=\frac{2}{3}$, we get

$$|\nabla u|^2+|\nabla f|^2 \geq \frac{1}{2}|\nabla u|^2+\frac{1}{3}|\nabla w|^2 \geq \frac{1}{3}(|\nabla u|^2+|\nabla w|^2). \quad (4.13)$$

Similarly, we have

$$\frac{1}{3}(u^2+w^2) \leq u^2+f^2 \leq 3(u^2+w^2). \quad (4.14)$$

As a consequence of (4.9), (4.13) and (4.14), we obtain

$$\begin{aligned} &DI(u,f)(u,f)-\frac{1}{6\lambda}\int_{\mathbb{R}^2}\left\{|\nabla u|^2+|\nabla w|^2\right\}\,\mathrm{d}x \\ &\geq \int_{\mathbb{R}^2}\left\{e^{v_0}(e^w-1)w+2(e^u-1)u+(e^{v_0}-1)w+\frac{h}{2\lambda}(u+w)\right\}\mathrm{d}x \\ &= \int_{\mathbb{R}^2}\left\{\left(e^{v_0}(e^w-1)+e^{v_0}-1+\frac{h}{2\lambda}\right)w+\left(2(e^u-1)+\frac{h}{2\lambda}\right)u\right\}\mathrm{d}x \\ &= \int_{\mathbb{R}^2}\left\{w\left(e^{v_0+w}-1+\frac{h}{2\lambda}\right)+u\left(2(e^u-1)+\frac{h}{2\lambda}\right)\right\}\mathrm{d}x \\ &\equiv M_1(w)+M_2(u). \end{aligned} \quad (4.15)$$

As in [20], we decompose w and u into their positive and negative parts, $w=w_+-w_-$ and $u=u_+-u_-$, where $q_+=\max\{q,0\}$ and $q_-=-\min\{q,0\}$ for $q\in\mathbb{R}$. Using the elementary inequality

$$e^t-1\geq t, \quad t\in\mathbb{R}, \quad (4.16)$$

we have

$$e^{v_0+w}-1+\frac{h}{2\lambda}\geq v_0+w+\frac{h}{2\lambda}, \quad (4.17)$$

which leads to

$$\begin{aligned} M_1(w_+) &\geq \int_{\mathbb{R}^2} w_+^2\mathrm{d}x+\int_{\mathbb{R}^2} w_+\left(v_0+\frac{h}{2\lambda}\right)\mathrm{d}x \\ &\geq \frac{1}{2}\int_{\mathbb{R}^2} w_+^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^2}\left(v_0+\frac{h}{2\lambda}\right)^2\mathrm{d}x. \end{aligned} \quad (4.18)$$

On the other hand, using the inequality

$$1-e^{-t}\geq\frac{t}{1+t}, \quad t\geq 0, \quad (4.19)$$

we have

$$w_-\left(1-\frac{h}{2\lambda}-e^{v_0-w_-}\right)=w_-\left(1-\frac{h}{2\lambda}+e^{v_0}(1-e^{-w_-})-e^{v_0}\right)$$
$$\geq w_-\left(1-\frac{h}{2\lambda}+e^{v_0}\frac{w_-}{1+w_-}-e^{v_0}\right)$$
$$=\frac{w_-^2}{1+w_-}\left(1-\frac{h}{2\lambda}\right)+\frac{w_-}{1+w_-}\left(1-e^{v_0}-\frac{h}{2\lambda}\right). \tag{4.20}$$

In view of (4.3), we see that we may choose $\tau>0$ large enough so that

$$\frac{h(x)}{\lambda}<1,\quad x\in\mathbb{R}^2. \tag{4.21}$$

Since $1-e^{v_0}$ and h both lie in $L^2(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2}\frac{w_-}{1+w_-}\left|1-e^{v_0}-\frac{h}{2\lambda}\right|\mathrm{d}x\leq\varepsilon\int_{\mathbb{R}^2}\frac{w_-^2}{1+w_-}\,\mathrm{d}x+C(\varepsilon), \tag{4.22}$$

where $\varepsilon>0$ may be chosen to be arbitrarily small. Combining (4.20) and (4.22), we obtain

$$M_1(-w_-)\geq\frac{1}{4}\int_{\mathbb{R}^2}\frac{w_-^2}{1+w_-}\,\mathrm{d}x-C_1(\varepsilon), \tag{4.23}$$

provided that $\varepsilon<\frac{1}{4}$. From (4.18) and (4.23), we get

$$M_1(w)\geq\frac{1}{4}\int_{\mathbb{R}^2}\frac{w^2}{1+|w|}\,\mathrm{d}x-C, \tag{4.24}$$

where in the sequel we use C to denote an irrelevant positive constant. Similar estimates may be made for $M_2(u)$. Thus, (4.15) gives us

$$DI(u,f)(u,f)-\frac{1}{6\lambda}\int_{\mathbb{R}^2}\left\{|\nabla u|^2+|\nabla w|^2\right\}\,\mathrm{d}x$$
$$\geq\frac{1}{4}\int_{\mathbb{R}^2}\left(\frac{u^2}{1+|u|}+\frac{w^2}{1+|w|}\right)\,\mathrm{d}x-C. \tag{4.25}$$

We now recall the well-known Gagliardo–Nirenberg inequality

$$\int_{\mathbb{R}^2}u^4\mathrm{d}x\leq 2\int_{\mathbb{R}^2}u^2\mathrm{d}x\int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x,\quad u\in W^{1,2}(\mathbb{R}^2). \tag{4.26}$$

