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Upper Bound for the Bethe–Sommerfeld Threshold and the Spectrum of the Poisson Random Hamiltonian in Two Dimensions

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Abstract. We consider the Schrödinger operator on $\mathbb{R}^2$ with a locally square-integrable periodic potential V and give an upper bound for the Bethe–Sommerfeld threshold (the minimal energy above which no spectral gaps occur) with respect to the square-integrable norm of V on a fundamental domain, provided that V is small. As an application, we prove the spectrum of the two-dimensional Schrödinger operator with the Poisson type random potential almost surely equals the positive real axis or the whole real axis, according as the negative part of the single-site potential equals zero or not. The latter result completes the missing part of the result by Ando et al. (Ann Henri Poincaré 7:145–160, 2006).

1. Introduction

1.1. Bethe–Sommerfeld Threshold

We consider the Schrödinger operator H on $\mathbb{R}^2$ given by

$$H = -\Delta + V,$$

and assume the following:

(V1) $V \in L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R})$.
(V2) V is periodic with respect to some lattice Γ (of rank 2).

Under (V1) and (V2), it is well known that $H|_{C_0^\infty(\mathbb{R}^2)}$ is essentially self-adjoint and we denote the unique self-adjoint extension by the same letter H. It is also well known that the spectrum of H consists of finitely or infinitely many closed intervals (energy bands) (see, e.g. [13]). The *Bethe–Sommerfeld conjecture* says the number of the spectral gaps is finite when the space dimension $d \geq 2$. This conjecture is proved by Skriganov [14,15] and Popov and Skriganov [11] in two-dimensional case, by Skriganov [16,17] in three-dimensional

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case, and by Helffer and Mohamed [3] in four-dimensional case. Skriganov [16] also proves the conjecture in arbitrary dimension when the period lattice is rational. In the above results they assume some smoothness or boundedness for the periodic potential. Karpeshina [4] proves the conjecture in the case V is singular (e.g. $V \in L^2_{\mathrm{loc}}$), in two- or three-dimensional case. Parnovski [9] also proves the conjecture in arbitrary dimension for smooth V, and Veliev [18] also gives an important contribution on this matter. For the detail, see the references in the above papers.

In the present paper, we shall give a refinement of Karpeshina's result [4] for singular potentials in two-dimensional case, as follows:

Theorem 1.1. *Assume* (V1) *and* (V2). *Let* Ω *be a fundamental domain of* Γ (*see Sect.* 2.1). *Then, there exist positive constants* $\epsilon = \epsilon(\Gamma)$ *and* $c = c(\Gamma)$ *such that*

$$\sigma(-\Delta + V) \supset [c\|V\|_{L^2(\Omega)}, \infty)$$

for any $V \in L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R})$ *periodic with respect to* Γ *and* $\|V\|_{L^2(\Omega)} \le \epsilon$.

It is easy to see that the constants $\epsilon(\Gamma)$ and $c(\Gamma)$ satisfy the scaling properties

$$\epsilon(\alpha\Gamma) = \epsilon(\Gamma)/\alpha, \quad c(\alpha\Gamma) = c(\Gamma)/\alpha \tag{1}$$

for any $\alpha > 0$, where $\alpha\Gamma = \{\alpha\gamma : \gamma \in \Gamma\}$ is the scaled lattice. Theorem 1.1 gives an upper bound for the Bethe–Sommerfeld threshold (the minimal energy above which no spectral gaps occur) with respect to $\|V\|_{L^2(\Omega)}$, for small V. To our knowledge, the upper bound of this type seems not obtained in the preceding literature. The closest result is the one by Skriganov [14, Theorem 2,3], which implies the conclusion of Theorem 1.1 with L^2-norm just replaced by L^∞-norm.[1]

The proof of Theorem 1.1 is based on Karpeshina's book [4]. We use the perturbation theory of the Bloch Laplacian $-\Delta_k$ on the Bloch wave subspace with quasi-momentum k. We prove that for a fixed small positive number μ and for any sufficiently large energy E, there always exists a quasi-momentum k such that E is a simple eigenvalue of $-\Delta_k$ and

$$|E - E'| \ge \mu \quad \text{for } E' \in \sigma(-\Delta_k)\backslash\{E\}. \tag{2}$$

The condition (2) is called the *μ-non diffraction condition*,[2] which enables us to apply the perturbation theory for simple eigenvalues. Further, we prove that for sufficiently large E there exists a quasi-momentum k satisfying (2) and the resolvent estimates

$$\|(-\Delta_k - z)^{-1}\| \le C, \quad \|V(-\Delta_k - z)^{-1}\|_{\mathrm{HS}} \le C\|V\|_{L^2(\Omega)}$$

for z on the circle $\{z \in \mathbb{C} : |z - E| = \mu/2\}$, where C is a constant independent of E, z, V, and $\|\cdot\|$ denotes the operator norm, $\|\cdot\|_{\mathrm{HS}}$ the Hilbert-Schmidt

[1] We can prove this statement by [14, Theorem 2,3] combined with a simple scaling argument as in the beginning part of the proof of Theorem 1.1.

[2] The name 'diffraction' comes from the condition for the diffraction of the plane wave inside the crystal, given by von Laue. For the detail, see [4].

norm. By a standard perturbative argument using the resolvent expansion with respect to $A = V(-\Delta_k - z)^{-1}$, we prove the value of the branch of the free band function λ_0 with $\lambda_0(k) = E$ changes at most $O(\|V\|_{L^2(\Omega)})$ by the perturbation by V, provided that $\|V\|_{L^2(\Omega)}$ is small. Combining this with the fact the Bethe–Sommerfeld conjecture trivially holds for $V = 0$, we conclude Theorem 1.1 holds.

The difference between our method and Karpeshina's is the following: Karpeshina uses the resolvent expansion with respect to

$$\widetilde{A} = (-\Delta_k - z)^{-1/2} V (-\Delta_k - z)^{-1/2},$$

and obtains the estimate for $\widetilde{A}$ using some decomposition of the Fourier space dependent on the Fourier coefficients of V (see, e.g. [4, (3.6.3)]). This method is applicable to more singular potentials than L^2_{loc}, but, however, also makes it difficult to see the dependence of the Bethe–Sommerfeld threshold with respect to $\|V\|_{L^2(\Omega)}$. The use of our operator A clarifies this point and also makes the proof simpler with the aid of a Chebyshev-like lemma (Lemma 4.2). Besides, our definition of the diffraction set is slightly different from Karpeshina's. In Karpeshina's definition, the number μ in (2) decays as R increases, but ours does not. This change gives us better resolvent estimates.

It is natural to ask whether an analogue of Theorem 1.1 holds in higher dimensions. However, as Karpeshina points out in [4, Section 4.1], there is a qualitative difference between the two-dimensional case and the higher dimensional case. As is well known, the density of states for the free Laplacian is a constant times $\lambda_+^{d/2-1}$, which is a constant if $d = 2$, and is an increasing function if $d \geq 3$. This fact makes it difficult to use the non-degenerate perturbation method when $d \geq 3$. In order to avoid this difficulty in the three-dimensional case, Karpeshina analyzes the perturbation of the degenerated eigenvalues by comparing them with those of some modelling operators. However, the method again depends on the distribution of the Fourier coefficients of each V, and the bound for the threshold with respect to L^2-norm is unknown at present. Further study is necessary in this direction.

1.2. Spectrum of the Schrödinger Operators with Poisson Type Random Potentials

We give an application of Theorem 1.1 to the spectral theory of the random Schrödinger operators. We consider the random Schrödinger operator on $\mathbb{R}^d$ $(d = 1, 2, 3, \ldots)$

$$H_\omega = -\Delta + V_\omega, \quad V_\omega(x) = \sum_{j=1}^{\infty} f(x - X_j(\omega)),$$

where $\omega \in X$ and $(X, \mathcal{F}, \mathbb{P})$ is a probability space, and f is a real-valued function called the *single site potential*. We assume the following:

(A1) f is a real-valued, measurable function satisfying

$$\sum_{n\in\mathbb{Z}^d}\left(\int_{n+[-\frac{1}{2},\frac{1}{2})^d}|f(x)|^p\mathrm{d}x\right)^{1/p}<\infty,$$

where

$$\begin{cases} p>2 & (d\le 3),\\ p>d/2 & (d\ge 4).\end{cases}$$

(A2) The random points $\{X_j(\omega)\}_{j=1}^{\infty}$ are the Poisson configuration with intensity measure $\rho\mathrm{d}x$, where ρ is a positive constant and $\mathrm{d}x$ is the Lebesgue measure. That is, the following conditions hold(see [1, Assumption (H3)]):

(i) For any $E_1, E_2, \ldots, E_n \subset \mathbb{R}^d$ disjoint Borel sets on $\mathbb{R}^d$, the random variables $\#\{j : X_j(\omega)\in E_k\}, k = 1,2,\ldots,n$ are mutually independent. Here we denote the cardinality of a set A by $\#A$.

(ii) If $E\subset\mathbb{R}$ is a Borel set with finite Lebesgue measure $|E|=\int_E \mathrm{d}x$,

$$\mathbb{P}(\#\{j : X_j(\omega)\in E\}=n)=\frac{(\rho|E|)^n}{n!}\mathrm{e}^{-\rho|E|},\quad n=0,1,2,\ldots.$$

H_ω describes the motion of electrons in amorphous materials where atoms are distributed randomly. Under the conditions (A1) and (A2), the operators H_ω are essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ almost surely, and measurable on X [2, Proposition V.3.2, Corollary V.3.4]. And $\{H_\omega\}_{\omega\in X}$ is an ergodic family of self-adjoint operators on $(X,\mathcal{F},\mathbb{P})$. It is well known that there exists a measurable set $X_0\subset X$ with probability one and a closed set $\Sigma\subset\mathbb{R}$ such that $\sigma(H_\omega)=\Sigma$ for every $\omega\in X_0$ (see e.g.[2, Proposition V.2.4]).

