

Critical Points of Wang–Yau Quasi-Local Energy

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Abstract. In this paper, we prove the following theorem regarding the Wang–Yau quasi-local energy of a spacelike two-surface in a spacetime: Let Σ be a boundary component of some compact, time-symmetric, space-like hypersurface Ω in a time-oriented spacetime N satisfying the dominant energy condition. Suppose the induced metric on Σ has positive Gaussian curvature and all boundary components of Ω have positive mean curvature. Suppose $H \leq H_0$ where H is the mean curvature of Σ in Ω and H_0 is the mean curvature of Σ when isometrically embedded in \mathbb{R}^3 . If Ω is not isometric to a domain in \mathbb{R}^3 , then

1. the Brown–York mass of Σ in Ω is a strict local minimum of the Wang–Yau quasi-local energy of Σ .
2. on a small perturbation $\tilde{\Sigma}$ of Σ in N , there exists a critical point of the Wang–Yau quasi-local energy of $\tilde{\Sigma}$.

1. Introduction and Statement of the Result

Let N be a spacetime, i.e., a Lorentzian manifold of dimension four. Suppose N is time orientable. Denote the Lorentzian metric on N by $\langle \cdot, \cdot \rangle$ and its covariant derivative by ∇^N . Let $\Sigma \subset N$ be an embedded, spacelike two-surface that is topologically a two-sphere. Suppose the mean curvature vector H of Σ in N is spacelike. Let σ be the induced metric on Σ and let K be the Gaussian curvature of (Σ, σ) .

Given a function τ on Σ such that $\hat{\sigma} = \sigma + d\tau \otimes d\tau$ is a metric of positive Gaussian curvature on Σ , by [19, Theorem 3.1] there exists an isometric embedding $X : (\Sigma, \sigma) \hookrightarrow \mathbb{R}^{3,1}$ such that τ is the time function of X , i.e.,

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$X = (\hat{X}, \tau)$, where $\hat{X} = (\hat{X}_1, \hat{X}_2, \hat{X}_3)$ is an isometric embedding of $(\Sigma, \hat{\sigma})$ in $\mathbb{R}^3 = \{(x, 0) \in \mathbb{R}^{3,1}\}$. The Wang–Yau quasi-local energy [18, 19], associated to such a time function τ , is given by

$$E_{wY}(\Sigma, \tau) = \frac{1}{8\pi} \left\{ \int_{\hat{\Sigma}} \hat{H} dv_{\hat{\Sigma}} - \int_{\Sigma} \left[\sqrt{1 + |\nabla\tau|^2} \cosh \theta |H| - \langle \nabla\tau, \nabla\theta \rangle - \langle V, \nabla\tau \rangle \right] dv_{\Sigma} \right\},$$

where

- $\hat{\Sigma} = \hat{X}(\Sigma)$, $\hat{H} > 0$ is the mean curvature of $\hat{\Sigma}$ in \mathbb{R}^3 , $dv_{\hat{\Sigma}}$ and dv_{Σ} are the volume forms of the metrics $\hat{\sigma}$ and σ .
- $\sinh \theta = \frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}$, ∇ and Δ are the gradient and the Laplacian operators of the metric σ , $|H| = \sqrt{\langle H, H \rangle}$.
- V is the tangent vector on Σ that is dual to the one form $\alpha_{e_3^H}^N(\cdot)$ defined by $\alpha_{e_3^H}^N(X) = \langle \nabla_X^N e_3^H, e_4^H \rangle$ for any X tangent to Σ . Here $e_3^H = -\frac{H}{|H|}$ and e_4^H is the future timelike unit normal to Σ that is orthogonal to e_3^H .

The Wang–Yau quasi-local mass of Σ [18, 19], which we denote by $m_{wY}(\Sigma)$, is then defined to be

$$m_{wY}(\Sigma) = \inf_{\tau} E_{wY}(\Sigma, \tau)$$

where the infimum is taken over all functions τ that are *admissible* (see [19, Definition 5.1] for the definition of admissibility).

Wang and Yau [19] show that a function τ is a critical point of $E_{wY}(\Sigma, \cdot)$ if and only if τ satisfies

$$- \left[\hat{H} \hat{\sigma}^{ab} - \hat{\sigma}^{ac} \hat{\sigma}^{bd} \hat{h}_{cd} \right] \frac{\nabla_b \nabla_a \tau}{\sqrt{1 + |\nabla\tau|^2}} + \operatorname{div}_{\Sigma} \left[\frac{\cosh \theta |H|}{\sqrt{1 + |\nabla\tau|^2}} \nabla\tau - \nabla\theta - V \right] = 0, \tag{1.1}$$

where $\hat{\sigma}, \hat{H}, \theta$ and V are defined as above, $\{a, b, c, d\}$ denote indices of local coordinates on Σ , \hat{h}_{ab} is the second fundamental form of $\hat{\Sigma}$ in \mathbb{R}^3 and $\operatorname{div}_{\Sigma}(\cdot)$ denotes the divergence operator on (Σ, σ) .

When the Gaussian curvature K of (Σ, σ) is positive, the function $\tau_0 = 0$ is admissible [19, Remark 1.1] and $E_{wY}(\Sigma, \tau_0) = m_{LY}(\Sigma)$, where $m_{LY}(\Sigma)$ is the Liu–Yau quasi-local mass of Σ [10, 11]. In this case, τ_0 is a critical point of $E_{wY}(\Sigma, \cdot)$ if and only if $\operatorname{div}_{\Sigma} V = 0$.

Now suppose Σ is one of the boundary components of a compact, time-symmetric, space-like hypersurface Ω in N , then $V = 0$ and $E_{wY}(\Sigma, \tau_0) = m_{BY}(\Sigma, \Omega)$, where $m_{BY}(\Sigma, \Omega)$ is the Brown–York mass of Σ in Ω [2, 3]. Considering the variational nature of $m_{wY}(\Sigma)$, one naturally wants to ask the following:

Question 1. Suppose Σ is a boundary component of a compact, time-symmetric, space-like hypersurface Ω in N , is the Brown–York mass $m_{BY}(\Sigma, \Omega)$ a local minimum value of the Wang–Yau quasi-local energy $E_{wY}(\Sigma, \cdot)$?

Question 2. Suppose Σ is a boundary component of a compact, time-symmetric, space-like hypersurface Ω in N , is the set of solutions to (1.1) open near the pair (Σ, τ_0) ? That is, suppose $\tilde{\Sigma} \subset N$ is another closed, embedded, spacelike two-surface which is a small perturbation of Σ , does there exist a solution τ to (1.1) with Σ replaced by $\tilde{\Sigma}$?

Our main result in this paper is the following theorem:

Theorem 1.1. *Let Σ be a boundary component of some compact, time-symmetric, spacelike hypersurface Ω in a time-oriented spacetime N satisfying the dominant energy condition. Suppose the induced metric σ on Σ has positive Gaussian curvature and all boundary components of Ω have positive mean curvature. Suppose*

$$H \leq H_0 \tag{1.2}$$

where H is the mean curvature of Σ in Ω and H_0 is the mean curvature of Σ when isometrically embedded in \mathbb{R}^3 . If Ω is not isometric to a domain in \mathbb{R}^3 , then

1. $m_{BY}(\Sigma, \Omega)$ is a strict local minimum of $E_{WY}(\Sigma, \cdot)$.
2. for $\tilde{\Sigma} \subset N$ near Σ , there is a solution τ to (1.1) for $\tilde{\Sigma}$.

We note that there are many types of surfaces Σ that satisfy the condition (1.2) of Theorem 1.1. Here we list a few of them:

- (i) $\Sigma = S_r$, where $S_r = \{|x| = r\}$ is a large coordinate sphere in a time-symmetric, asymptotically Schwarzschild (AS), spacelike slice $M \subset N$. Here a Riemannian 3-manifold M is called AS (with mass m) if there is a compact set $K \subset M$ such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus \{|x| \leq R\}$ for some R and the metric g on M with respect to the standard coordinates on \mathbb{R}^3 takes the form

$$g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij} + b_{ij}$$

where $|\partial^k b_{ij}| = O(r^{-2-k})$, $0 \leq k \leq 3$, $r = |x|$ and m is a constant. Direct calculation [see (5.1) in [6] for example] gives

$$H = \frac{2}{r} - \frac{4m}{r^2} + O(r^{-3}).$$

On the other hand, it was proved in [17] (the equation on the bottom of page 122) that

$$H_0 = \frac{2}{r} - \frac{2m}{r^2} + O(r^{-3}).$$

Therefore, $H < H_0$ for large r if M has positive mass m .

- (ii) Σ bounds a compact, time-symmetric spacelike slice Ω and Σ has constant positive Gaussian curvature and constant positive mean curvature H . In this case, by the results in [12, 17] one knows $H \leq H_0$ and $H = H_0$ if and only if Ω is isometric to a Euclidean round ball.

- (iii) Σ bounds a compact, time-symmetric spacelike slice Ω and Σ has positive Gaussian curvature and positive mean curvature. Suppose there exists a conformal diffeomorphism $f : \Omega \rightarrow \Omega_0$ between Ω and a domain Ω_0 in \mathbb{R}^3 such that $f^*(g_0)$ and g induce the same boundary metric on Σ and Σ has positive mean curvature in $(\Omega, f^*(g_0))$. Here g is the metric on Ω and g_0 is the Euclidean metric on Ω_0 . In this case, if one writes $g = u^4 f^*(g_0)$, it follows from the maximum principle (applied to u) that $H \leq H_0$ on Σ and $H = H_0$ precisely when Ω is isometric to Ω_0 .
- (iv) When viewed purely as a result on the Riemannian 3-manifold Ω , Theorem 1.1 applies to those Ω that are graphs over convex Euclidean domains. Precisely, let Σ be a strictly convex closed surface in \mathbb{R}^3 and let $\Omega_0 \subset \mathbb{R}^3$ be its interior. Let $f : \Omega_0 \rightarrow \mathbb{R}$ be a smooth function such that $f|_{\Sigma} = 0$. Let Ω be the graph of f in \mathbb{R}^4 with the induced metric and let H be the mean curvature of Σ in Ω . The calculation directly shows $H = \frac{1}{\sqrt{1+|\nabla f|^2}} H_0 \leq H_0$. The motivation to consider these Ω (with f chosen such that Ω has non-negative scalar curvature) comes from a recent work of Lam [8] on the graphs cases of the Riemannian positive mass theorem and Penrose inequality.

We should mention that related to (i) above, Chen et al. [4, Section 4] under the assumption of analyticity show that in asymptotically flat space-times, (1.1) has a formal power series solution, which is locally energy minimizing at all orders, for certain surfaces in an asymptotically flat hypersurface.

This paper is organized as follows: In Sect. 2, we compute the second variation of $E_{w_Y}(\Sigma, \tau)$ at $\tau_0 = 0$ and derive a sufficient condition for $\mathfrak{m}_{B_Y}(\Sigma, \Omega)$ to locally minimize $E_{w_Y}(\Sigma, \tau)$. In Sect. 3, we prove that the sufficient condition provided in Sect. 2 holds for those surfaces Σ satisfying the assumptions in Theorem 1.1. Hence, part (1) of Theorem 1.1 follows from Sects. 2 and 3. We note that, besides playing a key role in the proof of Theorem 1.1, Theorem 3.1 in Sect. 3 concerns analytical features of the boundary of compact Riemannian manifolds with non-negative scalar curvature, thus is of independent interest. In Sects. 4 and 5, we focus on part (2) of Theorem 1.1. The main idea there is to apply the Implicit Function Theorem (IFT). But to apply the IFT, we are confronted with the problem to show that the map F , sending a metric σ of positive Gaussian curvature on the two-sphere S^2 to the second fundamental form \mathbb{I} of the isometric embedding of (S^2, σ) in \mathbb{R}^3 , is a C^1 map between appropriate functional spaces. If σ is a $C^{k, \alpha}$ ($k \geq 2$) metric, by [15] one knows \mathbb{I} is a $C^{k-2, \alpha}$ symmetric tensor. We do not know whether F is C^1 from the $C^{k, \alpha}$ space to the $C^{k-2, \alpha}$ space. However, in Sect. 4, we prove that F is C^1 between $C^{k, \alpha}$ and $C^{k-3, \alpha}$ spaces for $k \geq 4$. This turns out to be sufficient to apply the IFT to obtain solutions to (1.1) because the metric $\hat{\sigma}$ in (1.1) involves $d\tau$ and (1.1) is a 4-th order differential equation of the function τ . In Sect. 5, we apply the result in Sects. 3, 4 and the IFT to prove the existence of critical points of $E_{w_Y}(\tilde{\Sigma}, \cdot)$ for surfaces $\tilde{\Sigma}$ nearby.

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2. Comparing $m_{BY}(\Sigma, \Omega)$ and $E_{WY}(\Sigma, \cdot)$

We start this section by computing the second variation of $E_{WY}(\Sigma, \cdot)$ at $\tau_0 = 0$, assuming τ_0 is a critical point for $E_{WY}(\Sigma, \cdot)$.

Proposition 2.1. *Let N be a time-oriented spacetime. Let $\Sigma \subset N$ be an embedded, spacelike two-surface that is topologically a two-sphere. Suppose the mean curvature vector H of Σ in N is spacelike. If $\tau_0 = 0$ is a critical point for $E_{WY}(\Sigma, \cdot)$, then the second variation of $E_{WY}(\Sigma, \cdot)$ at $\tau_0 = 0$ is given by*

$$\begin{aligned} &\delta^2 E_{WY}(\Sigma, \tau)|_{\tau=0}(\delta\tau) \\ &= \frac{1}{8\pi} \int_{\Sigma} \left[\frac{(\Delta\delta\tau)^2}{|H|} + (H_0 - |H|)|\nabla(\delta\tau)|^2 - \mathbb{I}\mathbb{I}_0(\nabla\delta\tau, \nabla\delta\tau) \right] dv_{\Sigma} \end{aligned} \tag{2.1}$$

where H_0 and $\mathbb{I}\mathbb{I}_0$ are the mean curvature and the second fundamental form of (Σ, σ) when isometrically embedded in \mathbb{R}^3 , and σ is the induced metric on Σ from N .

