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Recursion Between Mumford Volumes of Moduli Spaces

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Abstract. We propose a new proof, as well as a generalization of Mirzakhani's recursion for volumes of moduli spaces. We interpret those recursion relations in terms of expectation values in Kontsevich's integral, i.e., we relate them to a ribbon graph decomposition of Riemann surfaces. We find a generalization of Mirzakhani's recursions to measures containing all higher Mumford's κ classes, and not only κ_1 as in the Weil–Petersson case.

1. Introduction

Let

$$\mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_{g,n}(L_1,\ldots,L_n)) \tag{1.1}$$

be the volume (measured with Weil–Petersson's measure) of the moduli space of genus g curves with n geodesic boundaries of length $L_1,\ldots,L_n$. Maryam Mirzakhani found a good recursion relation [11,12] for those functions, allowing to compute all of them in principle. This relation has received several proofs [10,13], and we provide one more proof, more "matrix model oriented".

The main interest of our method is that it easily generalizes to a larger class of measures, containing all Mumford classes κ, which should also prove the result of Liu and Xu [10].

In fact, our recursion relations are those of [7], and they should be generalizable to a much larger set of measures, not only those based on Kontsevich's hyperelliptical spectral curve, but also rational spectral curves. For instance, they hold for the generalized Kontsevich integral, the spectral curve of which is not hyperelliptical, i.e., they should hopefully allow to compute also some sort of volumes of moduli spaces of stable maps with spin structures.

In [5], it was observed that after Laplace transform, Mirzakhani's recursion became identical to the solution of loop equations [7] for Kontsevich's matrix integral. Based on that remark, we are in a position to reprove Mirzakhani's result, and in fact we prove something more general.

Consider an arbitrary set of Kontsevich KdV times[1] $t_{2d+3}, d = 0, 1, \ldots, \infty$, we define their conjugated times $\tilde{t}_k, k = 0, 1, \ldots, \infty$, by:

$$f(z)=\sum_{a=1}^{\infty} \frac{(2a+1)!}{a!}\, \frac{t_{2a+3}}{2-t_3}\, z^a \quad \to \quad \tilde{f}(z)=-\ln(1-f(z))=\sum_{b=1}^{\infty} \tilde{t}_b\, z^b \quad (1.2)$$

Then we prove the following theorem:

Theorem 1.1. *Given a set of conjugated Kontsevich times $\tilde{t}_0, \tilde{t}_1, \tilde{t}_2, \ldots$, the following "Mumford volumes",*

$$W_{g,n}(z_1,\ldots,z_n) = 2^{-d_{g,n}}(t_3-2)^{2-2g-n} \sum_{d_0+d_1+\cdots+d_n=d_{g,n}} \sum_{k=1}^{d_0} \frac{1}{k!}$$
$$\times \sum_{b_1+\cdots+b_k=d_0, b_i>0} \prod_{i=1}^{n} \frac{2d_i+1!}{d_i!}\, \frac{dz_i}{z_i^{2d_i+2}} \prod_{l=1}^{k} \tilde{t}_{b_l} \left\langle \prod_{l=1}^{k} \kappa_{b_l} \prod_{i=1}^{n} \psi_i^{d_i} \right\rangle_{g,n} \quad (1.3)$$

where $d_{g,n} = 3g-3+n = \dim \mathcal{M}_{g,n}$, satisfy the following recursion relations (where $K = \{z_1,\ldots,z_n\}$):

$$W_{0,1} = 0 \qquad W_{0,2}(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2}$$

$$W_{g,n+1}(K, z_{n+1})$$
$$= \frac{1}{2} \operatorname*{Res}_{z\to 0} \frac{dz_{n+1}}{\left(z_{n+1}^2 - z^2\right)(y(z)-y(-z))dz} \Bigg[W_{g-1,n+2}(z,-z,K)$$
$$+ \sum_{h=0}^{g} \sum_{J\subset K} W_{h,1+|J|}(z,J)\, W_{g-h,1+n-|J|}(-z,K/J) \Bigg] \quad (1.4)$$

where

$$y(z) = z - \frac{1}{2} \sum_{k=0}^{\infty} t_{2k+3}\, z^{2k+1} \quad (1.5)$$

From Theorem 1.1, we obtain as an immediate consequence if $t_{2d+3} = -\frac{(2i\pi)^{2d}}{2d+1!} + 2\delta_{d,0}$, i.e., $\tilde{t}_1 = 4\pi^2$ and $\tilde{t}_k = 0$ for $k > 1$, and after Laplace transform:

Corollary 1.1. *The Weil–Petersson volumes satisfy Mirzakhani's recursions.*

The proof of Theorem 1.1 is detailed in the next sections, and it can be sketched as follows:

- We first define some $W_{g,n}(z_1,\ldots,z_n)$ which obey the recursion relations of [7], i.e., Eq. 1.4. In other words, we define them as the solution of the recursion, without knowing what they compute.

[1] Our definition of times t_k slightly differs from the usual one; we have $t_k = \frac{1}{N}\operatorname{Tr}\Lambda^{-k}$.

- We prove that those $W_{g,n}(z_1,\ldots,z_n)$ correspond to some expectation values in the Kontsevich integral $Z(\Lambda)=\int \mathrm{d}M\ \mathrm{e}^{-N\,\mathrm{Tr}\left(\frac{M^3}{3}-M\Lambda^2\right)}$, where $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, and $t_k=\frac{1}{N}\,\mathrm{Tr}\,\Lambda^{-k}$, of the form:
$$\frac{W_{g,n}(\lambda_{i_1},\ldots,\lambda_{i_n})}{\mathrm{d}x(\lambda_{i_1})\ldots\mathrm{d}x(\lambda_{i_n})}=(-1)^n\,\langle M_{i_1,i_1}\ldots M_{i_n,i_n}\rangle_c^{(g)} \tag{1.6}$$
- Then we expand $\langle M_{i_1,i_1}\ldots M_{i_n,i_n}\rangle$ into Feynman ribbon graphs, which are in bijection with a cell decomposition of $\mathcal{M}_{g,n}^{\mathrm{comb}}$ (like in Kontsevich's first works), and the value of each of those Feynman graphs is precisely the Laplace transform of the volume of the corresponding cell.
- the sum over all cells yields the expected result: the inverse Laplace transforms of $W_{g,n}$ are the volumes $V_{g,n}$, and, by definition, they satisfy the recursion relations.
- In fact, the volumes are first written in terms of the first Chern classes ψ_i in formula 2.31, and after some combinatorics, we find it more convenient to rewrite them in terms of Mumford κ classes.