Put

$$f_+(x)=\max\{f(x),0\},\quad f_-(x)=\max\{-f(x),0\}.$$

A naive observation tells us

(∗) if the negative part f_- of f vanishes, then the spectral set $\Sigma=[0,\infty)$; if not, then $\Sigma=\mathbb{R}$.

There are some discussions on the assertion (∗) in [5] and [10, Theorem 5.34], which does not, however, seem to be fully convincing to us. The assertion (∗) is also stated in [7] and [8] without proof.

When the dimension $d\neq 2$, (∗) is rigorously proved in [1] under the assumption (A1) and (A2). However, in the case $d=2, f_+=0$ and $f_-\neq 0$, their theorem [1, Theorem 1.2] needs the following additional condition:

$$\frac{1}{4\pi}\int_{\mathbb{R}^2} f_-(x)\mathrm{d}x\notin\mathbb{N},\tag{3}$$

for some technical reasons explained later.

Surprisingly, we can get rid of the above technical condition (3) using our Theorem 1.1.

Theorem 1.2. *Suppose $d = 2$ and the assumptions* (A1), (A2) *hold.*

(i) *If $f_- = 0$ (as an element of $L^1(\mathbb{R}^2)$), then $\Sigma = [0, \infty)$.*
(ii) *If $f_- \neq 0$, then $\Sigma = \mathbb{R}$.*

The part (i) is already proved in [1, Theorem 1.2]. Notice that we do not require the condition (3) in the part (ii). Thus, Theorem 1.2 completes the missing part of the proof of the statement $(*)$ in [1].

Let us explain why the condition (3) appears and how we use Theorem 1.1 in the proof of Theorem 1.2. The most fundamental tool to determine the spectral set Σ is the technique of the *the admissible potential*, developed by Kirsch and Martinelli [6]. This method asserts the union of the spectrum of some special potentials (admissible potentials) forms a dense subset of Σ. For example, [1] use the family of admissible potentials $\mathcal{A}_F$ given by

$$\mathcal{A}_F = \left\{ \sum_{j=1}^{n} f(x - u_j) : u_1, \ldots, u_n \in \mathbb{R}^d,\ n = 1, 2, \ldots \right\}, \tag{4}$$

and prove $\Sigma = \overline{\bigcup_{W \in \mathcal{A}_F} \sigma(-\Delta + W)}$. By using this fact, it is easy to show that $\Sigma \supset [0, \infty)$, since the admissible potential $W \in \mathcal{A}_F$ is relatively compact with respect to the negative Laplacian. In order to show "$\Sigma = \mathbb{R}$" when $f_- \neq 0$, they aim to deduce a contradiction, supposing that there exists $b \in \mathbb{R} \backslash \Sigma$. Then, for every $n = 1, 2, \ldots$, the number

$$\gamma(n) = \# \left\{ \text{eigenvalue of } -\Delta + \sum_{j=1}^{n} f(\cdot - u_j) \text{ less than } b \right\}$$

is continuous with respect to $u_1, \ldots, u_n \in \mathbb{R}^d$, so is a constant dependent only on n. They consider the limiting behavior of $\gamma(n)$ as $n \to \infty$ in the following two extremal cases. When $u_1 = \cdots = u_n$, the Weyl asymptotics yields

$$\gamma(n) = \begin{cases} n^{d/2} \frac{\tau_d}{(2\pi)^d} \int_{\mathbb{R}^d} (f_-(x))^{d/2} \mathrm{d}x (1 + o(1)) & (d \geq 2), \\ o(n) & (d = 1), \end{cases}$$

where τ_d is the volume of the d-dimensional unit ball. On the other hand, taking the limit $|u_j - u_k| \to \infty$ for every $j \neq k$, we have

$$\gamma(n) = n\gamma(1).$$

If $d \neq 2$ or $d = 2$ and (3) holds, then the two limiting behaviors are different and we reach a contradiction. Moreover, if $d = 2$, $f_+ \neq 0$ and $f_- \neq 0$, then we can replace $f(x)$ by $\tilde{f}(x) = f(x) + f(x - a)$ for some $a \in \mathbb{R}^2$ so that $\tilde{f}$ satisfies (3). Thus, the only remaining case is $d = 2$, $f_+ = 0$, $f_- \neq 0$ and (3) does not hold.

Instead of $\mathcal{A}_F$, we use the periodic admissible potentials $\mathcal{A}_P$ defined by

$$\mathcal{A}_P = \bigcup_{\Gamma} \Bigg\{ W(x) = \sum_{j=1}^{J} \sum_{\gamma \in \Gamma} f(x - x_j - \gamma) : \\ x_j \in \mathbb{R}^d \ (j = 1, \dots, J),\ J = 1, 2, \dots \Bigg\}, \tag{5}$$

where $\bigcup_\Gamma$ denotes the union over the all lattices Γ in $\mathbb{R}^d$. Then the proof proceeds as follows: Assume $d = 2$, $f_+ = 0$ and $f_- \neq 0$. Then

$$\hat{f}(0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(x) \mathrm{d}x < 0.$$

We define admissible potentials $V_K \in \mathcal{A}_P$ $(K = 1, 2, \dots)$ by

$$V_K(x) = \sum_{\gamma \in \Gamma} f\left(x - \frac{\gamma}{K}\right)$$

for some fixed lattice Γ. After a short computation using the Fourier series of V_K and the scaling $x = y/K$, we obtain

$$-\Delta_x + V_K(x) = K^2(-R_0 - \Delta_y + \widetilde{W}_K(y)),$$

where $\widetilde{W}_K$ is Γ-periodic with respect to y and

$$R_0 = -\frac{2\pi}{|\Omega|} \hat{f}(0) > 0, \quad \|\widetilde{W}_K\|_{L^2(\Omega)} \to 0 \text{ as } K \to \infty.$$

Here we use Theorem 1.1 for the operator $-\Delta_y + \widetilde{W}_K$, and conclude

$$\sigma(-\Delta + V_K) \supset [K^2(-R_0 + R), \infty)$$

for some $0 < R < R_0$ and sufficiently large K. This implies $\Sigma = \mathbb{R}$. This proof is nothing to do with the non-integer condition (3).

The rest of the paper is organized as follows: In Sect. 2, we review some basic facts in the Bloch theory. In Sect. 3, we prove Theorem 1.1. In Sect. 4, we prove some geometric lemmas used in the proof of Theorem 1.1, and in Sect. 5, we prove Theorem 1.2.

2. Bloch Theory

In this section, we shall introduce some basic notation for the lattice, and briefly review the matrix representation of the Bloch theory in the two-dimensional case. For the detail, see, e.g. [13].

A *lattice* Γ in $\mathbb{R}^2$ and its *fundamental domain* Ω are given by

$$\Gamma = \bigoplus_{j=1}^{2} \mathbb{Z} e_j, \quad \Omega = \left\{ \sum_{j=1}^{2} c_j e_j : -1/2 \le c_j < 1/2 \right\}$$

for some basis $\{e_j\}_{j=1}^2$ of $\mathbb{R}^2$. The basis $\{e_j^*\}_{j=1}^2$ satisfying $e_j \cdot e_{j'}^* = 2\pi \delta_{jj'}$ ($\cdot$ is the Euclidean inner product, $\delta_{jj'}$ is the Kronecker delta) is called the *dual* *basis* of $\{e_j\}_{j=1}^2$. The dual lattice Γ^* of Γ and its fundamental domain Ω^* are given by

$$\Gamma^* = \bigoplus_{j=1}^{2} \mathbb{Z}e_j^*, \quad \Omega^* = \left\{ \sum_{j=1}^{2} c_j e_j^* : -1/2 \le c_j < 1/2 \right\}.$$

A Γ-periodic function is naturally identified with a function on Ω. For $u \in L^2(\Omega)$, the Fourier series of u is given by

$$u(x) = \sum_{n\in\Gamma^*} u_n \mathrm{e}^{in\cdot x}, \quad u_n = \frac{1}{|\Omega|} \int_{\Omega} u(x)\mathrm{e}^{-in\cdot x}\mathrm{d}x.$$

Let Γ be a lattice in $\mathbb{R}^2, \Gamma^*$ the dual lattice of Γ, and Ω and Ω^* are the fundamental domains of Γ and Γ^*, respectively. Let $V \in L^2_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R})$ be a Γ-periodic potential. Define an operator H on $L^2(\mathbb{R}^2)$ by

$$H = -\Delta + V, \quad D(H) = H^2(\mathbb{R}^2),$$

where $H^2(\mathbb{R}^2)$ denotes the usual Sobolev space and $D(\cdot)$ denotes the operator domain. It is well known that H is self-adjoint, semi-bounded from below and decomposed as the constant fiber direct integral

$$H \simeq \int_{\Omega^*}^{\oplus} H_k \mathrm{d}k, \quad H_k = -\Delta_k + \widetilde{V},$$

where $\simeq$ means the unitary equivalence. The operator $-\Delta_k$ $(k \in \Omega^*)$ is a self-adjoint operator on $l^2(\Gamma^*)$ defined by

$$-\Delta_k u(n) = |n+k|^2 u(n) \quad (n \in \Gamma^*),$$
$$D(-\Delta_k) = \left\{ u \in l^2(\Gamma^*) : \sum_{n\in\Gamma^*} |n|^4 |u(n)|^2 < \infty \right\}.$$