Proof. The first variation of $E_{WY}(\Sigma, \cdot)$ was obtained by Wang and Yau in [19, Proposition 6.2] and is given by

$$\begin{aligned} \delta E_{WY}(\Sigma, \tau)(\delta\tau) &= \frac{1}{8\pi} \int_{\Sigma} \left\{ - \left[\hat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}(\hat{h}_{cd}) \right] \frac{\nabla_b \nabla_a \tau}{\sqrt{1 + |\nabla\tau|^2}} \right. \\ &\quad \left. + \operatorname{div}_{\Sigma} \left[\frac{\nabla\tau}{\sqrt{1 + |\nabla\tau|^2}} \cosh\theta|H| - \nabla\theta - V \right] \right\} \cdot \delta\tau \, dv_{\Sigma}. \end{aligned} \tag{2.2}$$

Let $\mathcal{H}(\tau)$ denote the functional

$$\begin{aligned} &- \left[\hat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}(\hat{h}_{cd}) \right] \frac{\nabla_b \nabla_a \tau}{\sqrt{1 + |\nabla\tau|^2}} \\ &+ \operatorname{div}_{\Sigma} \left[\frac{\nabla\tau}{\sqrt{1 + |\nabla\tau|^2}} \cosh\theta|H| - \nabla\theta - V \right]. \end{aligned} \tag{2.3}$$

Direct computation shows that the first variation of $\mathcal{H}(\cdot)$ at $\tau = 0$ is

$$\delta\mathcal{H}(\tau)|_{\tau=0}(\delta\tau) = -\langle H_0\sigma - \mathbb{I}\mathbb{I}_0, \nabla^2\delta\tau \rangle + \operatorname{div}_{\sigma}(|H|\nabla\delta\tau) + \Delta\left(\frac{\Delta\delta\tau}{|H|}\right), \tag{2.4}$$

where ∇^2 denotes the Hessian operator on (Σ, σ) . (2.1) now follows from (2.2), (2.4) and the fact that $H_0\sigma - \mathbb{I}\mathbb{I}_0$ is divergence free on (Σ, σ) . \square

Assuming the quadratic functional of $\delta\tau$ in (2.1) has certain positivity property, we show that $\tau_0 = 0$ is a strict local minimum point for $E_{WY}(\Sigma, \cdot)$.

Theorem 2.1. *Let Σ be a boundary component of some compact, time-symmetric, spacelike hypersurface Ω in a time-oriented spacetime N satisfying the dominant energy condition. Suppose the induced metric σ on Σ has positive*

Gaussian curvature and the mean curvature H of Σ in Ω is positive. Suppose in addition that there exists a constant $\beta > 0$ such that

$$\int_{\Sigma} \left[\frac{(\Delta\eta)^2}{H} + (H_0 - H)|\nabla\eta|^2 - \mathbb{I}\mathbb{I}_0(\nabla\eta, \nabla\eta) \right] dv_{\Sigma} \geq \beta \int_{\Sigma} (\Delta\eta)^2 dv_{\Sigma} \quad (2.5)$$

for all $\eta \in W^{2,2}(\Sigma)$, where H_0 and $\mathbb{I}\mathbb{I}_0$ are the mean curvature and the second fundamental form of (Σ, σ) when isometrically embedded in \mathbb{R}^3 . Then, for any constant $0 < \alpha < 1$, there exists a constant $\epsilon > 0$ depending only on σ, H and β , such that

$$E_{WY}(\Sigma, \tau) - \mathbf{m}_{BY}(\Sigma, \Omega) \geq \frac{\beta}{4} \int_{\Sigma} (\Delta\tau)^2 dv_{\Sigma} \quad (2.6)$$

for any smooth function τ with $\|\tau\|_{C^{3,\alpha}} < \epsilon$.

Proof. Let $X(\sigma)$ be a fixed isometric embedding of (Σ, σ) in \mathbb{R}^3 . By [15, p. 353], there exist positive constants C_1 and ϵ_1 , depending only on σ , such that if $\tilde{\sigma}$ is another $C^{2,\alpha}$ metric on Σ with $\|\tilde{\sigma} - \sigma\|_{C^{2,\alpha}} < \epsilon_1$, then $\tilde{\sigma}$ has positive Gaussian curvature and there exists an isometric embedding $X(\tilde{\sigma})$ of $(\Sigma, \tilde{\sigma})$ in \mathbb{R}^3 such that

$$\|X(\tilde{\sigma}) - X(\sigma)\|_{C^{2,\alpha}} \leq C_1 \|\tilde{\sigma} - \sigma\|_{C^{2,\alpha}}. \quad (2.7)$$

Now, let τ be any given smooth function with $\|\tau\|_{C^{3,\alpha}}^2 < \epsilon_1$. Let $\sigma(s) = \sigma + s^2 d\tau \otimes d\tau, 0 \leq s \leq 1$. Then

$$\|\sigma(s) - \sigma\|_{C^{2,\alpha}} \leq \|d\tau \otimes d\tau\|_{C^{2,\alpha}} \leq \|\tau\|_{C^{3,\alpha}}^2 < \epsilon_1.$$

Hence, $\sigma(s)$ has positive Gaussian curvature and there exists an isometric embedding $X(s)$ of $(\Sigma, \sigma(s))$ in \mathbb{R}^3 such that

$$\|X(s) - X(0)\|_{C^{2,\alpha}} \leq C_1 \|\tau\|_{C^{3,\alpha}}^2 \quad (2.8)$$

where $X(0) = X(\sigma)$. Let $H_0(s)$ and $\mathbb{I}\mathbb{I}_0(s)$ be the mean curvature and the second fundamental form of $X(s)(\Sigma)$. Let $dv_{\sigma(s)}$ be the volume form of $\sigma(s)$. For simplicity, denote $E_{WY}(\Sigma, \sigma(s))$ by $E_{WY}(s)$. By (2.2) (and also the fact $V = 0$), we have

$$\begin{aligned} & \frac{d}{ds} E_{WY}(s) \\ &= \frac{1}{8\pi} \int_{\Sigma} \left\{ - [H_0(s)\sigma^{ab}(s) - \sigma^{ac}(s)\sigma^{bd}(s)(\mathbb{I}\mathbb{I}_0(s))_{cd}] \frac{s\nabla_b \nabla_a \tau}{\sqrt{1 + s^2 |\nabla\tau|^2}} \right. \\ & \quad \left. + \operatorname{div}_{\Sigma} \left[\frac{s\nabla\tau}{\sqrt{1 + s^2 |\nabla\tau|^2}} H \cosh \theta - \nabla\theta \right] \right\} \tau \, dv_{\Sigma} \\ &= \frac{1}{8\pi} \int_{\Sigma} s [H_0(s)\sigma^{ab}(s) - \sigma^{ac}(s)\sigma^{bd}(s)(\mathbb{I}\mathbb{I}_0(s))_{cd}] \tau_a \tau_b \sqrt{1 + s^2 |\nabla\tau|^2} \, dv_{\Sigma} \end{aligned}$$

$$- \int_{\Sigma} \frac{s|\nabla\tau|^2}{\sqrt{1+s^2|\nabla\tau|^2}} H \cosh \theta \, dv_{\Sigma} - \int_{\Sigma} \theta \Delta\tau \, dv_{\Sigma} \Big\} \tag{2.9}$$

where we have used the facts that $H_0(s)\sigma(s) - \mathbb{I}_0(s)$ is divergence free with respect to $\sigma(s)$, $dv_{\sigma(s)} = (1 + s^2|\nabla\tau|^2)^{\frac{1}{2}} dv_{\Sigma}$ and

$$\frac{\nabla_b \nabla_a \eta}{1 + s^2|\nabla\tau|^2} = \nabla_b^s \nabla_a^s \eta \tag{2.10}$$

for any function η on Σ . Here ∇^s denotes the covariant derivative of $\sigma(s)$, and $\theta = \theta(s)$ is the function defined by

$$\sinh \theta = \frac{-s\Delta\tau}{H\sqrt{1+s^2|\nabla\tau|^2}}. \tag{2.11}$$

We estimate the expression in (2.9) term by term. First note that

$$\cosh \theta - 1 \leq \sinh^2 \theta, \quad |\theta - \sinh \theta| \leq |\sinh^3 \theta|, \quad \forall \theta \in \mathbb{R}. \tag{2.12}$$

Therefore,

$$\begin{aligned} \int_{\Sigma} \theta \Delta\tau \, dv_{\Sigma} &= \int_{\Sigma} \sinh \theta \Delta\tau \, dv_{\sigma_0} + \int_{\Sigma} (\theta - \sinh \theta) \Delta\tau \, dv_{\Sigma} \\ &= - \int_{\Sigma} \frac{s(\Delta\tau)^2}{H} \, dv_{\Sigma} + F_1 \end{aligned} \tag{2.13}$$

where

$$|F_1| \leq C_2 s^3 \|\tau\|_{C^{2,\alpha}}^2 \int_{\Sigma} [|\nabla\tau|^2 + (\Delta\tau)^2] \, dv_{\Sigma} \tag{2.14}$$

for some constant C_2 depending only on H . Similarly,

$$\begin{aligned} &\int_{\Sigma} \frac{s|\nabla\tau|^2}{\sqrt{1+s^2|\nabla\tau|^2}} H \cosh \theta \, dv_{\Sigma} \\ &= \int_{\Sigma} \frac{s|\nabla\tau|^2}{\sqrt{1+s^2|\nabla\tau|^2}} H \, dv_{\Sigma} + \int_{\Sigma} \frac{s|\nabla\tau|^2}{\sqrt{1+s^2|\nabla\tau|^2}} H (\cosh \theta - 1) \, dv_{\Sigma} \\ &= \int_{\Sigma} s|\nabla\tau|^2 H \, dv_{\Sigma} + F_2 \end{aligned} \tag{2.15}$$

where

$$|F_2| \leq C_3 s^3 \|\tau\|_{C^{2,\alpha}}^2 \int_{\Sigma} [|\nabla\tau|^2 + (\Delta\tau)^2] \, dv_{\Sigma} \tag{2.16}$$

for some constant C_3 depending only on H . Next, by (2.8) we have

$$\|\mathbb{I}_0(s) - \mathbb{I}_0\|_{C^{0,\alpha}} \leq C_4 \|\tau\|_{C^{3,\alpha}}^2 \tag{2.17}$$

for some constant C_4 depending only on σ . This, together with the fact that $\|\sigma(s) - \sigma\|_{C^{2,\alpha}} \leq \|\tau\|_{C^{3,\alpha}}^2$ implies

$$\int_{\Sigma} sH_0(s)\sigma^{ab}(s)\tau_a\tau_b\sqrt{1 + s^2|\nabla\tau|^2} dv_{\sigma_0} = \int_{\Sigma} sH_0|\nabla\tau|^2 dv_{\Sigma} + F_3 \tag{2.18}$$

where

$$|F_3| \leq C_5 s \|\tau\|_{C^{3,\alpha}}^2 \int_{\Sigma} |\nabla\tau|^2 dv_{\Sigma} \tag{2.19}$$

for some constant C_5 depending only on σ . Similarly,

$$\begin{aligned} &\int_{\Sigma} s\sigma^{ac}(s)\sigma^{bd}(s)(\mathbb{I}\mathbb{I}_0(s))_{cd}\tau_a\tau_b\sqrt{1 + s^2|\nabla\tau|^2} dv_{\sigma_0} \\ &= \int_{\Sigma} s\mathbb{I}\mathbb{I}_0(\nabla\tau, \nabla\tau) dv_{\Sigma} + F_4 \end{aligned} \tag{2.20}$$

where

$$|F_4| \leq C_6 s \|\tau\|_{C^{3,\alpha}}^2 \int_{\Sigma} |\nabla\tau|^2 dv_{\Sigma} \tag{2.21}$$

for some constant C_6 depending only on σ . By (2.5), (2.9) and (2.13)–(2.21), we have

$$\begin{aligned} \frac{d}{ds} E_{w_Y}(s) &= s \int_{\Sigma} \left[\frac{(\Delta\tau)^2}{H} + (H_0 - H)|\nabla\tau|^2 - \int_{\Sigma} \mathbb{I}\mathbb{I}_0(\nabla\tau, \nabla\tau) \right] dv_{\Sigma} \\ &\quad + F_1 + F_2 + F_3 + F_4 \\ &\geq s(\beta - C_7 \|\tau\|_{C^{3,\alpha}}^2) \int_{\Sigma} (\Delta\tau)^2 dv_{\Sigma} \end{aligned} \tag{2.22}$$

for some constant C_7 depending only on σ , where in the last step we have also used the fact [see (2.27) below] that

$$\lambda_1 \int_{\Sigma} |\nabla\tau|^2 dv_{\Sigma} \leq \int_{\Sigma} (\Delta\tau)^2 dv_{\Sigma} \tag{2.23}$$

with λ_1 being the first nonzero eigenvalue of the Laplacian of σ . Hence, if ϵ is chosen such that $0 < \epsilon^2 < \epsilon_1$ and $\beta - C_7\epsilon^2 > \frac{1}{2}\beta$, then we have

$$\frac{d}{ds} E_{w_Y}(s) \geq \frac{1}{2} s\beta \int_{\Sigma} (\Delta\tau)^2 dv_{\Sigma} \tag{2.24}$$

for any $0 \leq s \leq 1$ and for any smooth function τ with $\|\tau\|_{C^{3,\alpha}} < \epsilon$. In particular, this implies

$$E_{w_Y}(\Sigma, \tau) \geq E_{w_Y}(\Sigma, 0) + \frac{\beta}{4} \|\Delta\tau\|_{L^2}^2. \tag{2.25}$$

Theorem 2.1 is proved. □

The following corollary gives a simple condition in terms of σ and H that guarantees (2.5) in Theorem 1.1.