Then, we specialize our theorem to some choices of times t_k's, in particular the following:

- The first example is $t_{2d+3}=-\frac{(2i\pi)^{2d}}{2d+1!}+2\delta_{d,0}$, in which case $V_{g,n}$ the Laplace transform of $W_{g,n}$ are the Weil–Petersson volumes, and thus we recover Mirzakhani's recursions.
- Our second example is $t_k=\lambda^{-k}$, i.e., $\Lambda=\lambda\,\mathrm{Id}$, for which the Kontsevich integral reduces to a standard matrix model, and for which the $W_{g,n}$ are known to count triangulated maps, i.e., discrete surfaces with the discrete Regge metrics (metrics whose curvature is localized on vertices and edges). We are thus able to associate some class to that discrete measure on $\mathcal{M}_{g,n}$. And we have a formula which interpolates between the enumeration of maps and the enumeration of Riemann surfaces, in agreement with the spirit of 2d-quantum gravity in the 1980s [2,4,16].

2. Proof of the Theorem

2.1. Kontsevich's Integral

In his very famous work [9], Maxim Kontsevich introduced the following matrix integral as a generating function for intersection numbers

$$\begin{aligned} Z(\Lambda)&=\int \mathrm{d}M\ \mathrm{e}^{-N\,\mathrm{Tr}\left(\frac{M^3}{3}-M\left(\Lambda^2+t_1\right)\right)}\\ &=\mathrm{e}^{\frac{2N}{3}\,\mathrm{Tr}\,\Lambda^3+Nt_1\,\mathrm{Tr}\,\Lambda}\int \mathrm{d}M\ \mathrm{e}^{-N\,\mathrm{Tr}\left(\frac{M^3}{3}+M^2\Lambda-t_1M\right)} \end{aligned} \tag{2.1}$$

where the integral is a formal integral over hermitian matrices M of size N, and Λ is a fixed diagonal matrix

$$\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_n),\quad t_k=\frac{1}{N}\,\mathrm{Tr}\,\Lambda^{-k} \tag{2.2}$$

Throughout all this article, we shall assume $t_1 = 0$, since none of the quantities we are interested in here depend on t_1 (see symplectic invariance in [7], or see [3]).

In [7], a method to compute the topological expansion of such matrix integrals was developed. We first define the Kontsevich's spectral curve:

Definition 2.1. The spectral curve of $Z(\Lambda)$ is the rational plane curve of equation:

$$\mathcal{E}(x,y)=y^2-x-\frac{y}{N}\,\mathrm{Tr}\,\frac{1}{x-t_1-\Lambda^2}-\frac{1}{N}\left\langle \mathrm{Tr}\,\frac{1}{x-t_1-\Lambda^2}\,M\right\rangle^{(0)}=0 \quad (2.3)$$

i.e., it has the following rational uniformization

$$\mathcal{E}(x,y)=\begin{cases} x(z)=z^2+t_1 \\ y(z)=z+\frac{1}{2N}\,\mathrm{Tr}\,\frac{1}{\Lambda(z-\Lambda)}=z-\frac{1}{2}\sum_{k=0}^{\infty} t_{k+2}z^k \end{cases} \quad (2.4)$$

Then we define (i.e., the algebraic invariants of [7]):

Definition 2.2. We define the correlators:

$$W_{0,1}=0 \qquad W_{0,2}(z_1,z_2)=\frac{dz_1 dz_2}{(z_1-z_2)^2} \quad (2.5)$$

and we define by recursion on $2g-2+n$, the symmetric[2] form $W_{g,n+1}(z_0,z_1,\ldots,z_n)$ by (we write $K=\{z_1,\ldots,z_n\}$):

$$\begin{aligned} &W_{g,n+1}(K,z_{n+1}) \\ &\quad = \operatorname*{Res}_{z\to 0} \frac{z\,dz_{n+1}}{\left(z_{n+1}^2-z^2\right)(y(z)-y(-z))\mathrm{d}x(z)} \Bigg[W_{g-1,n+2}(z,-z,K) \\ &\qquad + \sum_{h=0}^{g}\sum_{J\subset K} W_{h,1+|J|}(z,J)\,W_{g-h,1+n-|J|}(-z,K/J)\Bigg] \end{aligned} \quad (2.6)$$

Then, if $\mathrm{d}\Phi = y\mathrm{d}x$, we define for $g>1$:

$$F_g=\frac{1}{2g-2}\operatorname*{Res}_{z\to 0}\Phi(z)W_{g,1}(z) \quad (2.7)$$

(there is a separate definition of F_g for $g=0,1$, but we shall not use it here).

We recall the result of [7] (which uses also [8]):

Theorem 2.1.

$$\ln Z=\sum_{g=0}^{\infty} N^{2-2g}F_g \quad (2.8)$$

Now, we prove the more elaborate result:

[2] The non-obvious fact that this is symmetric in its $n+1$ variables is proved by recursion in [7].

Theorem 2.2. *if* $i_1, \ldots, i_n$ *are* n *distinct integers in* $[1, N]$, *then:*

$$\boxed{\frac{W_n^{(g)}(\lambda_{i_1}, \ldots, \lambda_{i_n})}{\mathrm{d}x(\lambda_{i_1}) \ldots \mathrm{d}x(\lambda_{i_n})} = \langle M_{i_1,i_1} \ldots M_{i_n,i_n} \rangle_c^{(g)}} \tag{2.9}$$

where $\langle \cdot \rangle$ *means the formal expectation value with respect to the measure used to define* Z, *the subscript* c *means connected part or cumulant, and the subscript* (g) *means the* g^{th} *term in the* $1/N^2$ *topological expansion.*

In other words, the $W_{g,n}$ compute some expectation values in the Kontsevich integral, which are not the same as those computed by [3].

Proof. From Eq. 2.1, it is easy to see that:

$$N^{-n} \frac{\partial^n \ln Z}{\partial \lambda_{i_1} \ldots \partial \lambda_{i_n}} = 2^n \lambda_{i_1} \ldots \lambda_{i_n} \ \langle M_{i_1,i_1} \ldots M_{i_n,i_n} \rangle_c \tag{2.10}$$

i.e., to order N^{2-2g-n}:

$$\frac{\partial^n F_g}{\partial \lambda_{i_1} \ldots \partial \lambda_{i_n}} = 2^n \lambda_{i_1} \ldots \lambda_{i_n} \ \langle M_{i_1,i_1} \ldots M_{i_n,i_n} \rangle_c^{(g)} \tag{2.11}$$

Now, let us compute $\frac{\partial F_g}{\partial \lambda_i}$ with the method of [7].

