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High Frequency Resolvent Estimates for Perturbations by Large Long-range Magnetic Potentials and Applications to Dispersive Estimates

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Abstract. We prove optimal high-frequency resolvent estimates for self-adjoint operators of the form $G = (i\nabla + b(x))^2 + V(x)$ on $L^2(\mathbf{R}^n), n \geq 3$, where the magnetic potential $b(x)$ and the electric potential $V(x)$ are long-range and large. As an application, we prove dispersive estimates for the wave group $e^{it\sqrt{G}}$ in the case $n = 3$ for potentials $b(x), V(x) = O(|x|^{-2-\delta})$ for $|x| \gg 1$, where $\delta > 0$.

1. Introduction and Statement of Results

The purpose of the present paper is to study the high frequency behaviour of the resolvent of self-adjoint operators on $L^2(\mathbf{R}^n), n \geq 3$, of the form

$$G = (i\nabla + b(x))^2 + V(x),$$

where $b(x) = (b_1(x), \ldots, b_n(x))$ is a vector-valued magnetic potential and V is an electric potential, b_j and V being real-valued functions. To describe the class these functions belong to, we introduce the polar coordinates $r = |x|, w = \frac{x}{|x|} \in \mathbf{S}^{n-1}$. They are of the form $b(x) = b^L(x) + b^S(x), V(x) = V^L(x) + V^S(x)$, where b^L and V^L are $C^1(\mathbf{R}^+), \mathbf{R}^+ = (0, +\infty)$, functions with respect to the radial variable r. We suppose that there exist constants $C_V, C_b > 0, 0 < \delta \ll 1$ so that for all $(r, w) \in \mathbf{R}^+ \times \mathbf{S}^{n-1}$ we have:

$$\left|V^L(rw)\right| \leq C_V, \tag{1.1}$$

$$\partial_r V^L(rw) \leq C_V \psi_\delta(r), \tag{1.2}$$

$$\left|V^S(rw)\right| \leq C_V \langle r\rangle^{-1-\delta}, \tag{1.3}$$

$$\left|\partial_r^k b^L(rw)\right| \leq C_b r^{1-k} \psi_\delta(r), \quad k = 0, 1, \tag{1.4}$$

$$\left|b^S(rw)\right| \leq C_b \eta_\delta(r), \tag{1.5}$$

where $\psi_\delta(r) = r^{-1+\delta}\langle r\rangle^{-2\delta}, \eta_\delta(r) = r^\delta\langle r\rangle^{-1-2\delta}$. Finally, we suppose that the function $b^S(rw)$ is continuous in r uniformly in w. More precisely, we assume that the function $g_\delta(r,w) = b^S(rw)/\eta_\delta(r)$ satisfies

$$\begin{aligned}&\forall\epsilon > 0\,\exists\theta = \theta(\epsilon) > 0 \text{ so that } |g_\delta(r+\theta\sigma, w) - g_\delta(r,w)| \le \epsilon\\ &\quad\text{for all } r > 0, 0 < \sigma \le 1,\ w \in \mathbf{S}^{n-1}.\end{aligned} \tag{1.6}$$

Our first result is the following:

Theorem 1.1. *Under the assumptions* (1.1)–(1.6)*, for every* $\delta' > 0$ *there exist a positive constant* $C = C(\delta',\delta)$ *independent of* C_V *and* C_b*, and a positive constant* $\lambda_0 = \lambda_0(\delta',\delta,C_V,C_b)$ *so that for* $\lambda \ge \lambda_0, 0 < \varepsilon \le 1, 0 \le |\alpha_1|, |\alpha_2| \le 1$*, we have the estimate*

$$\left\|\langle x\rangle^{-\frac{1+\delta'}{2}}\partial_x^{\alpha_1}\left(G-\lambda^2 \pm i\varepsilon\right)^{-1}\partial_x^{\alpha_2}\langle x\rangle^{-\frac{1+\delta'}{2}}\right\|_{L^2\to L^2} \le C\lambda^{|\alpha_1|+|\alpha_2|-1}. \tag{1.7}$$

Moreover, if in addition we suppose that $b^S \equiv 0$ *and the functions* $b = b^L$ *and* $V = V^L + V^S$ *satisfy*

$$\left|\frac{\partial(r^2V^L(rw))}{\partial r}\right| \le \widetilde{C}r\psi_\delta(r), \tag{1.8}$$

$$\left|V^S(rw)\right| \le \widetilde{C}\langle r\rangle^{-2-\delta}, \tag{1.9}$$

$$\left|\partial_r^k b(rw)\right| \le \widetilde{C}r^{-k}\psi_\delta(r), \quad k = 0,1, \tag{1.10}$$

then for $\delta',\lambda,\varepsilon$ *as above and* $|\alpha_1|, |\alpha_2| \le 1$*, we have the estimate*

$$\left\|\langle x\rangle^{-\frac{3+\delta'}{2}}\partial_x^{\alpha_1}\left(G-\lambda^2 \pm i\varepsilon\right)^{-2}\partial_x^{\alpha_2}\langle x\rangle^{-\frac{3+\delta'}{2}}\right\|_{L^2\to L^2} \le C\lambda^{|\alpha_1|+|\alpha_2|-2}. \tag{1.11}$$

In fact, some of the conditions above can be weakened. Indeed, using Theorem 1.1 we prove the following:

Corollary 1.2. *Let* $b \in L^\infty(\mathbf{R}^n;\mathbf{R}^n), V \in L^\infty(\mathbf{R}^n;\mathbf{R})$ *satisfy*

$$\langle x\rangle^\delta|b(x)| + |V(x)| \le C, \quad \forall x \in \mathbf{R}^n, \tag{1.12}$$

with some constants $C > 0, 0 < \delta \ll 1$*. Suppose also that there exists a constant* $r_0 \gg 1$ *so that* $b = b^L + b^S, V = V^L + V^S$ *with functions* $b^L, b^S \in L^\infty(\mathbf{R}^n;\mathbf{R}^n), V^L, V^S \in L^\infty(\mathbf{R}^n;\mathbf{R}), b^L$ *and* V^L *belonging to* $C^1([r_0,+\infty))$ *with respect to the radial variable* r*, and satisfying*

$$\left|\partial_r b^L(rw)\right| + \left|\partial_r V^L(rw)\right| + \left|b^S(rw)\right| + \left|V^S(rw)\right| \le Cr^{-1-\delta} \tag{1.13}$$

for all $r \ge r_0, w \in \mathbf{S}^{n-1}$*. Finally, we suppose that the functions* $b^L(rw)$ *and* $b^S(rw)$ *are continuous with respect to* r *uniformly on* $[0,+\infty)\times\mathbf{S}^{n-1}$*. Then the estimate* (1.7) *holds true.*

These resolvent estimates are sharp in λ in the sense that we have the same for the free Laplacian. The estimate (1.7) is well known to hold for non-trapping compactly supported perturbations of the Laplacian (in which case it can be derived from the propagation of the singularities, e.g. see [12]) and in particular when $b, V \in C_0^\infty(\mathbf{R}^n), n \ge 2$. It is also proved in many situations for operators of the form $-\Delta_g + V$ under the non-trapping condition, where Δ_g denotes the (negative) Laplace–Beltrami operator on an infinite volume unbounded Riemannian manifold (e.g. see [11,12]). Note that without the non-trapping condition we have in general resolvent estimates with $O\left(e^{\gamma\lambda}\right), \gamma > 0$, in the right-hand side (see [2]). The estimate (1.7) is well known for operators $-\Delta + V$ on $\mathbf{R}^n$ for short-range potentials $V \in L^\infty(\mathbf{R}^n)$. In the case when the magnetic potential is not identically zero, it can also be easily proved for small short-range magnetic potentials (e.g. see [6]). For large short-range magnetic potentials $b(x)$ and electric potentials $V(x)$ the estimate (1.7) is proved in [8] (see Proposition 4.3) in all dimensions $n \geq 3$, provided $b(x)$ is a continuous function. For large long-range magnetic and electric potentials the estimate (1.7) is proved in [10], provided $b, V \in C^\infty(\mathbf{R}^n)$ and $\partial_x^\alpha b(x), \partial_x^\alpha V(x) = O_\alpha\left(\langle x\rangle^{-\delta-|\alpha|}\right), \delta > 0$. In fact, the method of [10] requires this condition for $|\alpha| \leq 2$, only. Note also that resolvent estimates like (1.7) play crucial role in the proof of uniform local energy, smoothing, Strichartz and dispersive estimates for the wave and the Schrödinger equations, which in turn explains the big interest in proving such kind of estimates in various situations. Therefore, the sharpness in λ is important as a loss in $\lambda \gg 1$ in the resolvent estimate produces a loss of derivatives in the applications mentioned above.

Clearly, we also have the following:

Corollary 1.3. *Let* $b = b^L + b^S, V = V^L + V^S$, *where* $b^L \in C^1(\mathbf{R}^n; \mathbf{R}^n), b^S \in C^0(\mathbf{R}^n; \mathbf{R}^n), V^L \in C^1(\mathbf{R}^n; \mathbf{R}), V^S \in L^\infty(\mathbf{R}^n; \mathbf{R})$ *satisfy*

$$\left|V^L(x)\right| + \langle x\rangle^{1+\delta} \sum_{|\alpha|=1} \left|\partial_x^\alpha V^L(x)\right| \leq C, \tag{1.14}$$

$$\left|V^S(x)\right| \leq C\langle x\rangle^{-1-\delta}, \tag{1.15}$$

$$\sum_{|\alpha|\leq 1} \langle x\rangle^{|\alpha|+\delta} \left|\partial_x^\alpha b^L(x)\right| \leq C, \tag{1.16}$$

$$\left|b^S(x)\right| \leq C\langle x\rangle^{-1-\delta}, \tag{1.17}$$

$$\forall\epsilon > 0\, \exists\theta = \theta(\epsilon) > 0 \text{ so that } \left|b^S(x + \theta y) - b^S(x)\right| \leq \epsilon\langle x\rangle^{-1-\delta} \text{ for all } x, y \in \mathbf{R}^n,\ |y| \leq 1, \tag{1.18}$$

with some constants $C > 0$ *and* $0 < \delta \ll 1$. *Then the estimate* (1.7) *holds true.*

As mentioned above, this result is proved in [8] in the case $b^L \equiv 0, V^L \equiv 0$ by a different method. Here we extend it to more general perturbations and provide a simpler proof.

Our strategy for proving (1.7) is based on the observation that if (1.7) holds for an operator G of the form above, it still holds if we perturb G by a small short-range magnetic potential and a large short-range electric potential. Thus, we first prove (1.7) in the case $b^S \equiv 0, V^S \equiv 0$. The method we use to do so is inspired from [2] where similar ideas were used to study the high-frequency behaviour of the resolvent of the Laplace–Beltrami operator on unbounded Riemannian manifolds perturbed by an electric potential (see also [11]). The presence of a magnetic potential, however, makes the analysis much more technical and harder. In the general case when b^S and V^S are not identically zero, we use the condition that b^S is continuous in r to approximate it by smooth in r long-range magnetic potentials having properties similar to those of b^L. Thus we decompose our perturbation as a large long-range part, small first-order short-range part and a large zero-order short-range part. Finally, we apply the argument above.