Corollary 2.1. *Let $N, \Omega, \Sigma, \sigma, H, H_0$ and \mathbb{I}_0 be given as in Theorem 2.1. Suppose the first non-zero eigenvalue λ_1 of the Laplacian of σ satisfies:*

$$\lambda_1 > H^{\max} (H^{\max} - \mathbb{I}_0^{\min}) \tag{2.26}$$

where $H^{\max} = \max_{\Sigma} H$ and \mathbb{I}_0^{\min} is the minimum of all the eigenvalues of \mathbb{I}_0 on (Σ, σ) . Then condition (2.5) holds, hence $\mathbf{m}_{BY}(\Sigma, \Omega)$ strictly locally minimizes $E_{WY}(\Sigma, \cdot)$

Proof. By Theorem 2.1, it suffices to show that there exists a constant $\beta > 0$ such that (2.5) holds for all $\eta \in W^{2,2}(\Sigma)$.

First, we note that

$$\lambda_1 \int_{\Sigma} |\nabla \eta|^2 dv_{\Sigma} \leq \int_{\Sigma} (\Delta \eta)^2 dv_{\Sigma}, \quad \forall \eta \in W^{2,2}(\Sigma). \tag{2.27}$$

To verify this, it suffices to assume $\int_{\Sigma} \eta dv_{\Sigma} = 0$. For such an η , we have

$$\begin{aligned} \int_{\Sigma} |\nabla \eta|^2 dv_{\Sigma} &= - \int_{\Sigma} \eta \Delta \eta dv_{\Sigma} \\ &\leq \left(\int_{\Sigma} \eta^2 dv_{\Sigma} \right)^{\frac{1}{2}} \left(\int_{\Sigma} (\Delta \eta)^2 dv_{\Sigma} \right)^{\frac{1}{2}} \\ &\leq \left(\lambda_1^{-1} \int_{\Sigma} |\nabla \eta|^2 dv_{\Sigma} \right)^{\frac{1}{2}} \left(\int_{\Sigma} (\Delta \eta)^2 dv_{\Sigma} \right)^{\frac{1}{2}} \end{aligned} \tag{2.28}$$

which implies (2.27).

Now suppose (i) holds. By the definition of \mathbb{I}_0^{\min} , we have

$$H_0 |\nabla \eta|^2 - \mathbb{I}_0(\nabla \eta, \nabla \eta) \geq \mathbb{I}_0^{\min} |\nabla \eta|^2.$$

Therefore,

$$\begin{aligned} &\int_{\Sigma} \left[\frac{(\Delta \eta)^2}{H} + (H_0 - H) |\nabla \eta|^2 - \mathbb{I}_0(\nabla \eta, \nabla \eta) \right] dv_{\Sigma} \\ &\geq \int_{\Sigma} \left[\frac{(\Delta \eta)^2}{H^{\max}} + (\mathbb{I}_0^{\min} - H^{\max}) |\nabla \eta|^2 \right] dv_{\Sigma} \\ &= \frac{1}{H^{\max}} \int_{\Sigma} [(\Delta \eta)^2 - (\lambda_1 - \delta) |\nabla \eta|^2] dv_{\Sigma} \\ &\geq \frac{\delta_1}{H^{\max}} \int_{\Sigma} (\Delta \eta)^2 dv_{\Sigma} \end{aligned} \tag{2.29}$$

where $\delta = \lambda_1 - H^{\max} (H^{\max} - \mathbb{I}_0^{\min}) > 0$, and $\delta_1 = \min\{1, \delta/\lambda_1\}$ which is positive. Hence, (2.5) is satisfied with $\beta = \delta_1/H^{\max}$. \square

We leave it to the interested readers to verify that those surfaces Σ in (i) and (ii) provided in Sect. 1 also satisfy the condition (2.26) in the above Corollary.

3. Strict Positivity of the Second Variation

We investigate the condition (2.5) in this section. Our main result is the following theorem:

Theorem 3.1. *Let (Ω, g) be a three-dimensional, compact Riemannian manifold of non-negative scalar curvature with boundary $\partial\Omega$. Suppose each component of $\partial\Omega$ has positive mean curvature. Let Σ be a component of $\partial\Omega$. Suppose the induced metric σ on Σ has positive Gaussian curvature and*

$$H \leq H_0 \tag{3.1}$$

where H is the mean curvature of Σ in Ω and H_0 is the mean curvature of Σ when isometrically embedded in \mathbb{R}^3 . If Ω is not isometric to a domain in \mathbb{R}^3 , then there exists a constant $\beta > 0$ such that

$$\int_{\Sigma} \left[\frac{(\Delta\eta)^2}{H} + (H_0 - H)|\nabla\eta|^2 - \mathbb{I}\mathbb{I}_0(\nabla\eta, \nabla\eta) \right] dv_{\Sigma} \geq \beta \int_{\Sigma} (\Delta\eta)^2 dv_{\Sigma} \tag{3.2}$$

for all $\eta \in W^{2,2}(\Sigma)$. Here $\mathbb{I}\mathbb{I}_0$ is the second fundamental form of Σ when isometrically embedded in \mathbb{R}^3 .

We divide the proof of Theorem 3.1 into a few steps. First, we consider the left side of (3.2) in the case that Ω is indeed a domain in \mathbb{R}^3 . That leads to a result concerning manifolds of non-negative Ricci curvature.

Proposition 3.1. *Let (Ω, g) be a compact Riemannian manifold of dimension $n \geq 3$. Suppose Ω has smooth boundary $\partial\Omega$ (possibly disconnected) which has positive mean curvature H . If g has non-negative Ricci curvature, then*

$$\int_{\partial\Omega} \left[\frac{(\Delta\eta)^2}{H} - \mathbb{I}\mathbb{I}(\nabla\eta, \nabla\eta) \right] dv_{\partial\Omega} \geq 0 \tag{3.3}$$

for any smooth function η on $\partial\Omega$. Here $\mathbb{I}\mathbb{I}$ is the second fundamental form of $\partial\Omega$ in (Ω, g) , ∇ and Δ are the gradient and Laplacian on $\partial\Omega$ and $dv_{\partial\Omega}$ is the volume form on $\partial\Omega$.

Moreover, equality in (3.3) holds for some η if and only if η is the boundary value of some smooth function u which satisfies $\nabla_{\Omega}^2 u = 0$ and $\text{Ric}(\nabla_{\Omega} u, \nabla_{\Omega} u) = 0$ on Ω . Here ∇_{Ω}^2 and ∇_{Ω} denote the Hessian and the gradient on (Ω, g) .

Proof. Given a smooth function η on $\partial\Omega$, let u be the harmonic function on (Ω, g) such that $u = \eta$ on $\partial\Omega$. By the Reilly formula [16, Equation (14)] (see

also [9, Theorem 8.1]), we have

$$-\int_{\partial\Omega} \left[\mathbb{I}(\nabla u, \nabla u) + 2\frac{\partial u}{\partial\nu} \Delta u + H \left(\frac{\partial u}{\partial\nu} \right)^2 \right] = \int_{\Omega} |\nabla_{\Omega}^2 u|^2 + \text{Ric}(\nabla_{\Omega} u, \nabla_{\Omega} u) \tag{3.4}$$

where $\text{Ric}(\cdot, \cdot)$ is the Ricci curvature of g . Here we omit the corresponding volume form in each integral.

Since $\text{Ric}(\cdot, \cdot) \geq 0$, (3.4) implies

$$\begin{aligned} \int_{\Sigma} \mathbb{I}(\nabla u, \nabla u) &\leq \int_{\Sigma} -2\frac{\partial u}{\partial\nu} \Delta u - H \left(\frac{\partial u}{\partial\nu} \right)^2 \\ &\leq \int_{\Sigma} \frac{(\Delta\eta)^2}{H} \end{aligned} \tag{3.5}$$

by the Cauchy-Schwarz inequality. Hence, (3.3) is proved.

Now suppose the equality in (3.3) holds, then the equalities in (3.5) must hold. In particular, we have

$$\int_{\Omega} |\nabla_{\Omega}^2 u|^2 + \text{Ric}(\nabla_{\Omega} u, \nabla_{\Omega} u) = 0, \tag{3.6}$$

which shows $\nabla_{\Omega}^2 u = 0$ and $\text{Ric}(\nabla_{\Omega} u, \nabla_{\Omega} u) = 0$ on Ω . On the other hand, if $\nabla_{\Omega}^2 u = 0$ on Ω , then

$$\Delta u + H \frac{\partial u}{\partial\nu} = 0 \text{ on } \Sigma \tag{3.7}$$

which shows the second equality in (3.5) must hold. If in addition $\text{Ric}(\nabla_{\Omega} u, \nabla_{\Omega} u) = 0$, then the first equality in (3.5) holds as well. Proposition 3.1 is proved. \square

Remark 3.1. We thank Michael Eichmair who brings Reilly’s formula (3.4) to our attention. (3.4) was derived by integrating the Bochner formula and expressing the boundary term $\frac{1}{2} \int_{\Sigma} \frac{\partial}{\partial\nu} |\nabla_{\Omega} u|^2$ as the left side of (3.4). In particular, Proposition 3.1 remains valid under the general assumption that the mean curvature H does not change sign on each component of $\partial\Omega$.

Specializing Proposition 3.1 to domains in \mathbb{R}^n , we have

Corollary 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n (n \geq 3)$ with a smooth connected boundary Σ . Suppose Σ has positive mean curvature H_0 . Let \mathbb{I}_0 be the second fundamental form of Σ in \mathbb{R}^n . Then*

$$\int_{\Sigma} \left[\frac{(\Delta\eta)^2}{H_0} - \mathbb{I}_0(\nabla\eta, \nabla\eta) \right] dv_{\Sigma} \geq 0 \tag{3.8}$$

for any smooth function η on Σ , where ∇ and Δ are the gradient and the Laplacian on Σ and dv_{Σ} is the volume form on Σ . Moreover, equality in (3.8)

holds for some η if and only if η is the restriction of a linear function to Σ , i.e., $\eta = a_0 + \sum_{i=1}^n a_i x^i$ for some constants a_0, a_1, \dots, a_n .

Remark 3.2. When $n = 3$ and Σ is a strictly convex surface in \mathbb{R}^3 , the inequality (3.8) can also be seen by considering the second variation of $E_{w_Y}(\Sigma, \cdot)$ for $\Sigma \subset \mathbb{R}^3 = \{(x, 0) \in \mathbb{R}^{3,1}\}$. In fact, by [19, Theorem A], $E_{w_Y}(\Sigma, \tau) \geq 0$ for any admissible function τ . Since Σ has positive Gaussian curvature, τ is admissible if $\|\tau\|_{C^{3,\alpha}}$ is sufficiently small [19, Remark 1.1]. Therefore, $E_{w_Y}(\Sigma, \tau) \geq 0$ for any such τ . On the other hand, it is obvious that $E_{w_Y}(\Sigma, 0) = 0$. Hence, (3.8) follows from (2.1).

Next, we derive an estimate of the left side of (3.2) for those η which are restriction of linear functions in \mathbb{R}^3 to Σ .

Proposition 3.2. *Let Ω be a three-dimensional Riemannian manifold. Let $\Sigma \subset \Omega$ be an embedded closed 2-surface that is diffeomorphic to a sphere. Suppose the mean curvature H of Σ in Ω is positive and the induced metric σ on Σ has positive Gaussian curvature. Let*

$$X = (X^1, X^2, X^3) : \Sigma \hookrightarrow \mathbb{R}^3$$

be an isometric embedding of (Σ, σ) into \mathbb{R}^3 . Given any constant a_0 and any constant unit vector $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, let $\eta = a_0 + \sum_{i=1}^3 a_i X^i$, then

$$\int_{\Sigma} \left[\frac{(\Delta\eta)^2}{H} + (H_0 - H)|\nabla\eta|^2 - \mathbb{I}\mathbb{I}_0(\nabla\eta, \nabla\eta) \right] dv_{\Sigma} \geq 8\pi m_{B_Y}(\Sigma, \Omega) \tag{3.9}$$

where H is the mean curvature of Σ in Ω , H_0 and $\mathbb{I}\mathbb{I}_0$ are the mean curvature and the second fundamental form of Σ when isometrically embedded in \mathbb{R}^3 .