Consider an infinitesimal variation of the matrix Λ: $\lambda_i \to \lambda_i + \delta\lambda_i$ (we assume $\delta t_1 = 0$). It translates into the following variations of the function $y(z)$:

$$\delta y(z) = \frac{1}{2Nz} \ \mathrm{Tr} \, \frac{\delta \Lambda}{(z - \Lambda)^2} \tag{2.12}$$

and thus the form:

$$\begin{aligned} -\delta y(z)\mathrm{d}x(z) &= d \left(\frac{1}{N} \ \mathrm{Tr} \, \frac{\delta\Lambda}{z - \Lambda} \right) \\ &= \operatorname*{Res}_{\zeta \to z} \frac{1}{(z-\zeta)^2} \frac{1}{N} \ \mathrm{Tr} \, \frac{\delta\Lambda}{\zeta - \Lambda} \\ &= -\sum_i \operatorname*{Res}_{\zeta \to \lambda_i} \frac{1}{(z-\zeta)^2} \frac{1}{N} \ \mathrm{Tr} \, \frac{\delta\Lambda}{\zeta - \Lambda} \end{aligned} \tag{2.13}$$

Then, using theorem 5.1 of [7], we have:

$$\begin{aligned} \delta F_g &= \sum_i \operatorname*{Res}_{\zeta \to \lambda_i} W_1^{(g)}(\zeta) \frac{1}{N} \ \mathrm{Tr} \, \frac{\delta\Lambda}{\zeta - \Lambda} \\ &= \sum_i \frac{W_1^{(g)}(\lambda_i)}{\mathrm{d}\lambda_i} \frac{\delta\lambda_i}{N} \end{aligned} \tag{2.14}$$

i.e.,

$$W_1^{(g)}(\lambda_i) = \langle M_{ii} \rangle^{(g)} \ \mathrm{d}x(\lambda_i) \tag{2.15}$$

And repeating the use of theorem 5.1 in [7] recursively we get the result. □

Example:

$$\langle M_{ii}\rangle^{(1)} = \frac{1}{16(2-t_3)}\left(\frac{1}{\lambda_i^5} + \frac{t_5}{(2-t_3)\lambda_i^3}\right) \longrightarrow \langle \operatorname{Tr} M\rangle^{(1)} = \frac{t_5}{8(2-t_3)^2} \quad (2.16)$$

2.2. Expectation Values and Ribbon Graphs

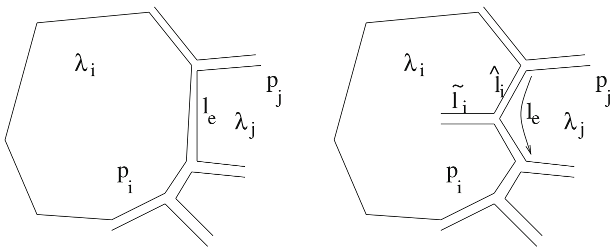
Let $i_1, \ldots, i_n$ be n distinct given integers $\in [1, \ldots, N]$. We want to compute:

$$\langle M_{i_1,i_1} \ldots M_{i_n,i_n}\rangle^{(g)} \quad (2.17)$$

Let us also choose n positive real perimeters $P_1, \ldots, P_n$

Let $\Gamma(g, n, m)$ be the set of trivalent oriented ribbon graphs of genus g, with n marked faces, and m unmarked faces. Each marked face $F = 1, \ldots, n$ carries the given index i_F, and each unmarked face f carries an index $i_f \in [1, \ldots, N]$.

Let us consider another set of graphs. Let $\Gamma^*(g, n, m)$ be the set of oriented ribbon graphs of genus g, with trivalent and 1-valent vertices, made of m unmarked faces bordered with only trivalent vertices, each of them carrying an index i_f, and n marked faces carrying the fixed index $i_F \in \{i_1, \ldots, i_n\}$, such that each marked face has one 1-valent vertex on its boundary. The unique trivalent vertex linked to the 1-valent vertex on each marked face corresponds to a marked point on the boundary of that face.



For any graph G in either $\Gamma(g, n, m)$ or $\Gamma^*(g, n, m)$, each edge e is bordered by two faces (possibly not different), and we denote the pair of their indices as $(e_{\text{left}}, e_{\text{right}})$.

Assume that $i_1, \ldots, i_n$ are distinct integers. The usual fat graph expansion of matrix integrals gives (cf. [2,4,9]):

$$\langle M_{i_1,i_1} \ldots M_{i_n,i_n}\rangle^{(g)} = N^{-m} \sum_m \sum_{G\in\Gamma^*_{g,n,m}} \sum_{\{i_f\}} \frac{(-1)^{\#\text{vertices}}}{\#\text{Aut}(G)} \prod_{e\in\text{edges}(G)} \frac{1}{\lambda_{e\,\text{left}} + \lambda_{e\,\text{right}}} \quad (2.18)$$

It is obtained by first expanding $\mathrm{e}^{-\frac{N}{3}\operatorname{Tr} M^3} = \sum_{v=0}^{\infty} \frac{N^v}{3^v\, v!}\,(-1)^v\,(\operatorname{Tr} M^3)^v$, and then computing each polynomial moment of the Gaussian measure $\mathrm{e}^{-N \operatorname{Tr} \Lambda M^2}$ with the help of Wick's theorem. Each $\operatorname{Tr} M^3$ corresponds to a trivalent vertex, each M_{ii} corresponds to a 1-valent vertex, and edges correspond to the "propagator" $\langle M_{ij}M_{kl}\rangle_{\text{Gauss}} = \frac{\delta_{il}\delta_{jk}}{N(\lambda_i+\lambda_j)}$. The result is best represented as a fat graph, the edges of which are double lines, carrying two indices. The indices are conserved along simple lines. The symmetry factor comes from the combination of $1/(3^v v!)$ and the fact that some graphs are obtained several times. Notice that $(-1)^v = (-1)^n$, because the total number of 1- and 3-valent vertices must be even.

Notice that the edge connected to the 1-valent vertex M_{i_F,i_F} gives a factor $1/2\lambda_{i_F}$, and the two edges on the boundary of face F, on each side of the 1-valent vertex, give a factor $1/(\lambda_{i_F}+\lambda_j)^2$ (where j is the index of the neighboring face), which can be written as:

$$\frac{1}{(\lambda_{i_F}+\lambda_j)^2} = \int_0^\infty \mathrm{d}l_e \int_0^{l_e} \mathrm{d}\hat{l}_i\ \mathrm{e}^{-l_e(\lambda_{i_F}+\lambda_j)} \tag{2.19}$$

and all other edges have a weight of the form:

$$\frac{1}{\lambda_{e\,\text{left}}+\lambda_{e\,\text{right}}} = \int_0^\infty \mathrm{d}l_e\ \mathrm{e}^{-l_e(\lambda_{e\,\text{left}}+\lambda_{e\,\text{right}})} \tag{2.20}$$

We are thus led to associate to each edge e a length $l_e \in \mathbf{R}^+$.