The operator $\widetilde{V}$ is defined by

$$\widetilde{V}u(n) = \sum_{m\in\Gamma^*} V_{n-m} u(m),$$

where V_{n-m} is the Fourier coefficient of V given by

$$V_n = \frac{1}{|\Omega|} \int_{\Omega} V(x)\mathrm{e}^{-in\cdot x}\mathrm{d}x.$$

The operator H_k is self-adjoint, lower semi-bounded on $l^2(\Gamma^*)$ with the domain $D(H_k) = D(-\Delta_k)$, and has compact resolvent. We enumerate the eigenvalues of H_k in an ascending order counting multiplicity

$$\lambda_1(k) \le \lambda_2(k) \le \cdots \le \lambda_j(k) \le \cdots \to \infty,$$

and call $\lambda_j(k)$ the *band function*. The band function $\lambda_j(k)$ is continuous on Ω^*, real-analytic in the region $\lambda_j(k) \ne \lambda_{j'}(k)$ for any other j', and can be

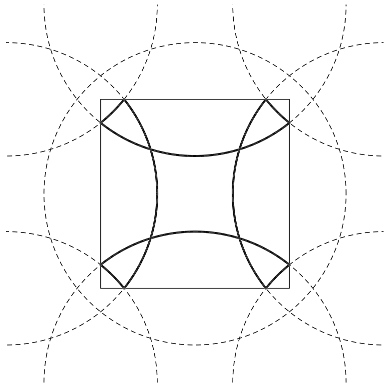

FIGURE 1. The isoenergetic surface S_R in the k-plane

extended as a Γ^*-periodic function. The spectrum $\sigma(H)$ is represented as

$$\sigma(H) = \bigcup_{j=1}^{\infty} I_j, \quad I_j = \bigcup_{k \in \Omega^*} \{\lambda_j(k)\}.$$

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 following the strategy of Karpeshina [4] and clarify the dependency of the threshold on $\|V\|_{L^2(\Omega)}$. The main ingredient of the proof consists of the geometrical consideration of the band function of the free Laplacian $-\Delta$, especially its isoenergetic surface.

We quote some terminology from Karpeshina's book [4] in a slightly modified form. First, we identify the product set $\Gamma^* \times \Omega^*$ with $\mathbb{R}^2$ by the one-to-one correspondence

$$\Gamma^* \times \Omega^* \ni (n, k) \mapsto \xi = n + k \in \mathbb{R}^2. \tag{6}$$

Define the *isoenergetic surface*

$$S_R = \{(n, k) \in \Gamma^* \times \Omega^* : |n + k| = R\}.$$

S_R is identified with the circle $S_R = \{\xi \in \mathbb{R}^2 : |\xi| = R\}$ via (6) (we use the same symbol for the subset of $\Gamma^* \times \Omega^*$ and the subset of $\mathbb{R}^2$, by abuse of notation). The k-plane projection of S_R and the ξ-plane image of S_R are shown in Figs. 1 and 2. Via (6), S_R becomes a measure space with the length measure $\mathrm{d}l = R\mathrm{d}\theta$, where θ is the angular coordinate on S_R given by $\xi = (R\cos\theta, R\sin\theta)$. We denote the length of a measurable subset S of S_R by $l(S)$.

For $\mu > 0$, define the *μ-diffraction set* $T_{R,\mu}$ by

$$T_{R,\mu} = \left\{(n, k) \in S_R : \left||n + k|^2 - |n' + k|^2\right| < \mu \text{ for some } n' \neq n\right\}.$$

Then the k-plane projection of $T_{R,\mu}$ is a neighborhood of the intersection points of the k-plane projection of S_R (see Fig. 1). Define the *μ-non diffraction set*

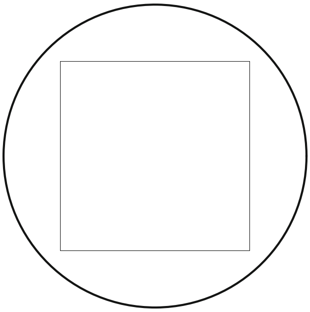

FIGURE 2. The isoenergetic surface S_R in the ξ-plane

$S_{R,\mu}$ by

$$S_{R,\mu} = S_R \backslash T_{R,\mu}.$$

For a given lattice Γ, we denote

$$L = \max_{k\in\partial\Omega^*} |k|, \quad L_0 = \min_{k\in\partial\Omega^*} |k|, \tag{7}$$

where $\partial\Omega^*$ is the boundary of the fundamental domain Ω^* of the dual lattice Γ^*. The values L and L_0 are the outside diameter of Ω^* and the inside diameter, respectively.

We show the following two lemmas in the next section:

Lemma 3.1. *For a lattice Γ in $\mathbb{R}^2$, define L and L_0 by (7). Then, there exists a constant $C_0 > 0$ dependent only on L, L_0 and $|\Omega^*|$ such that*

$$l(T_{R,\mu}) \leq C_0\sqrt{\mu}R \tag{8}$$

for every R, μ with $R \geq L$ and $0 < \mu \leq \mu_0$, where $\mu_0 = \min(L_0^2, 1)$.

Put

$$\mu_1 = \min(\mu_0, (2\pi/C_0)^2) \tag{9}$$

and assume $0 < \mu < \mu_1$. Lemma 3.1 implies

$$l(S_{R,\mu}) \geq (2\pi - C_0\sqrt{\mu})R > 0 \tag{10}$$

for every $R \geq L$, so $S_{R,\mu}$ is a non-empty set. For $(n,k) \in S_{R,\mu}$, $R^2 = |n+k|^2$ is a simple eigenvalue of $-\Delta_k$ and separated from other eigenvalues of $-\Delta_k$ at least by the distance μ. Thus, the circle

$$C_{R,\mu} = \{z \in \mathbb{C} : |z - R^2| = \mu/2\}$$

is contained in the resolvent set $\rho(-\Delta_k)$.

Lemma 3.2. *Let Γ be a lattice, L and L_0 given by (7), μ_1 given by (9), and μ satisfying $0 < \mu < \mu_1$. Then, for any $R \geq 2L$ and any $M > 0$, there exists a subset $S_{R,\mu,M}$ of $S_{R,\mu}$ such that the following* (i), (ii) *hold:*

(i) *Let* $N_{R,\mu,M}$ *be the* $\mu/(8R)$*-neighborhood of* $S_{R,\mu,M}$*, that is, the set of all* $(m,j) \in \Gamma^* \times \Omega^*$ *such that*

$$|(n+k)-(m+j)| \leq \frac{\mu}{8R} \tag{11}$$

for some $(n,k) \in S_{R,\mu,M}$*. Then, for every* $(m,j) \in N_{R,\mu,M}$*, the circle* $C_{R,\mu}$ *is contained in the resolvent set* $\rho(-\Delta_j)$*, the value* $|m+j|^2$ *is a simple eigenvalue of* $-\Delta_j$ *and the only point of* $\sigma(-\Delta_j)$ *inside* $C_{R,\mu}$*. Moreover, the estimates*

$$\|(-\Delta_j - z)^{-1}\| \leq \frac{7}{\mu}, \tag{12}$$

$$\|\widetilde{V}(-\Delta_j - z)^{-1}\|_{\mathrm{HS}} \leq M\|V\|_{L^2(\Omega)} \tag{13}$$

hold for every $(m,j) \in N_{R,\mu,M}$*, every* $z \in C_{R,\mu}$ *and every real-valued* Γ*-periodic function* $V \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$*, where* $\|\cdot\|$ *denotes the operator norm and* $\|\cdot\|_{\mathrm{HS}}$ *the Hilbert-Schmidt norm.*

(ii) *The length of* $S_{R,\mu,M}$ *is estimated below as*

$$l(S_{R,\mu,M}) \geq l(S_{R,\mu}) - \frac{C_1}{M^2}R, \tag{14}$$

where C_1 *is a positive constant dependent only on* Γ *and* μ*.*

Now we assume Lemma 3.1 and 3.2 hold and prove Theorem 1.1. The proof is similar to that of [4, section 2, 3], but we study the dependency on $\|V\|_{L^2(\Omega)}$ carefully.

Proof of Theorem 1.1. First we show that it is sufficient to show there exist constants $R_0 > 0$ and $\epsilon_0 > 0$ such that

$$\sigma(-\Delta + V) \supset [R_0, \infty) \tag{15}$$

for every Γ-periodic V with $\|V\|_{L^2(\Omega)} \leq \epsilon_0$. Suppose such (R_0, ϵ_0) exists and take Γ-periodic V with $\|V\|_{L^2(\Omega)} \leq \epsilon_0$. Then, there exists a positive integer K such that

$$\frac{\epsilon_0}{(2K)^2} < \|V\|_{L^2(\Omega)} \leq \frac{\epsilon_0}{K^2}.$$

We have by the scaling $x = Ky$

$$-\Delta_x + V(x) = \frac{1}{K^2}\left(-\Delta_y + W(y)\right), \quad W(y) = K^2 V(Ky).$$

The potential W is Γ-periodic and we have by the Γ-periodicity of V

$$\|W\|_{L^2(\Omega)} = K^2\|V\|_{L^2(\Omega)} \leq \epsilon_0.$$

Thus, we can apply (15) for the potential W and conclude

$$\sigma(-\Delta + V) \supset [R_0/K^2, \infty) \supset [(4R_0/\epsilon_0)\|V\|_{L^2(\Omega)}, \infty).$$

Thus, the conclusion of Theorem 1.1 holds with $\epsilon = \epsilon_0$ and $c = 4R_0/\epsilon_0$.