Proof. For such an η , Corollary 3.1 implies

$$\int_{\Sigma} \left[\frac{(\Delta\eta)^2}{H_0} - \mathbb{I}\mathbb{I}_0(\nabla\eta, \nabla\eta) \right] dv_{\Sigma} = 0. \tag{3.10}$$

Direct calculation shows

$$(\Delta\eta)^2 = (a \cdot \vec{H}_0)^2 \quad \text{and} \quad |\nabla\eta|^2 = 1 - (a \cdot \vec{H}_0)^2 H_0^{-2} \tag{3.11}$$

where \vec{H}_0 is the mean curvature vector of Σ when isometrically embedded in \mathbb{R}^3 . Therefore,

$$\begin{aligned} & \int_{\Sigma} \left[\frac{(\Delta\eta)^2}{H} + (H_0 - H)|\nabla\eta|^2 - \mathbb{I}\mathbb{I}_0(\nabla\eta, \nabla\eta) \right] dv_{\Sigma} \\ &= \int_{\Sigma} (a \cdot \vec{H}_0)^2 \left(\frac{1}{H} - \frac{2}{H_0} + \frac{H}{H_0^2} \right) + (H_0 - H) dv_{\Sigma} \\ &\geq \int_{\Sigma} (H_0 - H) dv_{\Sigma}. \end{aligned} \tag{3.12}$$

We are now ready to prove Theorem 3.1. □

Proof of Theorem 3.1. For convenience, we omit writing the volume form in each integral. Note that

$$\int_{\Sigma} \left[\frac{(\Delta\eta)^2}{H} + (H_0 - H)|\nabla\eta|^2 - \mathbb{I}_0(\nabla\eta, \nabla\eta) \right] = I_1(\eta, \eta) + I_2(\eta, \eta) \quad (3.13)$$

where

$$I_1(\eta, \eta) = \int_{\Sigma} \left[\frac{(\Delta\eta)^2}{H} - \frac{(\Delta\eta)^2}{H_0} + (H_0 - H)|\nabla\eta|^2 \right]$$

and

$$I_2(\eta, \eta) = \int_{\Sigma} \left[\frac{(\Delta\eta)^2}{H_0} - \mathbb{I}_0(\nabla\eta, \nabla\eta) \right].$$

By Corollary 3.1, we know $I_2(\eta, \eta) \geq 0$. By the assumption (3.1), we have $I_1(\eta, \eta) \geq 0$. Therefore,

$$\int_{\Sigma} \left[\frac{(\Delta\eta)^2}{H} + (H_0 - H)|\nabla\eta|^2 - \mathbb{I}_0(\nabla\eta, \nabla\eta) \right] \geq 0. \quad (3.14)$$

To prove (3.2), we argue by contradiction.

Suppose (3.2) is not true, then there exists a sequence of functions $\{\eta_k\} \subset W^{2,2}(\Sigma)$ with

$$\int_{\Sigma} \eta_k = 0 \text{ and } \int_{\Sigma} \eta_k^2 = 1 \quad (3.15)$$

such that

$$\int_{\Sigma} \left[\frac{(\Delta\eta_k)^2}{H} + (H_0 - H)|\nabla\eta_k|^2 - \mathbb{I}_0(\nabla\eta_k, \nabla\eta_k) \right] \leq \frac{1}{k} \int_{\Sigma} (\Delta\eta_k)^2. \quad (3.16)$$

By the interpolation inequality for Sobolev functions, we have

$$\begin{aligned} \int_{\Sigma} \frac{(\Delta\eta_k)^2}{H} &\leq \int_{\Sigma} [(H - H_0)|\nabla\eta_k|^2 + \mathbb{I}_0(\nabla\eta_k, \nabla\eta_k)] + \frac{1}{k} \int_{\Sigma} (\Delta\eta_k)^2 \\ &\leq C_1 + \int_{\Sigma} \frac{(\Delta\eta_k)^2}{2H} + \frac{1}{k} \int_{\Sigma} (\Delta\eta_k)^2. \end{aligned} \quad (3.17)$$

Here and below, $\{C_1, C_2, \dots\}$ denote positive constants independent on k . It follows from (3.17) that

$$\|\Delta\eta_k\|_{L^2(\Sigma)} \leq C_2. \quad (3.18)$$

By (3.15) and the usual L^p estimate, we then have

$$\|\eta_k\|_{W^{2,2}(\Sigma)} \leq C_3. \quad (3.19)$$

This implies that there exists a function $\eta \in W^{2,2}(\Sigma)$ such that

- a) η_k converges weakly to η in $W^{2,2}(\Sigma)$.
- b) η_k converges strongly to η in $W^{1,2}(\Sigma)$.

By (3.19) and a), b), one also easily verifies that

c) $\Delta\eta_k$ converges to $\Delta\eta$ weakly in $L^2(\Sigma)$.

Moreover, by (3.15) and b), η satisfies

$$\int_{\Sigma} \eta = 0 \quad \text{and} \quad \int_{\Sigma} \eta^2 = 1. \tag{3.20}$$

We now claim that

$$\int_{\Sigma} \left[\frac{(\Delta\eta)^2}{H} + (H_0 - H)|\nabla\eta|^2 - \mathbb{I}\mathbb{I}_0(\nabla\eta, \nabla\eta) \right] = 0. \tag{3.21}$$

To see this, we replace η by $\eta - \eta_k$ in (3.14) to obtain

$$\int_{\Sigma} \left[\frac{(\Delta(\eta - \eta_k))^2}{H} + (H_0 - H)|\nabla(\eta - \eta_k)|^2 - \mathbb{I}\mathbb{I}_0(\nabla(\eta - \eta_k), \nabla(\eta - \eta_k)) \right] \geq 0. \tag{3.22}$$

It follows from (3.16) and (3.22) that

$$\begin{aligned} \frac{1}{k} \int_{\Sigma} (\Delta\eta_k)^2 \, dv_{\Sigma} &\geq \int_{\Sigma} \left[\frac{(\Delta\eta_k)^2}{H} + (H_0 - H)|\nabla\eta_k|^2 - \mathbb{I}\mathbb{I}_0(\nabla\eta_k, \nabla\eta_k) \right] \\ &\geq \int_{\Sigma} \frac{2\Delta\eta_k \cdot \Delta\eta - (\Delta\eta)^2}{H} + (H_0 - H)(2\nabla\eta_k \cdot \nabla\eta - |\nabla\eta|^2) \\ &\quad + \int_{\Sigma} -2\mathbb{I}\mathbb{I}_0(\nabla\eta_k, \nabla\eta) + \mathbb{I}\mathbb{I}_0(\nabla\eta, \nabla\eta). \end{aligned} \tag{3.23}$$

Letting $k \rightarrow \infty$, by (3.19), a), b), c) and (3.23) we have

$$0 \geq \int_{\Sigma} \left[\frac{(\Delta\eta)^2}{H} + (H_0 - H)|\nabla\eta|^2 - \mathbb{I}\mathbb{I}_0(\nabla\eta, \nabla\eta) \right]. \tag{3.24}$$

This, together with (3.14), shows that

$$\int_{\Sigma} \left[\frac{(\Delta\eta)^2}{H} + (H_0 - H)|\nabla\eta|^2 - \mathbb{I}\mathbb{I}_0(\nabla\eta, \nabla\eta) \right] = 0. \tag{3.25}$$

Next, we claim that η must be the restriction of a linear function on Σ . Here we identify Σ with its image in \mathbb{R}^3 under the isometric embedding. To see this, first we note that η is a smooth function on Σ . That is because, by (3.14) and (3.25), η is a minimizer of the functional

$$\mathcal{F}(f) = \left[\frac{(\Delta f)^2}{H} + (H_0 - H)|\nabla f|^2 - \mathbb{I}\mathbb{I}_0(\nabla f, \nabla f) \right]$$

on $W^{2,2}(\Sigma)$. Hence, η is a weak solution to the Euler-Lagrange equation

$$\Delta \left(\frac{\Delta\eta}{H} \right) - \operatorname{div}((H_0 - H)\nabla\eta) + \operatorname{div}(\mathbb{I}\mathbb{I}_0(\cdot, \nabla\eta)) = 0. \tag{3.26}$$

Since the coefficients of (3.26) are assumed to be smooth, we know η is a smooth function by the standard elliptic regularity theory. Second, by (3.13), we have

$$0 = I_1(\eta, \eta) + I_2(\eta, \eta). \tag{3.27}$$

Since $I_1(\eta, \eta) \geq 0$ and $I_2(\eta, \eta) \geq 0$, we know $I_2(\eta, \eta) = 0$. By Corollary 3.1, we conclude that

$$\eta = a_0 + \sum_{i=1}^3 a_i x^i \tag{3.28}$$

for some constants a_0, a_1, a_2, a_3 . By (3.20) we further know that η is not a constant, hence $(a_1, a_2, a_3) \neq (0, 0, 0)$.

For such an η , Proposition 3.2 shows

$$\int_{\Sigma} \left[\frac{(\Delta\eta)^2}{H} + (H_0 - H)|\nabla\eta|^2 - \mathbb{I}\mathbb{I}_0(\nabla\eta, \nabla\eta) \right] \geq 8\pi m_{BY}(\Sigma, \Omega). \tag{3.29}$$

Therefore, by (3.25) we have

$$0 \geq 8\pi m_{BY}(\Sigma, \Omega) = \int_{\Sigma} (H_0 - H). \tag{3.30}$$

Since it is assumed $H_0 \geq H$ on Σ , we conclude that $H_0 = H$ everywhere on Σ .

To finish the proof, we apply the positive mass theorem to draw a contradiction. Let $N \subset \mathbb{R}^3$ be the exterior region of Σ . We attach N to the compact manifold Ω along Σ to get a Riemannian manifold M . The metric g_M on M has the feature that, though it may not be smooth across Σ , the mean curvatures of Σ from its both sides in M agree. We have the following two cases:

- When $\partial\Omega$ has only one component, i.e., $\Sigma = \partial\Omega$, we can apply Theorem 3.1 in [17] (or Theorem 2 in [12]) directly to conclude that Ω must be isometric to a domain in \mathbb{R}^3 . This is a contradiction to the assumption on Ω .
- When $\partial\Omega$ has more than one components, M has a nonempty boundary $\partial M = \partial\Omega \setminus \Sigma$, which by assumption has positive mean curvature (i.e., its mean curvature vector points inside M). In this case, one can modify the proof of Theorem 3.1 in [17] to show that Ω still must be isometric to a domain in \mathbb{R}^3 . Or one can proceed as in [13, Section 3.2] to draw a contradiction as follows: by minimizing area among surfaces in Ω that are homologous to Σ , we know there exists a closed minimal surface Σ_H in Ω having the property that there are no other closed minimal surface lying inside the region $\tilde{\Omega}$ bounded by Σ and Σ_H . By directly applying Lemma 2, 3, 4 in [13] and the Riemannian Penrose inequality [1, 7], we have

$$\text{the mass of } g_M \geq \sqrt{\frac{|\Sigma_H|}{16\pi}} > 0.$$

This contradicts the fact that M outside Σ is the exterior Euclidean region N , which has zero mass.

We conclude that (3.2) is true. Hence, Theorem 3.1 is proved. □

Part (1) of Theorem 1.1 now follows directly from Theorem 2.1 and Theorem 3.1.

4. Second Fundamental form of the Isometric Embedding

The rest of this paper is devoted to study of Question 2. As mentioned in the introduction, in order to apply the IFT, we want to verify that the map, which sends a metric σ (on the two-sphere S^2) of positive Gaussian curvature to the second fundamental form of the isometric embedding of (S^2, σ) into \mathbb{R}^3 , is a C^1 map between appropriate functional spaces. To do so, we follow closely the original work of Nirenberg [15].

First, we fix some notations. Let $\Sigma = S^2$. Given an integer $k \geq 2$ and a positive number $0 < \alpha < 1$, let

- $\mathcal{E}^{k,\alpha}$ = the space of $C^{k,\alpha}$ embeddings of Σ into \mathbb{R}^3
- $\mathcal{X}^{k,\alpha}$ = the space of $C^{k,\alpha}$ \mathbb{R}^3 -valued vector functions on Σ
- $\mathcal{S}^{k,\alpha}$ = the space of $C^{k,\alpha}$ symmetric $(0, 2)$ tensors on Σ
- $\mathcal{M}^{k,\alpha}$ = the space of $C^{k,\alpha}$ Riemannian metrics on Σ
- $\mathcal{M}_+^{k,\alpha}$ = open subset of $\mathcal{M}^{k,\alpha}$ with positive Gaussian curvature.

By the results in [15], for $k \geq 4$ and $\sigma \in \mathcal{M}_+^{k,\alpha}$, there is an isometric embedding $X(\sigma)$ of (Σ, σ) into \mathbb{R}^3 which is unique up to an isometry of \mathbb{R}^3 . Also, $X(\sigma)$ is necessarily in $\mathcal{E}^{k,\alpha}$ by [15]. Hence the following map is well-defined:

$$\mathcal{F} : \mathcal{M}_+^{k,\alpha} \rightarrow \mathcal{S}^{k-2,\alpha} \subset \mathcal{S}^{k-3,\alpha} \tag{4.1}$$

where $\mathcal{F}(\sigma) = \text{III}(X(\sigma))$ is the second fundamental form of $X(\sigma)(\Sigma)$ (pulled back via $X(\sigma)$ and viewed as an element in $\mathcal{S}^{k-2,\alpha}$). We want to study the smoothness of \mathcal{F} .