Therefore,

$$\begin{aligned}\langle M_{i_1,i_1}\dots M_{i_n,i_n}\rangle^{(g)} &= \frac{N^{-m}}{2^n\lambda_{i_1}\dots\lambda_{i_n}} \sum_m \sum_{G\in\Gamma^*_{g,n,m}} \sum_{\{i_f\}} \frac{(-1)^n}{\#\mathrm{Aut}(G)}\\ &\times \prod_{e\in\mathrm{edges}(G)} \int_0^\infty \mathrm{d}l_e\ \mathrm{e}^{-\sum_e l_e(\lambda_{e\,\text{left}}+\lambda_{e\,\text{right}})} \prod_{F=1}^n \int_0^{l_{e_F}} \mathrm{d}\hat{l}_F\end{aligned} \tag{2.21}$$

Now, we introduce the perimeters of each face P_F for marked faces, and p_f for unmarked ones.

Notice that each graph of $\Gamma^*_{g,n,m}$ projects on a graph of $\Gamma_{g,n,m}$ by removing the 1-valent vertex and its adjacent trivalent vertex, and keeping a marked point on the boundary of the face F. The sum of $\int\prod_F \mathrm{d}\hat{l}_F$ over graphs of $\Gamma^*_{g,n,m}$, which project to the same graph, corresponds to a sum of all possibilities of marking a point on the boundary of face F, i.e., a factor P_F, and thus removing the marked point. Therefore:

$$\begin{aligned}&\langle M_{i_1,i_1}\dots M_{i_n,i_n}\rangle^{(g)}\\ &= \frac{N^{-m}}{2^n\lambda_{i_1}\dots\lambda_{i_n}} \sum_m \sum_{G\in\Gamma_{g,n,m}} \sum_{\{i_f\}} \frac{(-1)^n}{\#\mathrm{Aut}(G)} \prod_f \int_0^\infty \mathrm{d}p_f\ \mathrm{e}^{-\sum_f \lambda_{i_f} p_f}\end{aligned}$$

$$\times \prod_F \int_0^\infty P_F \, \mathrm{d}P_F \, \mathrm{e}^{-\lambda_{i_F} P_F} \prod_e \int_0^\infty \mathrm{d}l_e \, \prod_f \delta\left(p_f - \sum_{e\in\partial f} l_e\right) \prod_{F=1}^n \delta\left(P_F - \sum_{e\in\partial F} l_e\right)$$
$$= \frac{1}{2^n \lambda_{i_1}\dots\lambda_{i_n}} \sum_m \sum_{G\in\Gamma_{g,n,m}} \frac{(-1)^n}{\#\mathrm{Aut}(G)}$$
$$\times \prod_f \int_0^\infty \mathrm{d}p_f \, \frac{1}{N}\, \mathrm{Tr}\,(\mathrm{e}^{-p_f\Lambda}) \prod_F \int_0^\infty P_F \, \mathrm{d}P_F \, \mathrm{e}^{-\lambda_{i_F} P_F} \quad \mathrm{Vol}\left(\pi_G^{-1}(P_F, p_f)\right) \tag{2.22}$$

where $\mathrm{Vol}\left(\pi_G^{-1}(P_F, p_f)\right)$ is the volume of the pullback of the ribbon graph G in $\mathcal{M}_{g,n+m}^{\mathrm{comb}}$:

$$\mathrm{Vol}\left(\pi_G^{-1}(P_F, p_f)\right) = \int \prod_e \mathrm{d}l_e \, \prod_f \delta\left(p_f - \sum_{e\in\partial f} l_e\right) \prod_{F=1}^n \delta\left(P_F - \sum_{e\in\partial F} l_e\right) \tag{2.23}$$

The number of integrations (i.e., after performing the δ) is $2d_{g,n+m} = \#\mathrm{edges} - \#\mathrm{faces} = 2(3g-3+n+m)$, which is the dimension of $\mathcal{M}_{g,n+m}$; therefore, $\prod_e \mathrm{d}l_e$ is a top-dimension volume form on $\mathcal{M}_{g,n+m}^{\mathrm{comb}} = \overline{\mathcal{M}}_{g,n+m} \times \mathbf{R}_+^{n+m}$, i.e.:

$$\prod_e \mathrm{d}l_e = \frac{\rho_{g,n+m}}{d_{g,n+m}!} \prod_F \mathrm{d}P_F \prod_f \mathrm{d}p_f \wedge \Omega^{d_{g,n+m}} \tag{2.24}$$

where Ω is the two-form on the strata $\pi_G^{-1}(P_F, p_f)$ of $\mathcal{M}_{g,n+m}^{\mathrm{comb}}$ such that:

$$\Omega = \sum_f p_f^2 \omega_f + \sum_F P_F^2 \omega_F \tag{2.25}$$

and where $\omega_f = \sum_{e<e'} d(l_e/p_f) \wedge d(l_{e'}/p_f)$ is the first Chern class of pullback of the cotangent bundle at the center of the face $\psi_f = c_1(\mathcal{L}_f)$.

Kontsevich [9] proved that the constant $\rho_{g,n+m}$ is given by:

$$\rho_{g,n+m} = 2^{g-1-2d_{g,n+m}} \tag{2.26}$$

Thus we have:

$$\mathrm{Vol}\left(\pi_G^{-1}(P_F, p_f)\right)$$
$$= \frac{\rho_{g,n+m}}{d_{g,n+m}!} \int_{\pi_G^{-1}(P_F,p_f)} \Omega^{d_{g,n+m}}$$
$$= \rho_{g,n+m} \sum_{\sum_f d_f + \sum_F d_F = d_{g,n+m}} \int_{\pi_G^{-1}(P_F,p_f)} \prod_f \frac{p_f^{2d_f}\, \psi_f^{d_f}}{d_f!} \prod_F \frac{P_F^{2d_F}\, \psi_F^{d_F}}{d_F!}$$
$$= \rho_{g,n+m} \sum_{\sum_f d_f + \sum_F d_F = d_{g,n+m}} \prod_f \frac{p_f^{2d_f}}{d_f!} \prod_F \frac{P_F^{2d_F}}{d_F!} \left\langle \prod_f \psi_f^{d_f} \prod_F \psi_F^{d_F} \right\rangle_G \tag{2.27}$$