We will use Theorem 1.1 to prove dispersive estimates for the wave group $\mathrm{e}^{it\sqrt{G}}$ for self-adjoint operators G as above in the case $n = 3$. More precisely, we are interested in generalizing the following three-dimensional dispersive estimate

$$\left\|\mathrm{e}^{it\sqrt{G_0}} G_0^{-1-\epsilon} \chi_a(\sqrt{G_0})\right\|_{L^1\to L^\infty} \le C_{a,\epsilon}|t|^{-1}, \quad \forall t \neq 0, \tag{1.19}$$

for every $a, \epsilon > 0$, where G_0 denotes the self-adjoint realization of the free Laplacian $-\Delta$ on $L^2(\mathbf{R}^3)$ and $\chi_a \in C^\infty(\mathbf{R}), \chi_a(\lambda) = 0$ for $\lambda \le a, \chi_a(\lambda) = 1$ for $\lambda \ge a+1$. We suppose that the magnetic potential b is $C^1(\mathbf{R}^+)$ with respect to the radial variable r, while no regularity is assumed on the electric potential V. We also suppose that there exist constants $C > 0$ and $0 < \delta \ll 1$ such that

$$|V(rw)| + |b(rw)| \le C\langle r\rangle^{-2-\delta}, \tag{1.20}$$

$$|b(rw)| \le Cr^\delta \quad \text{for} \quad r \le 1, \tag{1.21}$$

$$|\partial_r b(rw)| \le Cr^{-1+\delta}\langle r\rangle^{-1-2\delta}. \tag{1.22}$$

Clearly, the conditions of Theorem 1.1 are fulfilled (with $b^S \equiv 0, V^L \equiv 0$) for b and V satisfying (1.20), (1.21) and (1.22), so the estimates (1.7) and (1.11) are valid. When $n = 3$ we have the following

Theorem 1.4. *Under the assumptions* (1.20), (1.21) and (1.22), *there exists a constant* $a > 0$ *so that the following dispersive estimate holds*

$$\left\|\mathrm{e}^{it\sqrt{G}} G^{-3/2-\epsilon} \chi_a(\sqrt{G})\right\|_{L^1\to L^\infty} \le C_\epsilon|t|^{-1}, \quad \forall t \neq 0, \tag{1.23}$$

for every $\epsilon > 0$. *Moreover, for every* $\delta' > 0$ *there exists a constant* $a > 0$ *so that we have the estimate*

$$\left\|\mathrm{e}^{it\sqrt{G}} G^{-1-\epsilon} \chi_a(\sqrt{G})\langle x\rangle^{-3/2-\delta'}\right\|_{L^2\to L^\infty} \le C_{\epsilon,\delta'}|t|^{-1}, \quad \forall t \neq 0, \tag{1.24}$$

for every $\epsilon > 0$.

Remark. In fact, one can show that the estimates (1.23) and (1.24) hold true for every $a > 0$. Indeed, according to the results of [9] the condition (1.20) guarantees that the operator G has no embedded strictly positive eigenvalues, which in turn implies that the resolvent estimates (1.7) and (1.11) are valid for every $\lambda_0 > 0$ with constants $C > 0$ depending on λ_0.

The estimates (1.23) and (1.24) are not optimal–for example, in (1.23) there is a loss of one derivative. The desired result would be to prove the dispersive estimate

$$\left\|\mathrm{e}^{it\sqrt{G}} G^{-1-\epsilon} \chi_a(\sqrt{G})\right\|_{L^1\to L^\infty} \le C_\epsilon|t|^{-1}, \quad \forall t \neq 0, \tag{1.25}$$

for every $\epsilon > 0$ and some $a > 0$. When $b \equiv 0$ and for a large class of rough potentials V the estimate (1.25) follows from [7]. In higher dimensions $n \geq 4$ an analogue of (1.25) is proved in [1] for Schwartz class potentials V and in [3] for potentials $V \in C^{\frac{n-3}{2}}(\mathbf{R}^n), 4 \leq n \leq 7$, while in [13] dispersive estimates with a loss of $\frac{n-3}{2}$ derivatives are proved for potentials $V \in L^\infty(\mathbf{R}^n), V(x) = O\left(\langle x\rangle^{-\frac{n+1}{2}-\delta}\right), \delta > 0$. Proving (1.25) when the magnetic potential $b(x)$ is not identically zero, however, is a difficult and an open problem even if b is supposed small and smooth. Our conjecture is that (1.25) should hold for $b \in C_0^1(\mathbf{R}^3)$ and $V \in L^\infty(\mathbf{R}^3), V(x) = O\left(\langle x\rangle^{-2-\delta}\right), \delta > 0$, while in higher dimensions $n \geq 4$ we expect to have an optimal dispersive estimate (that is, without loss of derivatives) similar to (1.25) for $b \in C_0^{\frac{n-1}{2}}(\mathbf{R}^n)$ and $V \in C_0^{\frac{n-3}{2}}(\mathbf{R}^n)$. Note that dispersive estimates for the wave group with a loss of $\frac{n}{2}$ derivatives have been recently proved in [4] in all dimensions $n \geq 3$ for a class of potentials $b \in C^1(\mathbf{R}^n)$ and $V \in L^\infty(\mathbf{R}^n)$. Note also that an estimate similar to (1.24) is proved in [5] for a class of small potentials b and V still in dimension three.

Theorem 1.1 plays a crucial role in the proof of the dispersive estimates (1.23) and (1.24). For example, one can not use Corollary 1.3 instead, since a function $b(x)$ satisfying the conditions (1.20), (1.21) and (1.22) is not necessarily continuous in x. Finally, we expect that Theorem 1.4 can be extended to all dimensions $n \geq 3$ for potentials $b(x), V(x) = O\left(\langle x\rangle^{-\frac{n+1}{2}-\delta}\right)$.

2. Resolvent Estimates

Throughout this section C will stand for constants independent of C_V and C_b, while $\widetilde{C}$ will stand for constants which may depend on C_V and C_b. Both C and $\widetilde{C}$ may vary from line to line. Clearly, it suffices to prove the resolvent estimates for $0 < \delta' \ll \delta$. We will first consider the case $b^S \equiv 0, V^S \equiv 0$, so $b = b^L$ and $V = V^L$. Let $\alpha_1 = \alpha_2 = 0$. Clearly, in this case (1.7) follows from the a priori estimate

$$\left\|\psi_{\delta'}(|x|)^{1/2} f\right\|_{L^2(\mathbf{R}^n)} \leq C\lambda^{-1}\left\|\psi_{\delta'}(|x|)^{-1/2}\left(G - \lambda^2 \pm i\varepsilon\right) f\right\|_{L^2(\mathbf{R}^n)}, \tag{2.1}$$

for every $f \in H^2(\mathbf{R}^n)$. It suffices to consider the case "+" only. To prove (2.1) we will pass to polar coordinates $(r, w) \in \mathbf{R}^+ \times \mathbf{S}^{n-1}$. Recall that $L^2(\mathbf{R}^n) \cong L^2\left(\mathbf{R}^+ \times \mathbf{S}^{n-1}, r^{(n-1)/2}\mathrm{d}r\mathrm{d}w\right)$. Set $X = \left(\mathbf{R}^+ \times \mathbf{S}^{n-1}, \mathrm{d}r\mathrm{d}w\right), u = r^{(n-1)/2} f$,

$$P = \lambda^{-2} r^{(n-1)/2}\left(G - \lambda^2 + i\varepsilon\right) r^{-(n-1)/2}.$$

It is well known that

$$r^{(n-1)/2}\Delta r^{-(n-1)/2} = \partial_r^2 + \frac{\Delta_w - c_n}{r^2}, \tag{2.2}$$

where

$$c_n = \frac{(n-1)(n-3)}{4}$$

and Δ_w denotes the (negative) Laplace–Beltrami operator on $\mathbf{S}^{n-1}$ written in the coordinates w. It is easy to see that (2.1) follows from the estimate

$$\left\|\psi_{\delta'}(r)^{1/2}u\right\|_{H^1(X)} \le C\lambda \left\|\psi_{\delta'}(r)^{-1/2}Pu\right\|_{L^2(X)}, \tag{2.3}$$

where the norm in the left-hand side is defined as follows:

$$\left\|\psi_{\delta'}(r)^{1/2}u\right\|^2_{H^1(X)} = \left\|\psi_{\delta'}(r)^{1/2}u\right\|^2_{L^2(X)} + \left\|\psi_{\delta'}(r)^{1/2}\mathcal{D}_r u\right\|^2_{L^2(X)} + \left\|\psi_{\delta'}(r)^{1/2}r^{-1}\Lambda_w^{1/2}u\right\|^2_{L^2(X)},$$

where $\mathcal{D}_r = i\lambda^{-1}\partial_r, \Lambda_w = -\lambda^{-2}\Delta_w$. Throughout this section $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ will denote the norm and the scalar product in the Hilbert space $L^2(\mathbf{S}^{n-1})$. Hence $\|u\|^2_{L^2(X)} = \int_0^\infty \|u(r,\cdot)\|^2 \mathrm{d}r$. Using (2.2) one can easily check that the operator P can be written in the form:

$$\begin{aligned} P &= \mathcal{D}_r^2 + r^{-2}\widetilde{\Lambda}_w + \lambda^{-2}W(r,w) - 1 + i\varepsilon\lambda^{-2} \\ &\quad + \lambda^{-1}\sum_{j=1}^n w_j\left(b_j(rw)\mathcal{D}_r + \mathcal{D}_r b_j(rw)\right) \\ &\quad + \lambda^{-1}r^{-1}\sum_{j=1}^n \left(b_j(rw)Q_j(w,\mathcal{D}_w) + Q_j(w,\mathcal{D}_w)b_j(rw)\right), \\ W &= V(rw) + |b(rw)|^2 - i(n-1)r^{-1}\sum_{j=1}^n w_j b_j(rw), \end{aligned}$$

where $\widetilde{\Lambda}_w = \Lambda_w + \lambda^{-2}c_n, w_j = x_j/r, \mathcal{D}_w = i\lambda^{-1}\partial_w, Q_j(w,\mathcal{D}_w) = i\lambda^{-1} Q_j(w,\partial_w), Q_j(w,\xi) \in C^\infty(T^*\mathbf{S}^{n-1})$ are real-valued, independent of r and λ, and homogeneous of order 1 with respect to ξ. Decompose W as $W^L + W^S$, where

$$\begin{aligned} W^L &= V(rw) + |b(rw)|^2, \\ W^S &= -i(n-1)r^{-1}\sum_{j=1}^n w_j b_j(rw). \end{aligned}$$

It is easy to see that the assumptions (1.1), (1.2) and (1.4) imply

$$\left|W^L(r,w)\right| \le \widetilde{C}, \tag{2.4}$$

$$\partial_r W^L(r,w) \le \widetilde{C}\psi_\delta(r), \tag{2.5}$$

$$\left|W^S(r,w)\right| \le \widetilde{C}\psi_\delta(r). \tag{2.6}$$

Set

$$\begin{aligned} E(r) &= -\left\langle \left(r^{-2}\widetilde{\Lambda}_w - 1 + \lambda^{-2}W^L\right)u(r,w), u(r,w)\right\rangle + \|\mathcal{D}_r u(r,w)\|^2 \\ &\quad -2\lambda^{-1}r^{-1}\sum_{j=1}^n \mathrm{Re}\,\langle b_j(rw)Q_j(w,\mathcal{D}_w)u(r,w), u(r,w)\rangle. \end{aligned}$$