Given $\sigma \in \mathcal{M}_+^{k,\alpha}, k \geq 4$ and let $X = X(\sigma) \in \mathcal{E}^{k,\alpha}$ be an isometric embedding of (Σ, σ) . Let $\{(u, v)\}$ denote a fixed coordinate chart on Σ , let X_u, X_v denote the partial derivative of X with respect to u, v , and let $X_3 = X_u \wedge X_v / |X_u \wedge X_v|$ be the unit normal. The coefficients of the first and the second fundamental forms of X are denoted by E, F, G and L, M, N , respectively. Let $\Delta = \sqrt{EG - F^2}$ and let K, H be the Gaussian curvature, the mean curvature of $X(\Sigma)$ which are both positive. Let

$$\begin{pmatrix} l & m \\ m & n \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}^{-1}$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Note that

$$\begin{aligned} (X_3)_u &= AX_u + BX_v \\ (X_3)_v &= CX_u + DX_v \end{aligned}$$

By [15, Section 6-8], given any $\rho \in \mathcal{S}^{r,\alpha}$ and $r \geq 2$, there exists a uniquely determined $Y = \Phi(\sigma, \rho) \in \mathcal{X}^{s,\alpha}$ (which also depends on X), where $s = \min\{k-1, r\}$, such that Y is a solution of

$$2dX \cdot dY = \rho \tag{4.2}$$

and Y vanishes at a fixed point on Σ . Recall from [15] that Y is constructed in the following way:

STEP 1: Let ϕ be the unique solution of

$$\mathcal{L}(\phi_u, \phi_v) + H\phi = \mathcal{L}(c_1, c_2) - T \tag{4.3}$$

which is L^2 -orthogonal to the kernel of $\mathcal{L}(\phi_u, \phi_v) + H\phi$ which is spanned by the coordinates functions of X_3 . Here

$$\mathcal{L}(q_1, q_2) = \frac{1}{\Delta} \left(\frac{N}{K\Delta} q_1 - \frac{M}{K\Delta} q_2 \right)_u - \frac{1}{\Delta} \left(\frac{M}{K\Delta} q_1 - \frac{L}{K\Delta} q_2 \right)_v \tag{4.4}$$

$$c_1 = \frac{1}{\Delta} (\rho_{12;u} - \rho_{11;v}), \quad c_2 = \frac{1}{\Delta} (\rho_{22;u} - \rho_{21;v}) \tag{4.5}$$

$$T = \frac{1}{\Delta} (C\rho_{11} + (D - A)\rho_{12} - B\rho_{22}), \tag{4.6}$$

where $\rho_{ij;u}$ etc. are the covariant derivatives of ρ on (Σ, σ) .

Denote $\phi = \Psi(\sigma, \rho)$. Note that Ψ is linear in ρ .

STEP 2: $Y = \Phi(\sigma, \rho)$ is obtained by integrating:

$$\begin{aligned} Y_u &= \frac{1}{2\Delta^2} (\rho_{11}G - \rho_{12}F) X_u + \frac{1}{2\Delta^2} (\rho_{12}E - \rho_{11}F) X_v \\ &\quad + \frac{1}{2\Delta} (EX_v - FX_u) \phi + X_3 p_1 \end{aligned} \tag{4.7}$$

$$\begin{aligned} Y_v &= \frac{1}{2\Delta^2} (\rho_{12}G - \rho_{22}F) X_u + \frac{1}{2\Delta^2} (\rho_{22}E - \rho_{12}F) X_v \\ &\quad + \frac{1}{2\Delta} (FX_v - GX_u) \phi + X_3 p_2. \end{aligned}$$

where

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \frac{\Delta}{2} \begin{pmatrix} m & n \\ l & -m \end{pmatrix} \begin{pmatrix} \phi_u - c_1 \\ \phi_v - c_2 \end{pmatrix}. \tag{4.8}$$

In particular, Φ is linear in ρ . By (6.6) in [15], ϕ and Φ are also related by

$$\phi(u, v) = \frac{1}{\Delta} (X_v \cdot \Phi_u - X_u \cdot \Phi_v). \tag{4.9}$$

The following C^0 estimate of ϕ was proved in [14, Lemma 5.2].

Lemma 4.1. *Let $\sigma_0 \in \mathcal{M}_+^{5,\alpha}$. There exists positive numbers ϵ and C , depending only on σ_0 , such that if $\sigma \in \mathcal{M}_+^{4,\alpha}$ and $\|\sigma - \sigma_0\|_{C^{2,\alpha}} < \epsilon$, then for any $\rho \in \mathcal{S}^{r,\alpha}$, $r \geq 2$,*

$$\|\phi\|_{C^0} \leq C\|\rho\|_{C^{1,\alpha}}$$

where $\phi = \Psi(\sigma, \rho)$.

Let $\sigma \in \mathcal{M}_+^{k,\alpha}, k \geq 4$ and let $X = X(\sigma)$ be a given isometric embedding of (Σ, σ) . By [15, Section 5], for any $\tau \in \mathcal{M}_+^{k,\alpha}$ which is close to σ in the $C^{2,\alpha}$ norm, there exists an isometric embedding of (Σ, τ) in the form of $X + Y$ where Y is obtained as follows: Let $Y_0 = 0$ and $Y_m = \Phi(\sigma, \rho_{m-1})$, where $\rho_{m-1} = \tau - \sigma - (dY_{m-1})^2$, then $\{Y_m\}$ converges to Y in the $C^{2,\alpha}$ norm such that Y satisfies:

$$2dX \cdot dY = \tau - \sigma - (dY)^2. \tag{4.10}$$

Let us denote this particular solution Y to (4.10) by $Y(\sigma, \tau)$. Since both $X(\sigma)$ and $X(\sigma) + Y(\sigma, \tau)$ are in $\mathcal{E}^{k,\alpha}$, we know $Y(\sigma, \tau)$ is of $C^{k,\alpha}$.

In [14, Lemma 5.3], the following $C^{2,\alpha}$ estimate of Y was proved.

Lemma 4.2. *Let $\sigma^0 \in \mathcal{M}_+^{5,\alpha}$. There exists positive numbers δ, ϵ and C , depending only on σ^0 , with the following properties:*

Suppose $\sigma \in \mathcal{M}_+^{4,\alpha}$ satisfying

$$\|\sigma^0 - \sigma\|_{C^{2,\alpha}} < \delta.$$

Let $X(\sigma)$ be an isometric embedding of (Σ, σ) . Then for any $\tau \in \mathcal{M}_+^{2,\alpha}$ satisfying

$$\|\sigma - \tau\|_{C^{2,\alpha}} < \epsilon,$$

the solution $Y = Y(\sigma, \tau)$ to (4.10) satisfies

$$\|Y\|_{C^{2,\alpha}} \leq C\|\sigma - \tau\|_{C^{2,\alpha}}.$$

For the purpose in this paper, we want to obtain the corresponding $C^{k,\alpha}$ estimate ($k \geq 4$) of ϕ and Y . We have

Lemma 4.3. *Let $k \geq 4$ be an integer. Let $\sigma_0 \in \mathcal{M}_+^{k+1,\alpha}$. There exists positive numbers δ, ϵ and C , depending only on σ_0 , with the following properties:*

Suppose $\sigma \in \mathcal{M}_+^{k,\alpha} \cap B(\sigma_0, 1)$ where $B(\sigma_0, 1)$ is the open ball in $\mathcal{M}_+^{k,\alpha}$ with center at σ_0 and radius 1. Let $X(\sigma)$ be an isometric embedding of (Σ, σ) in \mathbb{R}^3 . Suppose

$$\|\sigma_0 - \sigma\|_{C^{k,\alpha}} < \delta.$$

Then for any $\tau \in \mathcal{M}_+^{k,\alpha} \cap B(\sigma_0, 1)$ satisfying

$$\|\sigma - \tau\|_{C^{k,\alpha}} < \epsilon,$$

the solution $Y = Y(\sigma, \tau)$ to (4.10) satisfies

$$\|Y\|_{C^{k,\alpha}} \leq C\|\sigma - \tau\|_{C^{k,\alpha}}.$$

Thus, if $X(\tau) = X(\sigma) + Y(\sigma, \tau)$ is the corresponding isometric embedding of (Σ, τ) , then

$$\|X(\sigma) - X(\tau)\|_{C^{k,\alpha}} \leq C\|\sigma - \tau\|_{C^{k,\alpha}}.$$

Proof. Let X_0 be a fixed isometric embedding of (Σ, σ_0) so that the origin is the center of the largest inscribed sphere of $X_0(\Sigma)$ in \mathbb{R}^3 . Let $\{(u, v)\}$ be a fixed coordinates chart of Σ and let $\Omega \subset \Sigma$ be an open set whose closure is covered by $\{(u, v)\}$. On Ω , we have $|X_0| \geq C$ and $K_0((X_0, (X_0)_1 \wedge (X_0)_2))^2 \geq C$.

Here and below, C always denote a positive constant depending only on σ_0 , K_0 denotes the Gaussian curvature of X_0 , and $(X_0)_1 = (X_0)_u$, etc.

By Lemma 4.2, there exist positive constants δ, ϵ and C , depending only on σ_0 , such that for any $\sigma, \tau \in \mathcal{M}_+^{k,\alpha}$ with $\|\sigma_0 - \sigma\|_{C^{k,\alpha}} < \delta$ and $\|\tau - \sigma\|_{C^{k,\alpha}} < \epsilon$, there exists an isometric embedding $X(\sigma)$ of (Σ, σ) such that

$$\|X(\sigma) - X_0\|_{C^{2,\alpha}} \leq C\|\sigma - \sigma_0\|_{C^{2,\alpha}} \tag{4.11}$$

and the solution $Y(\sigma, \tau)$ to (4.10) (with $X = X(\sigma)$) satisfies

$$\|Y(\sigma, \tau)\|_{C^{2,\alpha}} \leq C\|\tau - \sigma\|_{C^{2,\alpha}}. \tag{4.12}$$

For such given σ and τ , let $X(\tau) = X(\sigma) + Y(\sigma, \tau)$ and let $K(\sigma), K(\tau)$ be the Gaussian curvature of $X(\sigma), X(\tau)$. Assuming δ, ϵ are sufficiently small, by (4.11) and (4.12) we have

$$\begin{aligned} |X(\sigma)| &\geq C, & |X(\tau)| &\geq C, \\ K(\sigma)(\langle X(\sigma), (X(\sigma))_1 \wedge (X(\sigma))_2 \rangle)^2 &\geq C, \\ K(\tau)(\langle X(\tau), (X(\tau))_1 \wedge (X(\tau))_2 \rangle)^2 &\geq C. \end{aligned}$$

Here and below we always consider points in Ω .

Consider $\rho = \frac{1}{2}|X(\sigma)|^2$ as in [15, Section 3]. Let

$$\begin{aligned} \mathbf{A} &= \rho_{11} - \Gamma_{11}^1 \rho_1 - \Gamma_{11}^2 \rho_2 - E \\ \mathbf{B} &= \rho_{22} - \Gamma_{22}^1 \rho_1 - \Gamma_{22}^2 \rho_2 - G \\ \mathbf{C} &= \rho_{12} - \Gamma_{12}^1 \rho_1 - \Gamma_{12}^2 \rho_2 - F, \end{aligned}$$

where $\Gamma_{ij}^k, i, j, k \in \{1, 2\}$, are Christoffel symbols. By the equation (3.7) in [15],

$$\begin{aligned} \mathbf{AB} - \mathbf{C}^2 &= \Delta^2 K(\langle X(\sigma), X_3(\sigma) \rangle)^2 \\ &= K(\langle X(\sigma), X_1(\sigma) \wedge X_2(\sigma) \rangle)^2 \\ &\geq C. \end{aligned} \tag{4.13}$$

Differentiate this equation with respect to the i -th variable, we have

$$\mathbf{B}\rho_{i11} + \mathbf{A}\rho_{i22} - 2\mathbf{C}\rho_{i11} = \mathbf{P}, \tag{4.14}$$

where $\mathbf{P} = \mathbf{P}(\sigma, \partial\sigma, \partial\partial\sigma, \partial\partial\partial\sigma, \partial\rho, \partial\partial\rho, X(\sigma), \partial X(\sigma))$ is some fixed polynomial function of its arguments. Here we used a basic fact that the m -th derivatives of $X, m \geq 2$, can be expressed as a linear combination of X, X_1, X_2 with coefficients involving derivatives of σ, ρ of order at most m (see p. 348 in [15]). Now, since $\|\sigma\|_{C^{3,\alpha}}, \|\rho\|_{C^{2,\alpha}}, \|X(\sigma)\|_{C^{2,\alpha}}$ are all bounded, it follows from (4.13) and (4.14) that $\|\rho\|_{C^{3,\alpha}}$ is bounded. This in turn implies that $\|X(\sigma)\|_{C^{3,\alpha}}$ is bounded. Next, since the $\|\sigma\|_{C^{4,\alpha}}, \|\rho\|_{C^{3,\alpha}}, \|X(\sigma)\|_{C^{3,\alpha}}$ are bounded, we see $\|\rho\|_{C^{4,\alpha}}$ is bounded, which then implies $\|X(\sigma)\|_{C^{4,\alpha}}$ is bounded. Hence,

$$\|\rho\|_{C^{4,\alpha}} + \|X(\sigma)\|_{C^{4,\alpha}} \leq C.$$

Continue in this way and use the fact that $\|\sigma\|_{C^{k,\alpha}}$ is bounded, we conclude that

$$\|\rho\|_{C^{k,\alpha}} + \|X(\sigma)\|_{C^{k,\alpha}} \leq C. \tag{4.15}$$

Similarly, we have

$$\|\tilde{\rho}\|_{C^{k,\alpha}} + \|X(\tau)\|_{C^{k,\alpha}} \leq C, \tag{4.16}$$

where $\tilde{\rho} = \frac{1}{2}|X(\tau)|^2$, and $\tilde{\rho}$ satisfies

$$\tilde{\mathbf{B}}\tilde{\rho}_{i11} + \tilde{\mathbf{A}}\tilde{\rho}_{i22} - 2\tilde{\mathbf{C}}\tilde{\rho}_{i12} = \tilde{\mathbf{P}} \tag{4.17}$$

where $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{P}}$ are constructed in the same way as $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{P}$. By (4.14) and (4.17), we have

$$\begin{aligned} &\tilde{\mathbf{B}}(\tilde{\rho}_{i11} - \rho_{i11}) + \tilde{\mathbf{A}}(\tilde{\rho}_{i22} - \rho_{i22}) - 2\tilde{\mathbf{C}}(\tilde{\rho}_{i12} - \rho_{i12}) \\ &= \tilde{\mathbf{P}} - \mathbf{P} + (\mathbf{B} - \tilde{\mathbf{B}})\rho_{i11} + (\mathbf{A} - \tilde{\mathbf{A}})\rho_{i22} - 2(\mathbf{C} - \tilde{\mathbf{C}})\rho_{i12}. \end{aligned} \tag{4.18}$$