Next, we may assume the zeroth Fourier coefficient $V_0 = 0$. In fact, if (15) is proved for such case, then for any Γ-periodic V with $\|V\|_{L^2(\Omega)} \leq \min(\epsilon, \sqrt{|\Omega|}R_0)$ we have $|V_0| \leq R_0$ and

$$\sigma(-\Delta + V) \supset [R_0 + V_0, \infty) \supset [2R_0, \infty).$$

Let us find ϵ_0, R_0 satisfying the above conditions under the assumption $V_0 = 0$. Take μ with $0 < \mu < \mu_1$, and take M such that $2\pi - C_0\sqrt{\mu} - C_1/M^2 > 0$. By (10) and (14), $S_{R,\mu,M}$ is not empty for every $R \geq 2L$. Let $N_{R,\mu,M}$ as in Lemma 3.2. Take a Γ-periodic real-valued function V satisfying

$$\|V\|_{L^2(\Omega)} \leq \epsilon_1, \quad \epsilon_1 = \frac{1}{2M}. \tag{16}$$

For $(m, j) \in N_{R,\mu,M}$, put

$$A = A(j, z) = -\widetilde{V}(-\Delta_j - z)^{-1};$$

then

$$\|A\| \leq \|A\|_{\mathrm{HS}} \leq 1/2 \tag{17}$$

by (13) and (16). Take a real parameter α with $|\alpha| \leq 1$. For $z \in C_{R,\mu}$, consider the resolvent expansion

$$\begin{aligned}\left(-\Delta_j + \alpha\widetilde{V} - z\right)^{-1} &= ((I - \alpha A)(-\Delta_j - z))^{-1}\\ &= (-\Delta_j - z)^{-1}\sum_{p=0}^{\infty}\alpha^p A^p.\end{aligned}$$

By (12) and (17), the sum converges in the norm topology uniformly with respect to $z \in C_{R,\mu}, (m, j) \in N_{R,\mu,M}$ and $|\alpha| \leq 1$, and also continuous with respect to all the parameters.[3] This implies $C_{R,\mu} \subset \rho(-\Delta_j - \alpha\widetilde{V})$ for $(m, j) \in N_{R,\mu,M}$ and $|\alpha| \leq 1$. Then, the spectral projection $P_{j,\alpha}$ of the self-adjoint operator $-\Delta_j - \alpha\widetilde{V}$ corresponding to the spectrum inside $C_{R,\mu}$ is given by

$$\begin{aligned}P_{j,\alpha} &= -\frac{1}{2\pi i}\int_{C_{R,\mu}}(-\Delta_j + \alpha\widetilde{V} - z)^{-1}\mathrm{d}z\\ &= P_{j,0} - \frac{1}{2\pi i}\sum_{p=1}^{\infty}\alpha^p\int_{C_{R,\mu}}(-\Delta_j - z)^{-1}A(j, z)^p\mathrm{d}z.\end{aligned} \tag{18}$$

Let us show the right-hand side of (18) belongs to the trace class. Put

$$\begin{aligned}A_0(j, z) &= (I - P_{j,0})A(j, z)(I - P_{j,0})\\ &= -(I - P_{j,0})\widetilde{V}(-\Delta_j - z)^{-1}(I - P_{j,0}).\end{aligned}$$

Since $(I - P_{j,0})(-\Delta_j - z)^{-1}$ has no singularity inside $C_{R,\mu}$, we have

$$\int_{C_{R,\mu}}(-\Delta - z)^{-1}A_0(j, z)^p\mathrm{d}z = 0$$

[3] The topology of the set $N_{R,\mu,M}$ is given via the correspondence (6), so the continuity with respect to (m, j) means the continuity with respect to $\eta = m + j$.

for $p = 1, 2, \ldots$. By (18), we obtain

$$\begin{aligned} P_{j,\alpha} &= P_{j,0} - Q_{j,\alpha}, \\ Q_{j,\alpha} &= \frac{1}{2\pi i}\sum_{p=1}^{\infty}\alpha^p \int\limits_{C_{R,\mu}} (-\Delta_j - z)^{-1}(A^p - A_0^p)\mathrm{d}z. \end{aligned} \tag{19}$$

The (a, b)-component of the matrix $P_{j,0}\widetilde{V}P_{j,0}$ is $\delta_{aj}\delta_{bj}V_{j-j} = 0$, since $V_0 = 0$. Thus, $P_{j,0}\widetilde{V}P_{j,0} = 0$ and we have

$$A - A_0 = (I - P_{j,0})\widetilde{V}P_{j,0} + P_{j,0}\widetilde{V}(I - P_{j,0}). \tag{20}$$

Moreover, since the trace norm of the one-rank operator $Xu = \lambda(\psi, u)\phi$ ($\|\psi\| = \|\phi\| = 1$) is $|\lambda|$, we have

$$\|(I - P_{j,0})\widetilde{V}P_{j,0}\|_{\mathrm{tr}}^2 = \|P_{j,0}\widetilde{V}(I - P_{j,0})\|_{\mathrm{tr}}^2 \le \|\widetilde{V}P_{j,0}\|_{\mathrm{tr}}^2 = \|V\|_{L^2(\Omega)}^2/|\Omega|,$$

where $\|\cdot\|_{\mathrm{tr}}$ denotes the trace norm. Then we have by (20)

$$\|A - A_0\|_{\mathrm{tr}} \le 2\|V\|_{L^2(\Omega)}/\sqrt{|\Omega|},$$

and moreover

$$\|A^p - A_0^p\|_{\mathrm{tr}} \le \sum_{q=1}^{p}\|A^{p-q}(A - A_0)A_0^{q-1}\|_{\mathrm{tr}} \le p2^{2-p}\|V\|_{L^2(\Omega)}/\sqrt{|\Omega|}, \tag{21}$$

since $\|A_0\| \le \|A\| \le 1/2$ by (17). By (12), (17), and (21), the series (19) and also $\widetilde{V}$ times (19) uniformly converge in the trace class and

$$\|Q_{j,\alpha}\|_{\mathrm{tr}} \le C_2\|V\|_{L^2(\Omega)}, \tag{22}$$

$$\|\widetilde{V}Q_{j,\alpha}\|_{\mathrm{tr}} \le C_3\|V\|_{L^2(\Omega)}, \tag{23}$$

where C_2 and C_3 are some positive constants dependent only on μ and Γ. Now we assume

$$\|V\|_{L^2(\Omega)} \le \epsilon_2, \quad \epsilon_2 = \min\left(\epsilon_1, 1/(2C_2)\right). \tag{24}$$

Then by (22)

$$\|Q_{j,\alpha}\|_{\mathrm{tr}} \le \frac{1}{2}, \quad \frac{1}{2} \le \mathrm{tr}\, P_{j,\alpha} \le \frac{3}{2},$$

since $\mathrm{tr}\, P_{j,0} = 1$. Thus, $P_{j,\alpha}$ must be a one-rank projection operator, and $-\Delta_j + \alpha\widetilde{V}$ has unique eigenvalue $\mu_{j,\alpha}$ inside $C_{R,\mu}$. Then we can apply the analytic perturbation theory for simple eigenvalues and find normalized eigenfunctions $v_{j,\alpha}$ ($|\alpha| \le 1$) of $-\Delta_j + \alpha\widetilde{V}$ for the eigenvalue $\mu_{j,\alpha}$. The function $\alpha \mapsto \mu_{j,\alpha}$ is differentiable with respect to α for $|\alpha| < 1$ and continuous for $|\alpha| \le 1$. Then, by the Feynman–Hellmann theorem

$$\begin{aligned} \frac{\partial\mu_{j,\alpha}}{\partial\alpha} &= (\widetilde{V}v_{j,\alpha}, v_{j,\alpha})_{l^2(\Gamma^*)} = \mathrm{tr}\, P_{j,\alpha}\widetilde{V}P_{j,\alpha} = \mathrm{tr}\,\widetilde{V}P_{j,\alpha} \\ &= -\mathrm{tr}\left(\widetilde{V}Q_{j,\alpha}\right), \end{aligned} \tag{25}$$

since $\operatorname{tr} \widetilde{V} P_{j,0} = V_{j-j} = 0$. From (23) and (25), we have

$$|\mu_{j,0} - \mu_{j,\alpha}| \leq |\alpha| C_3 \|V\|_{L^2(\Omega)}$$

for every $|\alpha| \leq 1$. Since $\mu_{j,0} = |m+j|^2$, we conclude

$$\big||m+j|^2 - \mu_{j,1}\big| \leq C_3 \|V\|_{L^2(\Omega)}$$

for $(m,j) \in N_{R,\mu,M}$ and Γ-periodic V satisfying (24).