Consequently, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^2} u^2 \mathrm{d}x\right)^2 &= \left(\int_{\mathbb{R}^2} \frac{|u|}{1+|u|}(1+|u|)|u| \mathrm{d}x\right)^2 \\ &\le \int_{\mathbb{R}^2} \frac{u^2}{(1+|u|)^2}\mathrm{d}x \int_{\mathbb{R}^2}(1+|u|)^2|u|^2 \mathrm{d}x \\ &\le 2\int_{\mathbb{R}^2} \frac{u^2}{(1+|u|)^2}\mathrm{d}x \int_{\mathbb{R}^2}(u^2+u^4) \mathrm{d}x \\ &\le 4\int_{\mathbb{R}^2} \frac{u^2}{(1+|u|)^2}\mathrm{d}x \int_{\mathbb{R}^2} u^2 \mathrm{d}x \left(1+\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x\right) \\ &\le \frac{1}{2}\left(\int_{\mathbb{R}^2} u^2 \mathrm{d}x\right)^2 + C\left(1+\left[\int_{\mathbb{R}^2} \frac{u^2}{(1+|u|)^2}\mathrm{d}x\right]^4\right. \\ &\quad \left. + \left[\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x\right]^4\right). \end{aligned} \tag{4.27}$$

As a result of (4.27), we have

$$\left(\int_{\mathbb{R}^2} u^2 \mathrm{d}x\right)^{\frac{1}{2}} \le C\left(1+\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \int_{R^2} \frac{u^2}{(1+|u|)^2}\mathrm{d}x\right). \tag{4.28}$$

Applying (4.28) in (4.25), we arrive at

$$DI(u,f)(u,f) \ge C_1(\|u\|_{W^{1,2}(\mathbb{R}^2)} + \|w\|_{W^{1,2}(\mathbb{R}^2)}) - C_2. \tag{4.29}$$

Thus, using (4.11), (4.13) and (4.14) in (4.29), we conclude with the coercive lower bound

$$DI(u,f)(u,f) \ge C_1(\|u\|_{W^{1,2}(\mathbb{R}^2)} + \|f\|_{W^{1,2}(\mathbb{R}^2)}) - C_2. \tag{4.30}$$

With (4.30), we can now show that the existence of a critical point of the action functional (4.8) follows using a standard argument as in [41].

In fact, from (4.30), we can choose $R > 0$ large enough such that

$$\inf\{DI(u,f)(u,f) | u,f \in W^{1,2}(\mathbb{R}^2), \|u\|_{W^{1,2}(\mathbb{R}^2)} + \|f\|_{W^{1,2}(\mathbb{R}^2)} = R\} \ge 1 \tag{4.31}$$

(say). Now consider the minimization problem

$$\eta = \min\{I(u,f) | \|u\|_{W^{1,2}(\mathbb{R}^2)} + \|f\|_{W^{1,2}(\mathbb{R}^2)} \le R\}. \tag{4.32}$$

Let $\{(u_k, f_k)\}$ be a minimization sequence of (4.32). Without loss of generality, we may assume that $\{(u_k, f_k)\}$ weakly converges to an element (u, f) in $W^{1,2}(\mathbb{R}^2)$. The weakly lower semi-continuity of I implies that (u, f) solves

(4.32). To show that (u, f) is a critical pint of I, it suffices to see that it is an interior point. That is,

$$\|u\|_{W^{1,2}(\mathbb{R}^2)} + \|f\|_{W^{1,2}(\mathbb{R}^2)} < R. \tag{4.33}$$

Suppose otherwise that $\|u\|_{W^{1,2}(\mathbb{R}^2)} + \|f\|_{W^{1,2}(\mathbb{R}^2)} = R$. Then for $t \in (0, 1)$ the point $(1-t)(u, f)$ is interior which gives us

$$I((1-t)u, (1-t)f) \geq \eta = I(u, f). \tag{4.34}$$

On the other hand, we have

$$\begin{aligned}\lim_{t\to 0} \frac{I((1-t)(u,f)) - I(u,f)}{t} &= \frac{\mathrm{d}}{\mathrm{d}t} I((1-t)(u,f))|_{t=0} \\ &= -DI(u,f)(u,f) \leq -1.\end{aligned} \tag{4.35}$$

Consequently, if $t > 0$ is sufficiently small, (4.34) leads to

$$I((1-t)(u,f)) < I(u,f) = \eta, \tag{4.36}$$

which contradicts (4.34). Therefore, the existence of a critical point of I follows.

Note that the part in the integrand of I which does not involve the derivatives of u and f may be rewritten as

$$Q(u,f) = 2(e^u - 1) - u + e^{v_0 + f - u} - e^{v_0} + \left(\frac{h}{2\lambda} - 1\right) f, \tag{4.37}$$

whose Hessian is easily checked to be positive definite. Thus, the functional I is strictly convex. As a consequence, I can have at most one critical point (u, f) in the space $W^{1,2}(\mathbb{R}^2)$.

To proceed further, we now show that the following claim holds.

Claim: If $g \in W^{1,2}(\mathbb{R}^2)$, then $e^g - 1 \in L^2(\mathbb{R}^2)$.

We first recall the Sobolev embedding inequality in two dimensions [12]:

$$\|g\|_{L^k(\mathbb{R}^2)} \leq \left(\pi\left(\frac{k-2}{2}\right)\right)^{\frac{k-2}{2k}} \|g\|_{W^{1,2}(\mathbb{R}^2)}, \quad k \geq 2. \tag{4.38}$$

On the other hand, the MacLaurin series leads to

$$(e^g - 1)^2 = g^2 + \sum_{k=3}^{\infty} \frac{2^k - 2}{k!} g^k. \tag{4.39}$$

Combining the above with (4.38), we have, formally,

$$\|e^g - 1\|^2_{L^2(\mathbb{R}^2)} \leq \|g\|^2_{L^k(\mathbb{R}^2)} + \sum_{k=3}^{\infty} \frac{2^k - 2}{k!} \left(\pi \frac{k-2}{2}\right)^{\frac{k-2}{2}} \|g\|^k_{W^{1,2}(\mathbb{R}^2)}. \tag{4.40}$$

Setting

$$\alpha_k = \frac{2^k - 2}{k!} \left(\pi \frac{k-2}{2}\right)^{\frac{k-2}{2}} \|g\|^k_{W^{1,2}(\mathbb{R}^2)},$$

and applying the Stirling formula,

$$k! \sim \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k} \quad (k \to \infty), \tag{4.41}$$

we have

$$\sqrt[k]{\alpha_k} \sim \frac{\sqrt[k]{2^k-2}}{ke^{-1}(2k\pi)^{\frac{1}{2k}}}\left(\pi\frac{k-2}{2}\right)^{\frac{k-2}{2k}}\|g\|_{W^{1,2}(\mathbb{R}^2)}$$
$$\sim 2e\sqrt{\pi}\|g\|_{W^{1,2}(\mathbb{R}^2)}\left(\frac{k-2}{2k^2}\right)^{\frac{1}{2}} \to 0 \quad (k\to\infty). \tag{4.42}$$

Thus, we have shown that (4.40) is a convergent series, which verifies our claim.