By (4.12), (4.15) and (4.16), we have

$$\|\rho - \tilde{\rho}\|_{C^{2,\alpha}} + \|X(\sigma) - X(\tau)\|_{C^{2,\alpha}} \leq C\|\sigma - \tau\|_{C^{k,\alpha}}. \tag{4.19}$$

Now suppose for some integer l satisfying $2 \leq l < k$, we have

$$\|\rho - \tilde{\rho}\|_{C^{l,\alpha}} \leq C\|\sigma - \tau\|_{C^{k,\alpha}}.$$

By (4.15), (4.16) and (4.19), we then have

$$\|X(\sigma) - X(\tau)\|_{C^{l,\alpha}} \leq C\|\sigma - \tau\|_{C^{k,\alpha}},$$

where we also used the previously mentioned fact regarding writing the derivatives of $X(\sigma), X(\tau)$ in terms of those of $\rho, \tilde{\rho}$ (p. 348 in [15]). On the other hand, we have

$$\begin{aligned} \|\mathbf{A} - \tilde{\mathbf{A}}\|_{C^{l-2,\alpha}} &\leq C(\|\rho - \tilde{\rho}\|_{C^{l,\alpha}} + \|\sigma - \tau\|_{C^{l-1,\alpha}}), \\ \|\mathbf{B} - \tilde{\mathbf{B}}\|_{C^{l-2,\alpha}} &\leq C(\|\rho - \tilde{\rho}\|_{C^{l,\alpha}} + \|\sigma - \tau\|_{C^{l-1,\alpha}}), \\ \|\mathbf{C} - \tilde{\mathbf{C}}\|_{C^{l-2,\alpha}} &\leq C(\|\rho - \tilde{\rho}\|_{C^{l,\alpha}} + \|\sigma - \tau\|_{C^{l-1,\alpha}}), \end{aligned}$$

and

$$\|\mathbf{P} - \tilde{\mathbf{P}}\|_{C^{l-2,\alpha}} \leq C(\|\rho - \tilde{\rho}\|_{C^{l,\alpha}} + \|\sigma - \tau\|_{C^{l+1,\alpha}} + \|X(\sigma) - X(\tau)\|_{C^{l-1,\alpha}}).$$

Since $l + 1 \leq k$, by (4.18) we conclude

$$\|\rho - \tilde{\rho}\|_{C^{l+1,\alpha}} \leq C\|\sigma - \tau\|_{C^{k,\alpha}},$$

and therefore

$$\|X(\sigma) - X(\tau)\|_{C^{l+1,\alpha}} \leq C\|\sigma - \tau\|_{C^{k,\alpha}}$$

The result follows by induction. □

Lemma 4.4. *Let $\sigma_0 \in \mathcal{M}_+^{k+1,\alpha}$ ($k \geq 4$). Let $\epsilon > 0$ be as in Lemma 4.3. Suppose $\sigma \in \mathcal{M}_+^{k,\alpha}$ and $\|\sigma - \sigma_0\|_{C^{k,\alpha}} < \epsilon$. Let $X = X(\sigma)$ be an isometric embedding of (Σ, σ) into \mathbb{R}^3 . Given any $\rho \in \mathcal{S}^{r,\alpha}$ ($r \geq 2$), let $\phi = \Psi(\sigma, \rho)$ be the unique solution of (4.3) which is L^2 -orthogonal to the coordinates functions of the unit normal of $X(\sigma)$; let $Y = \Phi(\sigma, \rho)$ be the unique solution of (4.2) which vanishes at a fixed point on Σ and is obtained by integrating Y_u and Y_v defined by (4.7) and (4.8). There exist $C > 0$ depending only on σ_0 and ϵ such that*

$$\|\phi\|_{C^{s,\alpha}} \leq C\|\rho\|_{C^{r,\alpha}} \tag{4.20}$$

and

$$\|\Phi(\sigma, \rho)\|_{C^{s,\alpha}} \leq C\|\rho\|_{C^{r,\alpha}} \tag{4.21}$$

where $s = \min\{r, k - 1\}$.

Proof. Let X_0 be a fixed isometric embedding of (Σ, σ_0) . By Lemma 4.3, we may assume $X = X(\sigma)$ is chosen such that $\|X - X_0\|_{C^{k,\alpha}} \leq C$, where C depends only on σ_0 and ϵ . Recall that ϕ satisfies

$$\frac{1}{\Delta} \left(\frac{N}{K\Delta} \phi_u - \frac{M}{K\Delta} \phi_v \right)_u - \frac{1}{\Delta} \left(\frac{M}{K\Delta} \phi_u - \frac{L}{K\Delta} \phi_v \right)_v + H\phi = \mathcal{L}(c_1, c_2) - T. \tag{4.22}$$

By (4.4), (4.5) and (4.6), we have

$$\|\mathcal{L}(c_1, c_2) - T\|_{C^{s-2,\alpha}} \leq C\|\rho\|_{C^{r,\alpha}}. \tag{4.23}$$

Hence,

$$\|\phi\|_{C^{s,\alpha}} \leq C(\|\phi\|_{C^0} + \|\rho\|_{C^{r,\alpha}}). \tag{4.24}$$

Therefore, (4.20) holds by (4.24) and Lemma 4.1. Now (4.21) follows directly from (4.7), (4.8) and (4.20). \square

Now we are in a position to prove the main result of this section.

Theorem 4.1. *Let $\sigma_0 \in \mathcal{M}_+^{k+1,\alpha}$ ($k \geq 4$). There exists a constant $\kappa > 0$ such that the map*

$$\mathcal{F} : \mathcal{M}_+^{k,\alpha} \rightarrow \mathcal{S}^{k-3,\alpha}$$

defined by (4.1) is C^1 in $U = \{\sigma \in \mathcal{M}_+^{k,\alpha} \mid \|\sigma - \sigma_0\|_{k,\alpha} < \kappa\}$.

Proof. Let $\epsilon > 0$ and $\delta > 0$ be as in Lemma 4.3. We may assume that ϵ is so small that the open set $U_{2\epsilon} = \{\hat{\sigma} \in \mathcal{S}^{k,\alpha} \mid \|\hat{\sigma} - \sigma_0\|_{C^{k,\alpha}} < 2\epsilon\}$ in $\mathcal{S}^{k,\alpha}$ is indeed contained in $\mathcal{M}_+^{k,\alpha}$.

Let $\kappa > 0$ be chosen such that $\kappa < \min\{\epsilon, \delta\}$. Suppose $\sigma \in U$. Let $X = X(\sigma)$ be an isometric embedding of (Σ, σ) . Since $\kappa < \epsilon$, we may assume that $X(\sigma)$ is chosen such that $\|X(\sigma)\|_{C^{k,\alpha}} \leq C$, where C depends only on σ_0 and ϵ . Given any $\eta \in \mathcal{S}^{k,\alpha}$ such that $\|\eta\|_{C^{k,\alpha}} = 1$, consider $\sigma + t\eta \in \mathcal{M}_+^{k,\alpha}$ for $|t| < \epsilon$. Let $X(\sigma + t\eta) = X(\sigma) + Y(\sigma, \sigma + t\eta)$ be the (nearby) isometric embeddings of $(\Sigma, \sigma + t\eta)$. In what follows, we write $P = Y(\sigma, \sigma + t\eta)$. By (4.10), P satisfies

$$2dX \cdot dP = t\eta - (dP)^2. \tag{4.25}$$

Since $\kappa < \delta$ and $|t| < \epsilon$, by Lemma 4.3 we have

$$\|P\|_{C^{k,\alpha}} = \|X(\sigma + t\eta) - X(\sigma)\|_{C^{k,\alpha}} \leq C\|t\eta\|_{C^{k,\alpha}} \leq C|t| \tag{4.26}$$

where $C > 0$ is the constant in Lemma 4.3. In particular, C is independent on η .

Now let $Y = \Phi(\sigma, \eta)$ be the solution to

$$2dX \cdot dY = \eta. \tag{4.27}$$

By (4.25) and (4.27), we have

$$2dX \cdot (dP - t dY) = -(dP)^2 := \rho. \tag{4.28}$$

Since P is of $C^{k,\alpha}$, we know $\rho \in \mathcal{S}^{k-1,\alpha}$. By (4.26),

$$\|\rho\|_{C^{k-1,\alpha}} \leq Ct^2. \tag{4.29}$$

We claim that $P - tY = \Phi(\sigma, \rho)$. To see this, we first recall that $P = Y(\sigma, \sigma + t\eta) = \lim_{m \rightarrow \infty} Y_m$ in the $C^{2,\alpha}$ norm, where $Y_0 = 0, Y_m = \Phi(\sigma, \rho_{m-1})$ and $\rho_{m-1} = t\eta - (dY_{m-1})^2$. Next, let ϕ_m be the corresponding unique solution ϕ of (4.3) with ρ replaced by ρ_{m-1} . By (4.9), ϕ_m satisfies

$$\phi_m(u, v) = \frac{1}{\Delta} [X_v \cdot (Y_m)_u - X_u \cdot (Y_m)_v]. \tag{4.30}$$

Let ϕ_P be given by

$$\phi_P(u, v) = \frac{1}{\Delta} (X_v \cdot P_u - X_u \cdot P_v). \tag{4.31}$$

Since Y_m converges to P in the $C^{2,\alpha}$ norm, we see that ϕ_m converges to ϕ_P in the $C^{1,\alpha}$ norm. In particular, ϕ_P is L^2 -orthogonal to the coordinate functions of X_3 . On the other hand, by (6.15) in [15], ϕ_P is a solution to (4.3) with ρ replaced by $\tilde{\rho} = t\eta - (dP)^2$. Hence, by definition, we have $\phi_P = \Psi(\sigma, \tilde{\rho})$. Since P also vanishes at the fixed point where Y_m is set to vanish, we know that P is obtained by integrating P_u and P_v , which are given by (4.7) and (4.8) with ρ replaced by $\tilde{\rho}$ and with ϕ replaced by $\phi_P = \Psi(\sigma, \tilde{\rho})$. By definition, this shows $P = \Phi(\sigma, \tilde{\rho})$. Therefore, we have

$$P - tY = \Phi(\sigma, \tilde{\rho}) - t\Phi(\sigma, \eta) = \Phi(\sigma, \rho). \tag{4.32}$$

By Lemma 4.4 and (4.29), we then have

$$\|P - tY\|_{C^{k-1,\alpha}} \leq Ct^2 \tag{4.33}$$

or equivalently

$$\|X(\sigma + t\eta) - Z(t)\|_{C^{k-1,\alpha}} \leq Ct^2 \tag{4.34}$$

where $Z(t) = X(\sigma) + tY$.

Next, applying the fact that the second fundamental form $\mathbb{III}(Z)$ of any $Z \in \mathcal{E}^{m,\alpha} (m \geq 2)$, written in local coordinates, are polynomial functions of derivatives of Z of order at most 2, we see from (4.34) and the fact $\|X(\sigma + t\eta)\|_{C^{k,\alpha}} \leq C$ that

$$\|\mathbb{III}(X(\sigma + t\eta)) - \mathbb{III}(Z(t))\|_{C^{k-3,\alpha}} \leq Ct^2. \tag{4.35}$$

On the other hand, because the map $\rho \mapsto \Phi(\sigma, \rho)$ is linear from $\mathcal{S}^{k,\alpha}$ to $\mathcal{X}^{k-1,\alpha}$, and because $\|\Phi(\sigma, \eta)\|_{C^{k-1,\alpha}} \leq C$, there is a linear map $A : \mathcal{S}^{k,\alpha} \rightarrow \mathcal{S}^{k-3,\alpha}$ such that

$$\|\mathbb{III}(Z(t)) - \mathbb{III}(X(\sigma)) - tA(\eta)\|_{C^{k-3,\alpha}} \leq Ct^2. \tag{4.36}$$

By (4.35) and (4.36), we have

$$\|\mathbb{III}(X(\sigma + t\eta)) - \mathbb{III}(X(\sigma)) - tA(\eta)\|_{C^{k-3,\alpha}} \leq Ct^2 \tag{4.37}$$

for all $\eta \in \mathcal{S}^{k,\alpha}$ with $\|\eta\|_{C^{k,\alpha}} = 1$.

We want to compute $A(\eta)$ explicitly, which is simply $\frac{d}{dt}|_{t=0}\mathbb{I}\mathbb{I}(Z(t))$. Since $A(\eta)$ also depends on σ , we will denote it by $A^{(\sigma)}(\eta)$. Let $e_3(t) = \frac{Z_1(t)\wedge Z_2(t)}{|Z_1(t)\wedge Z_2(t)|}$ be the unit normal of $Z(t)$, where $Z_1 = \frac{\partial Z}{\partial u}$ and $Z_2 = \frac{\partial Z}{\partial v}$. Let $i, j \in \{1, 2\}$ and let Z_{ij} denote the corresponding second order derivative of Z . Then

$$\mathbb{I}\mathbb{I}(Z(t))_{ij} = -\langle e_3, Z_{ij} \rangle.$$

Hence

$$A(\eta)_{ij} = -\langle e_3(0), Y_{ij} \rangle - \left\langle \frac{de_3}{dt}, Z_{ij} \right\rangle \Big|_{t=0}.$$

Since $\frac{de_3}{dt} \perp e_3$, we may assume $\frac{de_3}{dt}|_{t=0} = c^i X_i$ for some coefficients c^i . Then

$$-\left\langle e_3, \frac{dZ_j}{dt} \Big|_{t=0} \right\rangle = \left\langle \frac{de_3}{dt} \Big|_{t=0}, X_j \right\rangle = c^i \sigma_{ij}.$$

Thus,

$$c^i = -\sigma^{ij} \langle e_3, Y_j \rangle.$$

Therefore

$$A^{(\sigma)}(\eta)_{ij} = -\langle X_3(\sigma), Y_{ij} \rangle + \sigma^{kl} \langle X_3(\sigma), Y_k \rangle \langle X_l, X_{ij} \rangle \tag{4.38}$$

where $X_3(\sigma)$ is the unit normal of $X(\sigma)$.