Put $\lambda(\eta) = \mu_{j,1}$ for $\eta = m+j$. Then $\lambda(\eta)$ is a continuous function defined on

$$D = \left\{\eta \in \mathbb{R}^2 : |\eta - \xi| \leq \frac{\eta}{8R}\right\}$$

for some $\xi = n+k, (n,k) \in S_{R,\mu,M}$,

$$\big||\eta|^2 - \lambda(\eta)\big| \leq C_3 \|V\|_{L^2(\Omega)}, \tag{26}$$

and the value $\lambda(\eta)$ coincides with the value of some band function for $-\Delta + V$. Since $|\xi| = R \geq 2L$ and $0 < \mu < \mu_1 < L$, the function $|\eta|^2$ takes the maximum

$$(R + \mu/8R)^2 = R^2 + \frac{\mu}{4} + \frac{\mu^2}{64R^2} > R^2 + \frac{\mu}{4}, \tag{27}$$

and the minimum

$$(R - \mu/8R)^2 = R^2 - \frac{\mu}{4} + \frac{\mu^2}{64R^2} < R^2 - \frac{\mu}{5} \tag{28}$$

at some boundary point of D. Finally, we assume

$$\|V\|_{L^2(\Omega)} \leq \epsilon_0, \quad \epsilon_0 = \min(\epsilon_2, \mu/(6C_3)). \tag{29}$$

The estimates (26), (27), (28) and (29) imply $\sigma(-\Delta + V) \supset [R^2 - \mu/30, R^2 + \mu/12]$ for every $R \geq 2L$. Thus, $\sigma(-\Delta + V) \supset [2L, \infty)$ for any Γ-periodic V satisfying (29). $\square$

4. Geometric Lemmas

We shall prove the lemmas in the previous section. The proof needs a detailed analysis of the geometrical structure of the band functions. In the proof, we use the following simple lemma several times. We omit the easy proof:

Lemma 4.1. *Let S be a measurable set of $\mathbb{R}^2$ such that the Lebesgue measure $|S|$ is positive and finite. Let Γ be a discrete set in $\mathbb{R}^2$ such that $(S+\gamma) \cap (S+\gamma') = \emptyset$ for every $\gamma, \gamma' \in \Gamma$ with $\gamma \neq \gamma'$, where $S + \gamma = \{s + \gamma : s \in S\}$. Let f be a non-negative measurable function on $U = \bigcup_{\gamma \in \Gamma}(S+\gamma)$ such that*

$$f(\gamma) \leq C f(x + \gamma)$$

for every $x \in S$ and $\gamma \in \Gamma$, where C is a positive constant independent of x and γ. Then,

$$\sum_{\gamma \in \Gamma} f(\gamma) \leq \frac{C}{|S|} \int_U f(x) \mathrm{d}x.$$

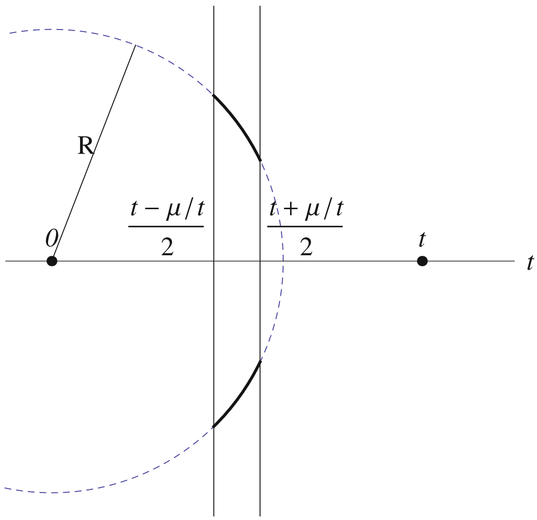

FIGURE 3. The isoenergetic surface S_R and the strip (30) in ξ-plane. The value $l_{R,\mu}(t)$ is the sum of the length of two thick-lined arcs

Proof of Lemma 3.1. Let $(n,k)\in S_R$ and put $\xi = n+k$. By the correspondence (6), the μ-diffraction set $T_{R,\mu}$ is identified with

$$\bigcup_{q\in\Gamma^*\setminus\{0\}} \left\{\xi\in\mathbb{R}^2 : |\xi| = R,\ \left|R^2-|\xi-q|^2\right| < \mu\right\}.$$

Since $|\xi| = R$, we have

$$|R^2-|\xi-q|^2| < \mu \Leftrightarrow \left|\xi\cdot\frac{q}{|q|}-\frac{|q|}{2}\right| < \frac{\mu}{2|q|}. \tag{30}$$

The right-hand side of (30) defines a strip of width $\mu/|q|$, orthogonal to the vector q and including the point $q/2$ in its center. By the rotational symmetry, the length of the intersection of the circle $\{|\xi| = R\}$ and the strip (30) is determined by R,μ and $t=|q|$ (see Fig. 3). We denote the length $l_{R,\mu}(t)$. Clearly,

$$l(T_{R,\mu}) \le \sum_{q\in\Gamma^*\setminus\{0\}} l_{R,\mu}(|q|). \tag{31}$$

We assume

$$0<\sqrt{\mu}\le L_0, \quad R\ge L. \tag{32}$$

Since $q\in\Gamma^*\setminus\{0\}$ satisfies $t=|q|\ge 2L_0>\sqrt{\mu}$, we have $(t-\mu/t)/2>0$. Figure 3 tells us

$$l_{R,\mu}(t) = \begin{cases} 0 & (t_1<t),\\ 2R\arccos g(t) & (t_2\le t\le t_1),\\ 2R(\arccos g(t)-\arccos f(t)) & (\sqrt{\mu}\le t<t_2),\end{cases}$$

where

$$f(t)=\frac{1}{2R}\left(t+\frac{\mu}{t}\right),\quad g(t)=\frac{1}{2R}\left(t-\frac{\mu}{t}\right),$$

and $t_1=R+\sqrt{R^2+\mu}$ and $t_2=R+\sqrt{R^2-\mu}$ are solutions of $g(t)=1$ and $f(t)=1$, respectively.

We divide the sum (31) into two parts,

$$l_1=\sum_{2L_0\le|q|\le t_2-L} l_{R,\mu}(|q|),\quad l_2=\sum_{t_2-L<|q|\le t_1} l_{R,\mu}(|q|).$$

First we consider l_1. For $\sqrt{\mu}\le t\le t_2-L$,

$$l_{R,\mu}(t)=2R\int_{g(t)}^{f(t)}\frac{1}{\sqrt{1-s^2}}\mathrm{d}s\le\frac{2\mu}{t}\cdot\frac{1}{\sqrt{1-f(t)^2}}. \tag{33}$$

By a simple calculation using (32), we can prove there exists $C_1=C_1(L_0,L)>0$ such that

$$\frac{1}{\sqrt{1-f(t)^2}}\le\frac{C_1}{\sqrt{1-(t/(2R))^2}}$$

for $2L_0\le t\le t_2-L$. By (33)

$$l_1\le\sum_{2L_0\le|q|\le t_2-L}\frac{C_1\mu}{|q|\sqrt{1-(|q|/(2R))^2}}.$$

We can also prove that there exists $C_2=C_2(L_0,L)>0$ such that

$$\frac{C_1\mu}{|q|\sqrt{1-(|q|/(2R))^2}}\le\frac{C_2\mu}{|j+q|\sqrt{1-(|j+q|/(2R))^2}}$$

for $2L_0\le|q|\le t_2-L$ and $j\in\Omega^*$. Since $\{\Omega^*+q\}_{2L_0\le|q|\le t_2-L}$ are disjoint sets contained in $\{L_0\le|\eta|\le 2R\}$, Lemma 4.1 implies

$$\begin{aligned}l_1&\le\frac{C_2}{|\Omega^*|}\sum_{2L_0\le|q|\le t_2-L}\int_{\Omega^*+q}\frac{\mu}{|\eta|\sqrt{1-(|\eta|/(2R))^2}}\mathrm{d}\eta\\&\le\frac{2\pi C_2}{|\Omega^*|}\int_0^{2R}\frac{\mu}{\sqrt{1-(t/(2R))^2}}\mathrm{d}t=C_3\mu R,\end{aligned} \tag{34}$$

where $C_3=2\pi^2C_2/|\Omega^*|$.

Next we consider l_2. First,

$$l_2\le\max_{\sqrt{\mu}\le t\le t_1} l_{R,\mu}(t)\cdot\#\{q\in\Gamma^*:t_2-L<|q|\le t_1\}. \tag{35}$$

It is easy to see that there exists $C_4=C_4(L_0,L,|\Omega^*|)>0$ such that

$$\#\{q\in\Gamma^*:t_2-L<|q|\le t_1\}\le C_4R. \tag{36}$$

By differentiation, we can check the maximum value is either $l_{R,\mu}(\sqrt{\mu})$ or $l_{R,\mu}(t_2)$ (see also Fig. 3). By (32) and (33), we have

$$l_{R,\mu}(\sqrt{\mu}) \leq \frac{2\sqrt{\mu}}{1-\mu/R^2} \leq C_5\sqrt{\mu}, \tag{37}$$

where $C_5 = C_5(L_0, L) > 0$. Next, put $\arccos g(t_2) = \theta$. Then

$$\cos\theta = g(t_2) = f(t_2) - \frac{\mu}{Rt_2} = 1 - \frac{\mu}{Rt_2},$$

since $f(t_2) = 1$. Thus, we have

$$\begin{aligned} l_{R,\mu}(t_2) &= 2R\theta \leq 2R\tan\theta = 2R\sqrt{\frac{1}{g(t_2)^2} - 1} \\ &= 2R\sqrt{\frac{1}{(1-\mu/(Rt_2))^2} - 1} \leq C_6\sqrt{\mu}, \end{aligned} \tag{38}$$

where $C_6 = C_6(L_0, L) > 0$. From (35), (36), (37) and (38), we conclude

$$l_2 \leq C_7 R\sqrt{\mu},$$

where $C_7 = \max(C_5, C_6)\cdot C_4$. This inequality and (34) imply (8) holds with $C_0 = C_2 + C_7$, for R and μ satisfying (32) and $\mu \leq 1$. □

In the proof of Lemma 3.2, the following lemma, similar to the Chebyshev inequality in the probability theory, plays a crucial role:

Lemma 4.2. *Let (X, m) be a measure space with the total measure $m(X)$ positive and finite. Let f be a non-negative square-integrable function on X. Then, for every $M > 0$, we have*

$$m(\{x \in X : f(x) > M\}) \leq \frac{1}{M^2}\int_X f(x)^2 dm(x).$$

Proof of Lemma 3.2. In this proof, we denote general positive universal constants (the constants independent of all the parameters) by C for simplicity, so C may change line by line. Take μ and R satisfying

$$0 < \mu < \mu_1, \quad R \geq L, \tag{39}$$

where μ_1 is given in (9). Let $(n, k) \in S_{R,\mu}$ and $(m, j) \in \Gamma^* \times \Omega^*$ satisfying (11). Put $\xi = n + k$ and $\eta = m + j$; then the eigenvalues of $-\Delta_k$ and $-\Delta_j$ are $\{|\xi + q|^2\}_{q\in\Gamma^*}$ and $\{|\eta + q|^2\}_{q\in\Gamma^*}$, respectively. By definition,

$$|\xi| = R, \quad |\xi - \eta| \leq \frac{\mu}{8R}, \tag{40}$$

$$\big||\xi|^2 - |\xi + q|^2\big| \geq \mu \tag{41}$$

for every $q \in \Gamma^*\backslash\{0\}$. Then we easily have by (39) and (40)

$$R^2 - \frac{1}{4}\mu \leq |\eta|^2 \leq R^2 + \frac{17}{64}\mu. \tag{42}$$

Moreover, we have by (39), (40), and (41)

$$\begin{cases} |\eta+q|^2 \ge R^2 + \left(1-\sqrt{2}/4\right)\mu & \text{if } |\xi+q|^2 \ge |\xi|^2+\mu, \\ |\eta+q|^2 \le R^2 - (47/64)\mu & \text{if } |\xi+q|^2 \le |\xi|^2-\mu. \end{cases} \tag{43}$$

The inequalities (42) and (43) show that $|\eta|^2 = |m+j|^2$ is the only, simple eigenvalue of $-\Delta_j$ inside $C_{R,\mu}$, and moreover

$$\inf_{q\in\Gamma^*} \min_{z\in C_{R,\mu}} \left|z - |\eta+q|^2\right| \ge \left(1-\frac{\sqrt{2}}{4}\right)\mu - \frac{\mu}{2} \ge \frac{\mu}{7}. \tag{44}$$

This shows the inequality (12).

Next we estimate the Hilbert-Schmidt norm of $\widetilde{V}(-\Delta_j - z)^{-1}$ for $z \in C_{R,\mu}$. Since the (a,b)-component of the matrix $\widetilde{V}(-\Delta_j - z)^{-1}$ is $V_{a-b}(|b+j|^2 - z)^{-1}$, we have

$$\begin{aligned} \|\widetilde{V}(-\Delta_j - z)^{-1}\|_{\mathrm{HS}}^2 &= \sum_{a,b\in\Gamma^*} |V_{a-b}|^2 \left||b+j|^2 - z\right|^{-2} \\ &= \frac{\|V\|_{L^2(\Omega)}^2}{|\Omega|} \sum_{q\in\Gamma^*} \left||\eta+q|^2 - z\right|^{-2} \end{aligned} \tag{45}$$

by the Plancherel theorem. In the sequel, we denote $R_s = \sqrt{R^2+s}$ for a real-number s. Since $\eta = m+j$ and $|m+j|^2$ is the only eigenvalue in $C_{R,\mu}$, the sum in (45) is estimated as

$$\begin{aligned} \sum_{q\in\Gamma^*} \left||\eta+q|^2 - z\right|^{-2} \le \frac{49}{\mu^2} &+ \sum_{|\xi+q|\ge R_\mu} \left((|\xi+q| - \mu/(8R))^2 - R^2 - \mu/2\right)^{-2} \\ &+ \sum_{|\xi+q|\le R_{-\mu}} \left(R^2 - \mu/2 - (|\xi+q| + \mu/(8R))^2\right)^{-2}, \end{aligned} \tag{46}$$

by (40) and (44). We denote the right-hand side of (46) by $F(\xi,R)$. We also divide two sums in (46) into four parts, (1) $|\xi+q| > R_\mu + L$, (2) $R_\mu + L \ge |\xi+q| \ge R_\mu$, (3) $R_{-\mu} \ge |\xi+q| \ge R_{-\mu} - L$, (4) $R_{-\mu} - L > |\xi+q|$, and denote them $F_1(\xi,R), F_2(\xi,R), F_3(\xi,R)$, and $F_4(\xi,R)$, respectively. By definition,

$$F(\xi,R) = \frac{49}{\mu^2} + \sum_{p=1}^{4} F_p(\xi,R). \tag{47}$$

First estimate $F_1(\xi,R)$. For $|\xi+q| > R_\mu + L$ and $y \in \Omega^*$, we have

$$\left((|\xi+q| - \mu/(8R))^2 - R^2 - \mu/2\right)^{-2} \le C\left((|\xi+q+y|)^2 - R^2 - \mu/2\right)^{-2} \tag{48}$$

by (39). Since $\{\xi + q + \Omega^*\}_{|\xi+q|>R_\mu+L}$ are disjoint sets contained in $\{\xi' \in \mathbb{R}^2 : |\xi'| \geq R_\mu\}$, we have by Lemma 4.1

$$F_1(\xi, R) \leq \frac{C}{|\Omega^*|} \int\limits_{|\xi'|\geq R_\mu} \left(|\xi'|^2 - (R^2 + \mu/2)\right)^{-2} \mathrm{d}\xi' \leq \frac{C}{|\Omega^*|}\mu^{-1}. \tag{49}$$

Similarly, we have

$$F_4(\xi, R) \leq \frac{C}{|\Omega^*|}\mu^{-1}. \tag{50}$$

It is not easy to estimate $F_2(\xi, R)$ and $F_3(\xi, R)$ directly, so we use Lemma 4.2 in the following way: We shall prove there exists some positive constant $C_0 = C_0(L, L_0, |\Omega^*|, \mu)$ such that

$$\int\limits_{|\xi|=R} F(\xi, R) R\mathrm{d}\theta \leq C_0 R \tag{51}$$

for every $R \geq 2L$,[4] where θ is the angular coordinate on $\{|\xi| = R\}$. Suppose (51) is proved. For any $M > 0$, put

$$S_{R,\mu,M} = \{\xi \in S_{R,\mu} : F(\xi, R) \leq M^2|\Omega|\}.$$

Then the conclusion of Lemma 3.2 follows from (45), (46) and Lemma 4.2.

Let us estimate

$$I_2(R) = \int\limits_{\{|\xi|=R\}} F_2(\xi, R) R\mathrm{d}\theta.$$

Using $R_\mu \leq |\xi + q| \leq R_\mu + L$, we can prove

$$\left((|\xi + q| - \mu/(8R))^2 - R^2 - \mu/2\right)^{-2} \leq C\left(|\xi + q|^2 - R^2 - \mu/2\right)^{-2}.$$

Thus,

$$I_2(R) \leq C \sum_{q\in\Gamma^*} I_{R,q}, \tag{52}$$

$$I_{R,q} = \int\limits_{|\xi|=R} \left(|\xi + q|^2 - R^2 - \mu/2\right)^{-2} \chi_{R_\mu \leq |\xi+q| \leq R_\mu+L} R\mathrm{d}\theta,$$

where χ_S denotes the characteristic function of the set S.

Let us write down the integral $I_{R,q}$ explicitly. When ξ moves on the circle $\{|\xi'| = R\}, \xi + q$ moves on the circle $\{|\xi' - q| = R\}$. So $I_{R,q}$ depends on how the circle $\{|\xi' - q| = R\}$ intersects the annulus $\{R_\mu \leq \xi' \leq R_\mu + L\}$ (see Fig. 4). By the rotational symmetry, $I_{R,q}$ depends only on R, μ and $t = |q|$. Put $t_1 = R_\mu + R$ and $t_2 = R_\mu + L - R$. Then, clearly

$$I_{R,q} = 0 \quad (t > t_1 + L \text{ or } t = 0).$$

[4] Remember we assume (39). The assumption $R \geq 2L$ is used only for $I_3(R)$.

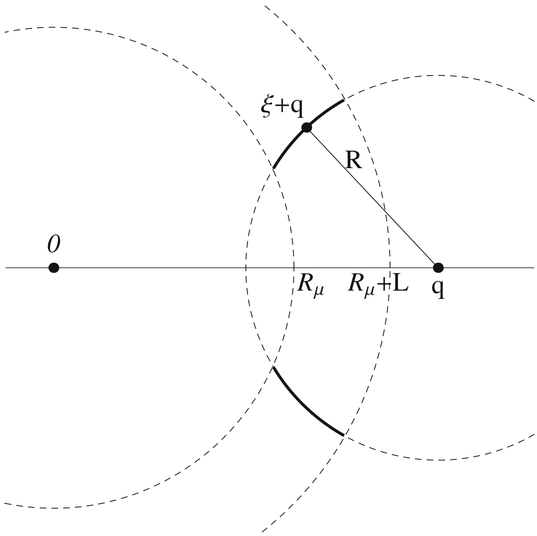

FIGURE 4. The orbit of $\xi + q$ as ξ moves on S_R

In other cases, we set $q = (t, 0)$ and calculate the integral $I_{R,q}$ by the change of variable from θ to $\rho = |\xi + q|$. Then $\rho^2 = R^2 + 2Rt\cos\theta + t^2$ and

$$I_{R,q} = \begin{cases} \int_{t-R}^{R_\mu+L} g(\rho)\mathrm{d}\rho & (t_1 + L \geq t > t_1), \\ \int_{R_\mu}^{R_\mu+L} g(\rho)\mathrm{d}\rho & (t_1 \geq t \geq t_2), \\ \int_{R_\mu}^{t+R} g(\rho)\mathrm{d}\rho & (t_2 > t > 0), \end{cases}$$

where

$$g(\rho) = \frac{4R\rho}{(\rho + R_{\mu/2})^2(\rho - R_{\mu/2})^2} \cdot \frac{1}{\sqrt{(t+R+\rho)(t+R-\rho)(\rho-t+R)(\rho+t-R)}}.$$