We now continue our work. Noting $v_0, h \in L^2(\mathbb{R}^2)$ and using the claim, we see that the right-hand side of (4.6) and (4.7) belongs to $L^2(\mathbb{R}^2)$. We may now apply the standard elliptic theory to (4.4) and (4.5) to infer that $u, w \in W^{2,2}(\mathbb{R}^2)$. In particular, u, w and $|\nabla u|, |\nabla w|$ approach zero as $|x| \to \infty$, which renders the validity of the boundary condition (4.1).

Finally, we derive the decay rates for u, v and $|\nabla u|, |\nabla v|$. Consider (3.8) and (3.9) outside the disk $D_R = \{x \in \mathbb{R}^2 | |x| < R\}$, where

$$R > \max\left\{|p_s|\,\middle|\, s = 1, 2, \dots, n\right\}.$$

We rewrite (3.8) and (3.9) in $\mathbb{R}^2 \backslash D_R$ as

$$\Delta u = \lambda(2e^u - e^v - 1), \tag{4.43}$$
$$\Delta v = \lambda(-2e^u + 3e^v - 1). \tag{4.44}$$

By computation, we have

$$\Delta(u^2+v^2) = 2(|\nabla u|^2 + |\nabla v|^2) + 4\lambda u(e^u - 1) + 6\lambda v(e^v - 1)$$
$$-2\lambda u(e^v - 1) - 4\lambda v(e^u - 1), \quad x \in \mathbb{R}^2\backslash D_R. \tag{4.45}$$

Noting $u, v \to 0$ as $|x| \to \infty$, for any $\varepsilon : 0 < \varepsilon < 1$, we can find a suitably large $R_\varepsilon > R$ so that

$$\Delta(u^2+v^2) \geq (1-\varepsilon)(4\lambda u^2 + 6\lambda v^2) - 6(1+\varepsilon)\lambda|uv|$$
$$\geq (1-\varepsilon)\lambda(u^2+v^2), \quad x \in \mathbb{R}^2\backslash D_{R_\varepsilon}. \tag{4.46}$$

Thus, using a comparison function argument and the property $u^2 + v^2 = 0$ at infinity, we can obtain a constant $C(\varepsilon) > 0$ to make

$$u^2(x) + v^2(x) \leq C(\varepsilon)e^{-\sqrt{(1-\varepsilon)\lambda}|x|} \tag{4.47}$$

valid.

Let ∂ denote any of the two partial derivatives, ∂_1 and ∂_2. Then (4.43) and (4.44) yield

$$\Delta(\partial u) = \lambda(2(\partial u)e^u - (\partial v)e^v), \tag{4.48}$$
$$\Delta(\partial v) = 3\lambda(\partial v)e^v - 2\lambda(\partial u)e^u. \tag{4.49}$$

By computation and then using the Cauchy inequality, we get

$$\begin{aligned}\Delta(|\nabla u|^2+|\nabla v|^2) &= 2[|\nabla(\partial_1 u)|^2+|\nabla(\partial_2 u)|^2+|\nabla(\partial_1 v)|^2+|\nabla(\partial_2 v)|^2]\\ &\quad +4\lambda|\nabla u|^2e^u+6\lambda|\nabla v|^2e^v-2\lambda\partial_1 u\partial_1 ve^v-2\lambda\partial_2 u\partial_2 ve^v\\ &\quad -4\lambda\partial_1 u\partial_1 ve^u-4\lambda\partial_2 u\partial_2 ve^u\\ &\ge 4\lambda|\nabla u|^2e^u+6\lambda|\nabla v|^2e^v-\lambda|\nabla u|^2(1+2e^{2u})\\ &\quad -\lambda|\nabla v|^2(2+e^{2v}),\quad x\in\mathbb{R}^2\backslash D_R. \end{aligned} \tag{4.50}$$

Therefore, as before, we conclude that for any $\varepsilon : 0<\varepsilon<1$, there is a $\widetilde{R}_\varepsilon > R$, so that

$$\Delta(|\nabla u|^2+|\nabla v|^2)\ge(1-\varepsilon)\lambda(|\nabla u|^2+|\nabla v|^2),\quad x\in\mathbb{R}^2\backslash D_{\widetilde{R}_\varepsilon}. \tag{4.51}$$

Noting the property $|\nabla u|^2+|\nabla v|^2=0$ at infinity, applying the comparison principle, we arrive at

$$|\nabla u|^2+|\nabla v|^2\le C(\varepsilon)e^{-\sqrt{(1-\varepsilon)\lambda}|x|},\quad |x|>R, \tag{4.52}$$

where $C(\varepsilon)>0$ is a constant.

Again, from (4.52), we see that $|\nabla u|,|\nabla v|=O(|x|^{-3})$ at infinity, which implies $|\nabla f|=O(|x|^{-3})$ at infinity. Therefore, in view of the divergence theorem, we have

$$\int_{\mathbb{R}^2}\Delta u\mathrm{d}x=\int_{\mathbb{R}^2}\Delta f\mathrm{d}x=0. \tag{4.53}$$

Integrating (4.6) and (4.7) over $\mathbb{R}^2$ and inserting (4.53) and the definition of h, we have

$$\lambda\int_{\mathbb{R}^2}(2e^u-e^{v_0+f-u}-1)\,\mathrm{d}x=0, \tag{4.54}$$

$$\lambda\int_{\mathbb{R}^2}(e^{v_0+f-u}-1)\,\mathrm{d}x=-2\pi n, \tag{4.55}$$

as stated in Theorem 2.1.

We may summarize our results as follows.