It is easy to see that the condition $f \in H^2(\mathbf{R}^n)$ implies that $E(\cdot) \in L^1(\mathbf{R}^+)$. We have the identity

$$\begin{aligned} E'(r) &:= \frac{\mathrm{d}E(r)}{\mathrm{d}r} \\ &= \frac{2}{r}\left\langle r^{-2}\widetilde{\Lambda}_w u(r,w), u(r,w)\right\rangle - \lambda^{-2}\left\langle \frac{\partial W^L}{\partial r} u(r,w), u(r,w)\right\rangle \\ &\quad -2\lambda^{-1}\sum_{j=1}^n \mathrm{Re}\left\langle \frac{\partial(b_j(rw)/r)}{\partial r} Q_j(w,\mathcal{D}_w)u(r,w), u(r,w)\right\rangle \\ &\quad -2\lambda^{-1}\sum_{j=1}^n \mathrm{Re}\left\langle w_j\frac{\partial b_j(rw)}{\partial r} u(r,w), \mathcal{D}_r u(r,w)\right\rangle \\ &\quad +2\lambda \mathrm{Im}\left\langle \widetilde{P}u(r,w), \mathcal{D}_r u(r,w)\right\rangle, \end{aligned}$$

where

$$\widetilde{P} = P - i\varepsilon\lambda^{-2} - \lambda^{-2}W^S,$$

and we have used that $\mathrm{Im}\,\langle b_j\mathcal{D}_r u, \mathcal{D}_r u\rangle = 0$. Observe now that by (1.4) we have

$$\left|\frac{\partial b(rw)}{\partial r}\right| \le \widetilde{C}\psi_\delta(r), \tag{2.7}$$

$$\left|\frac{\partial (b(rw)/r)}{\partial r}\right| \le \widetilde{C}r^{-3/2}\psi_\delta(r)^{1/2}. \tag{2.8}$$

Hence, using (2.5), (2.7) and (2.8), we obtain

$$\begin{aligned} E'(r) \ge \frac{2}{r}\left\langle r^{-2}\widetilde{\Lambda}_w u(r,w), u(r,w)\right\rangle - \gamma r^{-3}\sum_{j=1}^n \|Q_j(w,\mathcal{D}_w)u(r,w)\|^2 \\ -\lambda^{-1}\left\|\psi_\delta^{1/2}\mathcal{D}_r u(r,w)\right\|^2 - O_\gamma(\lambda^{-1})\left\|\psi_\delta^{1/2}u(r,w)\right\|^2 - 2\lambda M(r), \end{aligned} \tag{2.9}$$

$\forall\gamma > 0$ independent of λ and r, where

$$M(r) = \left|\left\langle \widetilde{P}u(r,w), \mathcal{D}_r u(r,w)\right\rangle\right|.$$

Since $\|Q_j(w,\mathcal{D}_w)u\| \le C\|\Lambda_w^{1/2}u\| \le C\|\widetilde{\Lambda}_w^{1/2}u\|$, taking γ small enough we can absorb the second term in the right-hand side of (2.9) by the first one and obtain

$$\begin{aligned} E'(r) \ge \frac{1}{r}\left\langle r^{-2}\widetilde{\Lambda}_w u(r,w), u(r,w)\right\rangle - \lambda^{-1}\left\|\psi_\delta^{1/2}\mathcal{D}_r u(r,w)\right\|^2 \\ -O(\lambda^{-1})\left\|\psi_\delta^{1/2}u(r,w)\right\|^2 - 2\lambda M(r). \end{aligned} \tag{2.10}$$

Using that $\widetilde{\Lambda}_w \geq 0$, we deduce from (2.10)

$$
\begin{aligned}
E(r) &= -\int\limits_r^\infty E'(t)\mathrm{d}t \\
&\leq \lambda^{-1}\left\|\psi_\delta^{1/2}\mathcal{D}_r u\right\|^2_{L^2(X)} + O(\lambda^{-1})\left\|\psi_\delta^{1/2}u\right\|^2_{L^2(X)} + 2\lambda\int\limits_0^\infty M(t)\mathrm{d}t. \quad (2.11)
\end{aligned}
$$

We also have

$$
\begin{aligned}
-E(r) &\leq \left\langle \left(r^{-2}\widetilde{\Lambda}_w - 1 + \lambda^{-2}W^L\right)u(r,w), u(r,w)\right\rangle \\
&\quad + O(\lambda^{-1})\sum_{j=1}^n \left\|r^{-1}Q_j(w,\mathcal{D}_w)u(r,w)\right\| \|u(r,w)\| \\
&\leq \left\langle \left(r^{-2}\widetilde{\Lambda}_w - \frac{1}{2}\right)u(r,w), u(r,w)\right\rangle \\
&\quad + O(\lambda^{-1})\left\|r^{-1}\widetilde{\Lambda}_w^{1/2}u(r,w)\right\| \|u(r,w)\| \\
&\leq 2\left\langle r^{-2}\widetilde{\Lambda}_w u(r,w), u(r,w)\right\rangle, \quad (2.12)
\end{aligned}
$$

provided λ is taken large enough. Let now $\psi(r) > 0$ be such that $\int_0^\infty \psi(r)\mathrm{d}r < +\infty$. Multiplying both sides of (2.11) by ψ and integrating from 0 to ∞, we get

$$
\begin{aligned}
\int\limits_0^\infty \psi(r)E(r)\mathrm{d}r &\leq O(\lambda^{-1})\left\|\psi_\delta^{1/2}\mathcal{D}_r u\right\|^2_{L^2(X)} + O(\lambda^{-1})\left\|\psi_\delta^{1/2}u\right\|^2_{L^2(X)} \\
&\quad + C\lambda\int\limits_0^\infty M(r)\mathrm{d}r. \quad (2.13)
\end{aligned}
$$

By (2.11) and (2.12), we also obtain

$$
\begin{aligned}
\int\limits_0^\infty \psi(r)\left|E(r)\right|\mathrm{d}r &\leq 2\left\|\psi^{1/2}r^{-1}\widetilde{\Lambda}_w^{1/2}u\right\|^2_{L^2(X)} + O(\lambda^{-1})\left\|\psi_\delta^{1/2}\mathcal{D}_r u\right\|^2_{L^2(X)} \\
&\quad + O(\lambda^{-1})\left\|\psi_\delta^{1/2}u\right\|^2_{L^2(X)} + C\lambda\int\limits_0^\infty M(r)\mathrm{d}r. \quad (2.14)
\end{aligned}
$$

In particular, (2.13) and (2.14) hold with $\psi = \psi_{\delta'}(r)$ for any $0 < \delta' \leq \delta$. It is easy also to check that

$$
\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(r\psi_{\delta'}(r)\right)\right| \leq O(\delta')\psi_{\delta'}(r),
$$

so we can use (2.14) to obtain

$$\int_0^\infty r\psi_{\delta'}(r)E'(r)\mathrm{d}r = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}r}\left(r\psi_{\delta'}(r)\right)E(r)\mathrm{d}r \le O(\delta')\int_0^\infty \psi_{\delta'}(r)\left|E(r)\right|\mathrm{d}r$$
$$\le O(\delta')\left\|\psi_{\delta'}^{1/2}r^{-1}\widetilde{\Lambda}_w^{1/2}u\right\|^2_{L^2(X)} + O(\lambda^{-1})\left\|\psi_{\delta}^{1/2}\mathcal{D}_r u\right\|^2_{L^2(X)}$$
$$+O(\lambda^{-1})\left\|\psi_{\delta}^{1/2}u\right\|^2_{L^2(X)} + C\lambda\int_0^\infty M(r)\mathrm{d}r. \tag{2.15}$$

Since $r\psi_{\delta'}(r) \le 1$, combining (2.10) and (2.15) and absorbing the $O(\delta')$ term by taking δ' small enough, we conclude

$$\left\|\psi_{\delta'}^{1/2}r^{-1}\widetilde{\Lambda}_w^{1/2}u\right\|^2_{L^2(X)} \le O(\lambda^{-1})\left\|\psi_{\delta}^{1/2}\mathcal{D}_r u\right\|^2_{L^2(X)}$$
$$+O(\lambda^{-1})\left\|\psi_{\delta}^{1/2}u\right\|^2_{L^2(X)} + C\lambda\int_0^\infty M(r)\mathrm{d}r. \tag{2.16}$$

On the other hand, in view of (2.4) we can choose λ big enough so that $1 - \lambda^{-2}W^L \ge 1/2$. Therefore, for $\lambda \gg 1$ we have the inequality

$$\int_0^\infty \psi_{\delta'}(r)E(r)\mathrm{d}r \ge \left\|\psi_{\delta'}^{1/2}\mathcal{D}_r u\right\|^2_{L^2(X)}$$
$$+\frac{1}{3}\left\|\psi_{\delta'}^{1/2}u\right\|^2_{L^2(X)} - 2\left\|\psi_{\delta'}^{1/2}r^{-1}\widetilde{\Lambda}_w^{1/2}u\right\|^2_{L^2(X)}. \tag{2.17}$$

By (2.13), (2.16) and (2.17), we conclude

$$\left\|\psi_{\delta'}^{1/2}u\right\|^2_{\widetilde{H}^1(X)} := \left\|\psi_{\delta'}^{1/2}\mathcal{D}_r u\right\|^2_{L^2(X)} + \left\|\psi_{\delta'}^{1/2}u\right\|^2_{L^2(X)} + \left\|\psi_{\delta'}^{1/2}r^{-1}\widetilde{\Lambda}_w^{1/2}u\right\|^2_{L^2(X)}$$
$$\le O(\lambda^{-1})\left\|\psi_{\delta}^{1/2}\mathcal{D}_r u\right\|^2_{L^2(X)} + O(\lambda^{-1})\left\|\psi_{\delta}^{1/2}u\right\|^2_{L^2(X)}$$
$$+C\lambda\int_0^\infty M(r)\mathrm{d}r. \tag{2.18}$$

Set

$$P^\sharp = \widetilde{P} + i\varepsilon\lambda^{-2} = P - \lambda^{-2}W^S(r,w),$$
$$M^\sharp(r) = \left|\left\langle P^\sharp u(r,w), \mathcal{D}_r u(r,w)\right\rangle\right|, \quad N(r) = \left|\left\langle P u(r,w), \mathcal{D}_r u(r,w)\right\rangle\right|.$$

In view of (2.6), we have

$$\lambda\int_0^\infty M^\sharp(r)\mathrm{d}r \le \lambda\int_0^\infty N(r)\mathrm{d}r + O(\lambda^{-1})\left\|\psi_{\delta}^{1/2}u\right\|^2_{L^2(X)}$$
$$+O(\lambda^{-1})\left\|\psi_{\delta}^{1/2}\mathcal{D}_r u\right\|^2_{L^2(X)}. \tag{2.19}$$