therefore

$$\begin{aligned}
&\prod_f \int_0^\infty \mathrm{d}p_f \ \frac{1}{N} \operatorname{Tr}\left(\mathrm{e}^{-p_f\Lambda}\right) \mathrm{Vol}\left(\pi_G^{-1}(P_F, p_f)\right) \\
&\quad = \rho_{g,n+m} \ \prod_f \int_0^\infty \mathrm{d}p_f \ \frac{1}{N} \operatorname{Tr}\left(\mathrm{e}^{-p_f\Lambda}\right) \sum_{\sum_f d_f + \sum_F d_F = d_{g,n+m}} \\
&\qquad \times \prod_f \frac{p_f^{2d_f}}{d_f!} \prod_F \frac{P_F^{2d_F}}{d_F!} \ \left\langle \prod_f \psi_f^{d_f} \prod_F \psi_F^{d_F} \right\rangle_G \\
&\quad = \rho_{g,n+m} \sum_{\sum_f d_f + \sum_F d_F = d_{g,n+m}} \\
&\qquad \times \prod_f \frac{2d_f!}{d_f!} \ \frac{1}{N} \operatorname{Tr}\left(\Lambda^{-(2d_f+1)}\right) \prod_F \frac{P_F^{2d_F}}{d_F!} \ \left\langle \prod_f \psi_f^{d_f} \prod_F \psi_F^{d_F} \right\rangle_G \\
&\quad = \rho_{g,n+m} \sum_{\sum_f d_f + \sum_F d_F = d_{g,n+m}} \\
&\qquad \times \prod_f \frac{2d_f!}{d_f!} \ t_{2d_f+1} \prod_F \frac{P_F^{2d_F}}{d_F!} \ \left\langle \prod_f \psi_f^{d_f} \prod_F \psi_F^{d_F} \right\rangle_G
\end{aligned} \tag{2.28}$$

and then, when we sum over all graphs (since we sum over graphs with m unmarked faces, we have to divide wrt the symmetry factor $m!$, like in [9]):

$$\begin{aligned}
\langle M_{i_1,i_1} \ldots M_{i_n,i_n} \rangle^{(g)} &= \frac{(-1)^n \rho_{g,n}}{2^n \lambda_{i_1} \ldots \lambda_{i_n}} \prod_F \int_0^\infty P_F \mathrm{d}P_F \mathrm{e}^{-\lambda_{i_F} P_F} \\
\times \sum_m \frac{1}{m!} \sum_{\sum d_f + d_F = d_{g,n} + m} & \prod_f \frac{2d_f!}{d_f!} \ \frac{t_{2d_f+1}}{4} \prod_F \frac{P_F^{2d_F}}{d_F!} \ \left\langle \prod_f \psi_f^{d_f} \prod_F \psi_F^{d_F} \right\rangle
\end{aligned} \tag{2.29}$$

Therefore, if we write:

$$\frac{W_{g,n}(\lambda_{i_1}, \ldots, \lambda_{i_n})}{\mathrm{d}\lambda_{i_1} \ldots \mathrm{d}\lambda_{i_n}} = \int_0^\infty \mathrm{d}P_1 \ldots \mathrm{d}P_n \ \prod_F P_F \, \mathrm{e}^{-\lambda_{i_F} P_F} \ V_{g,n}(P_1, \ldots, P_n) \tag{2.30}$$

we find that the inverse Laplace transform of $W_{g,n}$ is:

$$\boxed{\begin{aligned} &V_{g,n}(P_1, \ldots, P_n) = \rho_{g,n} \ \textstyle\sum_m \frac{(-1)^n}{m!} \sum_{\sum_1^m d_f + \sum_1^n d_F = d_{g,n}+m} \\ &\textstyle\prod_f \frac{2d_f!}{d_f!} \ \frac{t_{2d_f+1}}{4} \prod_F \frac{P_F^{2d_F}}{d_F!} \ \left\langle \prod_f \psi_f^{d_f} \prod_F \psi_F^{d_F} \right\rangle \end{aligned}} \tag{2.31}$$

where the intersection theory is computed on $\overline{\mathcal{M}}_{g,n+m}$.

Since we are interested only in the perimeters of the n marked faces, we may try to perform the integration over the m unmarked faces, i.e., we introduce the forgetful projection $\pi_{n+m\to n} : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$, which "forgets" the m remaining points. It is known [1,16] that the push forward of the classes $\psi_f^{d_f}$, can then be rewritten in terms of Mumford's [14] tautological classes κ_b on $\overline{\mathcal{M}}_{g,n}$, by the relation:

$$(\pi_{n+m\to n})_* \left(\psi_1^{a_1+1}\dots\psi_m^{a_m+1}\prod_F \psi_F^{d_F}\right) = \sum_{\sigma\in\Sigma_m} \prod_{c=\text{cycles of }\sigma} \kappa_{\sum_{i\in c} a_i} \prod_F \psi_F^{d_F} \tag{2.32}$$

Therefore, if we rewrite $d_f = a_f + 1$ we have:

$$\begin{aligned} \frac{1}{\rho_{g,n}} V_{g,n}(P_1,\dots,P_n) &= \sum_m \frac{(-1)^n}{m!} \sum_{\sum_1^m a_f + \sum_1^n d_F = d_{g,n}} \\ &\quad \times \prod_f \frac{2a_f+1!}{a_f!}\,\frac{t_{2a_f+3}}{2} \prod_F \frac{P_F^{2d_F}}{d_F!} \left\langle \prod_f \psi_f^{a_f+1} \prod_F \psi_F^{d_F}\right\rangle \\ &= \sum_m \frac{(-1)^n}{m!} \sum_{\sum_1^m a_f + \sum_1^n d_F = d_{g,n}} \sum_{\sigma\in\Sigma_m} \\ &\quad \times \prod_f \frac{2a_f+1!}{a_f!}\,\frac{t_{2a_f+3}}{2} \prod_F \frac{P_F^{2d_F}}{d_F!} \left\langle \prod_c \kappa_{\sum_c a_i} \prod_F \psi_F^{d_F}\right\rangle \\ &= (-1)^n \sum_{d_0+d_1+\dots+d_F=d_{g,n}} \prod_F \frac{P_F^{2d_F}}{d_F!} \sum_m \frac{1}{m!} \sum_{a_1+\dots+a_m=d_0,\, a_f\geq 0} \sum_{\sigma\in\Sigma_m} \\ &\quad \times \prod_f \frac{2a_f+1!}{a_f!}\,\frac{t_{2a_f+3}}{2} \left\langle \prod_c \kappa_{\sum_c a_i} \prod_F \psi_F^{d_F}\right\rangle \end{aligned} \tag{2.33}$$