Theorem 4.1. *For any distribution of the points $p_1,p_2,\ldots,p_n\in\mathbb{R}^2$, the system of nonlinear elliptic equations (3.8) and (3.9) subject to the boundary condition (4.1) has a unique solution. Furthermore, the solution satisfies the decay estimates*

$$u^2(x)+v^2(x)\le C(\varepsilon)e^{-\sqrt{(1-\varepsilon)\lambda}|x|}, \tag{4.56}$$

$$|\nabla u|^2(x)+|\nabla v|^2(x)\le C(\varepsilon)e^{-\sqrt{(1-\varepsilon)\lambda}|x|}, \tag{4.57}$$

for $x\in\mathbb{R}^2$ near infinity, where $\varepsilon\in(0,1)$ is arbitrary and $C(\varepsilon)>0$ is a constant.

In the next section, we turn our attention to the study of the doubly periodic case.

5. Solution on Doubly Periodic Domain

In this section, we consider solutions of (3.8) and (3.9) defined over a doubly periodic domain Ω. In order to get rid of the singular source terms, we introduce a background function v_0 satisfying

$$\Delta v_0 = -\frac{4\pi n}{|\Omega|} + 4\pi \sum_{s=1}^{n} \delta_{p_s}(x). \tag{5.1}$$

Using the new variable w so that $v = v_0 + w$, we can modify (3.8) and (3.9) into

$$\Delta u = \lambda(2e^u - e^{v_0+w} - 1), \tag{5.2}$$

$$\Delta u + \Delta w = 2\lambda(e^{v_0+w} - 1) + \frac{4\pi n}{|\Omega|}. \tag{5.3}$$

Note that, since the singularity of v_0 at p_s is of the type $\ln|x-p_s|^2$, the weight function e^{v_0} is everywhere smooth.

To proceed further, we take $u + w = f$. Then the governing system of equations become

$$\Delta u = \lambda(2e^u - e^{v_0+f-u} - 1), \tag{5.4}$$

$$\Delta f = 2\lambda(e^{v_0+f-u} - 1) + \frac{4\pi n}{|\Omega|}. \tag{5.5}$$

Integrating (5.5) and (5.4), we have

$$\int_\Omega e^{v_0+f-u}\mathrm{d}x = |\Omega| - \frac{2\pi n}{\lambda} \equiv C_1 > 0, \tag{5.6}$$

$$\int_\Omega e^u\mathrm{d}x = \frac{1}{2}\int_\Omega e^{v_0+f-u}\mathrm{d}x + \frac{1}{2}|\Omega| = \frac{1}{2}(C_1 + |\Omega|) \equiv C_2 > 0. \tag{5.7}$$

Of course, the conditions (5.6) and (5.7) imply that the existence of an n-vortex solution requires that $C_1 > 0$ and $C_2 > 0$, which is simply

$$|\Omega| - \frac{2\pi n}{\lambda} \equiv C_1 > 0, \tag{5.8}$$

since $C_1 > 0$ contains $C_2 > 0$.

We can prove that (5.8) is in fact sufficient for existence as well.

Theorem 5.1. *The system of the non-Abelian vortex equations* (5.4) *and* (5.5) *has a solution if and only if* (5.8) *holds or*

$$2\pi n < \lambda|\Omega|. \tag{5.9}$$

Furthermore, if a solution exists, it must be unique, which can be constructed through solving a multiply constrained minimization problem.

We use $W^{1,2}(\Omega)$ to denote the usual Sobolev space of scalar-valued or vector-valued Ω-periodic L^2-functions whose derivatives belong to $L^2(\Omega)$. We will prove Theorem 5.1 in terms of three lemmas as follows.

Lemma 5.2. *Consider the constrained minimization problem*

$$\min\{I(u,f)|(u,f)\in W^{1,2}(\Omega), J_k(u,f)=C_k, C_k>0, k=1,2\}, \tag{5.10}$$

where

$$I(u,f)=\int_\Omega \left\{\frac{1}{2}|\nabla u|^2+\frac{1}{4}|\nabla f|^2-\lambda u-\lambda f+\frac{2\pi n}{|\Omega|}f\right\}\mathrm{d}x, \tag{5.11}$$

$$J_1(u,f)=\int_\Omega e^{v_0}e^{f-u}\,\mathrm{d}x, \tag{5.12}$$

$$J_2(u,f)=\int_\Omega e^{u}\,\mathrm{d}x. \tag{5.13}$$

Then a solution of (5.10) *is a solution of Eqs.* (5.4) *and* (5.5).

Proof. It is clear that the Fréchet derivatives dJ_1, dJ_2 of the constraint functional are linearly independent.

Let (u,f) be a solution of (5.10). Then by standard elliptic regularity theory (u,f) must be smooth and there exist Lagrange multipliers $\lambda_1,\lambda_2\in\mathbb{R}$ so that

$$\Delta u=-\lambda-\lambda_1 e^{v_0}e^{f-u}+\lambda_2 e^{u}, \tag{5.14}$$

$$\Delta f=-2\lambda+2\lambda_1 e^{v_0}e^{f-u}+\frac{4\pi n}{|\Omega|}. \tag{5.15}$$

Integrating Eq. (5.15) and using $J_1(u,f)=C_1$, we obtain $\lambda_1=\lambda$ which means that (u,f) verifies Eq. (5.5). To recover Eq. (5.4), we use $J_2(u,f)=C_2$. By virtue of $\lambda_1=\lambda$ and integrating Eq. (5.14), we have $\lambda_2=2\lambda$.

In particular, (u,f) is a solution of Eqs. (5.4) and (5.5). The lemma is proven. □

The admissible set of the variational problem (5.10) will be denoted by

$$\mathcal{C}=\{(u,f)\in W^{1,2}(\Omega)|J_k(u,f)=C_k, k=1,2\}. \tag{5.16}$$

When (5.6) and (5.7) are satisfied, $C_1,C_2>0$. Thus $\mathcal{C}\neq\emptyset$.