We also have

$$\lambda \int_0^\infty M(r)\mathrm{d}r \leq \lambda \int_0^\infty M^\sharp(r)\mathrm{d}r + \varepsilon\lambda^{-1}\left(\|u\|^2_{L^2(X)} + \|\mathcal{D}_r u\|^2_{L^2(X)}\right), \tag{2.20}$$

$$\lambda \int_0^\infty N(r)\mathrm{d}r \leq \gamma^{-1}\lambda^2 \left\|\psi_{\delta'}^{-1/2} P u\right\|^2_{L^2(X)} + \gamma \left\|\psi_{\delta'}^{1/2}\mathcal{D}_r u\right\|^2_{L^2(X)}, \tag{2.21}$$

for every $\gamma > 0$ independent of λ. On the other hand, in view of (2.4) and (2.6), we have

$$\begin{aligned}\varepsilon\lambda^{-2}\|u\|^2_{L^2(X)} &= \mathrm{Im}\,\langle Pu, u\rangle_{L^2(X)} + (n-1)\lambda^{-2}\sum_{j=1}^n \left\langle r^{-1} w_j b_j(rw)u, u\right\rangle_{L^2(X)} \\ &\leq \left|\langle Pu, u\rangle_{L^2(X)}\right| + O(\lambda^{-2}) \left\|\psi_\delta^{1/2} u\right\|^2_{L^2(X)},\end{aligned} \tag{2.22}$$

$$\begin{aligned}\mathrm{Re}\,\langle Pu, u\rangle_{L^2(X)} &= \|\mathcal{D}_r u\|^2_{L^2(X)} + \left\|r^{-1}\widetilde{\Lambda}_w^{1/2} u\right\|^2_{L^2(X)} \\ &\quad + \left\langle \left(\lambda^{-2} W^L - 1\right) u, u\right\rangle_{L^2(X)} \\ &\quad + 2\lambda^{-1}\sum_{j=1}^n \mathrm{Re}\,\langle w_j b_j(rw)\mathcal{D}_r u, u\rangle_{L^2(X)} \\ &\quad + 2\lambda^{-1}\sum_{j=1}^n \mathrm{Re}\,\left\langle r^{-1} b_j(rw) Q_j u, u\right\rangle_{L^2(X)} \\ &\geq \|\mathcal{D}_r u\|^2_{L^2(X)} + \left\|r^{-1}\widetilde{\Lambda}_w^{1/2} u\right\|^2_{L^2(X)} - 2\|u\|^2_{L^2(X)} \\ &\quad - O(\lambda^{-1})\left(\|\mathcal{D}_r u\|^2_{L^2(X)} + \left\|r^{-1}\widetilde{\Lambda}_w^{1/2} u\right\|^2_{L^2(X)} + \|u\|^2_{L^2(X)}\right) \\ &\geq \frac{1}{2}\|\mathcal{D}_r u\|^2_{L^2(X)} - 2\|u\|^2_{L^2(X)},\end{aligned}$$

provided λ is taken large enough, which in turn implies

$$\|\mathcal{D}_r u\|^2_{L^2(X)} \leq 4\|u\|^2_{L^2(X)} + 2\left|\langle Pu, u\rangle_{L^2(X)}\right|. \tag{2.23}$$

Combining (2.20), (2.22) and (2.23), we get

$$\begin{aligned}\lambda \int_0^\infty M(r)\mathrm{d}r &\leq C\lambda \int_0^\infty M^\sharp(r)\mathrm{d}r + O(\lambda^{-1})\left\|\psi_\delta^{1/2} u\right\|^2_{L^2(X)} + 6\lambda\left|\langle Pu, u\rangle_{L^2(X)}\right| \\ &\leq C\lambda \int_0^\infty M^\sharp(r)\mathrm{d}r + C\gamma^{-1}\lambda^2\left\|\psi_{\delta'}^{-1/2} P u\right\|^2_{L^2(X)} \\ &\quad + (\gamma + O(\lambda^{-1}))\left\|\psi_{\delta'}^{1/2} u\right\|^2_{L^2(X)}.\end{aligned} \tag{2.24}$$

By (2.18), (2.19), (2.21) and (2.24), we conclude

$$\begin{aligned}&\left\|\psi_{\delta'}^{1/2}u\right\|_{\widetilde{H}^1(X)}\\&\quad\le \left(O_\gamma(\lambda^{-1})+C\gamma\right)^{1/2}\left\|\psi_{\delta'}^{1/2}u\right\|_{\widetilde{H}^1(X)}+C\gamma^{-1}\lambda\left\|\psi_{\delta'}^{-1/2}Pu\right\|_{L^2(X)},\end{aligned} \tag{2.25}$$

where we have used that $\psi_\delta \le \psi_{\delta'}$ for $\delta' \le \delta$. Now, taking γ small enough, independent of λ, C_V and C_b, and λ big enough, we can absorb the first term in the right-hand side of (2.25) to obtain (2.3). To prove (1.7) for all multi-indices $|\alpha_1|, |\alpha_2| \le 1$ we will use the following

Lemma 2.1. *If $|b(x)| + |V(x)| \le C' = Const$, then for every $s \in \mathbf{R}$ there exist constants $C > 0$ independent of b and V and $\lambda_0 > 0$ depending on C' so that for $\lambda \ge \lambda_0$ and $0 \le |\alpha_1|, |\alpha_2| \le 1$ we have the estimate*

$$\left\|\langle x\rangle^{-s}\partial_x^{\alpha_1}\left(G\pm i\lambda^2\right)^{-1}\partial_x^{\alpha_2}\langle x\rangle^{s}\right\|_{L^2\to L^2}\le C\lambda^{|\alpha_1|+|\alpha_2|-2}. \tag{2.26}$$

Proof. Without loss of generality we may suppose that $s \ge 0$. Let us first see that (2.26) is valid for the free operator G_0. This is obvious for $s = 0$. For $s > 0$ we will use the identity

$$\begin{aligned}&\left(G_0\pm i\lambda^2\right)^{-1}\langle x\rangle^{s}\\&\quad=\langle x\rangle^{s}\left(G_0\pm i\lambda^2\right)^{-1}+\left(G_0\pm i\lambda^2\right)^{-1}\left[\Delta,\langle x\rangle^{s}\right]\left(G_0\pm i\lambda^2\right)^{-1}.\end{aligned} \tag{2.27}$$

Since

$$[\Delta,\langle x\rangle^{s}]=O\left(\langle x\rangle^{s-1}\right)\partial_x+O\left(\langle x\rangle^{s-2}\right),$$

we obtain from (2.27) (with $|\alpha| \le 2$)

$$\begin{aligned}&\left\|\langle x\rangle^{-s}\partial_x^{\alpha}\left(G_0\pm i\lambda^2\right)^{-1}\langle x\rangle^{s}\right\|_{L^2\to L^2}\le\left\|\langle x\rangle^{-s}\partial_x^{\alpha}\langle x\rangle^{s}\left(G_0\pm i\lambda^2\right)^{-1}\right\|_{L^2\to L^2}\\&\quad+C\sum_{|\beta|\le 1}\left\|\langle x\rangle^{-s}\partial_x^{\alpha}\left(G_0\pm i\lambda^2\right)^{-1}\langle x\rangle^{s-1}\right\|_{L^2\to L^2}\left\|\partial_x^{\beta}\left(G_0\pm i\lambda^2\right)^{-1}\right\|_{L^2\to L^2}\\&\le C\lambda^{|\alpha|-2}+O(\lambda^{-1})\left\|\langle x\rangle^{-s}\partial_x^{\alpha}\left(G_0\pm i\lambda^2\right)^{-1}\langle x\rangle^{s-1}\right\|_{L^2\to L^2}.\end{aligned} \tag{2.28}$$

Iterating (2.28) a finite number of times and taking into account that the operator $\partial_x^{\alpha_2}$ commutes with the free resolvent, we get (2.26) for G_0. To prove (2.26) for the perturbed operator we will use the resolvent identity

$$\left(G\pm i\lambda^2\right)^{-1}=\left(G_0\pm i\lambda^2\right)^{-1}-\left(G\pm i\lambda^2\right)^{-1}\left(G-G_0\right)\left(G_0\pm i\lambda^2\right)^{-1}. \tag{2.29}$$

By (2.29) we get

$$
\begin{aligned}
&\sum_{|\alpha_1|,|\alpha_2|\le 1} \lambda^{-|\alpha_1|-|\alpha_2|} \left\| \langle x\rangle^{-s} \partial_x^{\alpha_1} \left(G \pm i\lambda^2\right)^{-1} \partial_x^{\alpha_2} \langle x\rangle^{s} \right\|_{L^2\to L^2} \\
&\quad \le \sum_{|\alpha_1|,|\alpha_2|\le 1} \lambda^{-|\alpha_1|-|\alpha_2|} \left\| \langle x\rangle^{-s} \partial_x^{\alpha_1} \left(G_0 \pm i\lambda^2\right)^{-1} \partial_x^{\alpha_2} \langle x\rangle^{s} \right\|_{L^2\to L^2} \\
&\qquad + C \sum_{|\alpha_1|,|\alpha_2|\le 1} \sum_{|\beta_1|+|\beta_2|\le 1} \lambda^{-|\alpha_1|-|\alpha_2|} \left\| \langle x\rangle^{-s} \partial_x^{\alpha_1} \left(G \pm i\lambda^2\right)^{-1} \partial_x^{\beta_1} \langle x\rangle^{s} \right\|_{L^2\to L^2} \\
&\qquad \times \left\| \langle x\rangle^{-s} \partial_x^{\beta_2} \left(G_0 \pm i\lambda^2\right)^{-1} \partial_x^{\alpha_2} \langle x\rangle^{s} \right\|_{L^2\to L^2} \\
&\quad \le C\lambda^{-2} + O(\lambda^{-1}) \\
&\qquad \times \sum_{|\alpha_1|,|\beta_1|\le 1} \lambda^{-|\alpha_1|-|\beta_1|} \left\| \langle x\rangle^{-s} \partial_x^{\alpha_1} \left(G \pm i\lambda^2\right)^{-1} \partial_x^{\beta_1} \langle x\rangle^{s} \right\|_{L^2\to L^2}. \qquad (2.30)
\end{aligned}
$$

Taking now λ big enough we can absorb the second term in the right-hand side of (2.30) and obtain (2.26). □

Let us see that (1.7) for all multi-indices α_1 and α_2 follows from (1.7) with $\alpha_1 = \alpha_2 = 0$ and Lemma 2.1. To this end, we will use the resolvent identity

$$
\begin{aligned}
\left(G - \lambda^2 \pm i\varepsilon\right)^{-1} &= \left(G - i\lambda^2\right)^{-1} + (\lambda^2 \mp i\varepsilon - i\lambda^2)\left(G - i\lambda^2\right)^{-2} \\
&\quad + (\lambda^2 \mp i\varepsilon - i\lambda^2)^2 \left(G - i\lambda^2\right)^{-1} \left(G - \lambda^2 \pm i\varepsilon\right)^{-1} \left(G - i\lambda^2\right)^{-1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left\| \langle x\rangle^{-\frac{1+\delta'}{2}} \partial_x^{\alpha_1} \left(G - \lambda^2 \pm i\varepsilon\right)^{-1} \partial_x^{\alpha_2} \langle x\rangle^{-\frac{1+\delta'}{2}} \right\|_{L^2\to L^2} \\
&\quad \le \left\| \partial_x^{\alpha_1} \left(G - i\lambda^2\right)^{-1} \partial_x^{\alpha_2} \right\|_{L^2\to L^2} \\
&\qquad + C\lambda^2 \left\| \partial_x^{\alpha_1} \left(G - i\lambda^2\right)^{-1} \right\|_{L^2\to L^2} \left\| \left(G - i\lambda^2\right)^{-1} \partial_x^{\alpha_2} \right\|_{L^2\to L^2} \\
&\qquad + C\lambda^4 \left\| \langle x\rangle^{-\frac{1+\delta'}{2}} \partial_x^{\alpha_1} \left(G - i\lambda^2\right)^{-1} \langle x\rangle^{\frac{1+\delta'}{2}} \right\|_{L^2\to L^2} \\
&\qquad \times \left\| \langle x\rangle^{-\frac{1+\delta'}{2}} \left(G - \lambda^2 \pm i\varepsilon\right)^{-1} \langle x\rangle^{-\frac{1+\delta'}{2}} \right\|_{L^2\to L^2} \\
&\qquad \times \left\| \langle x\rangle^{\frac{1+\delta'}{2}} \left(G - i\lambda^2\right)^{-1} \partial_x^{\alpha_2} \langle x\rangle^{-\frac{1+\delta'}{2}} \right\|_{L^2\to L^2} \\
&\quad \le C\lambda^{|\alpha_1|+|\alpha_2|-2} + C\lambda^{|\alpha_1|+|\alpha_2|} \left\| \langle x\rangle^{-\frac{1+\delta'}{2}} \left(G - \lambda^2 \pm i\varepsilon\right)^{-1} \langle x\rangle^{-\frac{1+\delta'}{2}} \right\|_{L^2\to L^2} \\
&\quad \le C\lambda^{|\alpha_1|+|\alpha_2|-1}.
\end{aligned}
$$

