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Poisson and Diffusion Approximation of Stochastic Master Equations with Control

Clément Pellegrini

Abstract. *Quantum Trajectories* are solutions of stochastic differential equations. Such equations are called *Stochastic Master Equations* and describe random phenomena in the continuous measurement theory of Open Quantum System. Many recent developments deal with the control of such models, i.e. optimization, monitoring and engineering. In this article, stochastic models with control are mathematically and physically justified as limits of concrete discrete procedures called *Quantum Repeated Measurements*. In particular, this gives a rigorous justification of the Poisson and diffusion approximations in quantum measurement theory with control.

1. Introduction

The study of the evolution of a small quantum system $\mathcal{H}_0$ undergoing an indirect and continuous measurement (the small system is in contact with an environment and the measurement is performed on the environment) is central in the Theory of Open Quantum System. Usually, the evolution of the system is described by classical stochastic differential equations called *Stochastic Master Equations*. Two characteristic examples are the *diffusive equation* and the *jump equation*.

1. The diffusive equation (Homodyne detection experiment) is given by

$$d\rho_t = L(\rho_t)\,dt + \Big(\rho_t C^\star + C\rho_t - \mathrm{Tr}\big[\rho_t(C + C^\star)\big]\rho_t\Big)dW_t, \tag{1}$$

where W_t describes a one-dimensional Brownian motion.

2. The jump equation (photon counting experiment) is

$$d\rho_t = L(\rho_t)dt + \left(\frac{\mathcal{J}(\rho_t)}{\mathrm{Tr}\big[\mathcal{J}(\rho_t)\big]} - \rho_t\right)\Big(d\tilde{N}_t - \mathrm{Tr}\big[\mathcal{J}(\rho_t)\big]\,dt\Big), \tag{2}$$

where $\tilde{N}_t$ is a counting process with stochastic intensity $\int_0^t \mathrm{Tr}[\mathcal{J}(\rho_s)]ds$.

More complicated models are described by jump-diffusion equations which need mixing of the both previous equations (see [6,33]). Solutions of such equations are called *Quantum Trajectories* and they describe the evolution of the state of the small system.

Recent progresses and developments in quantum optics and quantum information theory need a highest precision in experimentations using measurement [20] (sensitivity, miniaturization, optimization, etc.). This introduces the notion of control and monitoring of quantum systems. Two types of control are usually considered: deterministic and stochastic.

A basic example of deterministic control is the one of an atom monitored by a laser. In this context, the control is represented by the modification of the intensity of the laser. The word "deterministic" involves implicitly that the intensity is a non-random function of time. This setup is also called *Open Loop Control*. Experimentally, it is used to prepare systems in specific states.

Concerning stochastic control in the setup of continuous indirect measurement, an important class is called *Closed Loop Control* or *Feedback Control* [10, 37]. Here, depending on the information resulting from the measurement, one can choose special strategies of control. As a result of that measurement is random in quantum mechanics, the control becomes naturally random.

From a theoretical point of view, an important question is to lay out a mathematical setup to model the control in order to describe the evolution of controlled quantum systems.

Usually in the literature, in order to obtain and justify the stochastic master equations (1) and (2), Quantum Filtering theory [11] or Instrumental Process theory [8] is used. Such techniques are based on the Hilbertian formalism of Quantum Mechanics and on the theory of Stochastic Quantum Calculus. It uses analytic machinery and all the subtleties of the non-commutative character of quantum probability (conditional expectation in Von Neumann Algebra, partially observed system, etc.). The starting point is the description of interaction between a system and an environment in terms of quantum stochastic differential equations (also called *Hudson Parthasarathy Equations* [29]). In order to apply such frameworks in the control setup, the theory must satisfied the non-commutative character of the quantum mechanics theory. Even if it is satisfied, the derivation and the obtaining of stochastic master equations with control are far from being obvious and intuitive (see [13]).

Recently, in the framework of the description of the interaction of a small system with an environment (without measurement), in [4], the authors have introduced a discrete model of interaction: *Quantum Repeated Interactions*. The basic model is the one of a small system $\mathcal{H}_0$ in contact with an infinite chain of quantum system $\bigotimes_{j=1}^{\infty}\mathcal{H}$. One after the other, each copy of $\mathcal{H}$ interacts with $\mathcal{H}_0$ during a time h.

Such an approach of open quantum system yields a "good" and "useful" approximation model of continuous-time interaction models. Indeed by rescaling this interaction with respect to the time h, it is shown that the models of interaction (described by quantum stochastic differential equations) can be obtained as continuous limits (h goes to zero) of discrete models. In the measurement setup, this approach has been adapted in [31] and [32]. In these articles, it is shown that solutions of (1) and (2) can be obtained as continuous limits of discrete time models of quantum measurement. These models are called *Quantum Repeated Measurements*. The idea of discrete indirect measurements consists in performing a measurement of an observable of $\mathcal{H}$ after each interaction between $\mathcal{H}_0$ and a copy of $\mathcal{H}$. The evolution of the state of $\mathcal{H}_0$ is then described by a random discrete process called *discrete quantum trajectory*. In this case, the approach of the theory of stochastic master equations via approximation results is essentially based on classical probability theory (there are no problems of commutativity).

The main aim of this article is to adapt such techniques in the framework of control. The notion of control in the model of quantum repeated measurements is presented. Next, by adapting convergence results of [31] and [32], we obtain the description of stochastic master equations with control. Within this approach, all the problems concerning non commutativity are avoided and the physical justification of stochastic models is rigorous and intuitive.

This article is structured as follows:

The first section is devoted to present discrete models of quantum measurement with control theory. We remind the mathematical model of quantum repeated interactions. Next, we introduce an appropriate notion of control in this setup and by introducing the measurement principle, we obtain the description of *discrete quantum trajectory with control*. Next, in order to prepare final convergence results, we adapt and enlarge the asymptotic assumptions presented in [4] to the context of control. To investigate such problems, we focus on a central case in physical applications: a two-level atom in contact with a spin chain.

The second section is then devoted to continuous models. The main aim is to derive the equivalent of (1) and (2) with control. To this end, we apply the asymptotic assumptions on the two-level atom model of Sect. 1. We then obtain two different discrete evolution equations (in asymptotic form) describing the evolution of the state of $\mathcal{H}_0$. Each evolution equation describes the evolution of a discrete quantum trajectory with control for a specific observable. For each equation, we investigate the continuous limit equation and we show the convergence.

In the last section, we present an application of a deterministic control: an atom monitored by a laser. By modelling a suitable discrete model and by adapting the result of Sect. 2, we obtain the continuous stochastic model. Concerning applications of stochastic control, we refer for example to the existing literature of feedback control [10,25,36,37]. An important application of stochastic control is the concept of *Optimal Control* (see [28,34] for a classical approach and [14,15,38] for some quantum applications).

2. Discrete Controlled Quantum Trajectories

This section is devoted to the presentation of the model of discrete quantum trajectories in presence of external control.

2.1. Repeated Quantum Measurements with Control

In order to introduce the theory of control, we need to remind the general context of quantum repeated interactions.

A small system, represented by a Hilbert space $\mathcal{H}_0$, is in contact with an infinite chain of identical independent quantum systems. Each piece of the environment is represented by a Hilbert space $\mathcal{H}$ and interacts, one after the other, with $\mathcal{H}_0$ during a time interval h (a copy of $\mathcal{H}$ can represent an incoming photon or a measurement apparatus, etc.).

The space describing the first interaction between $\mathcal{H}_0$ and $\mathcal{H}$ is defined by the tensor product $\mathcal{H}_0 \otimes \mathcal{H}$. The evolution is given by a self-adjoint operator H_{tot} on the tensor product. This operator is called the total Hamiltonian and its general form is

$$H_{\text{tot}} = H_0 \otimes I + I \otimes H + H_{\text{int}},$$

where the operators H_0 and H are the free Hamiltonians of each system. The operator H_{int} represents the Hamiltonian of interaction. This allows to define a unitary-operator of evolution

$$U = e^{ih\,H_{\text{tot}}}.$$

In this way, the evolution of states of $\mathcal{H}_0 \otimes \mathcal{H}$, in the Schrödinger picture, is given by

$$\rho \mapsto U\,\rho\,U^\star.$$

After the first interaction, a second copy of $\mathcal{H}$ interacts with $\mathcal{H}_0$ in the same fashion and so on. For the whole sequence of interactions, the state space is described by

$$\mathbf{\Gamma} = \mathcal{H}_0 \otimes \bigotimes_{k\geq 1} \mathcal{H}_k, \tag{3}$$

where $\mathcal{H}_k$ denotes the kth copy of $\mathcal{H}$. The countable tensor product $\bigotimes_{k\geq 1} \mathcal{H}_k$ means the following. Consider that $\mathcal{H}$ is of finite dimension and that $\{e_0, e_1, \ldots, e_n\}$ is a fixed orthonormal basis of $\mathcal{H}$. The orthogonal projector on $\mathbb{C}e_0$ is denoted by $|e_0\rangle\langle e_0|$. This is the ground state (or vacuum state) of $\mathcal{H}$. The tensor product is taken with respect to e_0 (for more details, see [4]).

The unitary evolution describing the kth interaction is given by $\tilde{U}_k$ which acts like U on $\mathcal{H}_0 \otimes \mathcal{H}_k$ and acts like the identity operator on the rest of the space. If ρ is a state on $\mathbf{\Gamma}$, the effect of the kth interaction is then $\rho \mapsto \tilde{U}_k\,\rho\,\tilde{U}_k^\star$. Hence, the sequence of interactions is described by a sequence of unitary operators (V_k)

defined by $V_k = \tilde{U}_k \tilde{U}_{k-1} \dots \tilde{U}_1$, for all k. In the Schrödinger picture, the effect of k interactions is given by

$$\rho \mapsto V_k \, \rho \, V_k^{\star}.$$

Now, we are in the position to introduce the theory of control. An action of control consists in modifying the interaction at each new step depending on the previous step. Therefore, the operator $\tilde{U}_k$, describing the kth interaction, depends on two parameters. It depends on the time of interaction h and on a term u_{k-1}, which gives account of the control. The operator $\tilde{U}_k$ is then denoted by $\tilde{U}_k(h, u_{k-1})$.

The whole sequence $\mathbf{u} = (u_k)$ is called a *control strategy*. For instance, we leave non precise the definition of the terms (u_k) (it can depend either on some constraints, or on experimental conditions or on the evolution of the small system). In terms of $\mathbf{u}$, the k first interactions are then described by the unitary-operator $V_k^{\mathbf{u}}$

$$V_k^{\mathbf{u}} = \tilde{U}_k(h, u_{k-1}) \, \tilde{U}_{k-1}(h, u_{k-2}) \, \dots \, \tilde{U}_1(h, u_0). \tag{4}$$

Finally, the evolution in presence of control is given by

$$\rho \mapsto V_k^{\mathbf{u}} \; \rho \; (V_k^{\mathbf{u}})^{\star}. \tag{5}$$

Now, we are in the condition to describe repeated quantum measurements in the presence of control. Next, we make precise the definition of the control strategy.

Let us describe the basic procedure on each piece of the chain. Let A be an observable on $\mathcal{H}_k$ with spectral decomposition $A = \sum_{j=0}^{p} \lambda_j P_j$. Its natural ampliation, as an observable on $\mathbf{\Gamma}$, is given by

$$A^k := \bigotimes_{j=0}^{k-1} I \otimes A \otimes \bigotimes_{j \geq k+1} I. \tag{6}$$

The accessible data with a measurement are the eigenvalues of A^k and the result of the observation is random. If ρ is any state on $\mathbf{\Gamma}$, we observe λ_j with probability

$$P\big[\,\text{to observe } \lambda_j\,\big] = \mathrm{Tr}\big[\,\rho \, P_j^k\,\big], \quad j = 0, \dots, p,$$

where the operator P_j^k corresponds to the ampliation (6) of the eigenprojector P_j. If we have observed the eigenvalue λ_j, the *wave packet reduction principle* imposes the state after the measurement to be

$$\rho_j = \frac{P_j^k \, \rho \, P_j^k}{\mathrm{Tr}\big[\,\rho \, P_j^k\,\big]}.$$

Quantum repeated measurements are the combination of this previous principle and the successive interactions (5). After each interaction, a quantum measurement induces a random modification of the state of the system. It defines a discrete process which is called *discrete controlled quantum trajectory*. The description is as follows:

The initial state on $\mathbf{\Gamma}$ is chosen to be

$$\mu = \rho \otimes \bigotimes_{j \geq 1} \beta_j,$$

where ρ is any state on $\mathcal{H}_0$ and each $\beta_i = \beta$, where β is any state on $\mathcal{H}$. The state after k interactions is denoted by $\mu_k^{\mathbf{u}}$, and we have $\mu_k^{\mathbf{u}} = V_k^{\mathbf{u}}\ \mu\ (V_k^{\mathbf{u}})^\star$.