Lemma 5.3. *If the condition* (5.9) *holds, then* (5.10) *has a solution. In other words, the system* (3.8) *and* (3.9) *has a solution if and only if* (5.9) *is fulfilled.*

Proof. By virtue of Lemma 5.2, it is sufficient to show the existence of a minimizer of the constrained optimization problem (5.10).

We first proved that under the condition (5.8) or (5.9), the objective functional I is bounded from below on $\mathcal{C}$. For this purpose, we rewrite each $\eta\in W^{1,2}(\Omega)$ as follows

$$\eta=\underline{\eta}+\eta',$$

where $\underline{\eta}$ denotes the integral mean of η, $\underline{\eta}=\frac{1}{|\Omega|}\int_\Omega\eta\mathrm{d}x$ and $\int_\Omega\eta'\mathrm{d}x=0$. Hence, I may be put for $(u,f)\in\mathcal{C}$ in the form

$$I(u,f)=\int_\Omega\left\{\frac{1}{2}|\nabla u'|^2+\frac{1}{4}|\nabla f'|^2\right\}\mathrm{d}x-\lambda\underline{f}|\Omega|+2\pi n\underline{f}-\lambda\underline{u}|\Omega|. \tag{5.17}$$

Setting

$$\Lambda(\underline{u}, \underline{f}) = -(\lambda|\Omega| - 2\pi n)\underline{f} - \lambda|\Omega|\underline{u}, \tag{5.18}$$

we can derive from (5.6) and (5.7) the expressions

$$\underline{u} = \ln C_2 - \ln\left(\int\limits_\Omega e^{u'}\right), \tag{5.19}$$

$$\underline{f} = \ln(C_1C_2) - \ln\left(\int\limits_\Omega e^{u'}\right) - \ln\left(\int\limits_\Omega e^{v_0+f'-u'}\right). \tag{5.20}$$

Inserting (5.19) and (5.20) into (5.18), we have

$$\Lambda(\underline{u}, \underline{f}) = (2\lambda|\Omega| - 2\pi n)\ln\left(\int\limits_\Omega e^{u'}\right) + (\lambda|\Omega| - 2\pi n)\ln\left(\int\limits_\Omega e^{v_0+f'-u'}\right) + C_3,$$

where $C_3 = (2\pi n - \lambda|\Omega|)\ln C_1 + (2\pi n - 2\lambda|\Omega|)\ln C_2$.

Using Jensen's inequality, we get

$$\ln\left(\int\limits_\Omega \exp(v_0 + f' - u')\right) \geq \ln\left[|\Omega|\exp\left(\frac{1}{|\Omega|}\int\limits_\Omega (v_0 + f' - u')\right)\right]$$
$$= \ln\left[|\Omega|\exp\left(\frac{1}{|\Omega|}\int\limits_\Omega v_0\right)\right],$$
$$\ln\left(\int\limits_\Omega e^{u'}\right) \geq \ln\left[|\Omega|\exp\left(\frac{1}{|\Omega|}\int\limits_\Omega u'\right)\right] = \ln|\Omega|.$$

Noting (5.9), we have

$$\Lambda(\underline{u}, \underline{f}) \geq (2\lambda|\Omega| - 2\pi n)\ln|\Omega| + (\lambda|\Omega| - 2\pi n)\ln\left[|\Omega|\exp\left(\frac{1}{|\Omega|}\int\limits_\Omega v_0\right)\right] + C_3. \tag{5.21}$$

Inserting (5.21) into (5.17), we arrive at the coercive lower estimate

$$I(u, f) \geq \int\limits_\Omega \left\{\frac{1}{2}|\nabla u'|^2 + \frac{1}{4}|\nabla f'|^2\right\}\mathrm{d}x - C_4, \tag{5.22}$$

where $C_4 > 0$ is an irrelevant constant. From (5.22), we know that the existence of solution of (5.10) follows.

In fact, let $\{(u_j, f_j)\} \subset \mathcal{C}$ be a minimizing sequence of the variational problem (5.10) and set

$$\underline{f}_j = \frac{1}{|\Omega|}\int\limits_\Omega f_j\mathrm{d}x, \quad \underline{u}_j = \frac{1}{|\Omega|}\int\limits_\Omega u_j\mathrm{d}x. \tag{5.23}$$

Then, with $u'_j = u_j - \underline{u}_j$ and $f'_j = f_j - \underline{f}_j$, we have $\underline{u}'_j = 0$ and $\underline{f}'_j = 0$. In view of (5.22), we see that $\{(u'_j, f'_j)\}$ is bounded in $W^{1,2}(\Omega)$. Without loss of generality, we may assume that $\{(u'_j, f'_j)\}$ converges weakly in $W^{1,2}(\Omega)$ to an element (u', f') (say). The compact embedding

$$W^{1,2}(\Omega) \hookrightarrow L^p(\Omega), \quad p \geq 1, \tag{5.24}$$

then implies $(u'_j, f'_j) \to (u', f')$ in $L^p(\Omega)$ $(p \geq 1)$ as $j \to \infty$. In particular, $\underline{u}' = 0$ and $\underline{f}' = 0$.

Recall the Trudinger–Moser inequality [1]

$$\int_\Omega e^F \mathrm{d}x \leq C(\varepsilon) \exp\left(\left[\frac{1}{16\pi} + \varepsilon\right] \int_\Omega |\nabla F|^2 \mathrm{d}x\right), \quad F \in W^{1,2}(\Omega), \quad \underline{F} = 0, \tag{5.25}$$

where $C(\varepsilon) > 0$ is a constant. In view of (5.24) and (5.25), we see that the functionals defined by the right-hand side of (5.19) and (5.20) are continuous in u', f' with respect to the weak topology of $W^{1,2}(\Omega)$. Therefore, $\underline{u}_j \to \underline{u}, \underline{f}_j \to \underline{f}$ as $j \to \infty$, as given in (5.19) and (5.20). In other words, $(u, f) = (\underline{u} + u', \underline{f} + f')$ satisfies the constraints (5.6) and (5.7), and solves the constrained minimization problem (5.10). Thus, Lemma 5.2 is proven. □

Now we state the uniqueness of the solution to Eqs. (5.4) and (5.5) as follows.