We will now prove (1.7) in the general case. Let $\phi \in C_0^\infty(\mathbf{R}^+)$, $\phi \ge 0$, $\operatorname{supp}\phi \subset [0,1]$, $\int \phi(\sigma)\mathrm{d}\sigma = 1$, and given any $0 < \theta \le 1$, set

$$B_\theta(r,w) = \theta^{-1}\eta_\delta(r)\int\limits_{\mathbf{R}} g_\delta(r',w)\phi\left(\frac{r'-r}{\theta}\right)\mathrm{d}r'$$
$$= \eta_\delta(r)\int\limits_{\mathbf{R}} g_\delta(r+\theta\sigma,w)\phi(\sigma)\mathrm{d}\sigma,$$

$b_\theta^S(x) := B_\theta(|x|,\frac{x}{|x|})$. In view of the assumption (1.6), given any $\epsilon > 0$ there exists $\theta > 0$ so that for all $x \in \mathbf{R}^n$ we have

$$\left|b_\theta^S(x) - b^S(x)\right| \le \eta_\delta(|x|)\int\limits_{\mathbf{R}}\left|g_\delta\left(|x|+\theta\sigma,\frac{x}{|x|}\right) - g_\delta\left(|x|,\frac{x}{|x|}\right)\right|\phi(\sigma)\mathrm{d}\sigma$$
$$\le \epsilon\eta_\delta(|x|). \tag{2.31}$$

It is also clear that (1.5) implies the bounds

$$\left|b_\theta^S(rw)\right| \le \widetilde{C}\eta_\delta(r), \tag{2.32}$$
$$\left|\partial_r b_\theta^S(rw)\right| \le \widetilde{C}_\epsilon\psi_\delta(r). \tag{2.33}$$

We will use the above analysis and the fact that the constant C in the right-hand side of (1.7) depends only on the parameter δ', provided $0 < \delta' \le \delta$. In view of (1.1), (1.2), (1.4), (2.32) and (2.33), we can apply the already proved estimate (1.7) to the operator

$$G_1 = -\Delta + i(b^L + b_\theta^S)\cdot\nabla + i\nabla\cdot(b^L + b_\theta^S) + V^L + \left|b^L\right|^2$$

to get the estimate

$$\left\|\langle x\rangle^{-\frac{1+\delta'}{2}}\partial_x^{\alpha_1}\left(G_1-\lambda^2\pm i\varepsilon\right)^{-1}\partial_x^{\alpha_2}\langle x\rangle^{-\frac{1+\delta'}{2}}\right\|_{L^2\to L^2} \le C\lambda^{|\alpha_1|+|\alpha_2|-1} \tag{2.34}$$

for $\lambda \ge \lambda_0(\epsilon) > 0$ with a constant $C > 0$ independent of ϵ, ε and λ. On the other hand, in view of (1.3), (1.5) and (2.31), the difference $G - G_1$ is a first order differential operator of the form

$$G - G_1 = O\left(\epsilon\langle x\rangle^{-1-\delta}\right)\cdot\nabla + \nabla\cdot O\left(\epsilon\langle x\rangle^{-1-\delta}\right) + O\left(\langle x\rangle^{-1-\delta}\right).$$

Using this together with (2.34) and the resolvent identity

$$(G-\lambda^2\pm i\varepsilon)^{-1}$$
$$= (G_1-\lambda^2\pm i\varepsilon)^{-1} - (G_1-\lambda^2\pm i\varepsilon)^{-1}(G-G_1)(G-\lambda^2\pm i\varepsilon)^{-1},$$

we obtain

$$\left\|\langle x\rangle^{-\frac{1+\delta'}{2}}\left(G-\lambda^2\pm i\varepsilon\right)^{-1}\langle x\rangle^{-\frac{1+\delta'}{2}}\right\|_{L^2\to L^2}$$
$$\le \left\|\langle x\rangle^{-\frac{1+\delta'}{2}}\left(G_1-\lambda^2\pm i\varepsilon\right)^{-1}\langle x\rangle^{-\frac{1+\delta'}{2}}\right\|_{L^2\to L^2}$$
$$+ C\sum_{|\beta_1|+|\beta_2|\le 1}\epsilon^{|\beta_1|+|\beta_2|}\left\|\langle x\rangle^{-\frac{1+\delta'}{2}}\left(G_1-\lambda^2\pm i\varepsilon\right)^{-1}\partial_x^{\beta_1}\langle x\rangle^{-\frac{1+\delta}{2}}\right\|_{L^2\to L^2}$$
$$\times\left\|\langle x\rangle^{-\frac{1+\delta}{2}}\partial_x^{\beta_2}\left(G-\lambda^2\pm i\varepsilon\right)^{-1}\langle x\rangle^{-\frac{1+\delta'}{2}}\right\|_{L^2\to L^2}$$

$$\begin{aligned}
&\leq C\lambda^{-1} + C \sum_{|\beta_1|+|\beta_2|\leq 1} \epsilon^{|\beta_1|+|\beta_2|}\lambda^{|\beta_1|-1} \\
&\quad \times \left\| \langle x\rangle^{-\frac{1+\delta}{2}} \partial_x^{\beta_2} \left(G-\lambda^2 \pm i\varepsilon\right)^{-1} \langle x\rangle^{-\frac{1+\delta'}{2}} \right\|_{L^2\to L^2} \\
&\leq C\lambda^{-1} + O\left(\epsilon+\lambda^{-1}\right) \left\| \langle x\rangle^{-\frac{1+\delta}{2}} \left(G-\lambda^2 \pm i\varepsilon\right)^{-1} \langle x\rangle^{-\frac{1+\delta'}{2}} \right\|_{L^2\to L^2} \\
&\quad + O(\epsilon\lambda^{-1}) \sum_{|\beta_2|=1} \left\| \langle x\rangle^{-\frac{1+\delta}{2}} \partial_x^{\beta_2} (G-i\lambda^2)^{-1} \langle x\rangle^{\frac{1+\delta}{2}} \right\|_{L^2\to L^2} \\
&\quad \times \left(1+\lambda^2 \left\| \langle x\rangle^{-\frac{1+\delta}{2}} \left(G-\lambda^2 \pm i\varepsilon\right)^{-1} \langle x\rangle^{-\frac{1+\delta'}{2}} \right\|_{L^2\to L^2}\right) \\
&\leq C\lambda^{-1} + O\left(\epsilon+\lambda^{-1}\right) \left\| \langle x\rangle^{-\frac{1+\delta'}{2}} \left(G-\lambda^2 \pm i\varepsilon\right)^{-1} \langle x\rangle^{-\frac{1+\delta'}{2}} \right\|_{L^2\to L^2}, \qquad (2.35)
\end{aligned}$$

where we have used that $\delta' \leq \delta$ and Lemma 2.1. Taking $\epsilon > 0$ small enough, independent of λ, and λ big enough we can absorb the second term in the right-hand side of (2.35) and obtain (1.7) in the general case when $\alpha_1 = \alpha_2 = 0$. For all multi-indices α_1 and α_2 the estimate (1.7) follows from (1.7) with $\alpha_1 = \alpha_2 = 0$ and Lemma 2.1 in the same way as above.

To prove (1.11) we will use the commutator identity

$$\partial_r^2 + \frac{\Delta_w - c_n}{r^2} + \frac{1}{2}\left[r\partial_r, \partial_r^2 + \frac{\Delta_w - c_n}{r^2}\right] = 0. \qquad (2.36)$$

We obtain from (2.36) that the operators $\widetilde{G} = r^{(n-1)/2} G r^{-(n-1)/2}$ and $\widetilde{\Delta} = r^{(n-1)/2}\Delta r^{-(n-1)/2}$ satisfy the identity

$$\widetilde{G} + \frac{1}{2}\left[r\partial_r, \widetilde{G}\right] = \widetilde{G} + \widetilde{\Delta} + \frac{1}{2}\left[r\partial_r, \widetilde{G} + \widetilde{\Delta}\right] := \mathcal{Q}. \qquad (2.37)$$

We rewrite (2.37) as follows

$$\widetilde{G} - \lambda^2 + i\varepsilon + \frac{1}{2}\left[r\partial_r, \widetilde{G} - \lambda^2 + i\varepsilon\right] = -\lambda^2 + i\varepsilon + \mathcal{Q},$$

which yields the identity

$$\begin{aligned}
&\left(\widetilde{G} - \lambda^2 + i\varepsilon\right)^{-1} - \frac{1}{2}\left[r\partial_r, \left(\widetilde{G} - \lambda^2 + i\varepsilon\right)^{-1}\right] \\
&\quad = (-\lambda^2 + i\varepsilon)\left(\widetilde{G} - \lambda^2 + i\varepsilon\right)^{-2} + \left(\widetilde{G} - \lambda^2 + i\varepsilon\right)^{-1} \mathcal{Q} \left(\widetilde{G} - \lambda^2 + i\varepsilon\right)^{-1}. \qquad (2.38)
\end{aligned}$$

Set

$$\begin{aligned}
\widetilde{W}^L &= V^L(rw) + |b(rw)|^2, \\
\widetilde{W}^S &= V^S(rw) - i(n-1)r^{-1}\sum_{j=1}^n w_j b_j(rw).
\end{aligned}$$

Observe now that

$$\mathcal{Q} = \frac{1}{2r}\frac{\partial(r^2\widetilde{W}^L)}{\partial r} + \frac{1}{2}\left(\widetilde{W}^S + \partial_r r\widetilde{W}^S - r\widetilde{W}^S\partial_r\right)$$
$$+\frac{i}{2}\sum_{j=1}^{n} w_j\left(\frac{\partial(rb_j)}{\partial r}\partial_r + \partial_r\frac{\partial(rb_j)}{\partial r}\right)$$
$$+\frac{i}{2r}\sum_{j=1}^{n}\left(\frac{\partial(rb_j)}{\partial r}Q_j(w,\partial_w) + Q_j(w,\partial_w)\frac{\partial(rb_j)}{\partial r}\right).$$

It follows from the assumptions (1.4), (1.8), (1.9) and (1.10) that

$$\left|\frac{1}{r}\frac{\partial(r^2\widetilde{W}^L)}{\partial r}\right| + \left|\frac{\partial(rb)}{\partial r}\right| + \langle r\rangle\left|\widetilde{W}^S\right| \le \widetilde{C}\psi_\delta(r). \tag{2.39}$$