The probability space describing the experience is $\Sigma^{\mathbb{N}^\star}$, where $\Sigma = \{0, \dots, p\}$. The integers i correspond to the indexes of the eigenvalues of A. We endow $\Sigma^{\mathbb{N}^\star}$ with the cylinder σ-algebra $\mathcal{C}$ generated by the cylinder sets

$$\Lambda_{i_1,\dots,i_k} = \big\{\omega \in \Omega^{\mathbb{N}^\star} / \omega_1 = i_1, \dots, \omega_k = i_k\big\}, \quad k > 0.$$

Remarking that for all j, the unitary operator $\tilde{U}_j$ commutes with all the projectors P^k, for $k < j$, we can define the following operator:

$$\begin{aligned}\tilde{\mu}_k^{\mathbf{u}}(i_1, \dots, i_k) &= I \otimes P_{i_1} \otimes \dots \otimes P_{i_k} \otimes I \dots\ \mu_k^{\mathbf{u}}\ I \otimes P_{i_1} \otimes \dots \otimes P_{i_k} \otimes I \dots \\ &= P_{i_k}^k \dots P_{i_1}^1\ \ \mu_k^{\mathbf{u}}\ \ P_{i_1}^1 \dots P_{i_k}^k,\end{aligned}$$

for any set $\{i_1, \dots, i_k\}$. This is the non-normalized state corresponding to the successive observations of $\lambda_{i_1}, \dots, \lambda_{i_k}$. Now, we can define the probability measure on the cylinder sets

$$P\big[\,\Lambda_{i_1,\dots,i_k}\,\big] = P\big[\text{to observe } \lambda_{i1}, \dots, \lambda_{ik}\,\big] = \mathrm{Tr}\big[\,\tilde{\mu}_k^{\mathbf{u}}(i_1, \dots, i_k)\,\big].$$

This probability satisfies the Kolmogorov Consistency Criterion. Hence, this defines a unique probability measure on $\Sigma^{\mathbb{N}^\star}$. The discrete quantum trajectory with control strategy $\mathbf{u}$ on $\mathbf{\Gamma}$ is described by the following random sequence of states:

$$\begin{aligned}\tilde{\rho}_k^{\mathbf{u}} : \Sigma^{\mathbb{N}^\star} &\longrightarrow \mathcal{B}(\mathbf{\Gamma}) \\ \omega &\longmapsto \tilde{\rho}_k^{\mathbf{u}}(\omega_1, \dots, \omega_k) = \frac{\tilde{\mu}_k^{\mathbf{u}}(\omega_1, \dots, \omega_k)}{\mathrm{Tr}\big[\,\tilde{\mu}_k^{\mathbf{u}}(\omega_1, \dots, \omega_k))\,\big]}.\end{aligned}$$

From this description, the following result is obvious:

Proposition 1. *Let* $\mathbf{u}$ *be any strategy and let* $(\tilde{\rho}_k^{\mathbf{u}})$ *be the above random sequence of states. We have*

$$\tilde{\rho}_{k+1}^{\mathbf{u}}(\omega) = \frac{P_{\omega_{k+1}}^{k+1}\, \tilde{U}_{k+1}(h, u_k)\ \ \tilde{\rho}_k^{\mathbf{u}}(\omega)\ \ \tilde{U}_{k+1}^\star(h, u_k)\, P_{\omega_{k+1}}^{k+1}}{\mathrm{Tr}\left[\,\tilde{\rho}_k^{\mathbf{u}}(\omega)\ \tilde{U}_{k+1}^\star(h, u_k)\, P_{\omega_{k+1}}^{k+1}\, \tilde{U}_{k+1}(h, u_k)\right]},$$

for all $\omega \in \Sigma^{\mathbb{N}^\star}$ *and all* $k > 0$.

In general, one is only interested in the reduced state of the small system. This state is given by the partial trace operation. Let us recall what partial trace is. Let $\mathcal{Z}$ be any Hilbert space, the notation $\mathrm{Tr}_{\mathcal{Z}}[W]$ corresponds to the trace of any trace-class operator W on $\mathcal{Z}$.

Definition-Theorem 1. *Let $\mathcal{H}$ and $\mathcal{K}$ be any Hilbert spaces. Let α be a state on the tensor product $\mathcal{H} \otimes \mathcal{K}$. There exists a unique state η on $\mathcal{H}$ which is characterized by the property*

$$\mathrm{Tr}_{\mathcal{H}}[\,\eta\, X\,] = \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{K}}[\,\alpha(X \otimes I)\,],$$

for all $X \in \mathcal{B}(\mathcal{H})$. The state η is called the partial trace of α on $\mathcal{H}$ with respect to $\mathcal{K}$.

For any state α on $\mathbf{\Gamma}$, denote $\mathbf{E}_0[\alpha]$ the partial trace of α on $\mathcal{H}_0$ with respect to $\bigotimes_{k\geq 1} \mathcal{H}_k$. We then define the *reduced discrete controlled quantum trajectory* $(\rho_n^{\mathbf{u}})$ on $\mathcal{H}_0$ by putting

$$\rho_n^{\mathbf{u}}(\omega) = \mathbf{E}_0\left[\,\tilde{\rho}_n^{\mathbf{u}}(\omega)\,\right], \tag{7}$$

for all $\omega \in \Sigma^{\mathbb{N}^\star}$. The states, constituting the sequence $(\rho_n^{\mathbf{u}})$, are called *a posteriori* states. The following proposition is the equivalent of Proposition 1 for $(\rho_n^{\mathbf{u}})$.

Proposition 2. *Let $\mathbf{u}$ be any strategy and $(\rho_k^{\mathbf{u}})$ be the random sequence defined by (7). For all $k > 0$, we consider the unitary operator $U_k(h, u_{k-1})$ defined on $\mathcal{H}_0 \otimes \mathcal{H}$ and acting as $\tilde{U}_k(h, u_{k-1})$ on $\mathcal{H}_0 \otimes \mathcal{H}_k$. Hence, we have*

$$\rho_{k+1}^{\mathbf{u}}(\omega) = \mathbf{E}_0\left[\frac{I \otimes P_{\omega_{k+1}}\, U_{k+1}(h, u_k)\ \ (\rho_k^{\mathbf{u}}(\omega) \otimes \beta)\ \ U_{k+1}^\star(h, u_k)\, I \otimes P_{\omega_{k+1}}}{\mathrm{Tr}\left[\,(\rho_k^{\mathbf{u}}(\omega) \otimes \beta)\ U_{k+1}^\star(h, u_k)\ I \otimes P_{\omega_{k+1}}\ U_{k+1}(h, u_k)\right]}\right],$$

for all $\omega \in \Sigma^{\mathbb{N}^\star}$ and all $k > 0$.

Let us stress that in this proposition, the partial trace is taken along $\mathcal{H}$ (we have kept the same notation $\mathbf{E}_0$).

Remark. At this stage with Proposition 2, we can define clearly which kind of control strategies $\mathbf{u} = (u_k)$ we consider in this article. Previously, we have introduced these parameters in the quantum repeated interactions model without measurement. In this setup, one could suppose that the control is fixed and not linked with the procedure of measurement. However, a main aim of control theory is to choose a new strategy u_k at each step depending on the influence of the measurement. As a consequence, at time $k + 1$, we consider that the control parameter u_k depends on the past of discrete quantum trajectory, that is, we consider terms of form $u_k = u_k(h, \rho_k^{\mathbf{u}}, \rho_{k-1}^{\mathbf{u}}, \ldots, \rho_0^{\mathbf{u}})$ (in all the following formulas, we keep the notation u_k to enlighten the expressions). In this way, the control depends on the observations through their effects on the small system.

Two special cases of control are the deterministic and the Markovian controls. This is the topic of the following definition.

Definition 1. A control strategy $\mathbf{u}$, corresponding to a trajectory $(\rho_k^{\mathbf{u}})$, is called **deterministic** if there exists a function u from $\mathbb{R}$ to $\mathbb{R}^n$ such that for all k

$$u_k = u_k(h, \rho_k^{\mathbf{u}}, \ldots, \rho_0^{\mathbf{u}}) = u(kh).$$

A control strategy $\mathbf{u}$, corresponding to a trajectory $(\rho_k^{\mathbf{u}})$, is called **Markovian** if there exists a function u from $\mathbb{R} \times \mathcal{B}(\mathcal{H}_0)$ to $\mathbb{R}^n$ such that for all k

$$u_k = u_k(h, \rho_k^{\mathbf{u}}, \dots, \rho_0^{\mathbf{u}}) = u(kh, \rho_k^{\mathbf{u}}).$$

If for all k, we have $u_k = u_k(h, \rho_k^{\mathbf{u}}, \dots, \rho_0^{\mathbf{u}}) = u(\rho_k^{\mathbf{u}})$, then this is a homogeneous Markovian strategy.

The following theorem is an easy consequence of Proposition 2 and of the previous Definition.

Theorem 1. *Let $\mathbf{u}$ be either a deterministic or a Markovian strategy. Then on $(\Sigma^{\mathbb{N}^\star}, \mathcal{C}, P)$, the sequence $(\rho_n^{\mathbf{u}})$ is a Markov chain valued on the set of states of $\mathcal{H}_0$. More precisely, if $\rho_n^{\mathbf{u}} = \chi_n$ then $\rho_{n+1}^{\mathbf{u}}$ takes one of the values*

$$\mathbf{E}_0\left[\frac{I\otimes P_i\, U_{n+1}(h,u_n)\,(\chi_n\otimes\beta)\,U_{n+1}^\star(h,u_n)\; I\otimes P_i}{\mathrm{Tr}\big[\,U_{n+1}(h,u_n)\,(\chi_n\otimes\beta)\,U_{n+1}^\star(h,u_n)\,I\otimes P_i\,\big]}\right], \quad i = 0\dots p,$$

with probability $\mathrm{Tr}\left[\,U_{n+1}(h,u_n)\,(\chi_n\otimes\beta)\,U_{n+1}^\star(h,u_n)\,I\otimes P_i\,\right]$.

With the description of Theorem 1, we can express a discrete evolution equation describing the discrete quantum trajectory $(\rho_k^{\mathbf{u}})$. By putting

$$\mathcal{L}_i^{\mathbf{u},k}(\rho) = \mathbf{E}_0\big[\,I\otimes P_i\;U_k(h,u_{k-1})\,(\rho\otimes\beta)\,U_k^\star(h,u_{k-1})\,I\otimes P_i\,\big], \quad i = 0\dots p,$$

and $\mathbf{1}_i^k(\omega) = \mathbf{1}_i(\omega_k)$ for all $\omega \in \Sigma^{\mathbb{N}^\star}$, the discrete process $(\rho_k^{\mathbf{u}})$ then satisfies

$$\rho_{k+1}^{\mathbf{u}}(\omega) = \sum_{i=0}^{p} \frac{\mathcal{L}_i^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}}(\omega))}{\mathrm{Tr}\big[\,\mathcal{L}_i^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}}(\omega))\,\big]}\,\mathbf{1}_i^{k+1}(\omega), \tag{8}$$

for all $\omega \in \Sigma^{\mathbb{N}^\star}$ and all $k > 0$. In expression (8), if $\mathrm{Tr}[\mathcal{L}_i^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}})] = 0$, we consider that $\Big(\mathcal{L}_i^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}})/\mathrm{Tr}[\mathcal{L}_i^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}})]\Big)\mathbf{1}_i^{k+1}$ is equal to zero. This is consistent with the fact that $\mathbf{1}_i^{k+1} = 0$ almost surely in this case.

The following section is devoted to the study of (8) in a particular case of a two-level system in interaction with a spin chain.