Lemma 5.4. *If system* (5.4) *and* (5.5) *has a solution, then the solution must be unique.*

Proof. Consider the following functional,

$$J(u, f) = \frac{1}{2}||\nabla u||_2^2 + \frac{1}{4}||\nabla f||_2^2 + (-\lambda\underline{u} - \lambda\underline{f} + \frac{2\pi n}{|\Omega|}\underline{f})|\Omega| + \int_\Omega \left\{\lambda e^{v_0 + f - u} + 2\lambda e^u\right\} \mathrm{d}x.$$

It is straightforward to check by calculating the Hessian that J is strictly convex in $W^{1,2}(\Omega)$. Thus, J has at most one critical point. However, any solution of (5.4) and (5.5) must be a critical point of J. This proves the lemma. □

6. Further Extensions

In this section, we consider the multiple vortex equations (2.24)–(2.27) for the scalar fields χ and ϕ, both taking complex-valued, which are allowed to independently generate vortices with their respective prescribed zero sets

$$Z_\chi = \{q_1, q_2, \ldots, q_m\}, \quad Z_\phi = \{p_1, p_2, \ldots, p_n\}. \tag{6.1}$$

In such a context, we can similarly develop an existence and uniqueness theory for the solutions of the equations by the same variational methods. To see this, we observe that, with the prescribed zero sets given in (6.1) for the fields χ and ϕ and in terms of the variables $u = \ln|\chi|^2$ and $v = \ln|\phi|^2$, the governing system of nonlinear elliptic equations (3.8) and (3.9) is modified into

$$\Delta u = \lambda(2e^u - e^v - 1) + 4\pi \sum_{t=1}^{m} \delta_{q_t}(x), \tag{6.2}$$

$$\Delta(u+v) = 2\lambda(e^v - 1) + 4\pi \sum_{t=1}^{m} \delta_{q_t}(x) + 4\pi \sum_{s=1}^{n} \delta_{p_s}(x), \tag{6.3}$$

with the associated boundary condition

$$u, v \to 0 \quad \text{as } |x| \to \infty. \tag{6.4}$$

Parallel to Theorem 4.1, we have

Theorem 6.1. *The system of nonlinear elliptic equations* (6.2) *and* (6.3) *subject to the boundary condition* (6.4) *has a unique solution for which the boundary condition* (6.4) *may be achieved exponentially fast.*

In order to prove Theorem 6.1, we introduce the background functions as before,

$$u_0(x) = -\sum_{t=1}^{m} \ln(1 + \tau|x - q_t|^{-2}), \tag{6.5}$$

$$v_0(x) = -\sum_{s=1}^{n} \ln(1 + \tau|x - p_s|^{-2}), \quad \tau > 0. \tag{6.6}$$

Then

$$\Delta u_0 = -h_1(x) + 4\pi \sum_{t=1}^{m} \delta_{q_t}(x), \tag{6.7}$$

$$\Delta v_0 = -h_2(x) + 4\pi \sum_{s=1}^{n} \delta_{p_s}(x), \tag{6.8}$$

where

$$h_1(x) = 4\sum_{t=1}^{m} \frac{\tau}{(\tau + |x - q_t|^2)^2}, \tag{6.9}$$

$$h_2(x) = 4\sum_{s=1}^{n} \frac{\tau}{(\tau + |x - p_s|^2)^2}. \tag{6.10}$$

We set $u = u_0 + w_1, v = v_0 + w_2$, and $f = w_1 + w_2$. Then (6.2) and (6.3) become

$$\Delta w_1 = \lambda(2e^{u_0 + w_1} - e^{v_0 + f - w_1} - 1) + h_1(x), \tag{6.11}$$

$$\Delta f = 2\lambda(e^{v_0 + f - w_1} - 1) + h_1(x) + h_2(x). \tag{6.12}$$

It can be checked that (6.11) and (6.12) are the Euler–Lagrange equations of the action functional

$$I(w_1,f)=\int\limits_{\mathbb{R}^2}\left\{\frac{1}{2\lambda}|\nabla w_1|^2+\frac{1}{4\lambda}|\nabla f|^2+2\left(e^{u_0+w_1}-e^{u_0}\right)\right.$$
$$\left.+\left(e^{v_0+f-w_1}-e^{v_0}\right)+\left(\frac{h_1}{\lambda}-1\right)w_1+\left(\frac{h_1+h_2}{2\lambda}-1\right)f\right\}\mathrm{d}x. \tag{6.13}$$

It is clear that the functional I is C^1 over $W^{1,2}(\mathbb{R}^2)$ and strictly convex. We can use the methods in [20] and in the earlier study in the present paper to establish the coercive bounds

$$DI(w_1,f)(w_1,f)\geq C_1(\|w_1\|_{W^{1,2}(\mathbb{R}^2)}+\|f\|_{W^{1,2}(\mathbb{R}^2)})-C_2. \tag{6.14}$$

Therefore, it follows that the functional I has a unique critical point in $W^{1,2}(\mathbb{R}^2)$ which establishes the existence and uniqueness of a classical solution to the system of Eqs. (6.2) and (6.3) subject to the boundary condition (6.4).

We now turn our attention to the existence of multivortex solution over a doubly periodic domain Ω.

Take u_0 and v_0 over Ω to satisfy

$$\Delta u_0=-\frac{4\pi m}{|\Omega|}+4\pi\sum_{t=1}^{m}\delta_{q_t}(x),\quad \Delta v_0=-\frac{4\pi n}{|\Omega|}+4\pi\sum_{s=1}^{n}\delta_{p_s}(x). \tag{6.15}$$

Then setting $u=u_0+w_1, v=v_0+w_2$, we see that Eqs. (6.2) and (6.3) over the doubly periodic domain Ω become

$$\Delta w_1=\lambda(2e^{u_0+w_1}-e^{v_0+w_2}-1)+\frac{4\pi m}{|\Omega|}, \tag{6.16}$$

$$\Delta(w_1+w_2)=2\lambda(e^{v_0+w_2}-1)+\frac{4\pi}{|\Omega|}(m+n). \tag{6.17}$$

Theorem 6.2. *For the vortex equations* (6.16) *and* (6.17) *defined over a doubly periodic domain* Ω*, there is a solution if and only if the inequalities*

$$2\pi(m+n)<\lambda|\Omega|, \tag{6.18}$$

$$\pi(3m+n)<\lambda|\Omega|, \tag{6.19}$$

are satisfied. Moreover, if a solution exists, it must be unique.