By (2.38) and (2.39) we obtain

$$\begin{aligned}
&\lambda^2\left\|\langle r\rangle^{-1}\psi_{\delta'}(r)^{1/2}\left(\widetilde{G}-\lambda^2+i\varepsilon\right)^{-2}\psi_{\delta'}(r)^{1/2}\langle r\rangle^{-1}\right\|_{L^2(X)\to L^2(X)}\\
&\le \left\|\psi_{\delta'}(r)^{1/2}\left(\widetilde{G}-\lambda^2+i\varepsilon\right)^{-1}\psi_{\delta'}(r)^{1/2}\right\|_{L^2(X)\to L^2(X)}\\
&+O(\lambda)\sum_{\pm}\left\|\psi_{\delta'}(r)^{1/2}\mathcal{D}_r\left(\widetilde{G}-\lambda^2\pm i\varepsilon\right)^{-1}\psi_{\delta'}(r)^{1/2}\right\|_{L^2(X)\to L^2(X)}\\
&+O(\lambda)\sum_{\pm}\left\|\psi_{\delta'}(r)^{1/2}\left(\widetilde{G}-\lambda^2\pm i\varepsilon\right)^{-1}\psi_{\delta'}(r)^{1/2}\right\|_{L^2(X)\to L^2(X)}\\
&\times\left\|\psi_{\delta'}(r)^{1/2}\mathcal{D}_r\left(\widetilde{G}-\lambda^2\mp i\varepsilon\right)^{-1}\psi_{\delta'}(r)^{1/2}\right\|_{L^2(X)\to L^2(X)}\\
&+O(\lambda)\sum_{\pm}\left\|\psi_{\delta'}(r)^{1/2}\left(\widetilde{G}-\lambda^2\pm i\varepsilon\right)^{-1}\psi_{\delta'}(r)^{1/2}\right\|_{L^2(X)\to L^2(X)}\\
&\times\left\|\psi_{\delta'}(r)^{1/2}r^{-1}\Lambda_w^{1/2}\left(\widetilde{G}-\lambda^2\mp i\varepsilon\right)^{-1}\psi_{\delta'}(r)^{1/2}\right\|_{L^2(X)\to L^2(X)},
\end{aligned}\tag{2.40}$$

where we have used that $\psi_\delta \le \psi_{\delta'}$ for $\delta' \le \delta$. It is clear now that (1.11) with $\alpha_1 = \alpha_2 = 0$ follows from (1.7) and (2.40). Furthermore, it is easy to see that when $|\alpha_1| + |\alpha_2| \ge 1$ the estimate (1.11) follows from (1.7), (1.11) with $\alpha_1 = \alpha_2 = 0$ and Lemma 2.1. Indeed, we have

$$\begin{aligned}
&\left\|\langle x\rangle^{-\frac{3+\delta'}{2}}\partial_x^{\alpha_1}\left(G-\lambda^2\pm i\varepsilon\right)^{-2}\partial_x^{\alpha_2}\langle x\rangle^{-\frac{3+\delta'}{2}}\right\|_{L^2\to L^2}\\
&\le\left\|\langle x\rangle^{-\frac{3+\delta'}{2}}\partial_x^{\alpha_1}\left(G-i\lambda^2\right)^{-1}\langle x\rangle^{\frac{3+\delta'}{2}}\right\|_{L^2\to L^2}\\
&\times\left\|\langle x\rangle^{-\frac{3+\delta'}{2}}\left(G-i\lambda^2\right)\left(G-\lambda^2\pm i\varepsilon\right)^{-2}(G-i\lambda^2)\langle x\rangle^{-\frac{3+\delta'}{2}}\right\|_{L^2\to L^2}\\
&\times\left\|\langle x\rangle^{-\frac{3+\delta'}{2}}\partial_x^{\alpha_2}\left(G+i\lambda^2\right)^{-1}\langle x\rangle^{\frac{3+\delta'}{2}}\right\|_{L^2\to L^2}
\end{aligned}$$

$$
\begin{aligned}
&\leq O(\lambda^{|\alpha_1|+|\alpha_2|-4}) \sum_{k=0}^{2} \lambda^{2k} \left\| \langle x\rangle^{-\frac{3+\delta'}{2}} \left(G-\lambda^2 \pm i\varepsilon\right)^{-k} \langle x\rangle^{-\frac{3+\delta'}{2}} \right\|_{L^2\to L^2} \\
&\leq C\lambda^{|\alpha_1|+|\alpha_2|-2}.
\end{aligned}
$$

□

Proof of Corollary 1.2. We will first prove the assertion when $b(0)=0$. We will use Theorem 1.1 and the fact that the constant C in the right-hand side of (1.7) depends only on the parameter δ', provided $\delta' \leq \delta$ (an argument already used above in the case when $b^S \equiv 0, V^S \equiv 0$ and which is true in the general case). Since $b(0)=0$ and the function b is continuous in r, given any $\epsilon>0$ there is $0<\theta\leq 1$ so that $|b(x)|\leq\epsilon$ for $|x|\leq\theta$. Let $\zeta\in C_0^\infty(\mathbf{R}), 0\leq\zeta\leq 1, \zeta(\tau)=1$ for $|\tau|\leq 1/2, \zeta(\tau)=0$ for $|\tau|\geq 1$. We are going to apply Theorem 1.1 to the operator

$$G_2 = -\Delta + i(1-\zeta)(|x|/\theta)b(x)\cdot\nabla + i\nabla\cdot b(x)(1-\zeta)(|x|/\theta) + V(x) + |b(x)|^2 .$$

Let $\chi\in C^\infty(\mathbf{R}), 0\leq\chi\leq 1, \chi(r)=0$ for $r\leq r_0+1, \chi(r)=1$ for $r\geq r_0+2$. Set

$$
\begin{aligned}
&\widetilde{b}^L(x) = \chi(|x|)b^L(x), \quad \widetilde{V}^L(x) = \chi(|x|)V^L(x), \\
&\widetilde{b}^S(x) = (1-\zeta)(|x|/\theta)\left(b^S(x) + (1-\chi)(|x|)b^L(x)\right), \\
&\widetilde{V}^S(x) = V^S(x) + (1-\chi)(|x|)V^L(x) + \zeta(|x|/\theta)(2-\zeta(|x|/\theta))|b(x)|^2.
\end{aligned}
$$

It is easy to see that the operator G_2 is of the form

$$G_2 = \left(i\nabla + \widetilde{b}^L + \widetilde{b}^S\right)^2 + \widetilde{V}^L + \widetilde{V}^S,$$

and that the conditions of Corollary 1.2 imply that the functions $\widetilde{b}^L, \widetilde{b}^S, \widetilde{V}^L$ and $\widetilde{V}^S$ satisfy (1.1)–(1.6) with possibly a new constant $\delta>0$ independent of ϵ. Therefore, by Theorem 1.1 the operator G_2 satisfies the estimate (1.7) with a constant C in the right-hand side independent of ϵ. On the other hand, the difference $G-G_2$ is a first order differential operator of the form $O(\epsilon)\cdot\nabla+\nabla\cdot O(\epsilon)$ with coefficients supported in $|x|\leq 1$. Taking $\epsilon>0$ small enough, independent of λ, and proceeding in the same way as in the proof of (2.35) above, we obtain that the operator G satisfies (1.7), too.

Turn to the general case. Set $B(x) = b(x) - \nabla\varphi(x)$, where $\varphi(x) = \zeta(|x|)b(0)\cdot x$. Clearly, $B(0)=0$ and according to the analysis above the estimate (1.7) holds for the operator $(i\nabla+B)^2+V$. On the other hand, we have the identity

$$(i\nabla + b)^2 + V = \mathrm{e}^{i\varphi(x)}\left((i\nabla+B)^2+V\right)\mathrm{e}^{-i\varphi(x)},$$

which yields (1.7) for the operator $(i\nabla+b)^2+V$, too. □

3. Dispersive Estimates

Let $\varphi \in C_0^\infty((0,+\infty))$. It is easy to see that the estimates (1.23) and (1.24) follow from the following semi-classical dispersive estimates (e.g. see Sect. 2 of [3]).

Theorem 3.1. *Under the assumptions of Theorem 1.4, there exist constants $C, h_0 > 0$ such that for all $0 < h \le h_0, t \neq 0$, we have the estimate*

$$\left\| e^{it\sqrt{G}} \varphi(h\sqrt{G}) \right\|_{L^1 \to L^\infty} \le C h^{-3} |t|^{-1}. \tag{3.1}$$

Moreover, for every $\delta' > 0$ there exist $C, h_0 > 0$ such that for all $0 < h \le h_0, t \neq 0$, we have the estimate

$$\left\| e^{it\sqrt{G}} \varphi(h\sqrt{G}) \langle x \rangle^{-3/2-\delta'} \right\|_{L^2 \to L^\infty} \le C h^{-2} |t|^{-1}. \tag{3.2}$$

Proof. We are going to use the formula

$$e^{it\sqrt{G_0}} \varphi(h\sqrt{G_0}) = (\pi i)^{-1} \int_0^\infty e^{it\lambda} \varphi(h\lambda) \left(R_0^+(\lambda) - R_0^-(\lambda) \right) \lambda \mathrm{d}\lambda, \tag{3.3}$$

where $R_0^\pm(\lambda) = (G_0 - \lambda^2 \pm i0)^{-1}$ are the three-dimensional outgoing and incoming free resolvents with kernels given by

$$[R_0^\pm(\lambda)](x,y) = \frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|}.$$

We also have the formula

$$e^{it\sqrt{G}} \varphi(h\sqrt{G}) = (\pi i)^{-1} \int_0^\infty e^{it\lambda} \varphi(h\lambda) \left(R^+(\lambda) - R^-(\lambda) \right) \lambda \mathrm{d}\lambda, \tag{3.4}$$

where $R^\pm(\lambda) = (G - \lambda^2 \pm i0)^{-1}$ are the outgoing and incoming perturbed resolvents satisfying the relation

$$R^\pm(\lambda) - R_0^\pm(\lambda) = R_0^\pm(\lambda) L R^\pm(\lambda) =: T^\pm(\lambda) = T_1^\pm(\lambda) + T_2^\pm(\lambda), \tag{3.5}$$

where

$$T_1^\pm(\lambda) = R_0^\pm(\lambda) L R_0^\pm(\lambda), \quad T_2^\pm(\lambda) = R_0^\pm(\lambda) L R^\pm(\lambda) L R_0^\pm(\lambda),$$
$$L = G - G_0 = ib(x) \cdot \nabla + i\nabla \cdot b(x) + |b(x)|^2 + V(x).$$

In view of (3.3), (3.4) and (3.5) we can write

$$e^{it\sqrt{G}} \varphi(h\sqrt{G}) - e^{it\sqrt{G_0}} \varphi(h\sqrt{G_0}) = (i\pi h)^{-1} \int_0^\infty e^{it\lambda} \widetilde{\varphi}(h\lambda) T(\lambda) \mathrm{d}\lambda, \tag{3.6}$$

where we have put $\widetilde{\varphi}(\lambda) = \lambda\varphi(\lambda), T = T^+ - T^-$. It is easy to see that the estimates (3.1) and (3.2) follow from (3.6) and the following

Proposition 3.2. *The operator-valued functions* $T(\lambda) : L^1 \to L^\infty$ *and* $T(\lambda)\langle x\rangle^{-3/2-\delta'} : L^2 \to L^\infty$ *are* C^1 *for* λ *large enough and satisfy the estimates* (*with* $k = 0, 1$)

$$\left\|\partial_\lambda^k T(\lambda)\right\|_{L^1\to L^\infty} \le C\lambda, \tag{3.7}$$

$$\left\|\partial_\lambda^k T(\lambda)\langle x\rangle^{-3/2-\delta'}\right\|_{L^2\to L^\infty} \le C. \tag{3.8}$$

Proof. We will need the following properties of the three-dimensional free resolvent.