2.2. A Two-Level Atom

The physical situation is described by $\mathcal{H}_0 = \mathcal{H} = \mathbb{C}^2$. In this case, an observable A has two different eigenvalues, that is $A = \lambda_0 P_0 + \lambda_1 P_1$. Equation (8) becomes

$$\rho_{k+1}^{\mathbf{u}} = \frac{\mathcal{L}_0^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}})}{p_{k+1}^{\mathbf{u}}}\mathbf{1}_0^{k+1} + \frac{\mathcal{L}_1^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}})}{q_{k+1}^{\mathbf{u}}}\mathbf{1}_1^{k+1}, \tag{9}$$

where $p_{k+1}^{\mathbf{u}} = \mathrm{Tr}[\mathcal{L}_0^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}})] = 1 - q_{k+1}^{\mathbf{u}}$. Let us now introduce the centred and normalized random variables (X_k) defined by

$$X_{k+1} = \frac{\mathbf{1}_1^{k+1} - q_{k+1}^{\mathbf{u}}}{\sqrt{q_{k+1}^{\mathbf{u}}p_{k+1}^{\mathbf{u}}}},$$

for all $k > 0$. We define the associated filtration $(\mathcal{F}_k)$ on $\{0,1\}^{\mathbb{N}^\star}$ by putting $\mathcal{F}_k = \sigma(X_i, i \leq k)$, for all $k > 0$. By construction, we have $\mathbf{E}[X_{k+1}/\mathcal{F}_k] = 0$ and $\mathbf{E}[X_{k+1}^2/\mathcal{F}_k] = 1$. In terms of (X_k) the discrete controlled quantum trajectory satisfies

$$\begin{aligned}\rho_{k+1}^{\mathbf{u}} &= \mathcal{L}_0^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}}) + \mathcal{L}_1^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}}) \\ &\quad + \left[-\sqrt{\frac{q_{k+1}^{\mathbf{u}}}{p_{k+1}^{\mathbf{u}}}}\mathcal{L}_0^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}}) + \sqrt{\frac{p_{k+1}^{\mathbf{u}}}{q_{k+1}^{\mathbf{u}}}}\mathcal{L}_1^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}})\right] X_{k+1}. \end{aligned} \tag{10}$$

Equation (10) is a discrete version of a stochastic master equation with control. To go further, we need to express the terms $\mathcal{L}_i^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}})$ in an explicit way. To this end, we introduce a particular basis. Let $(e_0 = \Omega, e_1 = X)$ be the orthonormal basis of $\mathcal{H}_0 = \mathcal{H} = \mathbb{C}^2$. For the space $\mathcal{H}_0 \otimes \mathcal{H}$, we consider the following basis $\{\Omega\otimes\Omega, X\otimes\Omega, \Omega\otimes X, X\otimes X\}$. In this basis, the unitary operator can be written as a 2×2 block matrix

$$U_{k+1}(h,u_k) = \begin{pmatrix} L_{00}(kh,u_k) & L_{01}(kh,u_k) \\ L_{10}(kh,u_k) & L_{11}(kh,u_k)\end{pmatrix},$$

where each $L_{ij}(kh,u_k)$ are operators on $\mathcal{H}_0$. Now, we need to specify the reference state β of each copy of $\mathcal{H}$; we choose the ground state $\beta = |\Omega\rangle\langle\Omega|$. Let us also notice that the terms $\mathcal{L}_i^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}})$ depend on the expression of the eigenprojectors of the observable A. Finally, if the eigenprojector P_i is expressed as $P_i = (p_{kl}^i)_{0\leq k,l\leq 1}$ in the basis (Ω, X), we have

$$\begin{aligned}\mathcal{L}_i^{\mathbf{u},k+1}(\rho_k^{\mathbf{u}}) &= p_{00}^i L_{00}(kh,u_k)\,\rho_k^{\mathbf{u}}\,L_{00}^\star(kh,u_k) + p_{01}^i L_{00}(kh,u_k)\,\rho_k^{\mathbf{u}}\,L_{10}^\star(kh,u_k) \\ &\quad + p_{10}^i L_{10}(kh,u_k)\,\rho_k^{\mathbf{u}}\,L_{00}^\star(kh,u_k) + p_{11}^i L_{10}(kh,u_k)\,\rho_k^{\mathbf{u}}\,L_{10}^\star(kh,u_k).\end{aligned} \tag{11}$$

As the unitary evolution depends on h, the discrete quantum trajectory $(\rho_k^{\mathbf{u}})$ depends also on h. In Sect. 2, we consider the continuous time limits $(h \to 0)$ of the discrete processes $(\rho_k^{\mathbf{u}})$. The next subsection is devoted to present the asymptotic ingredients necessary to obtain the convergence results.

2.3. Description of Asymptotic Assumptions

In this section, we present suitable asymptotic conditions for the coefficients of the unitary operators $U_k(h,u_k)$ in order to have an effective continuous time limit. Let $h = 1/n$ be the time of interaction, we have for (U_k)

$$U_{k+1}(n,u_k) = \begin{pmatrix} L_{00}(k/n,u_k) & L_{01}(k/n,u_k) \\ L_{10}(k/n,u_k) & L_{11}(k/n,u_k)\end{pmatrix}.$$

In our context, the choice of the coefficients L_{ij} is an adaptation of the works of Attal-Pautrat in [4]. In their work, they consider only an evolution of the type

$$U_{k+1}(n) = \begin{pmatrix} L_{00}(n) & L_{01}(n) \\ L_{10}(n) & L_{11}(n)\end{pmatrix},$$

that is, a homogeneous evolution without control. They have shown that

$$V_{[nt]} = \tilde{U}_{[nt]}(n)\dots\tilde{U}_1(n)$$

converges (in operator theory) to a non-trivial process V_t, which is a solution of a quantum stochastic differential equation. Besides, this convergence is valid only if the coefficients $L_{ij}(n)$ obey certain normalization conditions. In their case, these coefficients must be of the form

$$L_{00}(n) = I + \frac{1}{n}\left(-iH_0 - \frac{1}{2}C^\star C\right) + \circ\left(\frac{1}{n}\right), \tag{12}$$

$$L_{10}(n) = \frac{1}{\sqrt{n}}C + \circ\left(\frac{1}{n}\right), \tag{13}$$

where H_0 is the Hamiltonian of $\mathcal{H}_0$ and C is any operator on $\mathbb{C}^2$. Hence, in the control context, the coefficients $L_{ij}(k/n, u_k)$ must follow similar expressions with non homogeneous terms. Let k be fixed, we put

$$\begin{aligned} L_{00}(k/n, u_k(n)) &= I + \frac{1}{n}\left(-iH_k(n,u_k) - \frac{1}{2}C_k^\star(n,u_k)C_k(n,u_k)\right) + \circ\left(\frac{1}{n}\right), \\ L_{10}(k/n, u_k(n)) &= \frac{1}{\sqrt{n}}C_k(n,u_k) + \circ\left(\frac{1}{n}\right), \end{aligned} \tag{14}$$

where $H_k(n,u_k)$ is a self-adjoint operator and $C_k(n,u_k)$ is an operator on $\mathbb{C}^2$. It is straightforward that the expression (12) of Attal-Pautrat is a particular case of the previous expression. Finally, we suppose that there exist some functions H and C such that

$$\begin{array}{rcl} H : \mathbb{R}^+ \times \mathbb{R} & \longrightarrow & \mathbb{H}_2(\mathbb{C}) \\ (t,s) & \longmapsto & H(t,s) \end{array} \quad \text{and} \quad \begin{array}{rcl} C : \mathbb{R}^+ \times \mathbb{R} & \longrightarrow & \mathbb{M}_2(\mathbb{C}) \\ (t,s) & \longmapsto & C(t,s) \end{array},$$

where $\mathbb{H}_2(\mathbb{C})$ denotes the set of self-adjoint operators on $\mathbb{C}^2$ and

$$H_k(n,u_k) = H(k/n, u_k), \qquad C_k(n,u_k) = C(k/n,u_k). \tag{15}$$

Furthermore, we suppose that all the $\circ$ are uniform in k.

Now, we shall express (10) with these asymptotic assumptions. Depending on the choice of the observable, we obtain two different behaviours.

1. If the observable A is diagonal in the basis (Ω, X), that is, it is of the form $A = \lambda_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain the asymptotic form for the probabilities

$$\begin{aligned} p_{k+1}^{\mathbf{u}}(n) &= 1 - \frac{1}{n}\mathrm{Tr}\Big[\mathcal{J}(k/n,u_k)(\rho_k^{\mathbf{u}}(n))\Big] + \circ\left(\frac{1}{n}\right), \\ q_{k+1}^{\mathbf{u}}(n) &= \frac{1}{n}\mathrm{Tr}\Big[\mathcal{J}(k/n,u_k)(\rho_k^{\mathbf{u}}(n))\Big] + \circ\left(\frac{1}{n}\right). \end{aligned}$$

The discrete equation (10) becomes

$$\begin{aligned}
&\rho^{\mathbf{u}}_{k+1}(n) - \rho^{\mathbf{u}}_k(n) \\
&\quad = \frac{1}{n}\Big(L(k/n,u_k)(\rho^{\mathbf{u}}_k(n)) + \circ(1)\Big) \\
&\qquad + \left(\frac{\mathcal{J}(k/n,u_k)(\rho^{\mathbf{u}}_k(n))}{\mathrm{Tr}\Big[\mathcal{J}(k/n,u_k)(\rho^{\mathbf{u}}_k(n))\Big]} - \rho^{\mathbf{u}}_k(n) + \circ(1)\right)\sqrt{q^{\mathbf{u}}_{k+1}(n)\, p^{\mathbf{u}}_{k+1}(n)}\; X_{k+1}(n),
\end{aligned} \tag{16}$$

where for all states ρ, we have defined

$$\begin{aligned}
&\mathcal{J}(t,s)(\rho) = C(t,s)\ \rho\ C^\star(t,s) \ \text{ and} \\
&L(t,s)(\rho) = -i\big[H(t,s),\rho\big] - \frac{1}{2}\big\{C^\star(t,s)C(t,s),\rho\big\} + \mathcal{J}(t,s)(\rho).
\end{aligned} \tag{17}$$

2. If the observable A is non diagonal in the basis (Ω, X), and if the eigen-projectors are expressed as $P_0 = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11}\end{pmatrix}$ and $P_1 = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11}\end{pmatrix}$, we have

$$\begin{aligned}
p^{\mathbf{u}}_{k+1} &= p_{00} + \frac{1}{\sqrt{n}}\mathrm{Tr}\Big[\rho^{\mathbf{u}}_k\ \big(p_{01}C(k/n,u_k) + p_{10}C^\star(k/n,u_k)\big)\Big] \\
&\quad + \frac{1}{n}\mathrm{Tr}\Big[\rho^{\mathbf{u}}_k\, p_{00}\ \big(C(k/n,u_k) + C^\star(k/n,u_k)\big)\Big] + \circ\left(\frac{1}{n}\right), \\
q^{\mathbf{u}}_{k+1} &= q_{00} + \frac{1}{\sqrt{n}}\mathrm{Tr}\Big[\rho^{\mathbf{u}}_k\ \big(q_{01}C(k/n,u_k) + q_{10}C^\star(k/n,u_k)\big)\Big] \\
&\quad + \frac{1}{n}\mathrm{Tr}\Big[\rho^{\mathbf{u}}_k\, q_{00}\ \big(C(k/n,u_k) + C^\star(k/n,u_k)\big)\Big] + \circ\left(\frac{1}{n}\right).
\end{aligned}$$

The discrete equation (10) becomes

$$\begin{aligned}
&\rho^{\mathbf{u}}_{k+1} - \rho^{\mathbf{u}}_k \\
&\quad = \frac{1}{n}\Big(L(k/n,u_k)(\rho^{\mathbf{u}}_k) + \circ\,(1)\Big) + \Big(e^{i\theta}C(k/n,u_k)\rho^{\mathbf{u}}_k + e^{-i\theta}\rho^{\mathbf{u}}_k C^\star(k/n,u_k) \\
&\qquad - \mathrm{Tr}\big[\rho^{\mathbf{u}}_k\ \big(e^{i\theta}C(k/n,u_k) + e^{-i\theta}C^\star(k/n,u_k)\big)\big]\ \rho^{\mathbf{u}}_k + \circ(1)\Big)\frac{1}{\sqrt{n}}\, X_{k+1}(n),
\end{aligned} \tag{18}$$

where θ is a real parameter. This parameter can be explicitly expressed with the coefficients of the eigenprojectors (P_i). By putting $C_\theta(k/n,u_k) = e^{i\theta}$ $C(k/n,u_k)$, we have the same form for (18) for all θ; then we consider in the following that $\theta = 0$. The expression of L is the same as in (17).