Using the facts that $\|Y\|_{k-1,\alpha} \leq C$ (Lemma 4.4) and $\|X(\sigma)\|_{k,\alpha} \leq C$, where both constants C depend only on σ_0 , we conclude from (4.38) that $A^{(\sigma)}$ is a bounded linear map from $\mathcal{S}^{k,\alpha}$ to $\mathcal{S}^{k-3,\alpha}$.

Next we want to prove that the map $\sigma \mapsto A^{(\sigma)}$ is continuous in the operator topology. Namely, for $\sigma_1 \in U$, we want to prove that

$$\lim_{\sigma \in U, \sigma \rightarrow \sigma_1} \sup_{\eta \in \mathcal{S}^{k,\alpha}, \|\eta\|_{C^{k,\alpha}}=1} \|A^{(\sigma)}(\eta) - A^{(\sigma_1)}(\eta)\|_{C^{k-3,\alpha}} = 0. \tag{4.39}$$

We first note that $A^{(\sigma)}$ does not depend on any particular choice of the embedding $X(\sigma)$. Suppose $\sigma_1 \in U$ and suppose $X(\sigma_1)$ is a fixed isometric embedding of σ_1 such that $\|X(\sigma_1)\|_{C^{k,\alpha}} \leq C$. By Lemma 4.3, for any $\sigma \in \mathcal{M}_+^{k,\alpha}$ with $\|\sigma - \sigma_1\|_{C^{k,\alpha}} < \epsilon - \kappa$, an isometric embedding $X(\sigma)$ can be chosen such that $X(\sigma) = X(\sigma_1) + P_1$, where $P_1 = Y(\sigma_1, \sigma)$ and

$$\|P_1\|_{C^{k,\alpha}} \leq C \|\sigma - \sigma_1\|_{C^{k,\alpha}}. \tag{4.40}$$

Here and below all the constants C depend only on σ_0 , but not on σ and η .

For any given $\eta \in \mathcal{S}^{k,\alpha}$ with $\|\eta\|_{C^{k,\alpha}} = 1$, let $Y^{(1)} = \Phi(\sigma_1, \eta)$ and $Y = \Phi(\sigma, \eta)$ be the solutions of

$$2dX(\sigma_1) \cdot dY^{(1)} = \eta$$

and

$$2dX(\sigma) \cdot dY = \eta.$$

In order to prove (4.39), by (4.38) and (4.40), it is sufficient to prove that

$$\|Y^{(1)} - Y\|_{C^{k-1,\alpha}} \leq C \|\sigma - \sigma_1\|_{C^{k,\alpha}}. \tag{4.41}$$

Let $\phi^{(1)} = \Psi(\sigma_1, \eta)$ and $\phi = \Psi(\sigma, \eta)$ be the functions that are used to construct $Y^{(1)}$ and Y . Then $\phi^{(1)}$ and ϕ satisfy two elliptic PDEs

$$a_{ij}^{(1)}\phi_{ij}^{(1)} + b_i^{(1)}\phi_i^{(1)} + c^{(1)}\phi^{(1)} = f^{(1)}$$

and

$$a_{ij}\phi_{ij} + b_i\phi_i + c\phi = f,$$

which correspond to (4.3) (where the metric and the embedding involved are given by σ_1 and $X(\sigma_1)$, σ and $X(\sigma)$ respectively, and ρ is replaced by η). By (4.4)–(4.6), (4.40) and the fact $\|X(\sigma_1)\|_{C^{k,\alpha}} \leq C$, we have

$$\|a_{ij}^{(1)}\|_{C^{k-2,\alpha}} + \|b_i^{(1)}\|_{C^{k-3,\alpha}} + \|c^{(1)}\|_{C^{k-2,\alpha}} \leq C \tag{4.42}$$

and

$$\begin{aligned} & \|a_{ij}^{(1)} - a_{ij}\|_{C^{k-2,\alpha}} + \|b_i^{(1)} - b_i\|_{C^{k-3,\alpha}} + \|c^{(1)} - c\|_{C^{k-2,\alpha}} + \|f^{(1)} - f\|_{C^{k-3,\alpha}} \\ & \leq C\|\sigma_1 - \sigma\|_{C^{k,\alpha}}. \end{aligned} \tag{4.43}$$

Hence

$$a_{ij}^{(1)}(\phi_{ij}^{(1)} - \phi_{ij}) + b_i^{(1)}(\phi_i^{(1)} - \phi_i) + c^{(1)}(\phi^{(1)} - \phi) = q \tag{4.44}$$

where $q = f^{(1)} - f + (a_{ij} - a_{ij}^{(1)})\phi_{ij} + (b_i - b_i^{(1)})\phi_i + (c - c^{(1)})\phi$. By (4.43) and Lemma 4.4, we have

$$\|q\|_{k-3,\alpha} \leq C\|\sigma_1 - \sigma\|_{C^{k,\alpha}}. \tag{4.45}$$

It follows from (4.42), (4.44), (4.45) and the Schauder estimates that

$$\|\phi^{(1)} - \phi\|_{C^{k-1,\alpha}} \leq C\left(\|\phi^{(1)} - \phi\|_{C^0} + \|\sigma_1 - \sigma\|_{C^{k,\alpha}}\right). \tag{4.46}$$

To estimate $\|\phi^{(1)} - \phi\|_{C^0}$, let x_1, x_2, x_3 be coordinate functions of the unit normal of $X(\sigma_1)$ and let y_1, y_2, y_3 be the unit normal of $X(\sigma)$. Define

$$\beta_i = \int_{\Sigma} x_i(\phi^{(1)} - \phi)d\sigma_1, \quad \omega_{ij} = \int_{\Sigma} x_i x_j d\sigma_1.$$

Since

$$\int_{\Sigma} x_i \phi^{(1)} d\sigma_1 = \int_{\Sigma} y_i \phi d\sigma = 0,$$

we have

$$|\beta_i| \leq C\|\sigma_1 - \sigma\|_{C^{k,\alpha}} \tag{4.47}$$

where we have also used (4.40) and Lemma 4.1. Since (ω_{ij}) has an inverse (ω^{ij}) , we let $\beta^i = \omega^{ij}\beta_j$. Then

$$\phi^{(1)} - \phi - \sum_k \beta^k x_k$$

is L^2 -orthogonal to each x_i . Moreover,

$$|\beta^i| \leq C\|\sigma_1 - \sigma\|_{C^{k,\alpha}} \tag{4.48}$$

and

$$\begin{aligned}
 & a_{ij}^{(1)}(\phi^{(1)} - \phi - \sum_k \beta^k x_k)_{ij} + b_i^{(1)}(\phi^{(1)} - \phi - \sum_k \beta^k x_k)_i \\
 & + c^{(1)}(\phi^{(1)} - \phi - \sum_k \beta^k x_k) = q_1
 \end{aligned}$$

where q_1 is some function satisfying $\|q_1\|_{C^{k-3,\alpha}} \leq C\|\sigma_1 - \sigma\|_{C^{k,\alpha}}$. By the integral expression of $\phi_{ij}^{(1)} - \phi_{ij} - \sum_k \beta^k x_k$ in terms of the Green’s function, see [15], we have

$$\|\phi^{(1)} - \phi - \sum_k \beta^k x_k\|_{C^0} \leq C\|\sigma_1 - \sigma\|_{C^{k,\alpha}}, \tag{4.49}$$

and therefore

$$\|\phi^{(1)} - \phi\|_{C^0} \leq C\|\sigma_1 - \sigma\|_{C^{k,\alpha}} \tag{4.50}$$

by (4.48). It follows from (4.46) and (4.50) that

$$\|\phi^{(1)} - \phi\|_{C^{k-1,\alpha}} \leq C\|\sigma_1 - \sigma\|_{C^{k,\alpha}}. \tag{4.51}$$

Finally, because $Y^{(1)}$ and Y are obtained by integrating $(Y^{(1)})_u, (Y^{(1)})_v$ and Y_u, Y_v which are determined by (4.7) with the corresponding $\phi^{(1)}$ and ϕ inserted, we conclude from (4.51) that (4.41) is true, hence the map $\sigma \mapsto A^{(\sigma)}$ is continuous in the operator topology. \square

5. Existence of Critical Points on Nearby Surfaces

We are now in a position to apply Theorem 4.1 and the IFT to study **Question 2**. Let Σ, N be given as in the Introduction, namely, Σ is a smoothly embedded, closed, spacelike two-surface, which is topologically a two-sphere, in a smooth time-oriented spacetime N . Suppose the mean curvature vector H of Σ in N is spacelike. Let σ be the induced metric on Σ from N . Suppose τ_0 is a $C^{k+1,\alpha}$ function on Σ with $k \geq 5$ such that $\sigma + d\tau_0 \otimes d\tau_0$ has positive Gaussian curvature and τ_0 is a solution to (1.1) on Σ .

To describe spacelike two-surfaces which are “close” to Σ , we use the exponential map $\exp^N(\cdot)$ associated to the Levi-Civita connection of the Lorentzian metric g on N . Precisely, we first fix a smooth future timelike normal vector field J on Σ which is orthogonal to H . Then $\{H, J\}$ form a basis for the normal bundle $(T\Sigma)^\perp$ of Σ . Let $\mathcal{B} = C^{k,\alpha}(\Sigma) \times C^{k,\alpha}(\Sigma)$, where $C^{k,\alpha}(\Sigma)$ is the Banach space of $C^{k,\alpha}$ functions on Σ . For any constant $a > 0$, let $B(a)$ be the open ball in \mathcal{B} centered at $(0, 0)$ with radius a . If a is sufficiently small, for any $f = (f_1, f_2) \in B(a)$, the map $F_f : \Sigma \rightarrow N$ defined by $F_f(x) = \exp^N(f_1(x)H(x) + f_2(x)J(x))$ is a $C^{k,\alpha}$ embedding, moreover $F_f(\Sigma)$ remains to be spacelike and has spacelike mean curvature vector H_f .

Consider the map $\mathcal{I} : B(a) \rightarrow \mathcal{M}^{k-1,\alpha}(\Sigma)$ given by $\mathcal{I}(f) = F_f^*(g)$, where $\mathcal{M}^{k-1,\alpha}(\Sigma)$ denotes the space of $C^{k-1,\alpha}$ Riemannian metrics on Σ . Let $U_{\tau_0}(a)$ be the open ball in $C^{k,\alpha}(\Sigma)$ centered at τ_0 with radius a . For a sufficiently small, we may also assume that $\mathcal{I}(f) + d\tau \otimes d\tau$ is a metric of positive Gaussian

curvature for all $f \in B(a)$ and $\tau \in U_{\tau_0}(a)$. Given such a small a , we define the map

$$\mathcal{H} : B(a) \times U_{\tau_0}(a) \longrightarrow C^{k-4,\alpha}(\Sigma)$$

where

$$\begin{aligned} \mathcal{H}(f, \tau) = & - \left[\hat{H} \hat{\sigma}^{ab} - \hat{\sigma}^{ac} \hat{\sigma}^{bd} (\hat{h}_{cd}) \right] \frac{\nabla_b \nabla_a \tau}{\sqrt{1 + |\nabla \tau|^2}} \\ & + \operatorname{div}_{\Sigma} \left[\frac{\nabla \tau}{\sqrt{1 + |\nabla \tau|^2}} \cosh \theta |H_f| - \nabla \theta - V_f \right] \end{aligned}$$

which is just the left side of (1.1) but with σ replaced by ${}^f\sigma = \mathcal{I}(f)$, H replaced by H_f and V replaced by V_f . Here the vector field V_f on Σ is understood as the pull back, through the embedding F_f , of the vector field dual to the one form $\alpha_{e_3^N}^N(\cdot)$ on $F_f(\Sigma)$.

Proposition 5.1. \mathcal{H} is a C^1 map.

Proof. Note that \mathcal{I} is a C^1 map. Hence, the map $(f, \tau) \mapsto \hat{\sigma} = {}^f\sigma + d\tau \otimes d\tau$ is C^1 from $B(a) \times U_{\tau_0}(a)$ to $\mathcal{M}^{k-1,\alpha}(\Sigma)$. By Theorem 4.1, the map $(f, \tau) \mapsto (\hat{h}_{cd})$ is C^1 from $B(a) \times U_{\tau_0}(a)$ to the space of $C^{k-4,\alpha}$ symmetric (0,2) tensors on Σ . Thus, to show \mathcal{H} is C^1 , it only remains to check that the map $f \mapsto \operatorname{div}_{\Sigma} V_f$ is C^1 from $B(a) \times U_{\tau_0}(a)$ to $C^{k-4,\alpha}(\Sigma)$.