In the special case when the scalar field χ has no zero, that is, $m=0$ in (6.18) and (6.19), we recover Theorem 5.1.

To proceed in the formalism of calculus of variations, we use the new variables $g=w_1, f=w_1+w_2$. Then (6.16) and (6.17) take the form

$$\Delta g=\lambda(2e^{u_0+g}-e^{v_0+f-g}-1)+\frac{4\pi m}{|\Omega|}, \tag{6.20}$$

$$\Delta f=2\lambda(e^{v_0+f-g}-1)+\frac{4\pi}{|\Omega|}(m+n). \tag{6.21}$$

Integrating these two equations and simplifying the results, we arrive at the constraints

$$\int_{\Omega} e^{v_0+f-g}\mathrm{d}x = |\Omega| - \frac{2\pi}{\lambda}(m+n) \equiv \alpha_1, \tag{6.22}$$

$$\int_{\Omega} e^{u_0+g}\mathrm{d}x = \frac{1}{2}\left(\alpha_1 + |\Omega| - \frac{4\pi m}{\lambda}\right) \equiv \alpha_2. \tag{6.23}$$

In order to show that the necessary condition $\alpha_1 > 0$, $\alpha_2 > 0$, which is exactly what stated in (6.18) and (6.19), is also sufficient for the existence of a solution, we recognize that Eqs. (6.20) and (6.21) are the Euler–Lagrange equations of the action functional

$$I(f,g) = \int_{\Omega} \left\{ \frac{1}{4\lambda}|\nabla f|^2 + \frac{1}{2\lambda}|\nabla g|^2 + 2e^{u_0+g} + e^{v_0+f-g} + \left(\frac{4\pi m}{\lambda|\Omega|} - 1\right) g + \left(\frac{2\pi(m+n)}{\lambda|\Omega|} - 1\right) f \right\} \mathrm{d}x. \tag{6.24}$$

Now decompose f,g into $f = f' + \underline{f}, g = g' + \underline{g}$ with $\underline{f}, \underline{g} \in \mathbb{R}$ and $\int_\Omega f'\mathrm{d}x = 0, \int_\Omega g'\mathrm{d}x = 0$. Thus, applying (6.22) and (6.23), we may rewrite (6.24) in the form

$$\begin{aligned} I(f,g) - \int_{\Omega} \left\{ \frac{1}{4\lambda}|\nabla f'|^2 + \frac{1}{2\lambda}|\nabla g'|^2 \right\} \mathrm{d}x &= \alpha_1 \ln\left(\int_{\Omega} e^{v_0+f'-g'}\,\mathrm{d}x\right) + 2\alpha_2 \ln\left(\int_{\Omega} e^{u_0+g'}\,\mathrm{d}x\right) \\ &\quad + \alpha_1(1-\ln\alpha_1) + 2\alpha_2(1-\ln\alpha_2). \end{aligned} \tag{6.25}$$

It is seen immediately that the right-hand side of (6.25) has a uniform lower bound in view of the Jensen inequality again. So the existence of a critical point of (6.24) subject to the constraints (6.22) and (6.23) follows as before. The uniqueness of a critical point of (6.24) results from the convexity of the functional.

Acknowledgements

The authors would like to thank the referee for helpful suggestions. The first author was partially supported by Henan Basic Science and Frontier Technology Program Funds (112300410054). The second author was partially supported by the Natural Science Foundation of China (11271052) and the Natural Science Foundation of Education Department of Henan Province (12A110004).

References

[1] Aubin, T.: Nonlinear Analysis on Manifolds: Monge–Ampére Equations. Springer, Berlin (1982)

[2] Aharony, O., Bergman, O., Jafferis, D.L., Maldacena, J.: $\mathcal{N} = 6$ superconformal Chern–Simons-matter theories, M2-branes and their gravity duals. J. High Energy Phys. **0810**, 091 (2008)

[3] Aldrovandi, L.G., Schaposnik, F.A.: Non-Abelian vortices in Chern–Simons theories and their induced effective theory. Phys. Rev. D **76**, 045010 (2007)

[4] Auzzi, R., Kumar, S.P.: Non-Abelian vortices at weak and strong coupling in mass deformed ABJM theory. J. High Energy Phys. **071**, 0910 (2009)

[5] Caffarelli, L.A., Yang, Y.: Vortex condensation in the Chern-Simons Higgs model: an existence theorem. Commun. Math. Phys. **168**, 321–336 (1995)

[6] Chae, D., Imanuvilov, O.Y.: The existence of non-topological multivortex solutions in the relativistic self-dual Chern–Simons theory. Commun. Math. Phys. **215**, 119–142 (2000)

[7] Chan, H., Fu, C.C., Lin, C.S.: Non-topological multi-vortex solutions to the self-dual Chern–Simons–Higgs equation. Commun. Math. Phys. **231**, 189–221 (2002)

[8] Desyatnikova, A.S., Kivshara, Y.S., Tornerb, L.: Optical vortices and vortex solitons. Prog. Optics **47**, 291–391 (2005)

[9] Dunne, G.: Self-Dual Chern–Simons Theories. Lecture Notes in Physics, vol. 36. Springer, Berlin (1995)

[10] Dunne G.: Aspects of Chern–Simons Theory. In: Aspects topologiques de la physique en basse dimension/Topological aspects of low dimensional systems (Les Houches 1999), EDP Sci., Les Ulis., pp. 177–263 (1999)

[11] Fisher, M.P.A.: Vortex-glass superconductivity: a possible new phase in bulk high-T_c oxides. Phys. Rev. Lett. **62**, 1415–1418 (1989)

[12] Gilbarg, D., Trudinger, N.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (1997)

[13] Gomis, J., Rodriguez–Gomez, D., Van Raamsdonk, M., Verlinde, H.A: massive study of M2-brane proposals. J. High Energy Phys. **0809**, 113 (2008)