In order to prepare the final convergence result, in each case, we can define a process $(\rho^{\mathbf{u}}_{[nt]})$ which satisfies

$$\begin{aligned}\rho^{\mathbf{u}}_{[nt]} &= \rho_0 + \sum_{i=0}^{[nt]-1} \left[\rho^{\mathbf{u}}_{i+1} - \rho^{\mathbf{u}}_i\right] \\ &= \rho_0 + \sum_{i=0}^{[nt]-1} \left[\mathcal{L}_0^{\mathbf{u},i+1}(\rho^{\mathbf{u}}_i) + \mathcal{L}_1^{\mathbf{u},i+1}(\rho^{\mathbf{u}}_i) - \rho^{\mathbf{u}}_i\right] \\ &\quad + \sum_{i=0}^{[nt]-1} \left[-\sqrt{\frac{q^{\mathbf{u}}_{i+1}}{p^{\mathbf{u}}_{i+1}}}\,\mathcal{L}_0^{\mathbf{u},i+1}(\rho^{\mathbf{u}}_i) + \sqrt{\frac{p^{\mathbf{u}}_{i+1}}{q^{\mathbf{u}}_{i+1}}}\,\mathcal{L}_1^{\mathbf{u},i+1}(\rho^{\mathbf{u}}_i)\right] X_{i+1} \\ &= \rho_0 + \sum_{i=0}^{[nt]-1} \frac{1}{n}\mathcal{Y}(i/n, u_i, \rho^{\mathbf{u}}_i) + \sum_{i=0}^{[nt]-1} \mathcal{Z}(i/n, u_i, \rho^{\mathbf{u}}_i) X_{i+1},\end{aligned} \tag{19}$$

for some functions $\mathcal{Y}$ and $\mathcal{Z}$ which depend on the descriptions (16) or (18).

In the next section, we show that the processes $(\rho^{\mathbf{u}}_{[nt]})$ converge to the solutions of particular stochastic differential equations.

3. Convergence to Continuous Models

In this section, starting from the description (19) with a Markovian strategy and following the asymptotic expressions (16) and (18), we show that discrete processes $(\rho^{\mathbf{u}}_{[nt]})$ converge in distribution to solutions of stochastic differential equations.

In this article, we obtain two different kinds of continuous equations which are similar to (1) and (2) with control.

1. If $(\rho^{\mathbf{u}}_t)$ denotes the state of a quantum system, the diffusive evolution is given by

$$d\rho^{\mathbf{u}}_t = L\big(t, u(t,\rho^{\mathbf{u}}_t)\big)(\rho^{\mathbf{u}}_t)\, dt + \Theta\big(t, u(t,\rho^{\mathbf{u}}_t)\big)(\rho^{\mathbf{u}}_t)\, \mathrm{d}W_t, \tag{20}$$

   where (W_t) describes a one-dimensional Brownian motion. The function L is expressed in (17) and Θ is defined by

$$\Theta(t,a)(\mu) = C(t,a)\mu + \mu C^\star(t,a) - \mathrm{Tr}\left[\mu\Big(C(t,a) + C^\star(t,a)\Big)\right]\mu, \tag{21}$$

   for all $t > 0$, for all a in $\mathbb{R}$ and all operators μ in $\mathbb{M}_2(\mathbb{C})$.
2. The evolution with jump is given by

$$\begin{aligned}\mathrm{d}\rho^{\mathbf{u}}_t &= L\big(t, u(t,\rho^{\mathbf{u}}_t)\big)(\rho^{\mathbf{u}}_t)\mathrm{d}t \\ &\quad + \left(\frac{\mathcal{J}\big(t, u(t,\rho^{\mathbf{u}}_t)\big)(\rho^{\mathbf{u}}_t)}{\mathrm{Tr}\Big[\mathcal{J}\big(t, u(t,\rho^{\mathbf{u}}_t)\big)(\rho^{\mathbf{u}}_t)\Big]} - \rho^{\mathbf{u}}_t\right)\Big(\mathrm{d}\tilde{N}_t - \mathrm{Tr}\Big[\mathcal{J}\big(t, u(t,\rho^{\mathbf{u}}_t)\big)(\rho^{\mathbf{u}}_t)\Big]\mathrm{d}t\Big),\end{aligned} \tag{22}$$

where $\tilde{N}_t$ is a counting process with stochastic intensity $\int_0^t \mathrm{Tr}\big[\mathcal{J}\big(s, u(s, \rho_s^{\mathbf{u}})\big)(\rho_s^{\mathbf{u}})\big]\mathrm{d}s$. The functions L and $\mathcal{J}$ are as in (17).

In a natural way, we call such equations *Controlled Stochastic Master Equations* and their solutions *Controlled Quantum Trajectories*.

For the moment, we do not speak about the regularity of the functions L, Θ and $\mathcal{J}$. This will be discussed when we deal with the question of existence and uniqueness of solutions. Let us start by studying the diffusive equation.

3.1. Existence, Uniqueness and Approximation of the Diffusive Equation with Control

In this section, we justify the diffusive model (20) of controlled stochastic master equations by proving that the solution of (20) is obtained from the limit of particular quantum trajectories $(\rho_{[nt]}^{\mathbf{u}})$. At the same time, we show that (20) admits a unique solution with values in the set of states.

Let us start with the problem of existence and uniqueness of a solution for (20). For the moment, let u be any measurable function which defines a Markovian strategy. Usually, in order to prove the existence and the uniqueness of a solution for a SDE of type (20), one imposes Lipschitz conditions [35]. However, even in the homogeneous case without control, such conditions are not satisfied. Indeed, in the homogeneous situation without control, for Θ we have

$$\Theta(t,a)(\mu) = \Theta(\mu) = C\mu + \mu C^\star - \mathrm{Tr}\left[\mu\big(C + C^\star\big)\right]\mu,$$

where the last term is quadratic in μ. Nevertheless, this function is C^∞ and then local Lipschitz (see [31,32] for similar reasoning). With control, the local Lipschitz condition is expressed as follows. For all integers $k > 0$ and all $x \in \mathbb{R}$, we define the function ϕ^k by

$$\phi^k(x) = -k\mathbf{1}_{]-\infty,-k[}(x) + x\mathbf{1}_{[-k,k]}(x) + k\mathbf{1}_{]k,\infty[}(x).$$

The function ϕ^k is called a truncation function. Its extension on the set of operator on $\mathbb{C}^2$ is given by

$$\tilde{\phi}^k(B) = \big(\phi^k(\mathrm{Re}(B_{kl})) + i\phi^k(\mathrm{Im}(B_{kl}))\big)_{0\le k,l\le 1}.$$

Now, let $T > 0$ and let $k > 0$, there exist constants $M^k(T)$ and $K^k(T)$ such that

$$\begin{aligned}
&\sup\{\|L(t,a)(\tilde{\phi}^k(\mu)) - L(t,a)(\tilde{\phi}^k(\rho))\|, \|\Theta(t,a)(\tilde{\phi}^k(\mu)) - \Theta(t,a)(\tilde{\phi}^k(\rho))\| \\
&\quad \le K^k(T)\|\mu - \rho\|, \\
&\sup\left\{\|L(t,a)(\tilde{\phi}^k(\rho))\|, \|\Theta(t,a)(\tilde{\phi}^k(\rho))\|\right\} \le M^k(T)(1 + \|\rho\| + \|a\|),
\end{aligned} \tag{23}$$

for all $t \le T$ and all $(\mu,\rho) \in \mathbb{M}_2(\mathbb{C})^2$ (these conditions are the global Lipschitz conditions for the functions L and Θ composed with $\tilde{\phi}^k$). As a consequence, we have the following existence and uniqueness theorem:

Theorem 2. *Let u be any measurable function. Let $k > 0$. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space which supports a standard Brownian motion (W_t). Assume that*

L and Θ *satisfy the conditions* (23)*. The stochastic differential equation*

$$\mathrm{d}\rho_t^{\mathbf{u},k} = L\left(t, u(t, \tilde{\phi}^k(\rho_t^{\mathbf{u},k})\right)\left(\tilde{\phi}^k(\rho_t^{\mathbf{u},k})\right)\mathrm{d}s + \Theta\left(t, u(t, \tilde{\phi}^k(\rho_t^{\mathbf{u},k}))\right) \times \left(\tilde{\phi}^k(\rho_t^{\mathbf{u},k})\right)\mathrm{d}W_t, \tag{24}$$

admits a unique continuous solution $(\rho_t^{\mathbf{u},k})$*.*

This theorem is just a consequence of the local Lipschitz condition (23) (cf [35]). The process $(\rho_t^{\mathbf{u},k})$ is called a truncated solution. In order to link the solution of (24) and the solution of (20), we define the random stopping time

$$T_k = \inf\left\{t > 0/\exists(ij),\, \mathrm{Re}(\rho_t^{\mathbf{u},k}(ij)) = k \text{ or } \mathrm{Im}(\rho_t^{\mathbf{u},k}(ij)) = k\right\}.$$

For any $k > 1$, we have $T_k > 0$ almost surely since ρ_0 is a state and the almost surely continuity of $(\rho_t^{\mathbf{u},k})$ (the coefficients of ρ_0 satisfy namely $|\rho_0(ij)| \leq 1$). Furthermore, on $[0, T_k[$, we have $\tilde{\phi}^k(\rho_t^{\mathbf{u},k}) = \rho_t^{\mathbf{u},k}$. Therefore, the process $(\rho_t^{\mathbf{u},k})$ satisfies

$$\rho_t^{\mathbf{u},k} = \rho_0 + \int_0^t L\Big(s, u(s, \rho_s^{\mathbf{u},k})\Big)(\rho_s^{\mathbf{u},k})\mathrm{d}s + \int_0^t \Theta\Big(s, u(s, \rho_s^{\mathbf{u},k})\Big)(\rho_s^{\mathbf{u},k})\mathrm{d}W_s, \tag{25}$$

for all $t < T_k$. Hence, the process $(\rho_t^{\mathbf{u},k})$ is the unique solution of (20) on $[0, T_k[$. Normally, in order to define a solution for all $t \geq 0$, the next step consists in showing that $\lim T_k = \infty$.

In our situation, we prove in fact that $T_k = \infty$, for all $k > 1$. This is provided by proving that the process $(\rho_t^{\mathbf{u},k})$ is valued in the set of states. Indeed, if the process $(\rho_t^{\mathbf{u},k})$ takes values in the set of states, we have $|\rho_t^{\mathbf{u},k}(ij)| \leq 1$, for all $t \geq 0$, $T_k = \infty$. It remains then to prove that (20) preserves the property of being a state. This result follows from the convergence. Indeed, let us assume that there is a discrete quantum trajectory $(\rho_{[nt]}^{\mathbf{u}})$ which converges in distribution to $(\rho_t^{\mathbf{u},k})$ (for some $k > 1$). Therefore, for all measurable functions $\mathcal{V}$ defined on $\mathbb{M}_2(\mathbb{C})$, we have

$$\mathcal{V}(\rho_{[nt]}^{\mathbf{u}}) \Longrightarrow \mathcal{V}(\rho_t^{\mathbf{u},k}),$$

where the symbol $\Longrightarrow$ denotes the convergence in distribution. We apply it for the functions $\mathcal{V}(\rho) = \mathrm{Tr}[\,\rho\,]$, for $\mathcal{V}(\rho) = \rho^\star - \rho$ and $\mathcal{V}_z(\rho) = \langle z, \rho z\rangle$, for all $z \in \mathbb{C}^2$. By definition, if ρ is a state, we have from the trace property $\mathrm{Tr}[\,\rho\,] = 1$, from self-adjointness $\rho^\star - \rho = 0$ and from positivity $\langle z, \rho z\rangle \geq 0$ for all $z \in \mathbb{C}^2$. As discrete quantum trajectories take values in the set of states, these properties are then conserved at the limit. The limit process $(\rho_t^{\mathbf{u},k})$ takes then values in the set of states. We shall now prove the convergence result.