Now, instead of summing over permutations, let us sum over classes of permutations, i.e., partitions $l_1 \geq l_2 \geq \dots \geq l_k > 0$, and we denote $|l| = \sum_i l_i = m$ the weight of the class and $|[l]|$ the size of the class:

$$|[l]| = \frac{|l|!}{\prod_i l_i \ \prod_j (\#\{i/\ l_i = j\})!} \tag{2.34}$$

The sum over the a's for each class gives:

$$\begin{aligned} &\frac{(-1)^n}{\rho_{g,n}} V_{g,n}(P_1,\dots,P_n) \\ &\quad = \sum_{d_0+d_1+\dots+d_F=d_{g,n}} \prod_F \frac{P_F^{2d_F}}{d_F!} \sum_k \sum_{l_1\geq l_2\geq\dots\geq l_k>0} \frac{|[l]|}{|l|!} \sum_{a_{i,j},\, i=1,\dots,k,\, j=1,\dots,l_i} \\ &\qquad \times \delta\left(\sum_{i,j} a_{i,j} - d_0\right) \prod_{i,j} \frac{2a_{i,j}+1!}{a_{i,j}!}\,\frac{t_{2a_{i,j}+3}}{2} \left\langle \prod_{i=1}^k \kappa_{\sum_{j=1}^{l_i} a_{i,j}} \prod_F \psi_F^{d_F}\right\rangle \end{aligned} \tag{2.35}$$

Since the summand is symmetric in the l_i's, the ordered sum over $l_1 \geq \ldots l_k$ can be replaced by an unordered sum (multiplying by $1/k!$, and by $\prod_i (\#\{i/\ l_i = j\})!$ in case some l_i coincide):

$$\begin{aligned}
&\frac{(-1)^n}{\rho_{g,n}} V_{g,n}(P_1,\ldots,P_n) \\
&= \sum_{d_0+d_1+\cdots+d_F=d_{g,n}} \prod_F \frac{P_F^{2d_F}}{d_F!} \sum_k \frac{1}{k!} \sum_{l_1,l_2,\ldots,l_k>0} \prod_{i=1}^k \frac{1}{l_i} \sum_{a_{i,j},i=1,\ldots,k,j=1,\ldots,l_i} \\
&\quad \times \delta\left(\sum_{i,j} a_{i,j} - d_0\right) \prod_{i,j} \frac{2a_{i,j}+1!}{a_{i,j}!} \frac{t_{2a_{i,j}+3}}{2} \left\langle \prod_{i=1}^k \kappa_{\sum_{j=1}^{l_i} a_{i,j}} \prod_F \psi_F^{d_F} \right\rangle \\
&= \sum_{d_0+d_1+\cdots+d_F=d_{g,n}} \prod_F \frac{P_F^{2d_F}}{d_F!} \sum_k \frac{1}{k!} \sum_{b_1+b_2+\cdots+b_k=d_0} \prod_{i=1}^k \tilde{t}_{b_i} \left\langle \prod_{i=1}^k \kappa_{b_i} \prod_F \psi_F^{d_F} \right\rangle
\end{aligned} \tag{2.36}$$

where

$$\tilde{t}_b = \sum_{l>0} \frac{1}{l} \sum_{a_1+\cdots+a_l=b} \prod_j \frac{2a_j+1!}{a_j!} \frac{t_{2a_j+3}}{2} \tag{2.37}$$

$\tilde{t}_b$ can be computed as follows: introduce the generating function

$$g(z) = \sum_{a=0}^{\infty} \frac{2a+1!}{a!} \frac{t_{2a+3}}{2} z^a \tag{2.38}$$

then $\tilde{t}_b$ is

$$\tilde{t}_b = \sum_{l>0} \frac{1}{l} \left(g^l\right)_b = (-\ln(1-g))_b \tag{2.39}$$

where the subscript b means the coefficient of z^b in the small z Taylor expansion of the corresponding function, i.e.,

$$-\ln(1-g(z)) = \sum_{b=0}^{\infty} \tilde{t}_b z^b = \tilde{g}(z), \quad 1-g(z) = \mathrm{e}^{-\tilde{g}(z)} \tag{2.40}$$

In fact, it is better to treat the $a=0$ and $b=0$ terms separately. Define:

$$f(z) = 1 - \frac{1-g(z)}{1-\frac{t_3}{2}} = \sum_{a=1}^{\infty} \frac{2a+1!}{a!} \frac{t_{2a+3}}{2-t_3} z^a \tag{2.41}$$

and

$$\tilde{f}(z) = -\ln(1-f(z)) = \tilde{g}(z) - \tilde{t}_0 = \sum_{b=1}^{\infty} \tilde{t}_b z^b \tag{2.42}$$

We have:

$$\tilde{t}_0 = -\ln\left(1 - \frac{t_3}{2}\right) \tag{2.43}$$

and $\tilde{t}_b$ is now a finite sum:

$$\tilde{t}_b = \sum_{l=1}^{b} \frac{(-1)^l}{l} \sum_{a_1+\dots+a_l=b, a_i>0} \prod_j \frac{2a_j+1!}{a_j!}\, \frac{t_{2a_j+3}}{t_3-2} \tag{2.44}$$

Using that $\kappa_0 = 2g-2+n$, we may also perform the sum over all vanishing b's. Let us change $k \to k+l$ where l is the number of vanishing b's, i.e.,

$$\begin{aligned}
&\frac{(-1)^n}{\rho_{g,n}} V_{g,n}(P_1,\dots,P_n)\\
&= \sum_{d_0+d_1+\dots+d_F=d_{g,n}} \prod_F \frac{P_F^{2d_F}}{d_F!} \sum_k \sum_l \frac{1}{k!l!} (\tilde{t}_0\kappa_0)^l\\
&\quad \times \sum_{b_1+b_2+\dots+b_k=d_0, b_i>0} \prod_{i=1}^{k} \tilde{t}_{b_i} \left\langle \prod_{i=1}^{k} \kappa_{b_i} \prod_F \psi_F^{d_F} \right\rangle\\
&= \mathrm{e}^{\tilde{t}_0\kappa_0} \sum_{d_0+d_1+\dots+d_F=d_{g,n}} \prod_F \frac{P_F^{2d_F}}{d_F!} \sum_k \frac{1}{k!}\\
&\quad \times \sum_{b_1+b_2+\dots+b_k=d_0, b_i>0} \prod_{i=1}^{k} \tilde{t}_{b_i} \left\langle \prod_{i=1}^{k} \kappa_{b_i} \prod_F \psi_F^{d_F} \right\rangle\\
&= \left(\frac{2}{2-t_3}\right)^{2g-2+n} \sum_{d_0+d_1+\dots+d_F=d_{g,n}} \prod_F \frac{P_F^{2d_F}}{d_F!} \sum_k \frac{1}{k!}\\
&\quad \times \sum_{b_1+b_2+\dots+b_k=d_0, b_i>0} \prod_{i=1}^{k} \tilde{t}_{b_i} \left\langle \prod_{i=1}^{k} \kappa_{b_i} \prod_F \psi_F^{d_F} \right\rangle
\end{aligned} \tag{2.45}$$