Lemma 3.3. *We have the estimates*

$$\left\|\partial_\lambda^k \left(R_0^+(\lambda) - R_0^-(\lambda)\right)\right\|_{L^1\to L^\infty} \le C\lambda, \quad k = 0, 1, \tag{3.9}$$

$$\left\|\partial_\lambda^k R_0^\pm(\lambda)\langle x\rangle^{-1/2-k-\delta'}\right\|_{L^2\to L^\infty} + \left\|\langle x\rangle^{-1/2-k-\delta'}\partial_\lambda^k R_0^\pm(\lambda)\right\|_{L^1\to L^2} \le C, \ \ k = 0, 1, \tag{3.10}$$

$$\left\|\partial_x^\alpha \partial_\lambda R_0^\pm(\lambda)\langle x\rangle^{-3/2-\delta'}\right\|_{L^2\to L^\infty} + \left\|\langle x\rangle^{-3/2-\delta'}\partial_\lambda R_0^\pm(\lambda)\partial_x^\alpha\right\|_{L^1\to L^2} \le C\lambda, \ \ |\alpha| = 1. \tag{3.11}$$

Moreover, if $|\alpha| = 1$, *given any* $\gamma > 0$ *independent of* λ *the operator* $\partial_x^\alpha R_0^\pm(\lambda)$ *can be decomposed as* $\mathcal{K}_{1,\alpha}^\pm(\lambda) + \mathcal{K}_{2,\alpha}$, *where*

$$\left\|\mathcal{K}_{1,\alpha}^\pm(\lambda)^*\langle x\rangle^{-1/2-\delta'}\right\|_{L^2\to L^\infty} + \left\|\langle x\rangle^{-1/2-\delta'}\mathcal{K}_{1,\alpha}^\pm(\lambda)\right\|_{L^1\to L^2} \le C_\gamma\lambda, \tag{3.12}$$

$$\left\|\mathcal{K}_{2,\alpha}^*\right\|_{L^\infty\to L^\infty} + \left\|\mathcal{K}_{2,\alpha}\right\|_{L^1\to L^1} \le \gamma. \tag{3.13}$$

Proof. The estimate (3.9) follows from the fact that the kernel of the operator

$$\partial_\lambda^k \left(R_0^+(\lambda) - R_0^-(\lambda)\right)$$

is $O(\lambda)$, while (3.10) follows from the fact that the kernel of the operator $\partial_\lambda^k R_0^\pm(\lambda)$ is $O\left(|x-y|^{k-1}\right)$ uniformly in λ. It is also easy to see that if $|\alpha| = 1$, the kernel of $\partial_x^\alpha \partial_\lambda R_0^\pm(\lambda)$ is $O(\lambda)$, which clearly implies (3.11). Furthermore, observe that the kernel of $\partial_x^\alpha R_0^\pm(\lambda)$ is equal to

$$\begin{aligned}
&\frac{\partial^\alpha |x-y|}{\partial x^\alpha}\left(\pm i\lambda\frac{\mathrm{e}^{\pm i\lambda|x-y|}}{|x-y|} - \frac{\mathrm{e}^{\pm i\lambda|x-y|}}{|x-y|^2}\right)\\
&\quad = \frac{\partial^\alpha |x-y|}{\partial x^\alpha}\left(\pm i\lambda\frac{\mathrm{e}^{\pm i\lambda|x-y|}}{|x-y|} + \frac{1-\mathrm{e}^{\pm i\lambda|x-y|}}{|x-y|^2} + \frac{\rho(|x-y|/\gamma') - 1}{|x-y|^2}\right)\\
&\qquad - \frac{\partial^\alpha |x-y|}{\partial x^\alpha}\frac{\rho(|x-y|/\gamma')}{|x-y|^2} := K_{1,\alpha}^\pm(x,y) + K_{2,\alpha}(x,y),
\end{aligned}$$

where $\gamma' > 0$ and $\rho \in C_0^\infty(\mathbf{R}), 0 \le \rho \le 1, \rho(\sigma) = 1$ for $|\sigma| \le 1, \rho(\sigma) = 0$ for $|\sigma| \ge 2$. Denote by $\mathcal{K}_{1,\alpha}^\pm(\lambda)$ (resp. $\mathcal{K}_{2,\alpha}$) the operator with kernel $K_{1,\alpha}^\pm$ (resp. $K_{2,\alpha}$). Clearly, $K_{1,\alpha}^\pm = O_{\gamma'}(\lambda)|x-y|^{-1}$, which implies (3.12). On the other hand, the left-hand side of (3.13) is upper bounded by

$$C\int_{\mathbf{R}^3}\frac{\rho(|x-y|/\gamma')}{|x-y|^2}\mathrm{d}y \leq C\int_{|z|\leq\gamma'}|z|^{-2}\mathrm{d}z \leq \widetilde{C}\gamma'. \tag{3.14}$$

Choosing $\gamma' = \gamma/\widetilde{C}$ we get (3.13). □

Using Theorem 1.1 and Lemma 3.3 together with (1.20) and the fact that the operator ∂_x^α commutes with the free resolvent, we obtain

$$\begin{aligned}
&\sum_{k=0}^{1}\left\|\partial_\lambda^k T(\lambda)\langle x\rangle^{-3/2-\delta'}\right\|_{L^2\to L^\infty} \leq \sum_{\pm}\left\|\frac{\mathrm{d}R_0^\pm(\lambda)}{\mathrm{d}\lambda}LR^\pm(\lambda)\langle x\rangle^{-3/2-\delta'}\right\|_{L^2\to L^\infty}\\
&\quad+\sum_{k=0}^{1}\left\|\left(R_0^+(\lambda)L\frac{\mathrm{d}^kR^+(\lambda)}{\mathrm{d}\lambda^k}-R_0^-(\lambda)L\frac{\mathrm{d}^kR^-(\lambda)}{\mathrm{d}\lambda^k}\right)\langle x\rangle^{-3/2-\delta'}\right\|_{L^2\to L^\infty}\\
&\leq C\sum_{\pm}\sum_{0\leq|\alpha_1|+|\alpha_2|\leq 1}\left\|\partial_x^{\alpha_1}\partial_\lambda R_0^\pm(\lambda)\langle x\rangle^{-3/2-\delta/2}\right\|_{L^2\to L^\infty}\\
&\quad\times\left\|\langle x\rangle^{-1/2-\delta/2}\partial_x^{\alpha_2}R^\pm(\lambda)\langle x\rangle^{-3/2-\delta'}\right\|_{L^2\to L^2}\\
&\quad+C\sum_{\pm}\sum_{k=0}^{1}\sum_{0\leq|\alpha|\leq 1}\left\|R_0^\pm(\lambda)\langle x\rangle^{-1/2-\delta/2}\right\|_{L^2\to L^\infty}\\
&\quad\times\left\|\langle x\rangle^{-3/2-\delta/2}\partial_x^\alpha\partial_\lambda^k R^\pm(\lambda)\langle x\rangle^{-3/2-\delta'}\right\|_{L^2\to L^2}\\
&\quad+C\sum_{\pm}\sum_{k=0}^{1}\sum_{|\alpha|=1}\left\|\mathcal{K}_{1,\alpha}^\mp(\lambda)^*\langle x\rangle^{-1/2-\delta/2}\right\|_{L^2\to L^\infty}\\
&\quad\times\left\|\langle x\rangle^{-3/2-\delta/2}\partial_\lambda^k R^\pm(\lambda)\langle x\rangle^{-3/2-\delta'}\right\|_{L^2\to L^2}\\
&\quad+C\sum_{k=0}^{1}\sum_{|\alpha|=1}\left\|\mathcal{K}_{2,\alpha}^*\right\|_{L^\infty\to L^\infty}\left\|\left(\partial_\lambda^k R^+(\lambda)-\partial_\lambda^k R^-(\lambda)\right)\langle x\rangle^{-3/2-\delta'}\right\|_{L^2\to L^\infty}\\
&\leq C_\gamma + O(\gamma)\sum_{k=0}^{1}\left\|\partial_\lambda^k T(\lambda)\langle x\rangle^{-3/2-\delta'}\right\|_{L^2\to L^\infty}
\end{aligned}\tag{3.15}$$

for every $\gamma > 0$. Taking γ small enough we can absorb the second term in the right-hand side of (3.15) and get (3.8). Let us see now that the operator $T_1 = T_1^+ - T_1^-$ satisfies (3.7). By Lemma 3.3 we have

$$\begin{aligned}
&\sum_{k=0}^{1}\left\|\partial_\lambda^k T_1(\lambda)\right\|_{L^1\to L^\infty} \leq \sum_{k=0}^{1}\left\|\frac{\mathrm{d}^kR_0^+(\lambda)}{\mathrm{d}\lambda^k}LR_0^+(\lambda)-\frac{\mathrm{d}^kR_0^-(\lambda)}{\mathrm{d}\lambda^k}LR_0^-(\lambda)\right\|_{L^1\to L^\infty}\\
&\quad+\left\|R_0^+(\lambda)L\frac{\mathrm{d}R_0^+(\lambda)}{\mathrm{d}\lambda}-R_0^-(\lambda)L\frac{\mathrm{d}R_0^-(\lambda)}{\mathrm{d}\lambda}\right\|_{L^1\to L^\infty}\\
&\leq C\sum_{\pm}\sum_{k=0}^{1}\sum_{|\alpha|\leq 1}\left\|\partial_x^\alpha\partial_\lambda^k R_0^\pm(\lambda)\langle x\rangle^{-3/2-\delta/2}\right\|_{L^2\to L^\infty}\left\|\langle x\rangle^{-1/2-\delta/2}R_0^\pm(\lambda)\right\|_{L^1\to L^2}
\end{aligned}$$

$$+C\sum_{\pm}\sum_{|\alpha|\le 1}\left\|R_0^{\pm}(\lambda)\langle x\rangle^{-1/2-\delta/2}\right\|_{L^2\to L^\infty}\left\|\langle x\rangle^{-3/2-\delta/2}\partial_x^\alpha\partial_\lambda R_0^{\pm}(\lambda)\right\|_{L^1\to L^2}$$
$$+C\sum_{\pm}\sum_{k=0}^{1}\sum_{|\alpha|=1}\left\|\partial_\lambda^k R_0^{\pm}(\lambda)\langle x\rangle^{-3/2-\delta/2}\right\|_{L^2\to L^\infty}\left\|\langle x\rangle^{-1/2-\delta/2}\mathcal{K}_{1,\alpha}^{\pm}(\lambda)\right\|_{L^1\to L^2}$$
$$+C\sum_{\pm}\sum_{|\alpha|=1}\left\|\mathcal{K}_{1,\alpha}^{\mp}(\lambda)^*\langle x\rangle^{-1/2-\delta/2}\right\|_{L^2\to L^\infty}\left\|\langle x\rangle^{-3/2-\delta/2}\partial_\lambda R_0^{\pm}(\lambda)\right\|_{L^1\to L^2}$$
$$+C\sum_{k=0}^{1}\sum_{|\alpha|=1}\left\|\partial_\lambda^k\left(R_0^{+}(\lambda)-R_0^{-}(\lambda)\right)\right\|_{L^1\to L^\infty}$$
$$\times\left(\|\mathcal{K}_{2,\alpha}\|_{L^1\to L^1}+\left\|\mathcal{K}_{2,\alpha}^*\right\|_{L^\infty\to L^\infty}\right)\le C\lambda. \tag{3.16}$$