Let us come back to the description (19) of discrete quantum trajectories. With the asymptotic expression (18) in the case of a non-diagonal observable A

and with a Markovian strategy, we have

$$\rho^{\mathbf{u}}_{[nt]} = \rho_0 + \sum_{k=1}^{[nt]-1} \frac{1}{n}\Big(L\big(k/n, u(k/n, \rho^{\mathbf{u}}_k)\big)(\rho^{\mathbf{u}}_k) + \circ\,(1)\Big) + \sum_{k=1}^{[nt]-1} \Big(\Theta\big(k/n, u(k/n, \rho^{\mathbf{u}}_k)\big)(\rho^{\mathbf{u}}_k) + \circ(1)\Big)\frac{1}{\sqrt{n}} X_{k+1}(n). \tag{26}$$

From this description, we can define the following processes and functions

$$\begin{aligned} W_n(t) &= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} X_k(n), \\ V_n(t) &= \frac{[nt]}{n}, \\ \rho^{\mathbf{u}}_n(t) &= \rho^{\mathbf{u}}_{[nt]}(n), \\ u_n(t, W) &= u([nt]/n, W), \\ \Theta_n(t, s) &= \Theta([nt]/n, s), \\ L_n(t, s) &= L([nt]/n, s), \end{aligned} \tag{27}$$

for all $t > 0$, for all $s \in \mathbb{R}$ and for all $W \in \mathbb{M}_2(\mathbb{C})$.

By observing that these processes and these functions are piecewise constant, we can describe the discrete quantum trajectory $(\rho^{\mathbf{u}}_n(t))$ as a solution of the following stochastic differential equation:

$$\begin{aligned} \rho^{\mathbf{u}}_n(t) &= \rho_0 + \int_0^t \Big[L_n\Big(s-, u_n(s-, \rho^{\mathbf{u}}_n(s-))\Big)\big(\rho^{\mathbf{u}}_n(s-)\big) + \circ(1)\Big]\, \mathrm{d}V_n(s) \\ &\quad + \int_0^t \Big[\Theta_n\Big(s-, u_n(s-, \rho^{\mathbf{u}}_n(s-))\Big)\big(\rho^{\mathbf{u}}_n(s-)\big) + \circ(1)\Big]\, \mathrm{d}W_n(s) \\ &= \rho_0 + \int_0^t \Big[L_n\Big(s-, u_n(s-, \tilde{\phi}^k(\rho^{\mathbf{u}}_n(s-))\Big)\big(\tilde{\phi}^k(\rho^{\mathbf{u}}_n(s-))\big) + \circ(1)\Big]\, \mathrm{d}V_n(s) \\ &\quad + \int_0^t \Big[\Theta_n\Big(s-, u_n(s-, \tilde{\phi}^k(\rho^{\mathbf{u}}_n(s-))\Big)\big(\tilde{\phi}^k(\rho^{\mathbf{u}}_n(s-))\big) + \circ(1)\Big]\, \mathrm{d}W_n(s), \end{aligned} \tag{28}$$

for all $k > 1$. Equation (28) appears then as a discrete version of (20). This approach has already been used in [32] and the final convergence result has been obtained by applying a convergence theorem of sequence of stochastic differential equations (Theorem of Kurtz and Protter [26, 27]). The main idea is to prove that the discrete process $(W_n(t))$ converges to a Brownian motion (W_t). Next, as L_n,

Θ_n and u_n converge to L, Θ and u, the result of Kurtz and Protter states that the solution of (28) converges to the solution of (20).

Actually in the non-homogeneous case of control, special conditions for the functions L, Θ and u must be satisfied to apply the Theorem of Kurtz and Protter. In order to present these requirements, we need to introduce some notations.

For all $T > 0$, we define $\mathcal{D}[0, T]$ the space of càdlàg process of $\mathbb{M}_2(\mathbb{C})$ endowed with the Skorohod topology.

Let $T_1[0, \infty)$ denote the set of non-decreasing mapping λ from $[0, \infty)$ to $[0, \infty)$ with $\lambda(0) = 0$ such that $\lambda(t + h) - \lambda(t) \leq h$ for all $t, h \geq 0$. For any function G defined from $\mathbb{R}^+ \times \mathbb{M}_2(\mathbb{C})$ to $\mathbb{M}_2(\mathbb{C})$, we define

$$\begin{aligned} \tilde{G} : \mathcal{D}[0, \infty) \times T_1[0, \infty) &\longrightarrow \mathcal{D}[0, \infty) \\ (X, \lambda) &\longmapsto G(X) \circ \lambda, \end{aligned}$$

such that for all $t \geq 0$, we have $G(X) \circ \lambda(t) = G(\lambda(t), X_{\lambda(t)})$. We introduce the two following conditions concerning a function $\tilde{G}$ and a sequence $\tilde{G}_n$ as above:

$(C1)$ For each compact subset $\mathcal{K} \in \mathcal{D}[0, \infty) \times T_1[0, \infty)$ and $t > 0$,
$$\sup_{(X,\lambda)} \sup_{s \leq t} \|\tilde{G}_n(X, \lambda)(s) - \tilde{G}(X, \lambda)(s)\| \to 0.$$

$(C2)$ For $(X_n, \lambda_n)_n \in \mathcal{D}[0, \infty) \times T_1[0, \infty) / \sup_{s \leq T} \|X_n(s) - X(s)\| \to 0$ and $\sup_{s \leq t} |\lambda_n(s) - \lambda(s)| \to 0$ for each $t > 0$ implies
$$\sup_{s \leq t} \|\tilde{G}(X_n, \lambda_n)(s) - \tilde{G}(X, \lambda)(s)\| \to 0. \tag{29}$$

In our context, we define for example $\tilde{L}_n$ by

$$\tilde{L_n}(X) \circ (\lambda)(t) = L_n(\lambda(t), u_n(\lambda(t), X_{\lambda(t)}))(X_{\lambda(t)}) + \circ(1),$$

for all $t > 0$, for all $\lambda \in T_1[0, \infty)$ and all càdlàg process (X_t). We consider the same definition for the other functions appearing in (28).

Furthermore, recall that the square-bracket $[X, X]$ is defined for a semimartingale by the formula

$$[X, X]_t = X_t^2 - 2 \int_0^t X_{s-} \mathrm{d}X_s.$$

We shall denote by $T_t(V)$ the total variation of a finite variation processes V on the interval $[0, t]$. Now, we are in the position to express the Theorem of Kurtz and Protter in our context.

Theorem 3. *Let ρ_0 be any state on $\mathcal{H}_0$. Let $(\rho_n^{\mathbf{u}}(t))$ be the discrete quantum trajectory satisfying*

$$\rho_n^{\mathbf{u}}(t) = \rho_0 + \int_0^t \Big[L_n\Big(s-, u_n(s-, \rho_n^{\mathbf{u}}(s-))\Big)\big(\rho_n^{\mathbf{u}}(s-)\big) + \circ(1)\Big] \mathrm{d}V_n(s)$$
$$+ \int_0^t \Big[\Theta_n\Big(s-, u_n(s-, \rho_n^{\mathbf{u}}(s-))\Big)\big(\rho_n^{\mathbf{u}}(s-)\big) + \circ(1)\Big] \mathrm{d}W_n(s). \quad (30)$$

Let $k > 1$ be any integer. Let $(\rho_t^{\mathbf{u},k})$ be the unique solution of

$$\rho_t^{\mathbf{u},k} = \rho_0 + \int_0^t L\Big(s, u(s, \tilde{\phi}^k(\rho_s^{\mathbf{u},k})\Big)\big(\tilde{\phi}^k(\rho_s^{\mathbf{u},k})\big)\mathrm{d}s$$
$$+ \int_0^t \Theta\Big(s, u(s, \tilde{\phi}^k(\rho_s^{\mathbf{u},k}))\Big)\big(\tilde{\phi}^k(\rho_s^{\mathbf{u},k})\big)\mathrm{d}W_s. \quad (31)$$

Suppose that the function u is sufficiently regular such that (L_n, L) and (Θ_n, Θ) composed with $\tilde{\phi}^k$ satisfy the conditions $(C1)$ and $(C2)$. Suppose that there exists a filtration $(\mathcal{F}_t^n)$ such that $(\rho_n(t))$ is $(\mathcal{F}_t^n)$ adapted. Suppose that (W_n, V_n) converges in distribution in the Skorohod topology to (W, V) where $V_t = t$, for all $t \geq 0$ and suppose that

$$\sup_n \Big\{\mathbf{E}\Big[[W_n, W_n]_t\Big]\Big\} < \infty,$$
$$\sup_n \Big\{\mathbf{E}\left[T_t(V_n)\right]\Big\} < \infty. \quad (32)$$

Hence, the process $(\rho_n(t))$ converges in distribution in $\mathcal{D}[0,T]$ for all $T > 0$ to the process $(\rho_t^{\mathbf{u},k})$.

Concerning the filtration $(\mathcal{F}_t^n)$, we define

$$\mathcal{F}_t^n = \sigma(X_i, i \leq [nt]). \quad (33)$$

The process $(\rho_n(t))$ is then clearly $(\mathcal{F}_t^n)$ adapted. Let us stress that the essential result concerns the convergence of the process $(W_n(t))$. Next, in a natural way this theorem expresses that if the discrete noise converges, under some technical assumptions, the discrete stochastic differential equation converges to the continuous one. The properties concerning the process $(W_n(t))$ follow from the next proposition (the property for $(V_n(t))$ is obvious).

Proposition 3. *Let $(\mathcal{F}_t^n)$ be the filtration defined by (33). We have the following properties:*

- *The process $(W_n(t))$ defined by (27) is a $(\mathcal{F}_t^n)$ martingale.* (34)
- $\lim_{n\to\infty} \mathbf{E}\big[|[W_n, W_n]_t - t|\big] = 0.$ (35)

As a consequence, the process $(W_n(t))$ *converges to a standard Brownian motion* (W_t)*. Furthermore, we have*

$$\sup_n \mathbf{E}\Big[[W_n, W_n]_t\Big] < \infty. \tag{36}$$

Concerning the fact that the properties (34) and (35) imply the convergence, this is a classical result in Probability Theory. This result is actually a generalisation of the Donsker invariant principle for dependant variables [18].

Proof. Thanks to the definition of the random variables (X_i), we have $\mathbf{E}\big[X_{i+1}/\mathcal{F}_i^n\big] = 0$, for all $i > 0$. This implies $\mathbf{E}\left[\sum_{i=[ns]+1}^{[nt]} X_i/\mathcal{F}_s^n\right] = 0$, for $t > s$ and we get the martingale property.

Concerning (35), we prove a L_2 convergence

$$\lim_{n\to\infty} \mathbf{E}\big[|[W_n, W_n]_t - t|^2\big] = 0,$$

which implies the L_1 convergence (35). To this end, we use the fact that $\mathbf{E}\left[X_i^2\right] = \mathbf{E}\left[\mathbf{E}[X_i^2/\sigma\{X_l, l < i\}]\right] = 1$. Hence, if $i < j$

$$\begin{aligned}
\mathbf{E}\left[(X_i^2 - 1)(X_j^2 - 1)\right] &= \mathbf{E}\left[(X_i^2 - 1)(X_j^2 - 1)/\sigma\{X_l, l < j\}]\right] \\
&= \mathbf{E}\left[(X_i^2 - 1)\right]\mathbf{E}\left[(X_j^2 - 1)\right] \\
&= 0.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathbf{E}\left[\left([W_n, W_n]_t - \frac{[nt]}{n}\right)^2\right] &= \frac{1}{n^2}\sum_{i=1}^{[nt]} \mathbf{E}\left[(X_i^2 - 1)^2\right] + \frac{1}{n^2}\sum_{i<j} \mathbf{E}\left[(X_i^2 - 1)(X_j^2 - 1)\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{[nt]} \mathbf{E}\left[(X_i^2 - 1)^2\right].
\end{aligned}$$

According to the fact that p_{00} and q_{00} are not equal to zero (because the observable A is not diagonal!), the terms $\mathbf{E}\left[(X_i^2 - 1)^2\right]$ are bounded uniformly in i. Then, we have

$$\lim_{n\to\infty} \mathbf{E}\left[\left([W_n, W_n]_t - \frac{[nt]}{n}\right)^2\right] = 0.$$

The result holds since $t \mapsto [nt]/n$ converges to $t \mapsto t$ in L_2.