Since $R_\mu \leq \rho \leq R_\mu + L$ in all cases and $\sqrt{\mu} < L \leq R \leq R_\mu \leq \sqrt{2}R$, we have

$$\frac{4R\rho}{(\rho + R_{\mu/2})^2\sqrt{t+R+\rho}} < \frac{2}{\sqrt{R}}.$$

Thus, we have

$$g(\rho) < \frac{2}{\sqrt{R}} \cdot \frac{1}{(\rho - R_{\mu/2})^2\sqrt{(t+R-\rho)(\rho-t+R)(\rho+t-R)}}. \tag{53}$$

We shall divide the sum (52) into six parts, (i) $t_1 + L \geq |q| \geq t_1$, (ii) $t_1 > |q| \geq t_1 - L/2$, (iii) $t_1 - L/2 > |q| \geq R_\mu + L/2$, (iv) $R_\mu + L/2 > |q| \geq t_2 + L/2$, (v) $t_2 + L/2 > |q| > t_2$, (vi) $t_2 \geq |q| \geq 2L_0$, and estimate each sum. In fact, the parts (i) and (ii) are treated similarly, so are (iii) and (iv), and so are (v) and (vi). Thus, we consider only the part (i), (iii), and (vi) for brevity.

First consider the part (i) $t_1 + L \geq |q| \geq t_1$. In this case $t + R - \rho \geq R, \rho + t - R \geq 2R$, so by (53) and the change of variable $s = \sqrt{\rho - t + R}$,

$$
\begin{aligned}
I_{R,q} &\leq \frac{\sqrt{2}}{R^{3/2}} \int_{t-R}^{R_\mu + L} \frac{1}{(\rho - R_{\mu/2})^2 \sqrt{\rho - t + R}} \mathrm{d}\rho \\
&\leq \frac{4\sqrt{2}}{R^{3/2}} \int_0^\infty \frac{1}{(s^2 + R_\mu - R_{\mu/2})^2} \mathrm{d}s \\
&= \frac{\sqrt{2}\pi}{R^{3/2}(R_\mu - R_{\mu/2})^{3/2}}.
\end{aligned} \tag{54}
$$

Since

$$
\frac{1}{R_\mu - R_{\mu/2}} = \frac{R_\mu + R_{\mu/2}}{\mu/2} \leq \frac{4\sqrt{2}R}{\mu}, \tag{55}
$$

we have from (54) and (55)

$$
I_{R,q} \leq \frac{C}{\mu^{3/2}} \quad (t_1 \leq |q| \leq t_1 + L).
$$

It is easy to verify

$$
\#\{q \in \Gamma^* : t_1 \leq |q| \leq t_1 + L\} \leq CLR/|\Omega^*|.
$$

Thus, we have

$$
\sum_{t_1 \leq |q| \leq t_1 + L} I_{R,q} \leq \frac{CL}{|\Omega^*|} \cdot \frac{R}{\mu^{3/2}}. \tag{56}
$$

Next consider the part (iii) $t_1 - L/2 > |q| \geq R_\mu + L/2$. In this case, we use the estimates $t + R - \rho \geq R/2, \rho + t - R \geq R$, and $\rho - t + R \geq t_1 - t$. Thus, we have from (53)

$$
I_{R,q} \leq \frac{C}{R^{3/2}} \frac{1}{\sqrt{t_1 - t}} \int_{R_\mu}^{R_\mu + L} \frac{1}{(\rho - R_{\mu/2})^2} \mathrm{d}\rho. \tag{57}
$$

The integral is estimated as

$$
\int_{R_\mu}^{R_\mu + L} \frac{1}{(\rho - R_{\mu/2})^2} \mathrm{d}\rho \leq \frac{1}{R_\mu - R_{\mu/2}} \leq \frac{4\sqrt{2}R}{\mu} \tag{58}
$$

by (55). Thus, we have

$$
I_{R,q} \leq \frac{C}{\mu} \cdot \frac{1}{R^{1/2}\sqrt{t_1 - |q|}} \quad (R_\mu + L/2 \leq |q| < t_1 - L/2).
$$

Let us apply Lemma 4.1. It is easy to see

$$
\frac{1}{t_1 - |q|} \leq \frac{C}{t_1 - |q + y|}
$$

for $y \in (1/2)\Omega^*$. Since $q + (1/2)\Omega^* \subset \{\xi' : R_\mu \le |\xi'| \le t_1\}$, we have

$$\sum_{R_\mu + L/2 \le |q| < t_1 - L/2} I_{R,q} \le \frac{C}{\mu R^{1/2}|\Omega^*|} \int_{R_\mu \le |\xi'| \le t_1} \frac{1}{\sqrt{t_1 - |\xi'|}} \mathrm{d}\xi'$$

$$\le \frac{C}{\mu R^{1/2}|\Omega^*|} \int_{R_\mu}^{t_1} \frac{t}{\sqrt{t_1 - t}} \mathrm{d}t$$

$$\le \frac{C}{|\Omega^*|} \cdot \frac{R}{\mu}. \tag{59}$$

Last, we consider the part (vi) $t_2 \ge |q| \ge 2L_0$. Then $\rho + t - R \ge 2L_0, \rho - t + R \ge R$ and

$$I_{R,q} \le \frac{C}{\sqrt{L_0} R} \int_{R_\mu}^{t+R} \frac{1}{\sqrt{t + R - \rho}(\rho - R_{\mu/2})^2} \mathrm{d}\rho. \tag{60}$$

We have the interval of the integral (60) into $R_\mu \le \rho \le (R_\mu + t + R)/2$ and $(R_\mu + t + R)/2 \le \rho \le t + R$. For $R_\mu \le \rho \le (R_\mu + t + R)/2$, we use

$$R_\mu - R = \frac{\mu}{R_\mu + R} \le \frac{L_0^2}{2R} \le \frac{L}{2}$$

and we have

$$t + R - \rho \ge \frac{t - (R_\mu - R)}{2} \ge \frac{3}{4} L_0. \tag{61}$$

So by (58)

$$\int_{R_\mu}^{(R_\mu + t + R)/2} g(\rho) \mathrm{d}\rho \le \frac{C}{L_0 R} \int_{R_\mu}^{(R_\mu + t + R)/2} \frac{1}{(\rho - R_{\mu/2})^2} \mathrm{d}\rho$$

$$\le \frac{C}{L_0 \mu}.$$

For $(R_\mu + t + R)/2 \le \rho \le t + R$, we have $\rho - R_{\mu/2} \ge (3/4)L_0$ by (61) and

$$\int_{(R_\mu + t + R)/2}^{t+R} g(\rho) \mathrm{d}\rho \le \frac{C}{L_0^{5/2} R} \int_{(R_\mu + t + R)/2}^{t+R} \frac{1}{\sqrt{t + R - \rho}} \mathrm{d}\rho$$

$$= \frac{C}{L_0^{5/2} R} \cdot \sqrt{\frac{t - (R_\mu - R)}{2}}$$

$$\le \frac{C\sqrt{L}}{L_0^{5/2} R} \le \frac{C}{L_0^3}.$$

Thus, we have

$$I_{R,q} \le \frac{C}{L_0} \left(\frac{1}{\mu} + \frac{1}{L_0^2} \right) \quad (t_2 \ge |q| \ge 2L_0).$$

Since

$$\#\{q \in \Omega^* : t_2 \geq |q| \geq 2L_0\} \leq C\frac{L^2}{|\Omega^*|},$$

we have

$$\sum_{t_2 \geq |q| \geq 2L_0} I_{R,q} \leq \frac{CL^2}{L_0|\Omega^*|}\left(\frac{1}{\mu} + \frac{1}{L_0^2}\right). \tag{62}$$

Summing up the inequalities (56), (59), (62), and the similar inequalities for the rest parts, we conclude

$$I_2(R) \leq C_2 R \tag{63}$$

for $R \geq L$, where $C_2 = C_2(L, L_0, |\Omega^*|, \mu)$ is some positive constant.

We can similarly prove

$$I_3(R) = \int_{|\xi|=R} F_3(\xi, R)R\mathrm{d}\theta \leq C_3 R \tag{64}$$

for $R \geq 2L$, for some positive constant $C_3 = C_3(L, L_0, |\Omega^*|, \mu)$ (we assume $R \geq 2L$ to avoid $R_{-\mu} - L$ being negative). And then (47), (49), (50), (63) and (64) imply the desired inequality (51). □

5. Proof of Theorem 1.2

In the last section, we consider the random Schrödinger operators H_ω of the Poisson type introduced in Sect. 1.2, and prove Theorem 1.2. First we prove the family $\mathcal{A}_P$ introduced in (5) is actually the "admissible potentials".

Lemma 5.1. *Let $\mathcal{A}_P$ as in (5). Put*

$$\Sigma = \overline{\bigcup_{W \in \mathcal{A}_P} \sigma(-\Delta + W)}. \tag{65}$$

Then, $\sigma(H_\omega) = \Sigma$ almost surely.