Let T be a smooth future timelike unit vector field on N . Let $\{(x^1, x^2)\}$ be any local coordinates on Σ . Let $v_b = (F_f)_* (\frac{\partial}{\partial x_a})$, $b = 1, 2$. Then

$$\begin{aligned} H_f &= ({}^f\sigma)^{ab} \nabla_{v_a}^N v_b - ({}^f\sigma)^{cd} \langle ({}^f\sigma)^{ab} \nabla_{v_a}^N v_b, v_c \rangle v_d, \\ V_f &= ({}^f\sigma)^{ab} \langle \nabla_{v_a}^N e_3^{H_f}, e_4^{H_f} \rangle v_b, \end{aligned}$$

where $e_3^{H_f} = -H_f/|H_f|$ and $e_4^{H_f} = w/\sqrt{-\langle w, w \rangle}$ with

$$w = T - ({}^f\sigma)^{ab} \langle T, v_a \rangle v_b - \langle T, e_3^{H_f} \rangle e_3^{H_f}.$$

From this it is easily seen that $f \mapsto \operatorname{div}_{\Sigma} V_f$ is a C^1 map. □

Lemma 5.1. Let $dv_{f\sigma}, dv_{\sigma}$ be the volume form of ${}^f\sigma, \sigma$ on Σ . Then, for any $(f, \tau) \in B(a) \times U_{\tau_0}(a)$,

$$\int_{\Sigma} \mathcal{H}(f, \tau) dv_{f\sigma} = 0.$$

Proof. It suffices to verify

$$\int_{\Sigma} \left[\hat{H} \hat{\sigma}^{ab} - \hat{\sigma}^{ac} \hat{\sigma}^{bd} (\hat{h}_{cd}) \right] \frac{\nabla_b \nabla_a \tau}{\sqrt{1 + |\nabla \tau|^2}} dv_{f\sigma} = 0. \tag{5.1}$$

Let $dv_{\hat{\sigma}}$ be the volume of $\hat{\sigma} = {}^f\sigma + d\tau \otimes d\tau$. Then

$$\sqrt{1 + |\nabla \tau|^2} dv_{\hat{\sigma}} = dv_{f\sigma}$$

and

$$\hat{\nabla}_b \hat{\nabla}_a \tau = \frac{1}{1 + |\nabla \tau|^2} \nabla_b \nabla_a \tau,$$

where $\nabla, \hat{\nabla}$ denote covariant derivatives of $f\sigma, \hat{\sigma}$ respectively. Hence

$$\left[\hat{H} \hat{\sigma}^{ab} - \hat{\sigma}^{ac} \hat{\sigma}^{bd} (\hat{h}_{cd}) \right] \frac{\nabla_b \nabla_a \tau}{\sqrt{1 + |\nabla \tau|^2}} dv_{f\sigma} = \left[\hat{H} \hat{\sigma}^{ab} - \hat{\sigma}^{ac} \hat{\sigma}^{bd} (\hat{h}_{cd}) \right] \hat{\nabla}_b \hat{\nabla}_a \tau dv_{\hat{\sigma}}$$

which implies (5.1) because $\hat{H} \hat{\sigma}^{ab} - \hat{\sigma}^{ac} \hat{\sigma}^{bd} \hat{h}_{cd}$ is divergence free with respect to $\hat{\sigma}$. \square

In what follows, we assume that $\tau_0 = 0$ is a solution to (1.1) on Σ . We give a sufficient condition that guarantees the existence of solutions to (1.1) on the nearby surfaces $F_f(\Sigma)$.

Theorem 5.1. *With the above assumptions and notations, suppose the induced metric σ on Σ has positive Gaussian curvature and the vector field V on Σ satisfies $\operatorname{div}_\Sigma V = 0$. Suppose in addition there exists a constant $C > 0$ such that*

$$\int_\Sigma \left[\frac{(\Delta \eta)^2}{|H|} + (H_0 - |H|) |\nabla \eta|^2 - \mathbb{I}_0(\nabla \eta, \nabla \eta) \right] dv_\Sigma \geq C \int_\Sigma (\Delta \eta)^2 dv_\Sigma \quad (5.2)$$

for all $\eta \in W^{2,2}(\Sigma)$, where H_0 and \mathbb{I}_0 are the mean curvature and the second fundamental form of (Σ, σ) when isometrically embedded in \mathbb{R}^3 . Then for any $k \geq 5$ and $0 < \alpha < 1$, there exists a small constant $a > 0$ such that, for any $f \in B(a)$, there exists a $C^{k,\alpha}$ solution τ to (1.1) on the $C^{k,\alpha}$ embedded surface $F_f(\Sigma)$.

Proof. Since $\operatorname{div}_\Sigma V = 0$, we know $\tau_0 = 0$ is a solution to (1.1) on Σ . For the given k and α , let $a > 0$ be sufficiently small such that the map \mathcal{H} is well defined on $B(a) \times U_{\tau_0}(a)$ with $\tau_0 = 0$. Let

$$D(a) = \{ \tau \in C^{k,\alpha}(\Sigma) \mid \|\tau\|_{C^{k,\alpha}} < a \text{ and } \int_\Sigma \tau dv_\Sigma = 0 \} \subset U_0(a)$$

and

$$C_0^{k-4,\alpha}(\Sigma) = \{ \phi \in C^{k-4,\alpha}(\Sigma) \mid \int_\Sigma \phi dv_\Sigma = 0 \}.$$

For $(f, \tau) \in B(a) \times D$, define

$$\mathcal{H}_0(f, \tau) = \frac{dv_{f\sigma}}{dv_\Sigma} \mathcal{H}(f, \tau).$$

By Proposition 5.1 and Lemma 5.1, \mathcal{H}_0 is a C^1 map from $B(a) \times D(a)$ to $C_0^{k-4,\alpha}(\Sigma)$.

Direct computations show that the partial derivative $D_\tau \mathcal{H}_0|_{(0,0)}$ of \mathcal{H}_0 at $(0, 0)$ with respect to τ is given by

$$D_\tau \mathcal{H}_0|_{(0,0)}(\eta) = -\langle H_0 \sigma - \mathbb{I}_0, \nabla^2 \eta \rangle + \operatorname{div}_\Sigma (|H| \nabla \eta) + \Delta \left(\frac{\Delta \eta}{|H|} \right) \quad (5.3)$$

for $\eta \in C_0^{k,\alpha}(\Sigma)$, the space of all $C_0^{k,\alpha}$ functions with zero integral on (Σ, σ) . Clearly, $D_\tau \mathcal{H}_0|_{(0,0)}$ is a bounded linear map from $C_0^{k,\alpha}(\Sigma)$ to $C_0^{k-4,\alpha}(\Sigma)$. We claim that, under the condition (5.2), $D_\tau \mathcal{H}_0|_{(0,0)}$ is a bijection. Once this claim can be verified, Theorem 5.1 will follow from the implicit function theorem.

To show $D_\tau \mathcal{H}_0|_{(0,0)}$ is a bijection, we let $\widetilde{W}^{2,2}(\Sigma)$ be the closed subspace of $W^{2,2}(\Sigma)$ consisting of those η with $\int_\Sigma \eta = 0$. Here and below, integrations and differentiations are taken with respect to the metric σ and we omit writing the volume form dv_Σ in the integrals. Consider the following bilinear form on the Hilbert space $\widetilde{W}^{2,2}(\Sigma)$:

$$\begin{aligned} B(\eta, \phi) &= \int_\Sigma D_\tau \mathcal{H}_0|_{(0,0)}(\eta)\phi \\ &= \int_\Sigma \frac{\Delta\eta\Delta\phi}{|H|} + (H_0\sigma - \mathbb{I}\mathbb{I}_0)(\nabla\eta, \nabla\phi) - |H|\langle\nabla\eta, \nabla\phi\rangle. \end{aligned} \tag{5.4}$$

Obvious B is bounded. That is $|B(\eta, \phi)| \leq C_1\|\eta\|_{W^{2,2}}\|\phi\|_{W^{2,2}}$ for some constant C_1 and for all $\eta, \phi \in \widetilde{W}^{2,2}(\Sigma)$.

By the L^p estimate [5, Theorem 9.11], there is a constant C_2 such that for all $\eta \in \widetilde{W}^{2,2}(\Sigma)$

$$\|\eta\|_{W^{2,2}} \leq C_2(\|\eta\|_{L^2} + \|\Delta\eta\|_{L^2}).$$

Since $\int_\Sigma \eta = 0$, by (2.27) we have

$$\int_\Sigma |\Delta\eta|^2 \geq \lambda_1^2 \int_\Sigma \eta^2$$

where $\lambda_1 > 0$ is the first nonzero eigenvalue of the Laplacian of σ . Hence, by (5.2), we have

$$B(\eta, \eta) \geq C_3\|\eta\|_{W^{2,2}}^2 \tag{5.5}$$

for some $C_3 > 0$ and for all $\eta \in \widetilde{W}^{2,2}(\Sigma)$, i.e., B is coercive. This readily implies that $D_\tau \mathcal{H}_0|_{(0,0)}$ is injective.

Now let f be an arbitrary element in $L^2(\Sigma)$ with $\int_\Sigma f = 0$. Define $T : \widetilde{W}^{2,2}(\Sigma) \rightarrow \mathbb{R}$ by

$$T(\phi) = \int_\Sigma \phi f.$$

Since T is a bounded linear functional on $\widetilde{W}^{2,2}(\Sigma)$, there exists an $\eta \in \widetilde{W}^{2,2}(\Sigma)$ such that

$$B(\eta, \phi) = T(\phi)$$

for all $\phi \in \widetilde{W}^{2,2}(\Sigma)$. That is to say,

$$\int_\Sigma \frac{\Delta\eta\Delta\phi}{|H|} + (H_0\sigma - \mathbb{I}\mathbb{I}_0)(\nabla\eta, \nabla\phi) - |H|\langle\nabla\eta, \nabla\phi\rangle = \int_\Sigma \phi f$$

for all $\phi \in \widetilde{W}^{2,2}(\Sigma)$. Integrating by parts, we have

$$\int_{\Sigma} \frac{\Delta\eta\Delta\phi}{|H|} = \int_{\Sigma} [\langle H_0\sigma - \mathbb{I}\mathbb{I}_0, \nabla^2\eta \rangle - \operatorname{div}_{\Sigma}(|H|\nabla\eta) + f] \phi \tag{5.6}$$

where we have used the fact that $H_0\sigma - \mathbb{I}\mathbb{I}_0$ is divergence free with respect to σ . This same fact also implies

$$\int_{\Sigma} \langle H_0\sigma - \mathbb{I}\mathbb{I}_0, \nabla^2\eta \rangle - \operatorname{div}_{\Sigma}(|H|\nabla\eta) + f = 0$$

because $\int_{\Sigma} f = 0$. Therefore, if we let

$$h = \langle H_0\sigma - \mathbb{I}\mathbb{I}_0, \nabla^2\eta \rangle - \operatorname{div}_{\Sigma}(|H|\nabla\eta) + f$$

which is in $L^2(\Sigma)$, then there exists $\psi \in W^{2,2}(\Sigma)$ such that $\Delta\psi = h$. Now we have

$$\int_{\Sigma} \frac{\Delta\eta\Delta\phi}{|H|} = \int_{\Sigma} \phi\Delta\psi = \int_{\Sigma} \psi\Delta\phi. \tag{5.7}$$

Hence

$$\int_{\Sigma} \left(\frac{\Delta\eta}{|H|} - \psi \right) \Delta\phi = 0 \tag{5.8}$$

for all $\phi \in \widetilde{W}^{2,2}(\Sigma)$. Recall that, for any $\zeta \in L^2(\Sigma)$ with $\int_{\Sigma} \zeta = 0$, there is $\phi \in \widetilde{W}^{2,2}(\Sigma)$ with $\Delta\phi = \zeta$. So (5.8) implies that

$$\int_{\Sigma} \left(\frac{\Delta\eta}{|H|} - \psi \right) \zeta = 0$$

for all $\zeta \in L^2(\Sigma)$ with $\int_{\Sigma} \zeta = 0$. Therefore,

$$\frac{\Delta\eta}{|H|} - \psi = C_4$$

for some constant C_4 . Since $\eta, \psi \in W^{2,2}(\Sigma)$, we know $\eta \in W^{4,2}(\Sigma)$ by [5, Theorem 9.19]. This, together with the fact that $\Delta\psi = h$, implies

$$\Delta \left(\frac{\Delta\eta}{|H|} \right) - \langle H_0\sigma - \mathbb{I}\mathbb{I}_0, \nabla^2\eta \rangle + \operatorname{div}_{\Sigma}(|H|\nabla\eta) = f. \tag{5.9}$$

If $f \in C_0^{k-4,\alpha}$, then it is easy to see that $\eta \in C_0^{k,\alpha}$ by bootstrap and the fact that $\eta \in \widetilde{W}^{4,2}(\Sigma)$. Hence, $D_{\tau}\mathcal{H}_0|_{(0,0)}$ is surjective. Theorem 5.1 is now proved. \square

Remark 5.1. Suppose Σ is a closed connected surface in $\mathbb{R}^n (n \geq 3)$ with second fundamental form $\mathbb{I}\mathbb{I}_0$ and positive mean curvature H_0 . By Corollary 3.1, the equation

$$\Delta \left(\frac{\Delta\eta}{H_0} \right) - \langle H_0\sigma - \mathbb{I}\mathbb{I}_0, \nabla^2\eta \rangle + \operatorname{div}_{\Sigma}(H_0\nabla\eta) = 0 \tag{5.10}$$

has a nontrivial kernel on Σ which consists of all functions $\eta = a_0 + \sum_{i=1}^n a_i x^i$, where a_0, a_1, \dots, a_n are arbitrary constants and x_1, \dots, x_n are coordinate functions on \mathbb{R}^n .

Part (2) of Theorem 1.1 now follows directly from Theorem 5.1 and Theorem 3.1.

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