Concerning the property (36), by definition of $[W_n, W_n]$, we have

$$[W_n, W_n]_t = W_n(t)^2 - 2\int_0^t W_n(s-)\mathrm{d}W_n(s) = \frac{1}{n}\sum_{i=1}^{[nt]} X_i^2.$$

Thus, we have

$$\mathbf{E}\big[[W_n, W_n]_t\big] = \frac{1}{n}\sum_{i=1}^{[nt]} \mathbf{E}[X_i^2] = \frac{1}{n}\sum_{i=1}^{[nt]} 1 = \frac{[nt]}{n}.$$

This implies $\sup_n \mathbf{E}\big[[W_n, W_n]_t\big] \le t < \infty$, and the proof is complete. □

As regards conditions $(C1)$ and $(C2)$, the assumption for the function u is satisfied, for example, when u is continuous. Indeed the local Lipschitz property (23) of L and Θ implies that the functions L_n, Θ_n, L and Θ, composed with the truncature $\tilde{\phi}^k$, satisfy conditions $(C1)$ and $(C2)$. Hence, we can express the final convergence theorem which concludes the section.

Theorem 4. *Suppose that the function u which defines a deterministic or Markovian control strategy is continuous. Hence, the process $(\rho_n^{\mathbf{u}}(t))$ describing the discrete controlled quantum trajectory for a non-diagonal observable converges to the unique solution $(\rho_t^{\mathbf{u}})$ of the stochastic differential equation*

$$\mathrm{d}\rho_t^{\mathbf{u}} = L\big(t, u(t, \rho_t^{\mathbf{u}})\big)(\rho_t^{\mathbf{u}})\mathrm{d}t + \Theta\big(t, u(t, \rho_t^{\mathbf{u}})\big)(\rho_t^{\mathbf{u}})\mathrm{d}W_t,$$

where (W_t) is an one-dimensional Brownian motion.

3.2. Existence, Uniqueness and Approximation of the Jump Equation with Control

In this section, we investigate the convergence of a discrete quantum trajectory which comes from repeated measurements of a diagonal observable.

In all this section, we fix a strategy $\mathbf{u}$ which defines a Markovian strategy. Furthermore, as in the diffusive case, we suppose that this strategy is continuous. Let A be any diagonal observable. With the use of description (16) and (19), the discrete quantum trajectory satisfies

$$\begin{aligned}\rho_{[nt]}^{\mathbf{u}} = \rho_0 + \sum_{k=0}^{[nt]-1} \frac{1}{n}\Bigg[& L\Big(k/n, u(k/n, \rho_k^{\mathbf{u}})\Big)(\rho_k^{\mathbf{u}}) - \mathcal{J}\Big(k/n, u(k/n, \rho_k^{\mathbf{u}})\Big)(\rho_k^{\mathbf{u}}) \\ & + \mathrm{Tr}\Big[\mathcal{J}\Big(k/n, u(k/n, \rho_k^{\mathbf{u}})\Big)(\rho_k^{\mathbf{u}})\Big]\,\rho_k^{\mathbf{u}} + \circ(1)\Bigg] \\ & + \sum_{k=0}^{[nt]-1}\left[\frac{\mathcal{J}\Big(k/n, u(k/n, \rho_k^{\mathbf{u}})\Big)(\rho_k^{\mathbf{u}})}{\mathrm{Tr}\Big[\mathcal{J}\Big(k/n, u(k/n, \rho_k^{\mathbf{u}})\Big)(\rho_k^{\mathbf{u}})\Big]} - \rho_k^{\mathbf{u}} + \circ(1)\right]\mathbf{1}_1^{k+1}. \end{aligned} \tag{37}$$

Following the idea presented in reference [31], we aim to show that the process $(\rho_{[nt]})$ converges $(n \to \infty)$ to a process (ρ_t) which satisfies

$$\begin{aligned}\rho_t^{\mathbf{u}} = \rho_0 + \int_0^t \Big[L\Big(s-, u(s-, \rho_{s-}^{\mathbf{u}})\Big)(\rho_{s-}^{\mathbf{u}}) + \mathrm{Tr}\Big[\mathcal{J}\Big(s-, u(s-, \rho_{s-}^{\mathbf{u}})\Big)(\rho_{s-}^{\mathbf{u}})\Big]\, \rho_{s-}^{\mathbf{u}} \\ - \mathcal{J}\Big(s-, u(s-, \rho_{s-}^{\mathbf{u}})\Big)(\rho_{s-}^{\mathbf{u}})\Big] \mathrm{d}s \\ + \int_0^t \int_{\mathbb{R}} \left[\frac{\mathcal{J}\Big(s-, u(s-, \rho_{s-}^{\mathbf{u}})\Big)(\rho_{s-}^{\mathbf{u}})}{\mathrm{Tr}\Big[\mathcal{J}\Big(s-, u(s-, \rho_{s-}^{\mathbf{u}})\Big)(\rho_{s-}^{\mathbf{u}})\Big]} - \rho_{s-}^{\mathbf{u}} \right] \\ \times \mathbf{1}_{0<x<\mathrm{Tr}[\mathcal{J}(s-,u(s-,\rho_{s-}^{\mathbf{u}}))(\rho_{s-}^{\mathbf{u}})]} N(\mathrm{d}s, \mathrm{d}x),\end{aligned} \tag{38}$$

where N is a Poisson point process on $\mathbb{R}^2$. As a consequence, if the process $(\rho_t^{\mathbf{u}})$ exists, this gives rise to the process $(\tilde{N}_t)$ defined by

$$\tilde{N}_t = \int_0^t \int_{\mathbb{R}} \mathbf{1}_{0<x<\mathrm{Tr}[\,\mathcal{J}(s-,u(s-,\rho_{s-}^{\mathbf{u}}))(\rho_{s-}^{\mathbf{u}})\,]} N(\mathrm{d}s, \mathrm{d}x) \tag{39}$$

which is a counting process with stochastic intensity $t \to \int_0^t \mathrm{Tr}[\,\mathcal{J}(s-, u(s-, \rho_{s-}^{\mathbf{u}}))(\rho_{s-}^{\mathbf{u}})]\mathrm{d}s$.

Remark. Let us stress that (38) provides a rigorous way to consider (22) (see [31] for a complete discussion regarding this topic). Indeed, with the expression (22) the driving process is not rigorously defined since it depends on the existence of the solution $(\rho_t^{\mathbf{u}})$. With (38), everything is defined in an intrinsic way.

Now, we consider (38) as the jump-model of continuous time measurement with control. It will be justified later as limit of discrete quantum trajectories.

For instance, let us deal with the problem of existence and uniqueness of a solution for this equation. Let us denote

$$\begin{aligned} R(t,a)(\rho) &= L(t,a)(\rho) + \mathrm{Tr}\big[\mathcal{J}(t,a)(\rho)\big]\rho - \mathcal{J}(t,a)(\rho), \\ Q(t,a)(\rho) &= \left(\frac{\mathcal{J}(t,a)(\rho)}{\mathrm{Tr}[\mathcal{J}(t,a)(\rho)]} - \rho \right) \mathbf{1}_{\mathrm{Tr}[\mathcal{J}(t,a)(\rho)]>0}, \end{aligned}$$

for all $t \geq 0$, for all $a \in \mathbb{R}$ and all states ρ. Let us stress that we consider $Q(t,a)(\rho) = 0$ if $\mathrm{Tr}[\mathcal{J}(t,a)(\rho)] = 0$. It is obvious that (38) is equivalent to

$$\begin{aligned}\rho_t^{\mathbf{u}} = \rho_0 + \int_0^t R\Big(s-, u(s-, \rho_{s-}^{\mathbf{u}})\Big)(\rho_{s-}^{\mathbf{u}})\mathrm{d}s \\ + \int_0^t \int_{\mathbb{R}} Q\Big(s-, u(s-, \rho_{s-}^{\mathbf{u}})\Big)(\rho_{s-}^{\mathbf{u}}) \mathbf{1}_{0<x<\mathrm{Tr}[\mathcal{J}(s-,u(s-,\rho_{s-}^{\mathbf{u}}))(\rho_{s-}^{\mathbf{u}})]} N(\mathrm{d}s, \mathrm{d}x).\end{aligned}$$

The existence and the uniqueness of a solution for such an equation relies again on the Lipschitz property of functions R an $\mathcal{J}$. As the diffusive case, only local Lipschitz property are satisfied. We use again a truncature method to transform (38) into an equation with Lipschitz property. The following remark concerns the technical assumptions ensuring that (38) admits a unique solution.

Remark. First, we suppose that R and $\mathcal{J}$ satisfy the local Lipschitz condition (23) defined in Sect. 2.1. Second, as the set of states is compact, we can suppose for the stochastic intensity that for all $T > 0$ there exists a constant $K(T)$ such that

$$\mathrm{Tr}\big[\mathcal{J}(t, u(t, X_t))(X_t)\big] \leq K(T),$$

for all $t \geq T$ and for all càdlàg process (X_t) with values in $\mathbb{M}_2(\mathbb{C})$. This previous condition implies the fact the stochastic intensity is bounded. Finally, in order to consider the stochastic differential equation for all càdlàg process, we consider the function

$$\tilde{Q}(t,a)(\rho) = \left(\frac{\mathcal{J}(t,a)(\rho)}{Re\Big(\mathrm{Tr}\big[\mathcal{J}(t,a)(\rho)\big]\Big)} - \rho \right) \mathbf{1}_{\mathrm{Re}(\mathrm{Tr}[\mathcal{J}(t,a)(\rho)])>0}, \tag{40}$$

and the stochastic differential equation

$$\begin{aligned} \rho_t^{\mathbf{u},k} = \rho_0 &+ \int_0^t R\Big(s-, u(s-, \tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})\Big)(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}} \tilde{Q}\Big(s-, u(s-, \tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})\Big)(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})) \\ &\times \mathbf{1}_{0<x<\mathrm{Re}(\mathrm{Tr}[\mathcal{J}(s-,u(s-,\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))]} N(\mathrm{d}s, \mathrm{d}x). \end{aligned} \tag{41}$$

where $\tilde{\phi}^k$ is a truncature function defined in Sect. 3.1. Let us notice that if ρ is a state

$$\mathrm{Re}\left(\mathrm{Tr}\big[\mathcal{J}(t,a)(\rho)\big]\right) = \mathrm{Tr}\big[\mathcal{J}(t,a)(\rho)\big] \geq 0,$$

for all $t \geq 0$ and for all $a \in \mathbb{R}$. As a consequence, if a solution of (41) takes values in the set of states, this is a solution of (38).

As in the diffusive case, we first show that the modified equation (41) admits a unique solution. Next, we prove that a discrete quantum trajectory converges in distribution to the solution of (41). The convergence result proves then that the solution of (41) is valued in the set of states. Concerning the existence and uniqueness of a solution of (41), we have the following theorem due to Jacod and Protter in [21]:

Theorem 5. *Let $(\Omega, \mathcal{F}, P)$ be a probability space of a Poisson point process N. The stochastic differential equation*

$$\begin{aligned}\rho_t^{\mathbf{u},k} = \rho_0 &+ \int_0^t R\Big(s-, u(s-, \tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})\Big)(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}} \tilde{Q}\Big(s-, u(s-, \tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})\Big)(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})) \\ &\times \mathbf{1}_{0<x<\mathrm{Re}(\mathrm{Tr}[\mathcal{J}(s-,u(s-,\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))]} N(\mathrm{d}s, \mathrm{d}x) \end{aligned} \tag{42}$$

admits a unique solution $(\rho_t^{\mathbf{u},k})$ defined for all $t \geq 0$. Furthermore, the counting process $(\overline{N}_t)$ defined by

$$\overline{N}_t = \int_0^t \int_{\mathbb{R}} \mathbf{1}_{0<x<\mathrm{Re}(\mathrm{Tr}[\mathcal{J}(s-,u(s-,\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))]} N(\mathrm{d}s, \mathrm{d}x)$$

allows to define the filtration $(\overline{\mathcal{F}}_t)$, where $\overline{\mathcal{F}}_t = \sigma\{\overline{N}_s, s \leq t\}$. Hence, the process

$$\overline{N}_t - \int_0^t \Big[\mathrm{Re}\Big(\mathrm{Tr}\Big[\mathcal{J}\Big(s-, u(s-, \tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k})\Big)(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))\Big]\Big)\Big]^+ \mathrm{d}s \tag{43}$$

is a $(\overline{\mathcal{F}}_t)$ martingale.