Notice that:

$$\rho_{g,n} 2^{2g-2+n} = 2^{-d_{g,n}} \tag{2.46}$$

thus

$$\begin{aligned}
&2^{d_{g,n}} (t_3-2)^{2g-2+n}\, V_{g,n}(P_1,\dots,P_n)\\
&= \sum_{d_0+d_1+\dots+d_F=d_{g,n}} \prod_F \frac{P_F^{2d_F}}{d_F!} \sum_k \frac{1}{k!} \sum_{b_1+b_2+\dots+b_k=d_0, b_i>0} \prod_{i=1}^{k} \tilde{t}_{b_i}\\
&\quad \times \left\langle \prod_{i=1}^{k} \kappa_{b_i} \prod_F \psi_F^{d_F} \right\rangle
\end{aligned} \tag{2.47}$$

Finally, we obtain Theorem 1.1. □

3. Examples

3.1. Some Examples

First, we give a few examples with general times t_ks.

Using formula 2.44, we have:

$$\tilde{t}_1 = -6\,\frac{t_5}{t_3-2},\quad \tilde{t}_2 = -60\,\frac{t_7}{t_3-2} + 18\,\frac{t_5^2}{(t_3-2)^2} \tag{3.1}$$

$$\tilde{t}_3 = -\frac{7!}{3!}\,\frac{t_9}{t_3-2} + \frac{3!5!}{2!}\,\frac{t_5t_7}{(t_3-2)^2} - \frac{3!^3}{3}\,\frac{t_5^3}{(t_3-2)^3},\ \ldots \tag{3.2}$$

Then we use Theorem 1.1 for some examples. In the examples that follow, the first expression is the definition Eq. 1.3, while the second expression results from the recursion equation 1.4.

$$W_{0,3}(z_1,z_2,z_3) = \frac{1}{t_3-2}\,\frac{dz_1\,dz_2\,dz_3}{z_1^2\,z_2^2\,z_3^2}\langle 1\rangle_0 = \frac{1}{t_3-2}\,\frac{dz_1\,dz_2\,dz_3}{z_1^2\,z_2^2\,z_3^2} \tag{3.3}$$

i.e.,

$$V_{0,3}(L_1,L_2,L_3) = \frac{1}{t_3-2},\quad \langle 1\rangle_0 = 1 \tag{3.4}$$

$$\begin{aligned} W_{1,1}(z) &= \frac{dz}{2(t_3-2)}\left(\frac{6}{z^4}\langle\psi\rangle_1 + \frac{\tilde{t}_1}{z^2}\langle\kappa_1\rangle_1\right)\\ &= \frac{dz}{8(t_3-2)}\left(\frac{1}{z^4} - \frac{t_5}{(t_3-2)z^2}\right) \end{aligned} \tag{3.5}$$

i.e.,

$$\langle\psi\rangle_1 = \frac{1}{24},\quad \langle\kappa_1\rangle_1 = \frac{1}{24} \tag{3.6}$$

$$\begin{aligned} &W_{1,2}(z_1,z_2)\\ &= \frac{dz_1dz_2}{4(t_3-2)^2\,z_1^6\,z_2^6}\left[\frac{5!}{2!}\left(z_1^4\langle\psi_2^2\rangle + z_2^4\langle\psi_1^2\rangle\right) + 3!^2 z_1^2z_2^2\langle\psi_1\psi_2\rangle\right.\\ &\quad\left. + \tilde{t}_1 z_1^2z_2^4\langle\kappa_1\psi_1\rangle + \tilde{t}_1 z_1^4z_2^2\langle\kappa_1\psi_2\rangle + \frac{1}{2}\tilde{t}_1^2 z_1^4z_2^4\langle\kappa_1^2\rangle + \tilde{t}_2 z_1^4z_2^4\langle\kappa_2\rangle\right]\\ &= \frac{dz_1dz_2}{8(t_3-2)^4z_1^6z_2^6}\left[(t_3-2)^2\left(5z_1^4+5z_2^4+3z_1^2z_2^2\right) + 6t_5^2z_1^4z_2^4\right.\\ &\quad\left. -(t_3-2)\left(6t_5z_1^4z_2^2 + 6t_5z_1^2z_2^4 + 5t_7z_1^4z_2^4\right)\right] \end{aligned} \tag{3.7}$$

i.e.,

$$\langle\kappa_1\psi_1\rangle_1 = \frac{1}{2},\quad \langle\kappa_1^2\rangle_1 = \frac{1}{8},\quad \langle\kappa_2\rangle_1 = \frac{1}{24} \tag{3.8}$$

The recursion equation 1.4 also gives:

$$\begin{aligned} W_{2,1}(z) = -\frac{dz}{128(2-t_3)^7z^{10}}&\left[252\,t_5^4z^8 + 12\,t_5^2z^6(2-t_3)(50\,t_7z^2 + 21\,t_5)\right.\\ &+z^4(2-t_3)^2\left(252\,t_5^2 + 348\,t_5t_7z^2 + 145\,t_7^2z^4 + 308\,t_5t_9z^4\right) \end{aligned}$$

$$+z^2(2-t_3)\left(203\,t_5+145\,z^2t_7+105\,z^4t_9+105\,z^6t_{11}\right)$$
$$+105\,(2-t_3)^4\Big]\,. \tag{3.9}$$

$$W_{4,0}(z_1,z_2,z_3,z_4)$$
$$= 12\,\frac{dz_1dz_2dz_3dz_4}{(t_3-2)^3\,z_1^2z_2^2z_3^2z_4^2}\,\left((t_3-2)\left(z_1^{-2}+z_2^{-2}+z_3^{-2}+z_4^{-2}\right)-t_5\right) \tag{3.10}$$

and so on.