Given a multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ such that $|\alpha|\le 1$, define the function b_α as follows: $b_0=\left(|b|^2+V\right)/2$, and if $|\alpha|=1,\alpha_j=1$, then $b_\alpha:=b_j$. The operator $T_2=T_2^+-T_2^-$ satisfies

$$\sum_{k=0}^{1}\left\|\partial_\lambda^k T_2(\lambda)\right\|_{L^1\to L^\infty}$$
$$\le\sum_{k_1+k_2+k_3\le 1}\left\|\sum_{\pm}\pm\frac{\mathrm{d}^{k_1}R_0^{\pm}(\lambda)}{\mathrm{d}\lambda^{k_1}}L\frac{\mathrm{d}^{k_2}R^{\pm}(\lambda)}{\mathrm{d}\lambda^{k_2}}L\frac{\mathrm{d}^{k_3}R_0^{\pm}(\lambda)}{\mathrm{d}\lambda^{k_3}}\right\|_{L^1\to L^\infty}$$
$$\le\sum_{k_1+k_2+k_3\le 1}\sum_{\substack{|\alpha_1|,|\alpha_2|,|\beta_1|,|\beta_2|\le 1,\\ |\alpha_1|+|\alpha_2|\le 1,|\beta_1|+|\beta_2|\le 1}}A_{k_1,k_2,k_3}^{\alpha_1,\alpha_2,\beta_1,\beta_2}(\lambda)=:\mathcal{A}(\lambda), \tag{3.17}$$

where

$$A_{k_1,k_2,k_3}^{\alpha_1,\alpha_2,\beta_1,\beta_2}(\lambda)$$
$$=\left\|\sum_{\pm}\pm\frac{\mathrm{d}^{k_1}R_0^{\pm}(\lambda)}{\mathrm{d}\lambda^{k_1}}\partial_x^{\alpha_1}b_{\alpha_1,\alpha_2}\partial_x^{\alpha_2}\frac{\mathrm{d}^{k_2}R^{\pm}(\lambda)}{\mathrm{d}\lambda^{k_2}}\partial_x^{\beta_2}b_{\beta_1,\beta_2}\partial_x^{\beta_1}\frac{\mathrm{d}^{k_3}R_0^{\pm}(\lambda)}{\mathrm{d}\lambda^{k_3}}\right\|_{L^1\to L^\infty},$$

where $b_{\alpha_1,\alpha_2}=b_{\alpha_1}$ if $\alpha_2=0, b_{\alpha_1,\alpha_2}=b_{\alpha_2}$ if $\alpha_1=0$. To bound these norms we will consider several cases.

Case 1. $\alpha_1=\beta_1=0$. By Theorem 1.1, Lemma 3.3 and (1.20), we have

$$A_{k_1,k_2,k_3}^{0,\alpha_2,0,\beta_2}(\lambda)\le C\sum_{\pm}\left\|\frac{\mathrm{d}^{k_1}R_0^{\pm}(\lambda)}{\mathrm{d}\lambda^{k_1}}\langle x\rangle^{-1/2-k_1-\delta/2}\right\|_{L^2\to L^\infty}$$
$$\times\left\|\langle x\rangle^{-1/2-k_2-\delta/2}\partial_x^{\alpha_2}\frac{\mathrm{d}^{k_2}R^{\pm}(\lambda)}{\mathrm{d}\lambda^{k_2}}\partial_x^{\beta_2}\langle x\rangle^{-1/2-k_2-\delta/2}\right\|_{L^2\to L^2}$$
$$\times\left\|\langle x\rangle^{-1/2-k_3-\delta/2}\frac{\mathrm{d}^{k_3}R_0^{\pm}(\lambda)}{\mathrm{d}\lambda^{k_3}}\right\|_{L^1\to L^2}$$
$$\le O(\lambda)\left\|\langle x\rangle^{-1/2-k_2-\delta/2}\partial_x^{\alpha_2}R^{\pm}(\lambda)^{1+k_2}\partial_x^{\beta_2}\langle x\rangle^{-1/2-k_2-\delta/2}\right\|_{L^2\to L^2}\le C\lambda. \tag{3.18}$$

Case 2. $|\alpha_1|+|\beta_1|\ge 1, k_1=1$ if $|\alpha_1|=1$ and $k_3=1$ if $|\beta_1|=1$. This case is treated in precisely the same way as Case 1.

Case 3. $k_1 = k_2 = 0, k_3 = 1, |\alpha_1| = 1, \alpha_2 = 0$. By Theorem 1.1, Lemma 3.3 and (1.20), we have

$$\begin{aligned}
A_{0,0,1}^{\alpha_1,0,\beta_1,\beta_2}(\lambda) \leq C \sum_{\pm} & \left\| \mathcal{K}_{1,\alpha_1}^{\mp}(\lambda)^* \langle x \rangle^{-1/2-\delta/2} \right\|_{L^2 \to L^\infty} \\
& \times \left\| \langle x \rangle^{-1/2-\delta/2} R^{\pm}(\lambda) \partial_x^{\beta_2} \langle x \rangle^{-1/2-\delta/2} \right\|_{L^2 \to L^2} \\
& \times \left\| \langle x \rangle^{-3/2-\delta/2} \partial_x^{\beta_1} \partial_\lambda R_0^{\pm}(\lambda) \right\|_{L^1 \to L^2} \\
& + C \left\| \mathcal{K}_{2,\alpha_1}^* \right\|_{L^\infty \to L^\infty} \left\| \sum_{\pm} \pm R^{\pm}(\lambda) \partial_x^{\beta_2} b_{\beta_1,\beta_2} \partial_x^{\beta_1} \partial_\lambda R_0^{\pm}(\lambda) \right\|_{L^1 \to L^\infty} \\
\leq C_\gamma \lambda + O(\gamma) & \left\| \sum_{\pm} \pm R_0^{\pm}(\lambda) \partial_x^{\beta_2} b_{\beta_1,\beta_2} \partial_x^{\beta_1} \partial_\lambda R_0^{\pm}(\lambda) \right\|_{L^1 \to L^\infty} \\
+ O(\gamma) & \left\| \sum_{\pm} \pm R_0^{\pm}(\lambda) L R^{\pm}(\lambda) \partial_x^{\beta_2} b_{\beta_1,\beta_2} \partial_x^{\beta_1} \partial_\lambda R_0^{\pm}(\lambda) \right\|_{L^1 \to L^\infty}.
\end{aligned} \tag{3.19}$$

In the same way as in the proof of (3.16) one can see that the second term in the right-hand side of (3.19) is $O(\lambda)$. On the other hand, it is clear that the third one is bounded by $O(\gamma)\mathcal{A}(\lambda)$. In other words, (3.19) yields

$$A_{0,0,1}^{\alpha_1,0,\beta_1,\beta_2}(\lambda) \leq C_\gamma \lambda + O(\gamma)\mathcal{A}(\lambda). \tag{3.20}$$

Case 4. $k_1 = 1, k_2 = k_3 = 0, |\beta_1| = 1, \beta_2 = 0$. This case is treated in the same way as Case 3.

Case 5. $k_1 = k_3 = 0, k_2 = 1, |\alpha_1| = |\beta_1| = 1, \alpha_2 = \beta_2 = 0$. By Theorem 1.1, Lemma 3.3 and (1.20), we have

$$\begin{aligned}
A_{0,1,0}^{\alpha_1,0,\beta_1,0}(\lambda) \leq C \sum_{\pm} & \left\| \mathcal{K}_{1,\alpha_1}^{\mp}(\lambda)^* \langle x \rangle^{-1/2-\delta/2} \right\|_{L^2 \to L^\infty} \\
& \times \left\| \langle x \rangle^{-3/2-\delta/2} \frac{\mathrm{d}R^{\pm}(\lambda)}{\mathrm{d}\lambda} \langle x \rangle^{-3/2-\delta/2} \right\|_{L^2 \to L^2} \\
& \times \left\| \langle x \rangle^{-1/2-\delta/2} \mathcal{K}_{1,\beta_1}^{\pm}(\lambda) \right\|_{L^1 \to L^2} \\
& + C \left\| \mathcal{K}_{2,\alpha_1}^* \right\|_{L^\infty \to L^\infty} \left\| \sum_{\pm} \pm \frac{\mathrm{d}R^{\pm}(\lambda)}{\mathrm{d}\lambda} b_{\beta_1,0} \partial_x^{\beta_1} R_0^{\pm}(\lambda) \right\|_{L^1 \to L^\infty} \\
& + C \left\| \mathcal{K}_{2,\beta_1} \right\|_{L^1 \to L^1} \left\| \sum_{\pm} \pm R_0^{\pm}(\lambda) \partial_x^{\alpha_1} b_{\alpha_1,0} \frac{\mathrm{d}R^{\pm}(\lambda)}{\mathrm{d}\lambda} \right\|_{L^1 \to L^\infty} \\
& + C \left\| \mathcal{K}_{2,\alpha_1}^* \right\|_{L^\infty \to L^\infty} \left\| \mathcal{K}_{2,\beta_1} \right\|_{L^1 \to L^1} \left\| \sum_{\pm} \pm \frac{\mathrm{d}R^{\pm}(\lambda)}{\mathrm{d}\lambda} \right\|_{L^1 \to L^\infty}.
\end{aligned} \tag{3.21}$$

By (3.9), (3.16) and (3.17), we have

$$\begin{aligned}
&\left\| \frac{\mathrm{d}(R^{+}(\lambda) - R^{-}(\lambda))}{\mathrm{d}\lambda} \right\|_{L^1 \to L^\infty} \\
&\quad \leq \left\| \frac{\mathrm{d}(R_0^{+}(\lambda) - R_0^{-}(\lambda))}{\mathrm{d}\lambda} \right\|_{L^1 \to L^\infty} + \left\| \frac{\mathrm{d}T(\lambda)}{\mathrm{d}\lambda} \right\|_{L^1 \to L^\infty} \\
&\quad \leq C\lambda + \left\| \frac{\mathrm{d}T_2(\lambda)}{\mathrm{d}\lambda} \right\|_{L^1 \to L^\infty} \leq C\lambda + \mathcal{A}(\lambda). \qquad (3.22)
\end{aligned}$$

Similarly, one can easily see that the second and the third terms in the right-hand side of (3.21) are bounded by $C\lambda + O(\gamma)\mathcal{A}(\lambda)$. Thus we obtain:

$$A_{0,1,0}^{\alpha_1,0,\beta_1,0}(\lambda) \leq C_\gamma \lambda + O(\gamma)\mathcal{A}(\lambda). \qquad (3.23)$$

Summing up the above inequalities we conclude:

$$\mathcal{A}(\lambda) \leq C_\gamma \lambda + O(\gamma)\mathcal{A}(\lambda). \qquad (3.24)$$

Taking $\gamma > 0$ small enough, independent of λ, we can absorb the second term in the right-hand side of (3.24) and conclude that $\mathcal{A}(\lambda) = O(\lambda)$. This together with (3.16) and (3.17) imply (3.7). □

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