In this theorem, the term $(x)^+$ denotes $\max(0, x)$. The martingale property (43) expresses that the process $(\int_0^t [\mathrm{Re}(\mathrm{Tr}[\mathcal{J}(s-, u(s-, \tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))])]^+ \mathrm{d}s)$ is the stochastic intensity of the counting process $(\overline{N}_t)$. We do not prove this theorem in detail. A theorem similar to Theorem 5 is treated completely in [31] for quantum trajectories without control. Here, we express the solution of (41) in a particular case. Suppose that there exists a constant K such that

$$\Big[\mathrm{Re}\Big(Tr\big[\mathcal{J}(t, u(t, X_t))(X_t)\big]\Big)\Big]^+ < K, \tag{44}$$

for all $t \geq 0$ and all càdlàg processes (X_t). With this property we can consider only the points of N contained in $\mathbb{R} \times [0, K]$. The random function

$$\mathcal{N}_t : t \to N(., [0, t] \times [0, K])$$

defines then a standard Poisson process with intensity K. Let $T > 0$, the Poisson Random Measure and the previous process generate on $[0, T]$ a sequence $\{(\tau_i, \xi_i), i \in \{1, \ldots, \mathcal{N}_t)\}\}$. Each τ_i represents the jump time of the process $(\mathcal{N}_t)$. Moreover, the random variables ξ_i are random uniform variables on $[0, K]$. Let $k > 1$ be a fixed integer, we can write the solution of (41) in the following way:

$$\begin{aligned}\rho_t^{\mathbf{u},k} &= \rho_0 + \int_0^t R\Big(s-, u(s-, \tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))\Big)(\tilde{\phi}^k(\rho_{s-}^{\mathbf{u},k}))\mathrm{d}s \\ &\quad + \sum_{i=1}^{\mathcal{N}_t} Q\Big(\tau_i-, u(\tau_i-, \tilde{\phi}^k(\rho_{\tau_i-}^{\mathbf{u},k})\Big)(\tilde{\phi}^k(\rho_{\tau_i-}^{\mathbf{u},k})) \\ &\quad \times \mathbf{1}_{0\leq\xi_i\leq(\mathrm{Re}(\mathrm{Tr}[\mathcal{J}(\tau_i-,u(\tau_i-,\tilde{\phi}^k(\rho_{\tau_i-}^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_{\tau_i-}^{\mathbf{u},k})]))^+} \\ \overline{N}_t &= \sum_{i=1}^{\mathcal{N}_t} \mathbf{1}_{0\leq\xi_i\leq(\mathrm{Re}(\mathrm{Tr}[\mathcal{J}(\tau_i-,u(\tau_i-,\tilde{\phi}^k(\rho_{\tau_i-}^{\mathbf{u},k})))(\tilde{\phi}^k(\rho_{\tau_i-}^{\mathbf{u},k})]))^+}\,.\end{aligned} \tag{45}$$

The expression (45) means that the solution of (41) is given by the solution of the ordinary differential equation

$$\mathrm{d}\rho_t^{\mathbf{u},k} = R\Big(t, u(t, \tilde{\phi}^k(\rho_t^{\mathbf{u},k}))\Big)(\tilde{\phi}^k(\rho_t^{\mathbf{u},k}))\,\mathrm{d}t$$

between the jump of the process $\overline{N}_t$. At the jump time of $\overline{N}_t$, a new initial condition is implemented by the value of the jump defined by the function Q. The process $\overline{N}_t$ corresponds to the number of points of the Poisson point process N included in the x axis and the curve

$$t \mapsto \Big[\mathrm{Re}\Big(\mathrm{Tr}\Big[\mathcal{J}\Big(t, u(t, \tilde{\phi}^k(\rho_t^{\mathbf{u},k}))\Big)(\tilde{\phi}^k(\rho_t^{\mathbf{u},k}))\Big]\Big)\Big]^+.$$

The general case is more technical, but can be expressed in the same way (see [21]).

Now, we investigate the convergence result. The starting point is to describe a discrete stochastic differential equation. To this end, from expression (37), we define

$$\begin{aligned}\rho_n^{\mathbf{u}}(t) &= \rho_{[nt]}^{\mathbf{u}}, \\ N_n(t) &= \sum_{k=1}^{[nt]} \mathbf{1}_1^k, \\ V_n(t) &= \frac{[nt]}{n}, \\ R_n(t,a)(\rho) &= R([nt]/n, a)(\rho), \\ Q_n(t,a)(\rho) &= Q([nt]/n, a)(\rho), \\ u_n(t,W) &= u([nt]/n, W),\end{aligned}$$

for all $t \geq 0$, for all $a \in \mathbb{R}$ and all $W \in \mathbb{M}_2(\mathbb{C})$. Hence, the process $(\rho_n^{\mathbf{u}}(t))$ satisfies the stochastic differential equation

$$\rho_n^{\mathbf{u}}(t) = \int_0^t \Big[R_n\Big(s-, u_n(s-, \rho_n^{\mathbf{u}}(s-)\Big)(\rho_n^{\mathbf{u}}(s-)) + \circ(1)\Big]\, \mathrm{d}V_n(s)$$
$$+ \int_0^t \Big[Q_n\Big(s-, u_n(s-, \rho_n^{\mathbf{u}}(s-))(\rho_n^{\mathbf{u}}(s-)\Big) + \circ(1)\Big]\, \mathrm{d}N_n(s).$$

Remark. In order to apply the Theorem of Kurtz and Protter, we should prove that $(N_n(t))$ converges to $(\tilde{N}_t)$. Actually, this result cannot be proved independently to the convergence of $(\rho_n^{\mathbf{u}}(t))$ to $(\rho_t^{\mathbf{u}})$. Indeed, the counting process $(\tilde{N}_t)$ is completely defined by its stochastic intensity. As the intensity depends on $(\rho_t^{\mathbf{u}})$, a result of convergence for $(N_n(t))$ and $(\tilde{N}_t)$ should involve one for $(\rho_n^{\mathbf{u}}(t))$ and $(\rho_t^{\mathbf{u}})$. Actually, we cannot apply a similar result of the form of Theorem 3.

Here, the convergence is obtained by using a random coupling method, that is, we realize the process $(\rho_{[nt]})$ in the probability space of the Poisson point process N. This method allows then to compare directly the continuous and discrete quantum trajectories in the same probability space. It is described as follows:

Remember that the random variables $(\mathbf{1}_1^k)$ satisfy

$$\begin{cases} \mathbf{1}_1^{k+1}(0) = 0 & \text{with probability} \\ & p_{k+1}(n) = 1 - \frac{1}{n}\mathrm{Tr}[\mathcal{J}(k/n, u(k/n, \rho_k^{\mathbf{u}}))(\rho_k^{\mathbf{u}})] + \circ\left(\frac{1}{n}\right) \\ \mathbf{1}_1^{k+1}(1) = 1 & \text{with probability} \\ & q_{k+1}(n) = \frac{1}{n}\mathrm{Tr}[\mathcal{J}(k/n, u(k/n, \rho_k^{\mathbf{u}}))(\rho_k^{\mathbf{u}})] + \circ\left(\frac{1}{n}\right), \end{cases}$$

for all $k > 0$. We define the sequence $(\tilde{\nu}_k)$ of random variables which are defined on the set of states by

$$\tilde{\nu}_{k+1}(\eta, \omega) = \mathbf{1}_{N(\omega, G_k(\eta))>0}, \tag{46}$$

for all $k > 0$, where $G_k(\eta) = \Big\{(t, u)/\frac{k}{n} \leq t < \frac{k+1}{n}, 0 \leq u \leq -n\ln\Big(\mathrm{Tr}\big[\mathcal{L}_0^{\mathbf{u},k+1}(n)(\eta)\big]\Big)\Big\}$. Let $\rho_0 = \rho$ be any state and $T > 0$, we define the process $(\tilde{\rho}_k)$ for $k < [nT]$ by the recursive formula

$$\tilde{\rho}_{k+1}^{\mathbf{u}} = \mathcal{L}_0^{\mathbf{u},k+1}(\tilde{\rho}_k^{\mathbf{u}}) + \mathcal{L}_1^{\mathbf{u},k+1}(\tilde{\rho}_k^{\mathbf{u}}) + \left[-\frac{\mathcal{L}_0^{\mathbf{u},k+1}(\tilde{\rho}_k^{\mathbf{u}})}{\mathrm{Tr}\big[\mathcal{L}_0^{\mathbf{u},k+1}(\tilde{\rho}_k^{\mathbf{u}})\big]} + \frac{\mathcal{L}_1^{\mathbf{u},k+1}(\tilde{\rho}_k^{\mathbf{u}})}{\mathrm{Tr}\big[\mathcal{L}_1^{\mathbf{u},k+1}(\tilde{\rho}_k^{\mathbf{u}})\big]}\right]$$
$$\times \Big(\tilde{\nu}_{k+1}(\tilde{\rho}_k^{\mathbf{u}}, .) - \mathrm{Tr}\big[\mathcal{L}_1^{\mathbf{u},k+1}(\tilde{\rho}_k^{\mathbf{u}})\big]\Big). \tag{47}$$

Thanks to the properties of the Poisson probability measure, the random variables $(\mathbf{1}_1^k)$ and $(\tilde{\nu}_k)$ have the same distribution. This involves the following property:

Proposition 4. *Let T be fixed. The discrete process $(\tilde{\rho}_k^{\mathbf{u}})_{k\leq[nT]}$ defined by (47) has the same distribution of the discrete quantum trajectory $(\rho_k^{\mathbf{u}})_{k\leq[nT]}$ defined by the quantum repeated measurement.*

The convergence result is then expressed as follows:

Theorem 6. *Let $T > 0$. Let $(\Omega, \mathcal{F}, P)$ be a probability space of a Poisson point process N. Let $(\tilde{\rho}^{\mathbf{u}}_{[nt]})_{0\leq t\leq T}$ be the discrete quantum trajectory defined by the recursive formula (47).*

Hence, for all $T > 0$ the process $(\tilde{\rho}^{\mathbf{u}}_{[nt]})_{0\leq t\leq T}$ converges in distribution in $\mathcal{D}[0, T]$ (for the Skorohod topology) to the process $(\rho^{\mathbf{u}}_t)$ solution of the stochastic differential equation

$$\rho^{\mathbf{u}}_t = \rho_0 + \int_0^t R\Big(s-, u(s-, \rho^{\mathbf{u}}_{s-})\Big)(\rho^{\mathbf{u}}_{s-})\mathrm{d}s$$
$$+ \int_0^t \int_{\mathbb{R}} Q\Big(s-, u(s-, \rho^{\mathbf{u}}_{s-})\Big)(\rho^{\mathbf{u}}_{s-})\mathbf{1}_{0<x<\mathrm{Tr}[\mathcal{J}(s-,u(s-,\rho^{\mathbf{u}}_{s-}))(\rho^{\mathbf{u}}_{s-})]}N(\mathrm{d}s, \mathrm{d}x).$$

This theorem relies on the fact that the process $(\tilde{\rho}^{\mathbf{u}}_{[nt]})$ satisfies the same asymptotic of the discrete quantum trajectory $(\rho^{\mathbf{u}}_{[nt]})$ [see (16)]. More details for such techniques can be found in [31] where the case without control is entirely developed (in particular, this result needs the convergence of the Euler scheme of the jump equation).

4. Example

This section is devoted to develop an application of quantum measurement with control.