[14] Gudnason, S.B., Jiang, Y., Konishi, K.: Non-Abelian vortex dynamics: effective world-sheet action. J. High Energy Phys. **1012**, 1008 (2010)

[15] Hindmarsh, M.B., Kibble, T.W.B.: Cosmic strings. Rep. Prog. Phys. **58**, 477–562 (1995)

[16] Hong, J., Kim, Y., Pac, P.Y.: Multivortex solutions of the Abelian Chern–Simons–Higgs theory. Phys. Rev. Lett. **64**, 2230–2233 (1990)

[17] Horvathy, P.A., Zhang, P.: Vortices in (Abelian) Chern–Simons gauge theory. Phys. Rep. **481**, 83–142 (2009)

[18] Jackiw, R., Lee, K., Weinberg, E.J.: Self-dual Chern–Simons solitons. Phys. Rev. D **42**, 3488–3499 (1990)

[19] Jackiw, R., Weinberg, E.J.: Self-dual Chern–Simons vortices. Phys. Rev. Lett. **64**, 2234–2237 (1990)

[20] Jaffe, A., Taubes, C.H.: Vortices and Monopoles. Birkhäuser, Boston (1980)

[21] Kumar, C.N., Khare, A.: Charged vortex of finite energy in nonAbelian gauge theories with Chern-Simons term. Phys. Lett. B **178**, 395–399 (1986)

[22] Lerida, F.N., Radu, E., Tchrakian, D.H.: Non-Abelian Chern–Simons–Higgs solutions in 2+1 dimensions. Phys. Rev. D **79**, 065036 (2008)

[23] Lieb, E.H., Yang, Y.: Non-Abelian vortices in supersymmetric gauge field theory via direct methods. Commun. Math. Phys. **313**, 445–478 (2012)

[24] Lin, C.S., Prajapat, J.V.: Vortex condensates for relativistic Abelian Chern–Simons model with two Higgs scalar fields and two gauge fields on a torus. Commun. Math. Phys. **288**, 311–347 (2009)

[25] Lin, C.S., Ponce, A.C., Yang, Y.: A system of elliptic equations arising in Chern–Simons field theory. J. Funct. Anal. **247**, 289–350 (2007)

[26] Lin, C.S., Yang, Y.: Non-Abelian multiple vortices in supersymmetric field theory. Commun. Math. Phys. **304**, 433–457 (2011)

[27] Lin, C.S., Yang, Y.: Sharp existence and uniqueness theorems for non-Abelian multiple vortex solutions. Nucl. Phys. B **846**, 650–676 (2011)

[28] Lozano, G.S., Marqués, D., Moreno, E.F., Schaposnik, F.A.: Non-Abelian Chern–Simons vortices. Phys. Lett. B **54**, 27–34 (2007)

[29] Mitrovi, V.F., Sigmund, E.E., Eschrig, M., Bachman, H.N., Halperin, W.P., Reyes, A.P., Kuhns, P., Moulton, W.G.: Spatially resolved electronic structure inside and outside the vortex core of a high temperature superconductor. Nature **413**, 501–504 (2001)

[30] Neshev, D.N., Alexander, T.J., Ostrovskaya, E.A., Kivshar, Y.S.: Observation of discrete vortex solitons in optically induced photonic lattices. Phys. Rev. Lett. **92**, 123903 (2004)

[31] Orlov, S.A.: Foundation of vortex gravitation, cosmology and cosmogony. Kathmandu University. J. Sci. Eng. Technol. **6**, 142–169 (2010)

[32] Paul, S.K., Khare, A.: Charged vortices in an Abelian Higgs model with Chern–Simons term. Phys. Lett. B **174**, 420–422 (1986). Errata: Phys. Lett. B **177**(453) (1986); Phys. Lett. B **182**, 415 (1986)

[33] Petrovic, M.S.: Vortex-induced rotating structures in optical photonic lattices. Optics Express **14**, 9415–9420 (2006)

[34] Schwinger, J.: A magnetic model of matter. Science **165**, 757–761 (1969)

[35] Spruck, J., Yang, Y.: Topological solutions in the self-dual Chern–Simons theory: existence and approximation. Ann. Inst. H. Poincaré Anal. Non linéaire **12**, 75–97 (1995)

[36] Spruck, J., Yang, Y.: The existence of non-topological solitons in the self-dual Chern–Simons theory. Commun. Math. Phys. **149**, 361–376 (1992)

[37] Tarantello, G.: Multiple condensate solutions for the Chern–Simons–Higgs theory. J. Math. Phys. **37**, 3769–3796 (1996)

[38] Tarantello, G.: Selfdual Gauge Field Vortices: An Analytical Approach. Birkhäuser, Boston (2008)

[39] de Vega, H.J., Schaposinik, F.A.: Electrically charged vortices in non-Abelian gauge theories with Chern–Simons term. Phys. Rev. Lett. **6**, 2564–2566 (1986)

[40] Wang, R.: The existence of Chern–Simons vortices. Commun. Math. Phys. **137**, 587–597 (1991)

[41] Yang, Y.: Solitons in Field Theory and Nonlinear Analysis. Springer Monographs in Mathematics. Springer, Berlin (2001)

[42] Zwanziger, D.: Exactly soluble nonrelativistic model of particles with both electric and magnetic charges. Phys. Rev. **176**, 1480–1488 (1968)

[43] Zwanziger, D.: Quantum field theory of particles with both electric and magnetic charges. Phys. Rev. **176**, 1489–1495 (1968)

Shouxin Chen, Ruifeng Zhang and Meili Zhu
School of Mathematics
Henan University
Kaifeng 475004, Henan
People's Republic of China
e-mail: chensx1982@gmail.com;
zrf615@henu.edu.cn;
meili.xiangyue@163.com

Shouxin Chen and Ruifeng Zhang
Institute of Contemporary Mathematics
Henan University
Kaifeng 475004, Henan
People's Republic of China

Communicated by Marcos Marino.
Received: May 23, 2012.
Accepted: September 10, 2012.