3.2. Specialization to the Weil–Petersson Measure

Now, we specialize to the Weil–Petersson spectral curve of [5]:

$$y(z)=\frac{1}{2\pi}\sin(2\pi z)\;\to\; t_{2d+3}=\frac{(2i\pi)^{2d}}{2d+1!}+2\delta_{d,0}\;\to\; f(z)=1-\mathrm{e}^{-4\pi^2 z} \tag{3.11}$$

so that:

$$\tilde f(z)=4\pi^2 z\;\to\;\tilde t_b=4\pi^2\,\delta_{b,1}+\delta_{b,0}\,\ln(-2) \tag{3.12}$$

therefore, each b_i must be 1, and we must have $k=d_0$, and we get:

$$\boxed{\begin{array}{l} V_{g,n}(P_1,\ldots,P_n)\\ =2^{-d_{g,n}}\sum_{d_0+d_1+\cdots+d_F=d_{g,n}}\frac{2^{d_0}}{d_0!}\prod_F\frac{P_F^{2d_F}}{d_F!}\,\langle(2\pi^2\kappa_1)^{d_0}\,\prod_F\psi_F^{d_F}\rangle\end{array}} \tag{3.13}$$

which is after Wolpert's relation [17], the Weil–Petersson volume since $2\pi^2\kappa_1$ is the Weil–Petersson Kähler form, and thus, we have rederived Mirzakhani's recursion relation.

3.3. Specialization to the κ_2 Measure

To illustrate our method, we consider the integrals with only κ_2:

$$V_{g,n}(P_1,\ldots,P_n)$$
$$=2^{-d_{g,n}}\sum_{2d_0+d_1+\cdots+d_F=d_{g,n}}\frac{1}{d_0!}\prod_F\frac{P_F^{2d_F}}{d_F!}\,\left\langle(\tilde t_2\kappa_2)^{d_0}\prod_F\psi_F^{d_F}\right\rangle \tag{3.14}$$

which correspond to the conjugated times

$$\tilde f(z)=\tilde t_2 z^2\quad\to\quad f(z)=\sum_{k=1}^{\infty}\frac{(-1)^k\,\tilde t_2^k}{k!}\,z^{2k} \tag{3.15}$$

i.e., $t_3=3$, and

$$t_{4a+3}=4(-1)^a\,\tilde t_2^a\,\frac{2a!}{a!(4a+1)!}-\delta_{a,0} \tag{3.16}$$

The corresponding spectral curve is:

$$y(z)=-\frac{z}{2}+2\sum_{k=1}^{\infty}(-\tilde t_2)^k\,\frac{2k!}{k!(4k+1)!}\,z^{4k+1} \tag{3.17}$$

with that spectral curve, the volumes $V_{g,n}$ satisfy the recursion of Theorem 1.1.

3.4. Specialization to Discrete Measure

Let us consider the example where $\Lambda = \lambda\,\mathrm{Id}$, which is particularly important because

$$Z = \int \mathrm{d}M\ \mathrm{e}^{-N\,\mathrm{Tr}\,\left(\frac{M^3}{3} - M\left(\lambda^2+\frac{1}{\lambda}\right)\right)} \propto \int \mathrm{d}M\ \mathrm{e}^{-\frac{N}{T}\,\mathrm{Tr}\,\left(\frac{1}{2}M^2 - \frac{M^3}{3}\right)} \tag{3.18}$$

where

$$T = -\frac{1}{8}\left(\lambda^2 + \frac{1}{\lambda}\right)^{-3/2} \tag{3.19}$$

i.e., Kontsevich integral reduces to the usual cubic one-matrix model, which is known to count triangulated maps [2].

In that case, we have:

$$t_k = \lambda^{-k} \tag{3.20}$$

thus for $b \geq 1$:

$$\tilde{t}_b = 2^b \lambda^{-2b} \sum_{l=1}^{b} \frac{1}{l}(1-2\lambda^3)^{-l} \sum_{a_1+\cdots+a_l=b, a_i>0} \prod_i (2a_i+1)!! \tag{3.21}$$

For instance we have:

$$V_{0,3}(L) = \frac{1}{\lambda^{-3}-2} \tag{3.22}$$

$$V_{1,1}(L) = \frac{1}{2}\,\frac{1}{\lambda^{-3}-2}\left(L\langle\psi_1\rangle_1 + \tilde{t}_1\langle\kappa_1\rangle_1\right) = \frac{-1}{8(2-\lambda^{-3})}\left(\frac{L}{6} + \frac{\lambda^{-5}}{2-\lambda^{-3}}\right) \tag{3.23}$$

where $\tilde{t}_1 = 6\lambda^{-2}\,(1-2\lambda^3)^{-1}$.

It would be interesting to understand how this relates to the discrete Regge measure on the set triangulated maps. In the case of triangulated maps, loop equations, i.e., the recursion equation 1.4 are known as Tutte's equations [15], which give a recursive manner to enumerate maps. This shows how general the recursion equation 1.4 is.

4. Other Properties

From the general properties of the invariants of [7], we immediately have the following properties:

- Integrability. The F_gs satisfy Hirota equations for KdV hierarchy. That property is well known and it motivated the first works on Witten–Kontsevich conjecture [9].
- Virasoro. The invariants of [7] were initially obtained in [6,7] from the loop equations, i.e., Virasoro constraints satisfied by $Z(\Lambda)$.
- From dilation equation, we have:

$$W_{g,n}(z_1,\ldots,z_n) = \frac{1}{2g+n-2}\ \mathop{\mathrm{Res}}_{z\to 0}\ \Phi(z)W_{g,n+1}(z_1,\ldots,z_n,z) \tag{4.1}$$

where $\mathrm{d}\Phi = y\mathrm{d}x$.

For the Weil–Petersson case, after Laplace transform this translates into [5]:

$$V_{g,n}(L_1,\dots,L_n)_{\mathrm{WP}} = \frac{1}{2g+n-2}\ \frac{\partial}{\partial L_{n+1}}\, V_{g,n+1}(L_1,\dots,L_n,2i\pi)_{\mathrm{WP}} \tag{4.2}$$

- It was also found in [7] how all those quantities behave at singular points of the spectral curve and thus obtained the so-called double scaling limit.
- The invariants constructed in [7] have many other good properties, and it would be interesting to explore their applications to algebraic geometry.

5. Conclusion

In this paper, we have shown how powerful the loop equation method is, and that the structure of the recursion equation 1.4 (i.e., Virasoro or W-algebra constraints) is very universal.

We have thus provided a new proof of Mirzakhani's relations, exploiting the numerous properties of the invariants introduced in [7]. However, the construction of [7] is much more general than that of Mirzakhani, since it can be applied to any spectral curve and not only to the Weil–Petersson curve $y = \frac{1}{2\pi}\sin(2\pi\sqrt{x})$. In other words, we have Mirzakhani-like recursions for other measures, and Theorem 1.1 gives the relationship between a choice of t_ks (i.e., a spectral curve) and a measure on moduli spaces. Moreover, the recursion relations always imply integrability and Virasoro.

It would be interesting to understand what the algebraic invariants $W_{g,n}$ defined by the recursion relation of [7] compute for an arbitrary spectral curve, not necessarily hyperelliptical or rational.

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