Here, we describe a discrete model of an atom monitored by a laser. A measurement is performed by a photon counter which detects the photon emission. The setup of repeated quantum interactions is described as follows:

The time of interaction is chosen to be $h = 1/n$. In this setup, we need three basis spaces. The atom system is represented by $\mathcal{H}_0$ equipped with a state ρ. The laser is represented by $(\mathcal{H}^l, \mu^l)$ and the photon counter by $(\mathcal{H}^c, \beta^c)$. Each Hilbert space is $\mathbb{C}^2$ endowed with the orthonormal basis (Ω, X) and the unitary operator is denoted by U. The compound system after the interaction is $\mathcal{H}_0 \otimes \mathcal{H}^l \otimes \mathcal{H}^c$, and the state after the interaction is

$$\alpha = U(\rho \otimes \mu^l \otimes \beta^c)U^\star.$$

The appropriate orthonormal basis of $\mathcal{H}_0 \otimes \mathcal{H}^l \otimes \mathcal{H}^c$, in this case, is $\Omega\otimes\Omega\otimes\Omega$, $X \otimes \Omega\otimes\Omega$, $\Omega\otimes X \otimes\Omega$, $X \otimes X \otimes\Omega, \Omega\otimes\Omega\otimes X$, $X \otimes\Omega\otimes X$, $\Omega\otimes X \otimes X$, $X \otimes X \otimes X$. The unitary operator is here considered as a 4×4 matrix $U = (L_{i,j}(n))_{0\leq i,j\leq 3}$, where each $L_{ij}(n)$ are operators on $\mathcal{H}_0$. Now, if the different states of the laser and the counter are of the form $\mu^l = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\beta^c = |\Omega\rangle\langle\Omega|$, the state $\alpha = (\alpha_{uv})_{0\leq u,v\leq 3}$ is given by

$$\alpha_{uv} = \Big(aL_{u0}(n)\rho + bL_{u1}(n)\rho\Big)L^{\star}_{v0}(n) + \Big(cL_{u0}(n)\rho + \mathrm{d}L_{u1}(n)\rho\Big)L^{\star}_{v1}(n). \quad (48)$$

The measurement is performed on the counter photon side. Let A denote any observable of $\mathcal{H}^c$ then $I \otimes I \otimes A$ denotes the corresponding observable on $\mathcal{H}_0 \otimes \mathcal{H}^l \otimes \mathcal{H}^c$. We perform a measurement and by partial trace operation with respect to $\mathcal{H}^l \otimes \mathcal{H}^c$, we obtain a new state on $\mathcal{H}_0$.

The control is rendered by the modification at each interaction of the intensity of the laser. This modification is taken into account by the reference state of the laser. The reference state at the kth interaction is denoted by μ^l_k. In the continuous case, the state of a laser is usually described by a coherent vector on a Fock space (see [11]). From works of Attal and Pautrat in approximation of Fock space ([1,30]), the state of the laser can be described by

$$\mu^l_k = \begin{pmatrix} a(k/n) & b(k/n) \\ c(k/n) & d(k/n) \end{pmatrix} = \frac{1}{1+|h(k/n)|^2}\begin{pmatrix} 1 & h(k/n) \\ \overline{h}(k/n) & |h(k/n)|^2 \end{pmatrix}. \quad (49)$$

The function h represents the evolution of the intensity of the laser and depends naturally on n.

Let ρ_k denote the state on $\mathcal{H}_0$ after k measurements. Hence, the state $\alpha^{k+1}(n) = \big(\alpha^{k+1}_{uv}(n)\big)_{0\leq u,v\leq 3} = U_{k+1}(n)(\rho_k \otimes \mu^l_k \otimes \beta^c)U^{\star}_{k+1}(n)$, after interaction, satisfies

$$\alpha^{k+1}_{uv}(n) = \Big(a(k/n)L_{u0}(n)\rho_k + b(k/n)L_{u1}(n)\rho_k\Big)L^{\star}_{v0}(n) + \Big(c(k/n)L_{u0}(n)\rho_k + d(k/n)L_{u1}(n)\rho_k\Big)L^{\star}_{v1}(n).$$

Remark. Let us stress that is not directly the framework of Sect. 1. Here, the control is namely not rendered by the modification of the unitary evolution. Moreover, the interacting system is described by $(\mathcal{H}^l \otimes \mathcal{H}^c, \mu^l_k \otimes \beta)$ and $\mu^l_k \otimes \beta$ is not of the form $|e_0\rangle\langle e_0|$ as in Sect. 1. In order to translate this setting in the case of discrete models of Sect. 1, one can use the G.N.S Representation theory of a finite dimensional Hilbert space ([23,24]). This theory allows to consider the state $\mu^l_k \otimes \beta$ as a state of the form $|e_0\rangle\langle e_0|$ in an enlarged Hilbert space. The G.N.S representation transforms then the expression of the operator U_k, and the control expressed in $\mu^l_k \otimes \beta$ is again expressed in the new expression of U_k (see [2] for more details). In our case, we do not use such a theory because it is more explicit to make directly computations to describe the discrete equation in asymptotic form.

Let us present the result. The principle of measurement is the same as in Sect. 1. The counting case is also given by a diagonal observable of $\mathcal{H}_c$. We shall focus on this case which renders the emission of photons [11]. The asymptotic properties for the unitary operator follows the asymptotic rules of Attal-Pautrat in [4]. Let $\delta_{ij} = 1$, if $i = j$, we denote $\epsilon_{ij} = 1/2(\delta_{0i} + \delta_{0j})$. The coefficients must follow the convergence condition

$$\lim_{n\to\infty} n^{\epsilon_{ij}}(L_{ij}(n) - \delta_{ij}I) = L_{ij}$$

where L_{ij} are operators on $\mathcal{H}_0$.

Let $P_0 = |\Omega\rangle\langle\Omega|$ and $P_1 = |X\rangle\langle X|$ be the eigenprojectors of a diagonal observable A. If ρ_k denotes the random state after k measurements we denote

$$\begin{aligned}\mathcal{L}_0^{k+1}(\rho_k) &= \mathbf{E}_0\Big[I\otimes I\otimes P_0\left(U_{k+1}(n)(\rho_k\otimes\mu_k^l\otimes\beta)U_{k+1}^\star(n)\right)I\otimes I\otimes P_0\Big]\\ &= \alpha_{00}^{k+1}(n)+\alpha_{11}^{k+1}(n),\\ \mathcal{L}_1^{k+1}(\rho_k) &= \mathbf{E}_0\Big[I\otimes I\otimes P_1\left(U_{k+1}(n)(\rho_k\otimes\mu_k^l\otimes\beta)U_{k+1}^\star(n)\right)I\otimes I\otimes P_1\Big]\\ &= \alpha_{22}^{k+1}(n)+\alpha_{33}^{k+1}(n).\end{aligned} \tag{50}$$

These are the two non-normalized states, the operator $\mathcal{L}_0^{k+1}(\rho_k)$ appears with probability $p_{k+1} = \mathrm{Tr}[\mathcal{L}_0^{k+1}(\rho_k)]$ and $\mathcal{L}_1^{k+1}(\rho_k)$ with probability $q_{k+1} = \mathrm{Tr}[\mathcal{L}_1^{k+1}(\rho_k)]$.

Following the approximations and the asymptotic description of Fock space developed by Attal-Pautrat, we put

$$h\left(\frac{k}{n}\right) = \frac{1}{\sqrt{n}}f\left(\frac{k}{n}\right) + \circ\left(\frac{1}{n}\right),$$

where f is a function from $\mathbb{R}$ to $\mathbb{C}$. As in Sect. 2, we assume that the intensity of the laser f is continuous.

With the same arguments of Sect. 1, the evolution of the discrete quantum trajectory is described by

$$\rho_k = \frac{\mathcal{L}_0^{k+1}(\rho_k)}{p_{k+1}} + \left[-\frac{\mathcal{L}_0^{k+1}(\rho_k)}{p_{k+1}} + \frac{\mathcal{L}_1^{k+1}(\rho_k)}{q_{k+1}}\right]\mathbf{1}_1^{k+1}. \tag{51}$$

For further use, convergence results will be established in the case $L_{01} = -L_{10}^\star$, and $L_{11} = L_{21} = L_{31} = L_{30} = 0$. Conditions about asymptotic of U and the fact that it is a unitary-operator imply that

$$L_{00} = -\left(iH + \frac{1}{2}\sum_{i=1}^{2} L_{i0}^\star L_{i0}\right). \tag{52}$$

As in Sect. 2.2, the convergence in this situation is expressed as follows:

Proposition 5. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space of a Poisson point process N on $\mathbb{R}^2$. The discrete quantum trajectory $(\rho_{[nt]})_{0\le t\le T}$ defined by the equation (51) weakly converges in $\mathcal{D}([0,T])$ for all T to the solution of the following stochastic differential equation*

$$\begin{aligned}\rho_t = \rho_0 &+ \int_0^t \Bigg(-i\big[H,\rho_{s-}\big] - \frac{1}{2}\left\{\sum_{i=1}^{2} L_{i0}^\star L_{i0}, \rho_{s-}\right\} + L_{10}\,\rho_{s-}\,L_{10}^\star\\ &+ \Big[\Big(\overline{f}(s-)L_{10}\rho_{s-} - f(s-)L_{10}^\star\Big), \rho_{s-}\Big] + \mathrm{Tr}\big[\,L_{20}\,\rho_{s-}\,L_{20}^\star\,\big]\rho_{s-}\Bigg)\mathrm{d}s\\ &+ \int_0^t\int_{\mathbb{R}} \left(-\rho_{s-} + \frac{L_{20}\,\rho_{s-}\,L_{20}^\star}{\mathrm{Tr}\big[\,L_{20}\,\rho_{s-}\,L_{20}^\star\,\big]}\right)\mathbf{1}_{0<x<\mathrm{Tr}[L_{20}\,\rho_{s-}\,L_{20}^\star]}N(\mathrm{d}x,\mathrm{d}s).\end{aligned} \tag{53}$$

Proof. For example, we have the following asymptotic result for $\mathcal{L}_0^{k+1}(\rho_k)$

$$\begin{aligned}\mathcal{L}_0(\rho_k) = \rho_k + \frac{1}{n}\Big(&L_{00}\rho + \rho L_{00}^\star + L_{10}\rho L_{10}^\star \\ &+ f\left(\frac{k}{n}\right)\left[L_{01}\rho + \rho L_{10}^\star\right] + \overline{f}\left(\frac{k}{n}\right)\left[L_{10}\rho + \rho L_{01}^\star\right]\Big) + \circ\left(\frac{1}{n}\right).\end{aligned}$$

The above expression, the conditions about the operators L_{ij} and the Theorem 6 prove the proposition. □

In this model, the control is deterministic. Before we give an application of stochastic control, let us briefly expose a use of the model of a laser monitoring an atom.

Consider the special case, where the Hamiltonian $H = 0$. Let us put

$$C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_{10} = k_l C, \quad L_{20} = k_c C,$$

with $|k_l|^2 + |k_c|^2 = 1$. The constant k_f and k_c are called decay rates [11].

Without control, the stochastic model of a two-level atom in presence of a photon counter [31] is given by

$$\begin{aligned}\mu_t = \mu_0 + \int_0^t &\left(-\frac{1}{2}\{C^\star C, \mu_{s-}\} + \mathrm{Tr}\left[C\mu_{s-}C^\star\right]\mu_{s-}\right)\mathrm{d}s \\ &+ \int_0^t\int_{\mathbb{R}} \left(-\mu_{s-} + \frac{C\mu_{s-}C^\star}{\mathrm{Tr}\left[C\mu_{s-}C^\star\right]}\right)\mathbf{1}_{0<x<\mathrm{Tr}[C\mu_{s-}C^\star]}N(\mathrm{d}x, \mathrm{d}s). \qquad (54)\end{aligned}$$

Let denote $\tilde{N}_t = \int_0^t\int_{\mathbb{R}} \mathbf{1}_{0<x<\mathrm{Tr}[C\mu_{s-}C^\star]}N(\mathrm{d}x, \mathrm{d}s)$ and $T = \inf\{t > 0;\ \tilde{N}_t > 0\}$. In [5] it is proved that

$$\mu_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |\Omega\rangle\langle\Omega|, \qquad (55)$$

for all $t > T$. Physically, it means that at most one photon appears on the photon counter. Mathematically, if we write (54) in the form

$$\mu_t = \int_0^t \Psi(\mu_{s-})\mathrm{d}s + \int_0^t \Phi(\mu_{s-})\mathrm{d}\tilde{N}_s,$$

then for $\mu = |\Omega\rangle\langle\Omega|$, we have $\Phi(\mu) = \Psi(\mu) = 0$. The state $|\Omega\rangle\langle\Omega|$ is then an invariant state of the dynamic.

In the presence of a laser, the control f gives rise to the term $[\overline{f}L_{10} - fL_{10}^\star, .] = [k_l\overline{f}C - \overline{k_l}fC^\star, .]$. Hence, if $\mu = |\Omega\rangle\langle\Omega|$, we still have $\Phi(\mu) = 0$, but we do not have anymore $\Psi(\mu) = 0$ and the property (55) is not satisfied. The state $|\Omega\rangle\langle\Omega|$ is no more an invariant state. As a consequence it is possible to observe more than one photon in the photon counter.

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Clément Pellegrini
School of Physics
National Institute for Theoretical Physics
University of KwaZulu-Natal
Private Bag X54001
Durban 4000
South Africa
e-mail: pelleg@math.univ-lyon1.fr

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