Proof. By [1, Theorem2.1],

$$\sigma(H_\omega) = \overline{\bigcup_{W_F \in \mathcal{A}_F} \sigma(-\Delta + W_F)} \tag{66}$$

almost surely, where $\mathcal{A}_F$ is given in (4). We show the right-hand side of (65) and that of (66) coincide. For a given $W_F \in \mathcal{A}_F$, we define $W_{P,k}(x) = \sum_{n\in\mathbb{Z}^d} W_F(x - kn)(k = 1, 2, \ldots)$. Clearly, $W_{P,k} \in \mathcal{A}_P$. Notice that $C_0^\infty(\mathbb{R}^d)$ is a common operator core for $-\Delta + W_{P,k}$ and $-\Delta + W_F$. By (A1) and [12, Theorem VIII.25], $-\Delta + W_{P,k}$ converges to $-\Delta + W_F$ in the strong resolvent sense. For given $\lambda \in \sigma(-\Delta + W_F)$, we can find a sequence $\lambda_k \in \sigma(-\Delta + W_{P,k})$ such that $\lambda_k \to \lambda$, by [12, Theorem VIII.24]. Thus, $\sigma(-\Delta + W_F) \subset \overline{\bigcup_{W_P \in \mathcal{A}_P} \sigma(-\Delta + W_P)}$. Conversely, we can approximate any $W_P \in \mathcal{A}_P$ by the functions $W_{F,k} \in \mathcal{A}_F$ and obtain $\sigma(-\Delta + W_P) \subset \overline{\bigcup_{W_F \in \mathcal{A}_F} \sigma(-\Delta + W_F)}$. □

Proof of Theorem 1.2. In the case $f_- = 0$ or $f_+ \neq 0$, the statement follows from [1, Theorem 1.2]. We assume $f_+ = 0$ and $f_- \neq 0$, and prove $\Sigma = \mathbb{R}$. Let $\hat{f}$ be the Fourier transform of f, that is,

$$\hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(x) \mathrm{e}^{-ix\xi} \mathrm{d}x.$$

Notice that (A1) implies $f \in L^1(\mathbb{R}^2)$, so $\hat{f}$ is a bounded, continuous function on $\mathbb{R}^2$ and $\hat{f}(0)$ is well defined. Moreover, $f_+ = 0$ and $f_- \neq 0$ imply

$$\hat{f}(0) < 0. \tag{67}$$

Let Γ be an arbitrary lattice and Γ^* its dual. For a positive integer K, put

$$V_K(x) = \sum_{\gamma \in \Gamma} f\left(x - \frac{\gamma}{K}\right). \tag{68}$$

By (A1), the sum (68) converges, $V_K \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$, and $V_K \in \mathcal{A}_P$. The period lattice of V_K is $(1/K)\Gamma$, so its Fourier series is given by

$$V_K(x) = \sum_{n \in \Gamma^*} b_n \mathrm{e}^{iKn \cdot x}, \quad b_n = \frac{K^2}{|\Omega|} \int_{(1/K)\Omega} V_K(x) \mathrm{e}^{-iKn \cdot x} \mathrm{d}x,$$

where Ω is the fundamental domain of Γ. Since the function $\mathrm{e}^{-iKn \cdot x}$ is $(1/K)\Gamma$-periodic with respect to x, we have

$$\begin{aligned} b_n &= \frac{K^2}{|\Omega|} \sum_{\gamma \in \Gamma} \int_{(1/K)\Omega} f\left(x - \frac{\gamma}{K}\right) \mathrm{e}^{-iKn \cdot x} \mathrm{d}x \\ &= \frac{K^2}{|\Omega|} \int_{\mathbb{R}^2} f(x) \mathrm{e}^{-iKn \cdot x} \mathrm{d}x \\ &= \frac{2\pi K^2}{|\Omega|} \hat{f}(Kn). \end{aligned}$$

Thus, we have

$$V_K(x) = \frac{2\pi K^2}{|\Omega|} \sum_{n \in \Gamma^*} \hat{f}(Kn) \mathrm{e}^{iKn \cdot x}.$$

Put

$$\begin{aligned} R_0 &= -\frac{2\pi}{|\Omega|} \hat{f}(0), \\ W_K(x) &= \frac{2\pi K^2}{|\Omega|} \sum_{n \in \Gamma^* \setminus \{0\}} \hat{f}(Kn) \mathrm{e}^{iKn \cdot x}. \end{aligned}$$

Then $R_0 > 0$ by (67) and we have

$$V_K = -R_0 K^2 + W_K.$$

Since the functions $\{e^{iKn\cdot x}\}_{n\in\Gamma^*}$ are orthogonal in $L^2(\frac{1}{K}\Omega)$, we have

$$\|W_K\|^2_{L^2(\frac{1}{K}\Omega)} = \frac{(2\pi)^2}{|\Omega|}K^2 \sum_{n\in\Gamma^*\setminus\{0\}} |\hat{f}(Kn)|^2. \tag{69}$$

Since $W_1 \in L^2(\Omega)$ by (A1), we have $\sum_{n\in\Gamma^*} |\hat{f}(n)|^2 < \infty$. Then (69) implies

$$\frac{1}{K^2}\|W_K\|^2_{L^2(\frac{1}{K}\Omega)} \to 0 \quad \text{as } K \to \infty. \tag{70}$$

Take a positive number R with $R < R_0$. By Theorem 1.1, we can take a small number ϵ such that

$$\sigma(-\Delta + V) \supset [R, \infty) \tag{71}$$

for every Γ-periodic potential V with $\|V\|_{L^2(\Omega)} < \epsilon$. By the scaling $x = y/K$, we have

$$-\Delta_x + W_K(x) = -K^2\Delta_y + W_K(y/K) = K^2(-\Delta_y + \widetilde{W}_K(y)), \tag{72}$$

where $\widetilde{W}_K(y) = W_K(y/K)/K^2$. The potential $\widetilde{W}_K(y)$ is Γ-periodic and by (70)

$$\|\widetilde{W}_K\|^2_{L^2(\Omega)} = \frac{1}{K^2}\|W_K\|^2_{L^2(\frac{1}{K}\Omega)} \to 0.$$

By (71) and (72), we see that for sufficiently large K

$$\sigma(-\Delta + W_K) \supset [K^2R, \infty).$$

Thus, we have

$$\Sigma \supset \sigma(-\Delta + V_K) \supset [(-R_0 + R)K^2, \infty)$$

for sufficiently large K, so $\Sigma = \mathbb{R}$. □

Remark. If $\hat{f}$ satisfies

$$|\hat{f}(n)| \le C|n|^{-\delta}$$

for some $C > 0, \delta > 2$ and every $n \in \Gamma^*\setminus\{0\}$, the proof can be done without Theorem 1.1. In this case $\|W_K\|_\infty \to 0$, and $\sigma(-\Delta+V_K)\cap[-R_0K^2, \infty)$ cannot have an open gap of width larger than $2\|W_K\|_\infty$. Thus, there is no spectral gap on the real line.

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References

[1] Ando, K., Iwatsuka, A., Kaminaga, M., Nakano, F.: The spectrum of Schrödinger Operators with Poisson type random potential. Ann. Henri Poincaré **7**, 145–160 (2006)

[2] Carmona, R., Lacroix, J.: Spectral Theory of Random Schrödinger Operators. Birkhäuser, Boston (1990)

[3] Helffer, B., Mohamed, A.: Asymptotics of the density of states for the Schrödinger operator with periodic electric potential. Duke Math. J. **92**, 1–60 (1998)

[4] Karpeshina, Y.E.: Perturbation Theory for the Schrödinger Operator with a Periodic Potential. Lecture Notes in Mathematics, vol. 1663. Springer, Berlin (1997)

[5] Kirsch, W.: Random Schrödinger operators. A course. In: Lecture Notes in Physics, vol. 345, pp. 264–370 (1989)

[6] Kirsch, W., Martinelli, F.: On the spectrum of Schrödinger operators with a random potential. Commun. Math. Phys. **85**, 329–350 (1982)

[7] Klopp, F.: A low concentration asymptotic expansion for the density of states of a random Schrödinger operator with Poisson disorder. J. Funct. Anal. **145**, 267–295 (1995)

[8] Klopp, F., Pastur, L.: Lifshitz tails for random Schrödinger operators with negative singular Poisson potential. Commun. Math. Phys. **206**, 57–103 (1999)

[9] Parnovski, L.: Bethe–Sommerfeld conjecture. Ann. Henri Poincaré **9**, 457–508 (2008)

[10] Pastur, L., Figotin, A.: Spectra of Random and Almost Periodic Operators. Springer, Berlin (1992)

[11] Popov, V.N., Skriganov, M.M.: Remark on the spectrum of the two-dimensional Schrödinger operator with the periodic potential. Zap. Nauchn. Sem. LOMI AN SSSR **109**, 131–133 (1981, Russian)

[12] Reed, M., Simon, B.: Methods of Modern Mathematical Physics I. Functional Analysis. Academic Press, New York (1980)

[13] Reed, M., Simon, B.: Methods of Modern Mathematical Physics IV. Analysis of Operators. Academic Press, New York (1978)

[14] Skriganov, M.M.: On the Bethe–Sommerfeld conjecture. Soviet Math. Dokl. **20**(1), 89–90 (1979)

[15] Skriganov, M.M.: Proof of the Bethe–Sommerfeld conjecture in dimension two. Soviet Math. Dokl. **20**(5), 956–959 (1979)

[16] Skriganov, M.M.: Geometrical and arithmetical methods in the spectral theory of the multi-dimensional periodic operators. Proc. Steklov Math. Inst. **171** (1984)

[17] Skriganov, M.M.: The spectrum band structure of the three-dimensional Schrödinger operator with periodic potential. Invent. Math. **80**, 107–121 (1985)

[18] Veliev, O. A.: On the spectrum of multidimensional periodic operators, theory of Functions, functional analysis and their applications. Kharkov University **49**, 17–34 (1988, in Russian)

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